

# Stable Marriage on Networks

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## ABSTRACT

We consider a stable marriage problem on a network, where two groups of agents (men and women) form the network and each only knows its neighbors. Our objective is to design a novel mechanism that motivates agents to invite their neighbors to participate in the matching game if they are not already in it and provides agents with more selections in the enlarged game. The difficulty is that invitees may make the inviters' allocation worse off, which occurs if we apply the classic Deferred Acceptance (DA) mechanism. To induce mutual invitations, one idea is to add restrictions on DA, allowing each agent to propose only to their neighbors, which ensures that everyone is eager to invite all their neighbors. The drawback is that agents can never match with non-neighbors. To combat this, we propose Dynamic Deferred Acceptance (DDA), which enables matching among non-neighbors through a closed alliance construction and a sharing process. We demonstrate several impossibility results regarding incentive compatibility, stability, and efficiency, and then construct the theoretical boundaries for our model. Finally, we prove that DDA is the first mechanism to achieve all these desirable properties.

## KEYWORDS

Algorithmic game theory, Mechanism design, Two-sided matching, Social network

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## 1 INTRODUCTION

The stable marriage problem, introduced by Gale and Shapley [5], is a classic matching problem that seeks stable pairs between two fixed groups of men and women. In the traditional setting, a general assumption is that every agent knows all agents it can match, which means the information is global. However, in reality, agents only have local information about their potential partners due to limited social connections. Targeting this problem, dating or recruitment platforms (such as nannies and housekeeping services) try to incentivize existing users to diffuse the matching information on their social networks, so that the user market of both sides, those who provide and those who ask for the service, is expanded without advertisement fees. In this expanded market, participants will have more selections, and the platforms can propose a more

satisfying match. Therefore, it is crucial to model and study the stable marriage problem on networks and design mechanisms to reach desirable properties.

There has been a line of research studying mechanism design on networks, including auctions, cooperative games, and matching, which utilizes agents' social networks to enlarge the game and provide better outcomes [18]. The key challenge is that participants have no incentives to invite each other because they are competing for resources in the games. Besides, existing solutions for the traditional (static) settings cannot provide such incentives.

In particular, some works studied matching problems on networks, including one-sided matching [9, 15] and one-to-many two-sided matching where only one side forms a network [3, 6]. However, their solutions cannot apply to our setting. One reason is that the participants have larger action spaces in our setting, for their neighbors consist of both sides of the agents instead of only one.

In the stable marriage problem, stability is a crucial concept. A matching outcome is stable if no pair of agents would rather match with each other than with their current partners. However, in the network setting, this stability notion cannot coexist with the incentives for invitations. Therefore, we define new versions of stability notions that take into account agents' network structure. We show that the best stability can be achieved under incentive compatibility is Stable-1, where no pair of adjacent agents (on the social network) forms a blocking pair.

Besides stability, efficiency is also a well-studied concept that evaluates whether the matching allocation can be further improved. In this paper, we establish a series of impossibility results regarding stability and efficiency. These results reveal the trade-offs and limitations of achieving incentive compatibility for invitation in the new setting. To get an achievable efficiency notion, we identify a special structure on networks called bipartite fully-connected coalitions (BFC), where every man and every woman in this coalition are neighbors. We restrict the improvement group to be BFC and define efficiency notions accordingly.

We construct the theoretical boundaries for our setting, regarding incentive compatibility, stability, and efficiency. Then, we propose the first mechanism that incentivizes invitation and enables matching with non-neighbors, named Dynamic Deferred Acceptance (DDA). DDA ensures stability and incentives for invitation by restricting matching within neighbors in each round, and it provides matching opportunities for stranger participants by constantly updating the neighbor sets of the unmatched participants when some are matched and leave the market. The idea is that those already matched have no interest in the match of the others, and thus, they can share their remaining neighbors so that these unmatched agents can potentially improve their matching. However, participants may all prefer to delay their matching to benefit from

this sharing process, meaning that they have incentives to misreport information to get a better allocation. To ensure truthfulness, we define a strategy-proof order for agents to determine who to match first, and propose a new method called dynamic alliance to identify a special group of agents and construct matching in each round, satisfying the other desirable properties. To summarize, our main contributions are as follows.

- We formulate the stable marriage problem in a social network setting, where the goal is to incentivize all participants from both sides to invite others to enlarge the market.
- We extend the traditional desirable properties in the network setting. Furthermore, we demonstrate several impossibility results and construct the theoretical boundaries regarding incentive compatibility, stability, and efficiency.
- We propose a novel mechanism called Dynamic Deferred Acceptance to reach the theoretical boundaries.

## 2 RELATED WORK

Mechanism design over social networks is a popular research topic that explores various game-theoretic settings with a special consideration on agents' social networks [10, 17]. Matching without monetary transfers is a more challenging problem in this context, as the mechanism cannot use payment to incentivize invitations. For one-sided matching, Kawasaki et al. [9] proposed a modified TTC mechanism that induces truthful invitations on tree networks. You et al. [16] studied one-sided matching with existing tenants over social networks. Yang et al. [15] defined a stability notion compatible with social network settings and presented the Leave and Share mechanism, which improves the overall welfare from a cardinal perspective by a sharing process. Song et al. [13] constructed the theoretical boundaries regarding stability and optimality and proposed Connected Trading Cycles to reach them.

