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RESEARCH ARTICLE

Robust upper bounds for American put options

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Abstract

In this paper, we develop robust and model-free upper bounds for American put option prices. Our bounds have all of those appealing features of the upper bounds for European options provided in DeMarzo et al. (2016, Robust option pricing: Hannan and Blackwell meet Black and Scholes, Journal of Economic Theory, 410-434) but cover more popular derivatives in practice. Numerical and empirical investigations illustrate the performance of our method.

KEYWORDS

American option pricing, gradient strategies, regret minimization, robust upper bounds

JEL CLASSIFICATION

C73, D81, G13

1 | INTRODUCTION

Option pricing has been an important topic in finance for decades. Various celebrated model-based approaches, such as the Black–Scholes–Merton model (Black & Scholes, 1973), the Heston stochastic volatility model (Heston, 1993), the jump-diffusion model (Kou, 2002), among others, have been proposed to price financial options traded on either exchanges or over-the-counter markets (see Broadie and Detemple (2004) for an excellent survey). Those models have proved successful in generating tractable or numerically efficient pricing formulas which partially fit certain stylized empirical facts observed in financial markets. Contrarily, the model risk, as an inevitable issue for any model-based method, has attracted a great deal of attention among researchers and practitioners, especially after the financial crisis in 2008. This concern substantially motivates a growing literature on model-independent bounds for option prices, which can be deliberately used for evaluating the accuracy of different models and facilitating general risk management practice. Research progress on bounds for prices of European-style options is comprehensively reviewed in Kahalé (2017), who also develops new bounds delivered numerically through convex programming based on optimal super-/subreplication strategies. Another closely related work is Kahalé (2016), in which lower bounds on discretely monitored variance swaps in terms of a continuum of European call option prices with the same maturity are obtained. These results, however, cannot cover the early-exercise feature of American options.

An American option[1] allows its holders to exercise their rights before the maturity and thus offer more flexibility than a European option. Most of the equity options, as well as the very popular OEX® S&P 100 index options[2], traded on the Chicago Board Options Exchange, are early exercisable and hence are American options. Chen and Yeh (2002) first develop analytical upper bounds for an American option based on its payoff at maturity with some conditions on prespecified risk neutral pricing measures. Chung and Chang (2007) generalize and tighten Chen and Yeh (2002)'s bounds. Chaudhury (2006) obtains more general upper bounds with similar constraints on risk-neutral pricing measures. Recently, Hobson and Neuberger (2017)[3] establish upper bounds on an American option via prices of a family of European call options written on the same underlying asset.

In this paper, we derive robust upper bounds for American put option price in terms of a quadratic functional of the underlying asset's log returns, by extending the results on European option price bounds in DeMarzo, Kremer, and Yishay (2016). Their idea is based on the Blackwell approachability (Blackwell, 1956) and no-regret learning theory (Hannan, 1957). They construct a financial trading model where an investor manages a portfolio containing a stock and a bond. A no-regret gradient trading strategy is designed such that the value of the portfolio at the expiration date of the option is no worse than a certain fraction of the value from buying and holding either the stock or the bond. A model-free upper bound for the price of a European option is then obtained.

We utilize two approaches to derive upper bounds for the price of an American put option. The first approach constructs a dynamic portfolio to directly obtain the upper bound with a functional of the asset's returns. The second approach bounds the price of an American option from above by building up a bridge between the price of a European option and that of an American option. In line with DeMarzo et al. (2016), our bounds are new to the literature on early-exercisable contingent claim pricing problems, and are robust in the sense that we do not assume the continuity of stock price paths, the completeness of markets, or any specification on the pricing kernel.

The rest of this paper is organized as follows: Section 2 introduces a discrete-time trading framework and revisits DeMarzo et al. (2016)'s gradient trading strategy. Section 3 collects our main results, and the performance is illustrated in Section 4 via numerical and empirical studies. Section 5 concludes the paper.

2 | THE GRADIENT TRADING STRATEGY

We follow the model in DeMarzo et al. (2016) and slightly generalize it to incorporate nonzero risk-free interest rate and dividend yield. Consider a discrete-time financial trading model with N periods from time 0 to time T, where a period is indexed by $n \in \{1, ..., N\}$ with equal length $\Delta t = T/N$. There are only two assets in the market: A stock and a bond. The stock, with an initial price S_0 at time 0, has a known continuously compounded dividend yield d. Denote by $r_{n,s}$ the stock return in the nth period such that $r_{n,s} = (S_n/S_{n-1})e^{\frac{dT}{N}} - 1$, and its corresponding log return is $\pi_{n,s} = \ln(1 + r_{n,s})$. Let $R = \{r_{1,s}, r_{2,s}, ..., r_{N,s}\}$ be a stock return path, and the set of all possible paths is represented by Φ_T. Furthermore, there exists a risk-free bond expiring at the end of the Nth period. For simplicity, we assume the initial price of the bond is one. Denote by r_b the bond return in each period such that the bond price B_n is equal to $B_{n-1}(1 + r_b)$, while its corresponding log return is $\pi_{n,b} = \ln(1 + r_b)$. Let r be the continuously compounded risk-free interest rate, thus $B_N = (1 + r_b)^N = e^{rT}$.

An investor in this model manages a dynamic portfolio consisting of the stock and the bond. This portfolio has an initial value $G_0 = 1$ and its value at the end of the nth period is denoted by $G_n(R)$ along through the sample path R, which is further written as G_n for short if no confusion occurs. At the beginning of the nth period, she invests $w_n \in [0, 1]$ portion of G_{n-1} in the stock and the other $1 - w_n$ portion in the bond. DeMarzo et al. (2016) generalize the regret-minimizing gradient trading strategy developed in Hart and Mas-Colell (2000) to obtain the portfolio allocation. Assuming that both d and r are zero, they define a regret vector,

$$L_n = \begin{bmatrix} \ln \frac{S_n}{S_0} - \ln G_n \\ \ln B_n - \ln G_n \end{bmatrix} + x = \begin{bmatrix} \ln \frac{S_n}{S_0} - \ln G_n + x_1 \\ \ln B_n - \ln G_n + x_2 \end{bmatrix},$$

[1] In this paper, we do not differentiate the terms "American option" and "Bermudan option," since our primary focus is a discrete-time trading model where the time intervals between two consecutive exercise opportunities can be arbitrarily small. Formally, an American option can be exercised at any time before its maturity, while a Bermudan option can be exercised in a finite set of predetermined time points before its maturity.

[2] According to the webpage of Chicago Board Options Exchange via http://www.cboe.com/products/stock-index-options-spx-rut-msci-ftse/s-p-100-index-options-oex-xeo (last accessed on August 21, 2017), "More than one billion OEX options have been traded, making the OEX one of the most popular equity portfolio management tools in history."

[3] We thank an anonymous referee who mentioned this reference and Kahalé (2016) to us.

where $x = (x_1, x_2)$ is a nonnegative parameter representing an initial fictitious loss, and is used to polish the performance of the strategy. The first (resp., second) element of L_n essentially measures the loss of wealth resulting from choosing the dynamic portfolio compared to a buy-and-hold strategy in the stock (resp., bond) itself, shifted by x_1 (resp., x_2). The gradient trading strategy is then constructed based on the regret vector. At the end of the $(n-1)$th period, if there is no regret, namely $\ln G_{n-1} \geq \max\left(\ln(S_{n-1}/S_0) + x_1, \ln B_{n-1} + x_2\right)$, w_n is chosen arbitrarily. Otherwise, w_n is specified as

$$w_n = \frac{L_{n-1,1}^+}{L_{n-1,1}^+ + L_{n-1,2}^+} \in [0, 1],$$

where $L_{n-1,i}^+ = \max(L_{n-1,i}, 0)$. They show that the portfolio value at maturity, that is, G_N, is no less than a certain fraction of the maximum of the stock and the bond, which further implies an upper bound for the price of a European option.

3 | UPPER BOUNDS FOR AMERICAN OPTIONS

We mainly focus on American options that can be exercised only once at the end of each time interval in the discrete-time trading model above. In particular, we exploit a similar gradient strategy to derive robust upper bounds for American put options, while it is straightforward to apply our methodology in the case of American call options. All prices involved in the sequel are derived under the common assumption that there are no risk-free arbitrage opportunities in the financial market. We first present an upper bound on the price of an American put option by constructing a dynamically adjusted portfolio with the gradient trading strategy directly.

Lemma 1 *The price of a European put option with stock price S_0, strike price K and expiration date T satisfies*

$$P^E(S_0, K, T|\Phi_T) \leq K e^{\frac{1}{2}\left[\ln\frac{S_0}{K} - (r+d)T\right] + \frac{1}{2}\sqrt{2q^2(\Phi_T) + \left[\ln\frac{S_0}{K} + (r-d)T\right]^2}} - S_0 e^{-dT},$$

where $q^2(\Phi_T)$ is the maximal quadratic variation of the excess log returns among all of the possible stock return paths in Φ_T:

$$q^2(\Phi_T) = \sup_{R\in\Phi_T} q^2(R) = \sup_{R\in\Phi_T} \sum_{n=1}^{N} \left[\ln(1 + r_{n,s}) - \ln(1 + r_b)\right]^2.$$

Lemma 1 slightly extends DeMarzo et al. (2016)'s derivation for their Proposition 6 to allow for nonzero risk free rate and dividend yield. Its detailed proof is provided in the Appendix.

Theorem 2 *The price of an American put option with initial stock price S_0, strike price K and expiration date T satisfies*

$$P^A(S_0, K, T|\Phi_T) \leq K e^{\frac{1}{2}\left(\ln\frac{S_0}{k} - dT\right) + \frac{1}{2}\sqrt{2q^2(\Phi_T) + \left(\ln\frac{S_0}{k} - dT\right)^2}} - S_0 e^{-dT},$$

where $q^2(\Phi_T)$ is defined in Lemma 1.

Proof The idea is to construct a dynamically adjusted portfolio such that its value is at least K for any $t = n\Delta t$ before the time T and at least $\max\{S_N, K\}$ at T. Consider the following two portfolios:

Portfolio I: Long an American put option with strike price K and expiration date T and simultaneously long e^{-dT} shares of the stock.
Portfolio II: invest $I (I \geq 0)$ into a dynamically adjusted portfolio following the gradient strategy in the proof of Lemma 1.

If the American option has not been exercised early, Portfolio I will have a payoff of $\max\{S_N, K\}$ at time T. If it is exercised early at some time $t_0 = n_0\Delta t < T$, Portfolio I will have a payoff of $K - \left[1 - e^{-d(T-t_0)}\right] S_{n_0}$ at t_0, which is not greater than K. On the other hand, according to the inequality (A1) in the proof of Lemma 1, for any fixed (x_1, x_2) and time $t = n\Delta t$ for $n \in \{0,\ldots,N-1\}$, Portfolio II is worth I^*G_n which satisfies

$$I^*G_n \geq \max\{I^*\frac{e^{x_1-\sqrt{q^2(\Phi_t)+x_1^2+x_2^2}+dt}}{S_0}S_n, \quad I^*e^{x_2-\sqrt{q^2(\Phi_t)+x_1^2+x_2^2}+rt}\}. \tag{1}$$

If (x_1, x_2, I) is carefully chosen such that the payoff of Portfolio II satisfies two conditions:

C1. It is greater than $\{S_N, K\}$ at time T, and
C2. It is greater than K at any time $n\Delta t$ for $n \in \{0,\ldots,N-1\}$,
Portfolio I should be worth less than Portfolio II at time 0. As a result, $I - S_0e^{-dT}$ should be viewed as an upper bound for the price of the American put option $P^A(S_0, K, T|\Phi_T)$.