The stable marriage problem is a well-known two-sided matching problem with many applications, such as matching users on dating platforms, allocating students to colleges [5], and assigning papers to reviewers [1]. Several variants of the stable marriage problem have been studied theoretically [4, 14], as well as the trade-offs and impossibility results among different desirable properties [11].

To the best of our knowledge, no previous work has addressed two-sided matching considering strategic invitation of both sides agents on social networks. Some works have incorporated social connections into two-sided matching [2, 7, 8], but they did not consider invitation as a strategic behavior of the agents. Cho et al. [3]'s and Hamasaki et al. [6]'s work is closely related to ours, but they focused on a school choice problem where the schools are fixed and known beforehand, and only the students are in the social networks and require invitation incentives. While this assumption is reasonable for school choice problems, it is more general and realistic to allow both sides to invite each other in other applications of the networked stable marriage problem. Therefore, in this paper, we propose a model where both men and women are on the same social network and they can invite same-sex or opposite-sex participants via social connections.

## 3 PROBLEM FORMULATION

We consider a stable marriage problem over the social network as a one-to-one two-sided matching problem. The social network is denoted by an undirected graph  $G = (N, E)$  where  $N$  is partitioned into two disjoint sets of agents, men and women. Specifically,  $N = M \cup W$ , with  $M = \{m_1, m_2, \dots, m_{|M|}\}$  denoting the set of men and  $W = \{w_1, w_2, \dots, w_{|W|}\}$  the set of women. We define agents  $i$  and  $j$  are neighbors if there exists an edge  $(i, j) \in E$ . Let  $r_i \subseteq N$  represent agent  $i$ 's neighbor set. Every man  $m_i \in M$  has a strict preference  $\succ_{m_i}$  over  $W \cup \{m_i\}$ , where  $w_j \succ_{m_i} w_k$  means that  $m_i$  prefers  $w_j$  compared with  $w_k$ . If  $m_i \succ_{m_i} w_j$ , we say  $w_j$  is *unacceptable* to  $m_i$ . Similarly, every woman  $w_i \in W$  has a strict preference  $\succ_{w_i}$  over  $M \cup \{w_i\}$ .

Initially, only a subset of agents, denoted as  $N_0 \subseteq N$ , participate in the game. The participants can further invite their neighbors to join in the matching. We model agent  $i$  invites  $j$  to participate as  $i$  reports  $j$  in its neighbor set. In practice, only invited agents will ultimately take part in the matching process, and whenever an agent  $i$  joins the game, the others will have preferences towards  $i$ . Although uninvited agents do not participate, the model assumes that all agents can report and have preferences towards everyone, even those who have not yet joined, so that key matching properties, like incentive compatibility, can be properly defined and analyzed.

Each agent  $i \in N$  has a type profile  $\theta_i = (\succ_i, r_i)$ . The profile of all  $n = |N|$  agents is  $\theta = (\theta_1, \dots, \theta_n) = (\succ, r)$ , where  $\succ = (\succ_1, \dots, \succ_n)$ ,  $r = (r_1, \dots, r_n)$ . We use  $\theta_{-i}$  to represent the profile of all others except  $i$ , so  $\theta = (\theta_i, \theta_{-i})$ .  $\Theta$  represents the space of profiles of all agents, and  $\Theta = (\Theta_i, \Theta_{-i})$ . The type of each agent is private information and cannot be accessed externally. The only way to obtain this information is through the agents' reports, which may differ from the actual information. We define the report profile of agent  $i$  as  $\theta'_i = (\succ'_i, r'_i)$ ,  $r'_i \subseteq r_i$ , and the report profile of all agents as  $\theta' = (\theta'_1, \dots, \theta'_n)$ .

Given a report profile  $\theta'$ , we construct a directed graph  $G(\theta') = (N(\theta'), E(\theta'))$ , where there is an edge  $(i, j) \in E(\theta')$  if and only if  $j \in r'_i$ . Therefore, the directed graph  $G(\theta')$  is constructed based on the reported neighbors, which may not be the original social network  $G$  if someone misreported.

**DEFINITION 1 (TWO-SIDED MATCHING).** *A direct-revelation two-sided matching mechanism is defined as  $\pi = (\pi_i)_{i \in N}$ , where  $\pi_i : \Theta \rightarrow N$  and  $\pi$  satisfies that  $\forall i \in M(W)$ ,  $i$  belongs to one of two conditions: (1)  $\pi_i(\theta') = j \succ_i i$ ,  $\pi_j(\theta') = i$ , where  $j \in W(M)$  is  $i$ 's partner. (2)  $\pi_i(\theta') = i$ , which means  $i$  remains single. For an allocation  $\pi(\theta')$ , if  $\forall i \in N$ ,  $\pi_i(\theta') \succeq_i i$ , we say  $\pi_i(\theta')$  is acceptable to  $i$ , and the allocation  $\pi(\theta')$  is acceptable<sup>1</sup>.*

As previously mentioned, only agents who receive proper invitations can participate in the matching. We say that an agent  $i$  is *qualified* under  $\theta'$  if and only if there exists at least one directed path from some agent in  $N_0$  to  $i$  in  $G(\theta')$  (a proper invitation chain from an initial participant). We use  $\mathcal{Q}(\theta')$  to represent the set of all qualified agents under  $\theta'$ . Next, we define a special version of two-sided matching, called diffusion two-sided matching, which works on qualified agents only.

<sup>1</sup>For simplicity, we use  $\pi_i(\theta') \succeq_i i$  to represent  $\pi_i(\theta') \succ_i i$  or  $\pi_i(\theta') = i$ .

DEFINITION 2 (DIFFUSION TWO-SIDED MATCHING). A diffusion direct-revelation two-sided matching mechanism  $\mathcal{M}$  is a two-sided matching mechanism,  $\pi = (\pi_i)_{i \in N}$ , such that for all reported type profile  $\theta'$ , it satisfies:

- (1) for all unqualified agents  $i \notin Q(\theta')$ ,  $\pi_i(\theta') = i$ .
- (2) for all qualified agents  $i \in Q(\theta')$ ,  $\pi_i(\theta')$  is independent of the reports of all unqualified agents.

That is to say, a diffusion two-sided matching mechanism can only construct matchings among the qualified agents, because it is not aware of the unqualified agents nor their reports, so these agents stay single,  $\pi_i = i, \forall i \notin Q(\theta')$ .

## 4 PROPERTIES AND IMPOSSIBILITY RESULTS

In this section, we define and analyze the key properties of a diffusion two-sided matching mechanism, and then construct the theoretical boundaries. See Appendix A for more illustrations.

### 4.1 Incentive Compatibility and Stability

Firstly, we define incentive compatibility (IC), which means that each agent has a dominant strategy to report its type truthfully. In our model, the type profile of an agent consists of two parts: preference and neighbor set. The incentive for inviting all neighbors to join the game is crucial because it ensures the dissemination of information and determines the possible selections in matching.

DEFINITION 3 (INCENTIVE COMPATIBILITY). A diffusion two-sided matching mechanism  $\pi$  is incentive compatible for men (women)<sup>2</sup> IC4M(W) if for all  $i \in M(W)$ , all  $\theta_i \in \Theta_i$  and all  $\theta' \in \Theta$ , we have  $\pi_i(\succ_i, r_i, \theta'_i) \succeq_i \pi_i(\succ'_i, r'_i, \theta'_i)$ .

IC4M(W) means that each agent in  $M(W)$  will truthfully report type, whatever others are. Previous study in the traditional setting has shown that no stable matching mechanism ensures that reporting true preferences is a dominant strategy for every agent [11]. To get an achievable IC notion, we define ICI4M(W) where only truthful invitation is required, i.e., IC4M(W)  $\implies$  ICI4M(W).

DEFINITION 4 (INCENTIVE COMPATIBILITY FOR INVITATION). A diffusion two-sided matching mechanism  $\pi$  is **incentive compatible for invitation** for men (women) (ICI4M(W)) if for all  $i \in M(W)$ , all  $\theta_i \in \Theta_i$  and all  $\theta' \in \Theta$ , there exist  $(\succ'_i, r_i) \in \Theta_i$  s.t.  $\pi_i(\succ'_i, r_i, \theta'_i) \succeq_i \pi_i(\succ_i, r'_i, \theta'_i)$ .

Namely, ICI4M(W) ensures that there always exists a dominant strategy where the neighbor set report is truthful and the preference report may not be truthful. Especially, ICI4M(W) is strengthened to IC4M(W) if  $\succ'_i$  is always equal to  $\succ_i$ .

Another essential property is stability. In traditional two-sided matching, a blocking pair consists of two agents (a man and a woman) who prefer each other to their current partners. A two-sided matching allocation is stable if there are no blocking pairs.

In the social network setting, it is natural to require two agents to know each other before they can form a blocking pair. To become acquainted, two non-neighboring agents need the information

transmission of their neighbors and other agents. Following a similar motivation, Hofer and Wagner [8] proposed local stability where there exist no blocking pairs in friends' friends. We generalize this idea by defining blocking pairs with the distance between the two agents. Intuitively, if there exists a directed path from agent  $m$  to  $w$ ,  $m$  will have the potential to know  $w$  by the information diffusion along the path. The longer the path, the harder  $m$  and  $w$  can know each other. Hence, we define blocking pairs within distance of  $k$  and the corresponding stability notion.

DEFINITION 5 (BLOCKING PAIR WITHIN DISTANCE OF  $k$ ). Given an allocation  $\pi(\theta')$ , we say a man  $m_i \in M$  and a woman  $w_j \in W$  is a blocking pair within the distance of  $k$  if (1)  $w_j \succ_{m_i} \pi_{m_i}(\theta')$  and  $m_i \succ_{w_j} \pi_{w_j}(\theta')$  and (2) in  $G(\theta')$ , both the shortest path length from  $m_i$  to  $w_j$  and that from  $w_j$  to  $m_i$  are no more than  $k$ . Especially, if only one shortest path length is no more than  $k$ , we say  $m_i$  and  $w_j$  is an asymmetrical blocking pair within distance of  $k$ .