To achieve this, consider the following optimization problem:

$$\begin{aligned} \min_{x_1,x_2,I} \quad & I \\ s.\,t. \quad & I \quad * e^{x_1-\sqrt{q^2(\Phi_T)+x_1^2+x_2^2}+dT} \geq S_0, \\ & I \quad * e^{x_2-\sqrt{q^2(\Phi_T)+x_1^2+x_2^2}} \geq K. \end{aligned}$$

Define $g(t, z) := x_2 - \sqrt{z + x_1^2 + x_2^2} + rt$. Because $q^2(\Phi_t)$ increases in t, it follows that $g(t, q^2(\Phi_t)) \geq g(t, q^2(\Phi_T)) \geq g(0, q^2(\Phi_T))$ for any $t \geq 0$. Thus, any feasible solution to the above optimization problem must satisfy

$$I^*e^{x_2-\sqrt{q^2(\Phi_t)+x_1^2+x_2^2}+rt} \geq K,$$

for any $t \in \{0, \Delta t,\ldots,(N-1)\Delta t, T\}$ and

$$I^*\frac{e^{x_1-\sqrt{q^2(\Phi_T)+x_1^2+x_2^2}+dT}}{S_0}S_N \geq S_N,$$

which makes the payoff of Portfolio II satisfy the conditions C1 and C2 above, according to (1).
Then, we can focus on the above optimization problem. By the Karush–Kuhn–Tucker conditions, we get its optimal solution (x_1^*, x_2^*, I^*) as follows:

$$\begin{aligned} x_1^* &= \frac{1}{2}\left(\ln\frac{S_0}{K} - dT\right) + \frac{1}{2}\sqrt{2q^2(\Phi_T) + \left(\ln\frac{S_0}{K} - dT\right)^2}, \\ x_2^* &= -\frac{1}{2}\left(\ln\frac{S_0}{K} - dT\right) + \frac{1}{2}\sqrt{2q^2(\Phi_T) + \left(\ln\frac{S_0}{K} - dT\right)^2}, \\ I^* &= Ke^{\frac{1}{2}\left(\ln\frac{S_0}{k}-dT\right)+\frac{1}{2}\sqrt{2q^2(\Phi_T)+\left(\ln\frac{S_0}{k}-dT\right)^2}}. \end{aligned}$$

Therefore, we have

$$P^A(S_0, K, T|\Phi_T) \leq Ke^{\frac{1}{2}\left(\ln\frac{S_0}{k}-dT\right)+\frac{1}{2}\sqrt{2q^2(\Phi_T)+\left(\ln\frac{S_0}{k}-dT\right)^2}} - S_0e^{-dT}.$$

□

Alternatively, we can also manage to utilize the upper bound for a European option's price in DeMarzo et al. (2016) purely as an input for that on an American option's price. Thus, we need to bound the price of an American option from above by that of a corresponding European option, as accomplished in the following Lemma 3. Its proof is in the Appendix.

Lemma 3 *Let $P^A(S_0, K, T)$ be the price of an American put option with stock price S_0, strike price K and expiration date T, and $P^E\left(S_0, K\mathrm{e}^{(r-d)^+T}, T\right)$ be the corresponding European put option's price with strike price $K\mathrm{e}^{(r-d)^+T}$. Then, we have*

$$P^A(S_0, K, T) \le \mathrm{e}^{\min\{d,r\}T} P^E\left(S_0, K\mathrm{e}^{(r-d)^+T}, T\right).$$

Lemma 3 has some interesting implications. For the case of $r \ge d = 0$, since

$$P^E(S_0, e^{rT}K, T) - P^E(S_0, K, T) \le (1 - e^{-rT})K,$$

we can bound the early exercise premium of an American put option from above by

$$P^A(S_0, K, T) - P^E(S_0, K, T) \le P^E(S_0, e^{rT}K, T) - P^E(S_0, K, T) \le (1 - \mathrm{e}^{-rT})K.$$

On the other hand, when $d \ge r = 0$, $P^A(S_0, K, T) \le P^E(S_0, K, T)$ by Lemma 3, which revisits a well-known fact that an American put option will not be early exercised if $r = 0$.

Combining Lemma 1 and Lemma 3, we have the following result.

Theorem 4 *Given $q^2(\Phi_T)$ defined in Lemma 1, the price of an American put option with initial stock price S_0, strike price K and expiration date T satisfies:*

i. If $r \ge d \ge 0$,

$$P^A(S_0, K, T|\Phi_T) \le K\mathrm{e}^{\frac{1}{2}\ln\frac{S_0}{k} + \frac{1}{2}\sqrt{2q^2(\Phi_T) + \left(\ln\frac{S_0}{k}\right)^2}} - S_0.$$

ii. If $d > r \ge 0$,

$$P^A(S_0, K, T|\Phi_T) \le K\mathrm{e}^{\frac{1}{2}\left[\ln\frac{S_0}{k} + (r-d)T\right] + \frac{1}{2}\sqrt{2q^2(\Phi_T) + \left[\ln\frac{S_0}{k} + (r-d)T\right]^2}} - S_0\mathrm{e}^{(r-d)T}.$$

We have obtained two upper bounds for the price of an American put option from the Theorem 2 and the Theorem 4, respectively. They coincide when $r = 0$ or $d = 0$. For general cases, it is straightforward to get a smaller upper bound by taking the minimum of the two[4].