In this paper, we slightly abuse the definition and say that an agent  $i$  forms a blocking pair with itself if  $\pi_i$  is unacceptable for  $i$ .

DEFINITION 6 (STABILITY WITHIN DISTANCE OF  $k$ ). We say a mechanism  $\pi$  is stable within distance of  $k$  (Stable- $k$ ) if for all type profiles  $\theta'$ , there are no blocking pairs within the distance of  $k$  for  $\pi(\theta')$ . We use Stable- $k_a$  to denote the stability notion where there exist no asymmetrical blocking pairs. Especially, if there are no blocking pairs within the distance of  $k$  for every  $k$ , we say  $\pi$  is Stable.

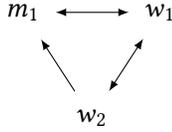
It is clear that Stable  $\implies$  Stable- $k \implies$  Stable- $(k-1)$  ( $k \geq 2$ ) and Stable- $k_a \implies$  Stable- $k$ . Stable is equivalent to the stability definition in the classical stable marriage problem, which does not take into account the social networks. When  $k = 1$ , Stable-1 means that there are no blocking pairs in neighbors for the allocation  $\pi$ . Stable-1 is equivalent to Stable when the social network is a directed complete graph. Hofer and Wagner's local stability, where there are no blocking pairs among friends' friends, is equal to Stable-2.

According to Roth [11], in the traditional setting, no Stable matching can ensure incentive compatibility for both sides. Consider the scenario where the social network is fully connected where Stable-1 is identical to Stable, the impossibility result still holds. This means no diffusion matching mechanism is Stable-1, IC4M, and IC4W. Therefore, we weaken the incentive compatibility notion to ICI4M(W). Next, we will prove that the weakened notion is not compatible with a stronger stability notion compared to Stable-1.

THEOREM 1 (IMPOSSIBILITY FOR STABLE-1<sub>a</sub> OR STABLE-2 WITH ICI4M(W)). There does not exist a diffusion two-sided matching mechanism that satisfies incentive compatibility for invitation for men (women) (ICI4M(W)) and (1) stability within distance of 1 for asymmetrical blocking pairs (Stable-1<sub>a</sub>) or (2) stability within distance of 2 (Stable-2).

PROOF. Consider the example given in Figure 1. The only Stable-1<sub>a</sub> allocations is  $((m_1, w_2), (w_1, w_1))$  (Stable-2 allocation is the same). In this case,  $w_1$  will not invite  $w_2$ , so that  $w_2$  will be unqualified and the only Stable-1<sub>a</sub> (Stable-2) allocation is  $(m_1, w_1)$ . Additionally, if  $w_1$  truthfully report her neighbor set she cannot improve her matching, because no matter how she misreports her preference,  $((m_1, w_2), (w_1, w_1))$  is always the only Stable-1<sub>a</sub> (Stable-2) allocation. Hence, no mechanism can simultaneously satisfy ICI4W and

<sup>2</sup>To simplify the presentation, we use parentheses to indicate that changing M/men to W/women in the statement is also correct. We will use similar expressions frequently throughout this paper.



**Figure 1: The initial agents  $N_0$  is  $\{m_1, w_1\}$ . The truthful preferences are  $m_1 : (w_2 \succ_{m_1} w_1 \succ_{m_1} m_1)$ ,  $w_1 : (m_1 \succ_{w_1} w_1)$ ,  $w_2 : (m_1 \succ_{w_2} w_2)$ .**

Stable-1<sub>a</sub> or ICI4W and Stable-2. Symmetrically, there is no mechanism satisfying ICI4M and Stable-1<sub>a</sub> or ICI4M and Stable-2 from the example in which all agents in figure 1 reverse their gender.  $\square$

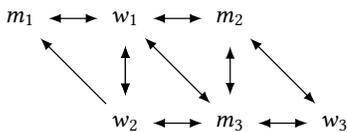
Theorem 1 shows that given ICI4M(W), the stability property cannot achieve Stable-2 or Stable-1<sub>a</sub>. Thus, we have to weaken the stability to Stable-1. Recall that IC4M and IC4W are not achievable, so we defined a weakened IC notion for one side as ICI4W(M). In the next subsection, we will discuss the best efficiency for a diffusion two-sided matching mechanism that satisfies IC4M(W), ICI4W(M), and Stable-1, thereby constructing the theoretical boundaries.

## 4.2 Efficiency

Since Stable-1 is the best possible stability notion compatible with IC4M(W) and ICI4W(M), it is obvious that some blocking pairs formed by non-neighbors cannot be eliminated by an incentive-compatible diffusion two-sided matching. That is to say, we sacrifice some desirable pairs, in terms of preferences, to ensure truthful invitation. In traditional models, an allocation is efficient means no group of agents on one-side can improve by a reassignment of their partners. To define an efficiency notion compatible with IC4M(W) and ICI4W(M), we restrict the improving group and their current partners to have a special network structure, where every man (woman) is connected to every woman (man). We define these agents as a bipartite fully-connected coalition.