Corollary 5 *An upper bound for the price of an American put option is given by the minimum of the upper bounds in Theorem 2 and Theorem 4.*

Note that our results can be applied to American options with arbitrarily small intervals between two consecutive exercise opportunities. In the case when the American option can be continuously exercised, our results also hold as long as the limit of $q^2(\Phi_T)$ exists when Δt approaches zero[5].

4 | EMPIRICAL ANALYSIS

The performance of the upper bound in the above Corollary 5 is tested in this section from two perspectives. First, we compare it with the true price in the conventional binomial-tree framework. Furthermore, S&P 100 options data-based empirical investigations are conducted. These results show that our upper bounds are empirically meaningful in general.

[4]One could probably further reduce the upper bounds by taking the minimum with other upper bounds in the literature when all assumptions are satisfied.

[5]We refer to the Section 2.6 in DeMarzo et al. (2016) for related discussion. For example, they show that the limit of $q^2(\Phi_T)$ in the continuous setting exists if S_t is a semi-martingale.

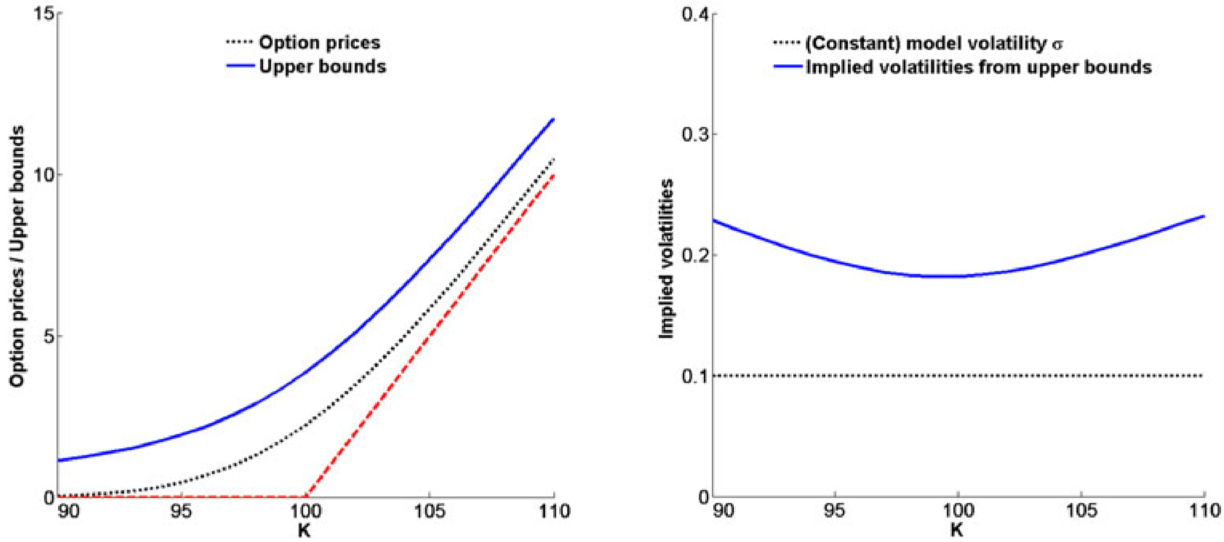

FIGURE 1 Upper bounds versus option prices when $r > d$ [Color figure can be viewed at wileyonlinelibrary.com]

4.1 | The binomial tree setting

Consider a standard binomial tree model with N time steps before expiration date T and each step has a length of $\Delta t = T/N$. During each time step, the stock's gross return can either be $u = e^{\sigma\sqrt{\Delta t}}$ or $d = e^{-\sigma\sqrt{\Delta t}}$. Denote by R_k any return path with k upward movements and $N - k$ downward movements in the stock price from time 0 to T. By definition, for any $R_k \in \Phi_T = \{R_0, R_1, ..., R_N\}$,

$$\begin{aligned} q^2(R_k) &= \sum_{n=1}^{N} [\ln \frac{S_n}{S_{n-1}} + (d-r)\Delta t]^2, \\ &= k[\sigma\sqrt{\Delta t} + (d-r)\Delta t]^2 + (N-k)[-\sigma\sqrt{\Delta t} + (d-r)\Delta t]^2, \\ &= N\sigma^2\Delta t + N(d-r)^2(\Delta t)^2 + 2(2k-N)(d-r)\sigma(\Delta t)^{\frac{3}{2}}, \\ &= \sigma^2 T + (d-r)^2 T\Delta t + 2(2k\Delta t - T)(d-r)\sigma\sqrt{\Delta t}. \end{aligned}$$

It follows that $q^2(\Phi_T) = q^2(R_N)$ when $d > r$ and $q^2(\Phi_T) = q^2(R_0)$ otherwise. Then, we can directly compare the upper bound in Corollary 5 with the benchmark option price generated by the binomial tree method with $N = 10,000$ steps (denoted by "Option price" in the following figures). We set $S_0 = 100$, $K \in [90, 110]$, $T = 3$ months, $d = 0.04$, $\sigma = 0.1$, and $r = 0.06$ (when $r > d$) or $r = 0.02$ (when $r < d$), respectively. The dashed curve for $(K - S_0)^+$ with varying K in the following Figures serves as a lower bound on the option price.