**DEFINITION 7 (BIPARTITE FULLY-CONNECTED COALITION).** *Given the reported social network  $G(\theta') = (N(\theta'), E(\theta'))$ , we say a group of agents  $C \subseteq N(\theta')$  is a **bipartite fully-connected coalition (BFC)** if  $\langle i, j \rangle \in E(\theta')$  and  $\langle j, i \rangle \in E(\theta')$  for all  $i \in M \cap C$  and  $j \in W \cap C$ , i.e., the complete bipartite graph formed by  $C$  is a spanning subgraph of the induced subgraph  $G_C(\theta')$ . If there exists a spanning subgraph of  $G_C(\theta')$  that only lacks one edge to be a complete bipartite graph, we say  $C$  is an **almost bipartite fully-connected coalition**.*

In Figure 2, the agents' set  $\{w_3, w_1, m_3, m_2\}$  and  $\{w_1, w_2, m_3\}$  are both BFC, but  $\{m_1, w_1, w_2\}$  and  $\{w_1, w_2, m_2\}$  are not. Moreover,  $\{m_1, w_1, w_2\}$  is an almost BFC, for it only lacks one edge  $\langle m_1, w_2 \rangle$  to form a BFC. Especially, a BFC can also form an almost BFC.



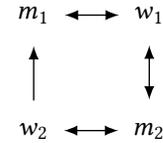
**Figure 2: An example for bipartite fully-connected coalition.**

In stable marriage problems, stability is an important property, and it is worthwhile to define the efficiency notion over all the stable allocations. In the networked setting, we define the efficiency notion over the Stable-1 allocations.

**DEFINITION 8 (LOCAL OPTIMAL STABLE-1 ALLOCATION).** *Given a profile  $\theta$ , we say a Stable-1 allocation is local optimal for men (women) if for any bipartite fully-connected coalition  $C$ , **no other Stable-1 allocation  $\pi'(\theta)$  can locally improve**, i.e.,  $\forall i \in N \setminus C$ ,  $\pi'_i(\theta) = \pi_i(\theta)$  and  $\forall i \in C$ , we have  $\pi'_i(\theta) \succeq_i \pi_i(\theta)$ , and there exists at least one agent  $j \in C$  with  $\pi'_j(\theta) \succ_j \pi_j(\theta)$ .*

This property means the local optimal Stable-1 allocation  $\pi(\theta)$  is weakly preferred by every man (woman) in  $C$  to every other Stable-1 allocation formed by locally reassignment. More specifically, in those locally changed Stable-1 allocations, every agent  $i$  of one side either gets a less desirable match than  $\pi_i(\theta)$  or gets the same partner at both allocations. The next theorem will show though we define the efficiency notion on a rather restricted structure: a bipartite fully connected coalition, it cannot be further relaxed.

**PROPOSITION 1.** *If relaxing the restriction on the improved coalition to **almost bipartite fully-connected coalitions**, there does not exist an ICI4W(M) mechanism which outputs a local optimal Stable-1 allocation (under almost BFC) for men (women) for any profile  $\theta$ .*



**Figure 3: The initial agents  $N_0$  is  $\{m_1, w_1\}$ . The preferences are  $m_1 : (w_2 \succ_{m_1} w_1 \succ_{m_1} m_1)$ ,  $m_2 : (w_1 \succ_{m_2} w_2 \succ_{m_2} m_2)$ ,  $w_1 : (m_1 \succ_{w_1} m_2 \succ_{w_1} w_1)$ ,  $w_2 : (m_2 \succ_{w_2} m_1 \succ_{w_2} w_2)$ .**

**PROOF.** Consider the example given in Figure 3. The only local optimal Stable-1 allocation (under almost bipartite fully-connected coalition) for men is  $((w_1, m_2), (w_2, m_1))$ . If  $w_1$  misreports her neighbor set as  $\{m_1\}$  instead of  $\{m_1, m_2\}$ ,  $\{w_2, m_2\}$  will be unqualified in the game, and the only local optimal Stable-1 allocation for men will be  $((w_1, m_1))$ , which is better for  $w_1$ . Hence, if the restriction on improved coalition is relaxed to almost BFC, no ICI4W diffusion matching mechanism can output local optimal Stable-1 allocations for men for any profile. The symmetrical theorem can be proved if all agents in Figure 3 reverse their gender.  $\square$

Furthermore, we consider the Pareto Efficiency property, which compares an allocation with all acceptable allocations (including unstable allocations). We first directly apply the traditional definition of Pareto Efficiency to the networked setting.

**DEFINITION 9 (PARETO EFFICIENCY).** *A diffusion two-sided matching mechanism  $\pi$  is (1) **strongly Pareto Efficient for men (women) (sPE4M(W))** if for all type profile  $\theta$ , there is **no other acceptable allocation  $\pi'(\theta)$  such that for each  $i \in M(W)$ ,  $\pi'_i(\theta) \succeq_i \pi_i(\theta)$ , and there exists at least one agent  $j$ ,  $\pi'_j(\theta) \succ_j \pi_j(\theta)$** . (2) **weakly Pareto Efficient for men (women) (wPE4M(W))** if for all type profile  $\theta$ , there is **no other acceptable allocation  $\pi'(\theta)$  such that for each  $i \in M(W)$ ,  $\pi'_i(\theta) \succ_i \pi_i(\theta)$** .*

It is clear that  $\text{sPE4M(W)} \implies \text{wPE4M(W)}$ . Roth [11] showed that Stable and  $\text{sPE4M(W)}$  are incompatible and that the DA mechanism is  $\text{wPE4M(W)}$  in traditional setting (without network). However, in our setting, we find that  $\text{wPE4M(W)}$  cannot coexist with  $\text{ICI4W(M)}$  and Stable-1.