Figure 1 illustrates the results with $r > d$. In the left panel, our upper bound is a convex function of the strike price, which is akin to the shape of the option price. Furthermore, as K increases, the gap between the upper bound and the option price increases first and then decreases, and reaches its largest when K is about 102. In the right panel, the corresponding implied volatilities are plotted with respect to the strike prices, which are computed by the binomial-tree model with the same specification in pricing the option. The implied volatility curve exhibits a stylized volatility smile. Figure 2 with $r < d$ shows similar patterns.

4.2 | Empirical analysis based on S&P 100 options data

The performance of our bound in the Corollary 5 is further checked via empirical data. We collect from Bloomberg the last quoted bid and ask prices of each S&P 100 put option on each trading day from October 10, 2017 to January 9, 2018 for the out-of-sample comparison. The average of the bid and ask prices are treated as the put option price. Following Barone-Adesi, Engle, and Mancini (2008), any option with a time-to-maturity less than 16 calendar days or more than 365 calendar days is excluded, which leaves 7,875 observations of the S&P 100 put options in the

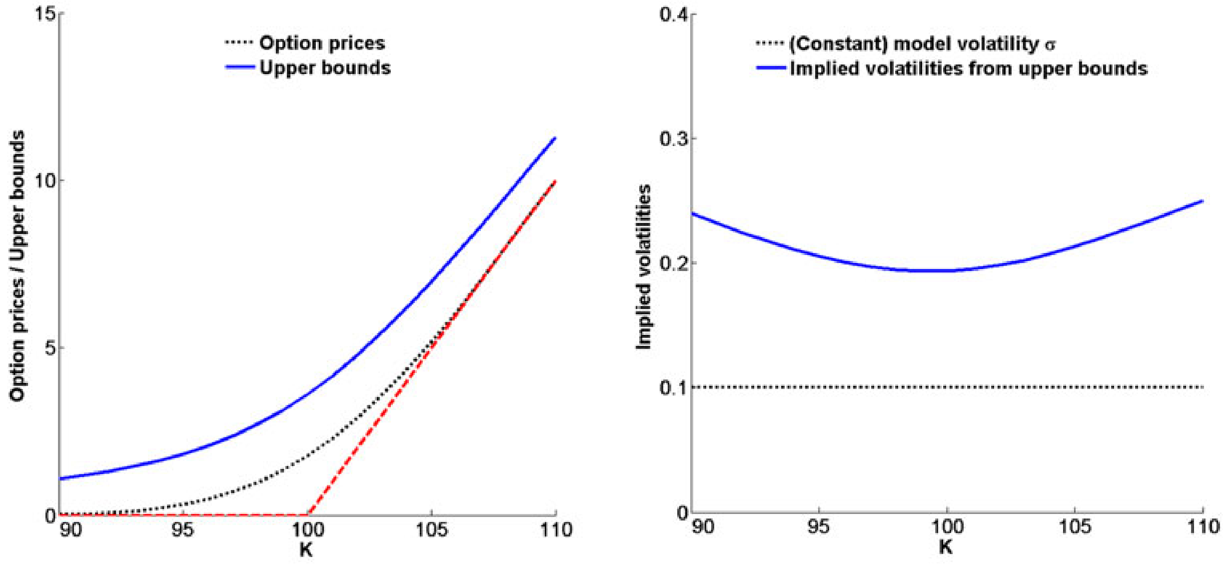

FIGURE 2 Upper bounds versus option prices when $r < d$ [Color figure can be viewed at wileyonlinelibrary.com]

sample period. The daily US Treasury yield curve rate data were downloaded from the website of the US department of the Treasury.[6]

At any trading date t_0 in the out-of-sample period, for an S&P 100 put option which expires at a future date t_1, the values of r, d and $q^2(\Phi_T)$ are determined in the following way. Define T as the number of trading days between t_0 and t_1. The linearly interpolated treasury yield rate from t_0 to t_1 is taken as the risk-free rate r. According to Harvey and Whaley (1992), the continuous dividend yield d for the S&P 100 index from t_0 to t_1 can be computed by

$$d = \ln \frac{S_0 + \sum_{i=1}^{T} D_i \mathrm{e}^{\frac{r_i(T-i)}{252}}}{S_0} \bigg/ \frac{T}{252},$$

where S_0 is the index level at t_0, and D_i and r_i are the cash dividend on the index and the linearly interpolated treasury yield rate at the ith trading day after t_0, respectively. Since D_i is unknown at date t_0, d is estimated by averaging the historical dividend yields of the corresponding period over the past three years. For example, three dividend yields for the periods from October 10, 2014 to March 16, 2015, from October 10, 2015 to March 16, 2016, and from October 10, 2016 to March 16, 2017, are averaged to estimate the dividend yield for the period from October 10, 2017 to March 16, 2018. Related cash dividend data and the closing prices of S&P 100 index used in such computation were also obtained from Bloomberg. To estimate $q^2(\Phi_T)$, the past $\mathcal{H}$ daily index returns before t_0 is used, where $\mathcal{H} = T + 99$. Denote the return path as $(r_{1,s}, r_{2,s}, ..., r_{\mathcal{H},s})$. It is partitioned into 100 sub-paths. Denote the set of these subpaths by $\Phi_T := \{(r_{1,s}, ..., r_{T,s}), (r_{2,s}, ..., r_{T+1,s}), ..., (r_{\mathcal{H}-T+1,s}, ..., r_{\mathcal{H},s})\}$, where each subpath has T daily returns. For each subpath $R \in \Phi_T$, $q^2(R)$ is computed by its definition. The largest $q^2(R)$ over all subpaths is selected as the estimation to $q^2(\Phi_T)$.

Next, define the moneyness of an option, that is m, as the ratio of its strike price to the underlying asset price. Following Alcock and Auerswald (2010), an S&P 100 put option is called in-the-money (ITM) if $m \in [1, 025, 1.075)$, deeply ITM (DITM) if $m \geq 1.075$, at-the-money (ATM) if $m \in [0.975, 1.025)$, out-of-the-money (OTM) if $m \in [0.925, 0.975)$, and deeply OTM (DOTM) if $m < 0.925$, respectively. In terms of maturity, the S&P 100 put options are classified into three categories: Short-term options (maturity $\leq$ 42 trading days), medium-term options (42 trading days < maturity $\leq$ 105 trading days), and long-term options (maturity > 105 trading days). As a result, all the put options in our sample can be divided into 15 categories by a combination of moneyness and maturity.

Descriptive statistics of the options data is summarized in Table 1. In Panel A, the mean, standard deviation, maximum and minimum of the option prices, and the moneyness and maturity in trading days are reported. The mean and standard deviation of the option prices and the number of observations for each moneyness-maturity category are

[6]https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield (last accessed on January 12, 2018).

TABLE 1 Description of S&P 100 put options data

Panel A				
	Mean	**Std**	**Max**	**Min**
Option price	24.70	32.40	205.75	0.28
Moneyness	0.94	0.11	1.17	0.64
Maturity in trading days	75.34	52.14	250	12
Obs.	7,875			

Panel B							
		Maturity in trading days					
		Short-term		**Medium-term**		**Long-term**	
Moneyness		**Mean**	**Std**	**Mean**	**Std**	**Mean**	**Std**
DOTM	Price	1.96	1.06	3.46	2.45	8.36	6.47
	Obs.	84		1605		1086	
OTM	Price	3.49	1.87	9.92	4.10	25.78	8.29
	Obs.	836		564		216	
ATM	Price	14.08	7.93	23.69	8.00	42.08	10.68
	Obs.	1019		570		221	
ITM	Price	49.93	12.36	62.03	15.09	73.27	13.76
	Obs.	440		545		217	
DITM	Price	111.31	15.37	116.87	17.48	127.95	25.63
	Obs.	21		291		160	

Note. ATM: at-the-money; DITM: deeply in-the-money; DOTM: deeply out-of-the-money; ITM: in-the-money; OTM: out-of-the-money.

reported in Panel B. Among the 7,875 observations, DOTM, OTM, ATM, and ITM options account for 35.24%, 20.52%, 22.98%, and 15.26% respectively, and short-term and medium-term options account for 30.48% and 45.40%, respectively.

Let $\hat{P}$ be an upper bound on an S&P 100 put option and P be its market price. Define the mean absolute pricing error (MAPE) and the mean relative pricing error (MRPE) as $MAPE = (\sum_{j=1}^{J} (\hat{P}_j - P_j))/j$ and $MRPE = (\sum_{j=1}^{J} (\hat{P}_j - P_j))/(\sum_{j=1}^{J} P_j)$, respectively, where J is the number of observations. The average of our upper bounds, the $MAPE$, and the $MRPE$ for each moneyness-maturity category are documented in Table 2.

TABLE 2 Performance of the upper bound

		Maturity in trading days		
Moneyness		**Short-term**	**Medium-term**	**Long-term**
DOTM	Average bound	5.53	5.87	10.26
	MAPE	3.57	2.42	1.89
	MRPE (%)	181.74	69.98	22.64
OTM	Average bound	9.65	15.04	29.03
	MAPE	6.16	5.12	3.25
	MRPE (%)	176.79	51.62	12.62
ATM	Average bound	25.93	34.82	52.22
	MAPE	11.85	11.12	10.13
	MRPE (%)	84.19	46.95	24.08
ITM	Average bound	59.86	75.11	91.51
	MAPE	9.93	13.08	18.24
	MRPE (%)	19.88	21.09	24.90
DITM	Average bound	117.82	126.60	149.86
	MAPE	6.51	9.73	21.91
	MRPE (%)	5.85	8.33	17.12

Note. ATM: at-the-money; DITM: deeply in-the-money; DOTM: deeply out-of-the-money; ITM: in-the-money; MAPE: mean absolute pricing error; MRPE: the mean relative pricing error; OTM: out-of-the-money.

Consistent with the average option price, the average upper bound for each moneyness-maturity category increases in both strike price and maturity. For short-term and medium-term options, the *MAPE* increases first and then decreases as moneyness moves from DOTM to DITM, which is similar to the pattern observed in §4.1. Meanwhile, an increasing relationship between the *MAPE* and moneyness is observed for long-term options.

At least three observations about the *MRPE* are worth noting. First, for short-term and medium-term options, the *MRPE* decreases significantly when moneyness moves from DOTM to DITM. Specifically, with increasing moneyness, the *MRPE* decreases from 181.74% to 5.85% for the short-term options, and decreases from 69.98% to 8.33% for the medium-term options. However, the *MRPE* for long-term options, with a range from 12.62% to 24.90%, has no monotonicity. Second, *MRPE* and *MAPE* change in the same direction with respect to the maturity, given any fixed moneyness. Last but not the least, our upper bound performs much better for ITM options than OTM ones, especially in the short-term case. It is interesting to see that the *MRPE* for short-term DITM options could be as low as 5.85%, which means our upper bound is very tight in this case. In addition, our upper bound is still useful for OTM options. Such options typically lack liquidity with large bid-ask spreads and volatile prices. This underlies that a large *MRPE* in this case is consistent with the robustness of our upper bound.