**THEOREM 2 (IMPOSSIBILITY FOR  $\text{ICI4W(M)}$ , STABLE-1,  $\text{wPE4M(W)}$ ).** *There does not exist a diffusion matching mechanism that satisfies incentive compatibility of invitation for women (men) ( $\text{wICI4W(M)}$ ), stability within the distance of 1 (Stable-1), and weakly Pareto Efficiency for men (women) ( $\text{wPE4M(W)}$ ).*

**PROOF.** Look back to the example in Figure 3, the Stable-1 and  $\text{wPE4M(W)}$  allocation is  $((m_1, w_2), (m_2, w_1))$ . However,  $w_1$  can misreport and do not invite  $m_2$ , so that only  $m_1, w_1$  are in the matching. Hence the only Stable-1 and  $\text{wPE4M(W)}$  allocation is  $(m_1, w_1)$  where  $w_1$  has a better partner.  $\square$

To get a reachable efficiency notion compatible with  $\text{ICI4M(W)}$  and Stable-1, we define a restricted version of weakly Pareto Efficiency that only considers whether men (women) in bipartite fully-connected coalitions can improve their matching.

**DEFINITION 10 (LOCAL WEAKLY PARETO EFFICIENCY).** *A diffusion two-sided matching mechanism  $\pi$  is local weakly Pareto Efficient for men (women) ( $\text{wPE4M(W)-L}$ ) if for any bipartite fully-connected coalition  $C$ , **no other** acceptable allocation  $\pi'(\theta)$  can locally improve, i.e.,  $\forall i \in N \setminus C, \pi'_i(\theta) = \pi_i(\theta)$  and  $\forall i \in C$ , we have  $\pi'_i(\theta) \succ_i \pi_i(\theta)$ .*

It is clear that  $\text{sPE4M(W)} \implies \text{wPE4M(W)-L}$  but  $\text{wPE4M(W)-L}$  and  $\text{wPE4M(W)}$  do not imply each other. The difference is that  $\text{wPE4M(W)-L}$  ensures no local improvements, while  $\text{wPE4M(W)}$  ensures that all agents from one side cannot improve their matching together, without constraints on their social structure. Next, we prove that the efficiency notion defined over Stable-1 allocations can imply  $\text{wPE4M(W)-L}$ .

**THEOREM 3.** *If a mechanism outputs an local optimal Stable-1 allocation for men (women)  $\pi(\theta)$  for any profile  $\theta$ , the mechanism is local weakly Pareto Efficient for men (women).*

**PROOF.** Given any profile  $\theta$ , let  $C$  be any bipartite fully-connected coalition. Suppose there exists a locally improved allocation  $\pi'$  such that all men in  $C$  have better partners. Since the improvement is a locally reassignment, we have  $\pi_i \in C$  and  $\pi'_i \in C$ , for all  $i \in C \cap M$ . Previous work [12] shows that in the traditional setting, man-optimal stable matching is weakly Pareto efficient for the men among all acceptable allocations. Given that  $\pi(\theta)$  is a local optimal Stable-1 allocation where the improvement coalitions are restricted to BFC, and in BFC, Stable-1 is identical to Stable in traditional settings. We can safely extend the traditional results and say  $\pi(\theta)$  is a weakly Pareto efficient allocation when the improvement coalitions are restricted to BFC, i.e.,  $\text{wPE4M-L}$ .

In other words, there is no acceptable allocation  $\pi'(\theta)$  such that for each  $i \in M \cap C, \pi'_i(\theta) \succ_i \pi_i(\theta)$ . Thus, the theorem is proven. Symmetrically, the women side is similar to prove.  $\square$

Similar to the local optimal Stable-1 allocation, we also can not relax the improvement coalition in local weakly Pareto Efficient from bipartite fully-connected to almost bipartite fully-connected

coalitions. To prove this, we can use the same process illustrated in Theorem 1 with Figure 3.

**PROPOSITION 2.** *If relaxing the coalition in local optimal Stable-1 allocation from bipartite fully-connected to almost BFC, there does not exist a diffusion matching mechanism that satisfies weakly incentive compatibility for invitation for women (men) ( $\text{ICI4W(M)}$ ), stability within the distance of 1 (Stable-1), and local weakly Pareto Efficiency (under almost BFC) for men (women) ( $\text{wPE4M(W)-L}$ ).*

From the above impossibility results, the best efficiency notion for an  $\text{IC4M(W)}$ ,  $\text{ICI4W(M)}$  and Stable-1 mechanism is local optimal Stable-1 for men (women). This also infers that the mechanism satisfies  $\text{wPE4M(W)-L}$ . Based on the previous work [11], given Stable-1, incentive compatibility cannot be improved from  $\text{IC4M(W)}$ ,  $\text{ICI4W(M)}$  to  $\text{IC4M(W)}$ ,  $\text{IC4W(M)}$ . Due to Theorem 1, given  $\text{ICI4W(M)}$ , stability cannot be improved from Stable-1 to Stable-2 or Stable-1<sub>a</sub>. Finally, due to Theorem 2 and Proposition 2, given  $\text{IC4M(W)}$ ,  $\text{ICI4W(M)}$ , and Stable-1, the best efficiency notion over all accepted allocations is  $\text{wPE4M(W)-L}$ , and that over Stable-1 allocations is local optimal Stable-1 for men (women). In the next section, we will propose a mechanism to satisfy these properties.