5 | CONCLUSIONS AND FUTURE WORK

By extending the work in DeMarzo et al. (2016), we propose robust upper bounds for the price of an American put option. Numerical experiments and empirical analysis show that our upper bounds have meaningful performance. Our work also leaves a few interesting questions which deserve further investigation. For instance, what is the optimal upper bound in the robust setting if it exists? It is also challenging to improve the tightness of our upper bounds for OTM options.

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APPENDIX

Proof of Lemma 1

Proof Assume that a portfolio with the initial value of \$1 is dynamically adjusted by a gradient trading strategy, which specifies at the end of the nth period the weight of the stock $w_{n+1} \in [0, 1]$ as follows.

$$w_{n+1} = \frac{L_{n,1}^+}{L_{n,1}^+ + L_{n,2}^+},$$

where $L_{n,i}^+ = \max(L_{n,i}, 0)$ and L_n is the regret vector which is defined as

$$L_n = \begin{bmatrix} \ln \frac{S_n}{S_0} + \frac{ndT}{N} - \ln G_n + x_1 \\ \ln B_n - \ln G_n + x_2 \end{bmatrix}.$$

If $L_{n,1}^+ = L_{n,2}^+ = 0$, w_{n+1} is chosen arbitrarily.

Denote by $\pi_{n,g}$ the log return of the portfolio in the nth period such that $\pi_{n,g} = \ln[1 + w_n r_{n,s} + (1 - w_n) r_b]$, and define ΔL_n as

$$\Delta L_n = L_n - L_{n-1} = \begin{bmatrix} \pi_{n,s} - \pi_{n,g} \\ \pi_{n,b} - \pi_{n,g} \end{bmatrix},$$

where $\pi_{n,s}$ and $\pi_{n,b}$ are defined at the beginning of Section 2.

Direct calculation verifies

$$\|L_n^+\|^2 = \|(L_{n-1} + \Delta L_n)^+\|^2 \le \|L_{n-1}^+ + \Delta L_n\|^2 = \|L_{n-1}^+\|^2 + \|\Delta L_n\|^2 + 2L_{n-1}^+ \cdot \Delta L_n$$

and

$$L_{n-1}^+ \cdot \Delta L_n = \left(L_{n-1,1}^+ + L_{n-1,2}^+\right)[w_n \pi_{n,s} + (1 - w_n)\pi_{n,b} - \pi_{n,g}].$$

From the concavity of the ln function,

$$\begin{aligned} \pi_{n,g} &= \ln[1 + w_n r_{n,s} + (1 - w_n) r_b] \\ &\ge w_n \ln[1 + r_{n,s}] + (1 - w_n)\ln[1 + r_b] \\ &= w_n \pi_{n,s} + (1 - w_n)\pi_{n,b}, \end{aligned}$$

then we have $L_{n-1}^+ \cdot \Delta L_n \le 0$.

It follows that

$$\|L_n^+\|^2 \leq \|L_{n-1}^+\|^2 + \|\Delta L_n\|^2.$$

Iterating the above inequality recursively will lead to

$$\|L_N^+\|^2 \leq \sum_{n=1}^{N} \|\Delta L_n\|^2 + \|L_0^+\|^2.$$

Since

$$\begin{aligned}
\|\Delta L_n\|^2 &= (\pi_{n,s} - \pi_{n,g})^2 + (\pi_{n,b} - \pi_{n,g})^2, \\
&= \pi_{n,s}^2 + \pi_{n,b}^2 + 2\pi_{n,g}^2 - 2(\pi_{n,s} + \pi_{n,b})\pi_{n,g}, \\
&= (\pi_{n,s} - \pi_{n,b})^2 + 2[\pi_{n,g}^2 + \pi_{n,s}\pi_{n,b} - (\pi_{n,s} + \pi_{n,b})\pi_{n,g}], \\
&= (\pi_{n,s} - \pi_{n,b})^2 + 2(\pi_{n,g} - \pi_{n,s})(\pi_{n,g} - \pi_{n,b}), \\
&\leq (\pi_{n,s} - \pi_{n,b})^2,
\end{aligned}$$

we have

$$\|L_N^+\|^2 \leq \sum_{n=1}^{N} (\pi_{n,s} - \pi_{n,b})^2 + x_1^2 + x_2^2.$$

Define $q^2(R) = \sum_{n=1}^{N} (\pi_{n,s} - \pi_{n,b})^2$. Because $L_{N,i} \leq \max(L_{N,1}, L_{N,2}) \leq \|L_N^+\|$, we get

$$\ln \frac{S_N}{S_0} + dT - \ln G_N + x_1 = L_{N,1} \leq \sqrt{q^2(R) + x_1^2 + x_2^2},$$

and

$$\ln B_N - \ln G_N + x_2 = L_{N,2} \leq \sqrt{q^2(R) + x_1^2 + x_2^2},$$

leading to

$$G_N \geq \max\left\{\frac{e^{x_1 - \sqrt{q^2(R) + x_1^2 + x_2^2} + dT}}{S_0} S_N, \quad e^{x_2 - \sqrt{q^2(R) + x_1^2 + x_2^2}} B_N\right\}.$$

Define $q^2(\Phi_T) = \sup_{R \in \Phi_T} q^2(R)$. We get, for any $R \in \Phi_T$,

$$G_N \geq \max\left\{\frac{e^{x_1 - \sqrt{q^2(\Phi_T) + x_1^2 + x_2^2} + dT}}{S_0} S_N, \quad e^{x_2 - \sqrt{q^2(\Phi_T) + x_1^2 + x_2^2} + rT}\right\}. \tag{A1}$$

Now consider the following two portfolios at time 0:

Portfolio I: Long one European put option on stock with strike price K and expiration date T and simultaneously long e^{-dT} shares of the stock.
Portfolio II: Invest I into the dynamically adjusted portfolio above.