## 5 MECHANISM DESIGN

In this section, we propose a novel mechanism named as **Dynamic Deferred Acceptance (DDA)**. Firstly, we show that directly applying classic matching mechanism DA in this new model cannot ensure incentive compatibility for invitation. The reason is that DA does not take agents' network relationship into consideration, and the invitee may compete with its inviter for the same partner thus hindering truthful invitation. To avoid competition and provide incentives for invitation, an intuitive method is to limit the propose side in DA to only propose to their neighbors. This is identical to assuming non-neighbors are unacceptable for every proposer when it comes to the execution of the mechanism. However, the drawback is that agents can never match with non-neighbors.

To conquer this problem, we propose DDA that provides agents with opportunities to match with non-neighbors. It is inspired by Yang et al. [15]'s work in one-sided matching on networks, where they introduce a *sharing process* to improve the matching allocation. In two-sided matching on networks, we propose a new method called *dynamic alliance* and further construct the Dynamic Deferred Acceptance mechanism to satisfy  $\text{IC4M(W)}$ ,  $\text{ICI4W(M)}$ , Stable-1,  $\text{wPE4M(W)-L}$ , and local optimal Stable-1 for men (women). In this paper, we present a men-propose version of DDA, and the women-propose version is symmetrical.

### 5.1 DA Does Not Incentivize Invitation

We first introduce DA, a classic matching mechanism that satisfies several desirable properties in the traditional setting. Next, we prove that DA fails to satisfy  $\text{ICI4W(M)}$  in the social network setting.

**DEFINITION 11 (DEFERRED ACCEPTANCE ALGORITHM (DA)).** *In each round, each man proposes to his favorite woman who has not rejected him yet. Each woman who has received at least two proposals keeps her favorite proposal and rejects the rest. Repeat the process until no man has a woman to propose to or each woman has at most one*

proposal. Then, the algorithm terminates and each man is assigned to a woman who has not rejected his proposal or remains single.

In the traditional setting, Roth [11] has proven that the men-propose version of DA is acceptable, IC4M (reporting true **preference** is a dominant strategy for men), Stable, and wPE4M. In the next theorem, we prove that DA cannot satisfy ICI4W(M), which only requires truthful invitation.

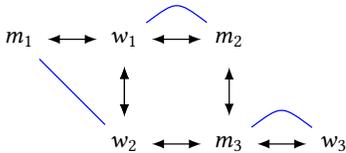
**THEOREM 4.** *The men-propose version of DA does not satisfy ICI4W(M) in the network setting.*

**PROOF.** Roth [11] shows that the DA mechanism satisfies Stable. Since Stable implies Stable-2, we know that DA satisfies Stable-2. In Theorem 1, we proved that no diffusion matching mechanism can satisfy both Stable-2 and ICI4W(M). Thus, we can conclude that DA does not satisfy ICI4W(M).  $\square$

DA fails in the social network setting because it ignores the competition between inviters and invitees, especially when these agents are from the same side. To avoid competition, we can limit everyone to only proposing to neighbors.

Consider the simple example in Figure 4. Agents' preferences are presented in Table 1. The allocation of DA is  $((m_1, w_2), (m_2, w_1), (m_3, w_3))$  shown by blue lines. If we restrict agents' selection space to neighbors,  $m_1$  can only propose to  $w_1$ , and thus the allocation result is  $((m_2, w_1), (m_3, w_3))$ ,  $w_2$  and  $m_1$  remain single.

From this example, it is clear that only allowing agents to match with their neighbors sacrifices efficiency and undermines the incentives for agents to invite others or participate in the matching. This contradicts our goal, which is to construct a more desirable matching that could encourage agents to join and motivate existing participants to spread information simultaneously. Looking back to this example,  $w_2$  and  $m_1$ , who remain single, could achieve a better result by matching with each other. To allow agents can match with friends' friends, we propose a novel mechanism called DDA that can attain a superior allocation and provide stronger incentives for agents to invite others. For the example in Figure 4, DDA will allow  $w_2$  to match with  $m_1$ , which is the same as DA.