At time T, portfolio I earns $\max\{S_N, K\}$ while Portfolio II is worth greater than $\frac{I}{S_0}\mathrm{e}^{x_1-\sqrt{q^2(\Phi_T)+x_1^2+x_2^2}+dT}\max\left\{S_N, \mathrm{e}^{x_2-x_1-(d-r)T}S_0\right\}$. If (x_1, x_2, I) is carefully chosen such that the payoff of Portfolio II at time T is greater than $\max\{S_N, K\}$, Portfolio II should have a greater initial investment cost than portfolio I. As a result, $I - S_0\mathrm{e}^{-dT}$ should be viewed as an upper bound for the price of the European put option $P^E(S_0, K, T|\Phi_T)$. To obtain an upper bound as tight as possible, (x_1, x_2, I) is selected to satisfy:

$$\begin{aligned} \min_{x_1,x_2,I} \quad & I \\ s.\,t. \quad & I * \mathrm{e}^{x_1-\sqrt{q^2(\Phi_T)+x_1^2+x_2^2}+dT} \geq S_0, \\ & I * \mathrm{e}^{x_2-\sqrt{q^2(\Phi_T)+x_1^2+x_2^2}+rT} \geq K. \end{aligned}$$

Solving the optimization problem by Karush–Kuhn–Tucker Conditions, we get the optimal solution (x_1^*, x_2^*, I^*):

$$\begin{aligned} x_1^* &= \tfrac{1}{2}\left[\ln\tfrac{S_0}{K} + (r-d)T\right] + \tfrac{1}{2}\sqrt{2q^2(\Phi_T) + \left[\ln\tfrac{S_0}{K} + (r-d)T\right]^2}, \\ x_2^* &= -\tfrac{1}{2}\left[\ln\tfrac{S_0}{K} + (r-d)T\right] + \tfrac{1}{2}\sqrt{2q^2(\Phi_T) + \left[\ln\tfrac{S_0}{K} + (r-d)T\right]^2}, \\ \text{and} \quad I^* &= K\mathrm{e}^{\frac{1}{2}\left[\ln\frac{S_0}{k}-(r+d)T\right]+\frac{1}{2}\sqrt{2q^2(\Phi_T)+\left[\ln\frac{S_0}{k}+(r-d)T\right]^2}}. \end{aligned}$$

Therefore, the price of the European put option $P^E(S_0, K, T|\Phi_T)$ satisfies

$$P^E(S_0, K, T|\Phi_T) \leq K\mathrm{e}^{\frac{1}{2}\left[\ln\frac{S_0}{k}-(r+d)T\right]+\frac{1}{2}\sqrt{2q^2(\Phi_T)+\left[\ln\frac{S_0}{k}+(r-d)T\right]^2}} - S_0\mathrm{e}^{-dT}.$$

□

Proof of Lemma 3

Proof We start by proving the inequality when $r \geq d \geq 0$. Consider an American put option with strike price K and a portfolio of e^{dT} shares of European put options with strike price $\mathrm{e}^{(r-d)T}K$. Both options have the same expiration date T and the underlying stock pays dividend yield d. At time T, if the American option has not been exercised early, its payoff is $\max\{K - S_N, 0\}$, while the European option portfolio pays $\mathrm{e}^{dT}\max\{\mathrm{e}^{(r-d)T}K - S_N, 0\} \geq \max\{K - S_N, 0\}$. Contrarily, if the American option is exercised early at some time $t = n\Delta t$, its payoff at t should be equal to $K - S_n$, while the European option portfolio is worth

$$\begin{aligned} \mathrm{e}^{dT}P^E(S_n, \mathrm{e}^{(r-d)T}K, T-t) &\geq \mathrm{e}^{dT}[\mathrm{e}^{(r-d)T-r(T-t)}K - \mathrm{e}^{-d(T-t)}S_n] \\ &= \mathrm{e}^{rt}K - \mathrm{e}^{dt}S_n \geq \mathrm{e}^{dt}(K - S_n) \geq K - S_n. \end{aligned}$$

Hence, when $r \geq d \geq 0$, the American put option is worth no more than the European put option portfolio. It follows that $P^A(S_0, K, T) \leq \mathrm{e}^{dT}P^E(S_0, \mathrm{e}^{(r-d)T}K, T)$.

In the case of $d > r \geq 0$, we consider one American put option with strike price K and a portfolio of e^{rT} shares of corresponding European put options with the same parameters. Similarly, if the American option is not exercised early, its payoff at time T must be no greater than that of the European put option portfolio. Moreover, if the American option is exercised early at some time $t = n\Delta t$, it has a payoff of $K - S_n$ at time t, while the European option portfolio is worth

$$\begin{aligned} \mathrm{e}^{rT}P^E(S_n, K, T-t) &\geq \mathrm{e}^{rT}[\mathrm{e}^{-r(T-t)}K - \mathrm{e}^{-d(T-t)}S_n] \\ &= \mathrm{e}^{rt}[K - \mathrm{e}^{(r-d)(T-t)}S_n] \geq K - S_t. \end{aligned}$$

Therefore, the American put option will be worth no more than the European put option portfolio. Then, we have $P^A(S_0, K, T) \leq \mathrm{e}^{rT}P^E(S_0, K, T)$. □