**Figure 4:**  $N_0 = \{w_1\}$ . The blue lines represent DA's allocation.

| W     | Preferences                         | M     | Preferences                         |
|-------|-------------------------------------|-------|-------------------------------------|
| $w_1$ | $m_3 \succ m_2 \succ w_1 \succ m_1$ | $m_1$ | $w_1 \succ w_2 \succ w_3 \succ m_1$ |
| $w_2$ | $m_3 \succ m_1 \succ m_2 \succ w_2$ | $m_2$ | $w_1 \succ w_3 \succ w_2 \succ m_2$ |
| $w_3$ | $m_2 \succ m_3 \succ m_1 \succ w_3$ | $m_3$ | $w_3 \succ w_1 \succ w_2 \succ m_3$ |

**Table 1:** The preferences of each agent in Figure 4.

## 5.2 The DDA Mechanism

In one-sided matching problems, Yang et al. [15] improve the matching result by a sharing process. In short, whenever a group of agents get matched and leave, their unmatched neighbors become new neighbors of each other, so these agents will have more selections in matching. However, when we attempt to incorporate sharing into our mechanism design, we encounter several challenges: (1) sharing cannot naturally coexist with ICI4M(W), and (2) achieving Stable-1 under sharing is difficult.

To tackle these challenges, we introduce a new method named *dynamic alliance* to facilitate the sharing process. It starts with only one agent in an alliance and dynamically adds other agents desired by agents in the alliance through a propose-and-reject process similar to DA. This dynamic process terminates when no new agent is desired by agents in the alliance, and the alliance will not change anymore (i.e., each man in the alliance is assigned to a woman who has not rejected his proposal or he remains single). Now, agents in the alliance can leave the market with their partners following the accepted proposal and share all their unmatched neighbors.

To formulate the dynamic alliance method, we define two new operations based on the original propose and reject process in DA.

**DEFINITION 12 (PROPOSAL AND ENFORCEMENT OPERATION).** *Agent  $i$  in alliance  $A$  makes a proposal to  $i$ 's favorite neighbor  $j$ , who has not rejected him yet. Especially,  $j$  can be  $i$  and  $i$  remains single. If  $j \notin A$ , update alliance  $A$  by including agent  $j$ .*

**DEFINITION 13 (REFUSAL AND ENFORCEMENT OPERATION).** *Agent  $j$  in alliance  $A$  tentatively accepts the best  $i \in A$  from all receiving proposals of neighbors and refuses others. Especially,  $i$  can be  $j$  when  $j$  chooses to remain single or receive no proposal. Update alliance  $A$  by including all neighbors of  $j$  whom  $j$  prefers over  $i$ .*

Next, we define an order for all agents to determine the first agent of each alliance.

**DEFINITION 14.** *An ordering of agents is a one-to-one function  $\mathcal{P} : \mathbb{N}^+ \rightarrow \mathbb{N}$ , where agent  $\mathcal{P}(i)$  is the  $i^{\text{th}}$  agent in  $\mathcal{P}$ . Agents in  $\mathcal{P}$  are sorted in ascending order by the length of the shortest path from the initial agents set  $N_0$  to them. Especially, for any agent  $i \in N_0$ , its shortest path length is 0. When multiple agents have the same shortest path length, we use a random tie-breaking to determine the order.*

Now, we are ready to present the formal description of the Dynamic Deferred Acceptance mechanism, which combines the dynamic alliance method and a sharing process.

DDA operates through multiple rounds of iterations, each consisting of a dynamic alliance method and a sharing process. In each round, DDA identifies an alliance  $A$  and matches all agents in  $A$  with a partner within  $A$ . Each round of the dynamic alliance method starts from the unmatched agent with the minimum order in  $\mathcal{P}$  as the *source* of  $A$ . Note that both sides of the agents can be the source. In this process, any agent in  $M \cap A$  can propose to his most preferred neighbor  $j$  who has not rejected him, and  $j$  is added to  $A$  if  $j \notin A$ . Similarly, any agent  $j$  in  $W \cap A$ , who accepts a proposal from  $k \in M \cap A$ , will add her neighbors, whom she ranks higher than  $k$  and are not currently in  $A$ , to join  $A$ . If the source is a woman  $j$ , she will add all her male neighbors  $i \succ_j j, i \in M \cap r_j$  to join  $A$  because she is single (This will be the case in the example we show

### Dynamic Deferred Acceptance (DDA)

Input:  $\theta', N_0$

- (1) Initialize order  $\mathcal{P}$
- (2) Initialize  $A$  by the agent who is still in the market and has the minimum index in order  $\mathcal{P}$ , i.e.,  $\mathcal{P}(1)$ .
- (3) (*Dynamic Alliance*) Repeat steps 3a, 3b until the following three conditions hold:
  - (i)  $\forall i \in M \cap A$  is successfully assigned an agent  $j \in W \cap A$  or  $i$  remains single with no agent to propose,
  - (ii)  $\forall j \in W \cap A$ ,  $j$  accepts a proposal or remains single,
  - (iii) Alliance  $A$  is not updated in this iteration.
  - (a) Each agent  $i \in M \cap A$  conducts proposal and enforcement operation.
  - (b) Each agent  $j \in W \cap A$  conducts refusal and enforcement operation.
- (4) (*Sharing Process*)
  - (a) All agents in alliance  $A$  leave the matching market with their assigned partners (return  $\pi_i$  for all  $i \in A$ ). Remove  $A$  in  $\mathcal{P}$ .
  - (b) Let  $f(A) = \{i | \langle j, i \rangle \in E(\theta'), j \in A\}$ . Add pairwise bidirectional edges between all agents of  $f(A)$  to update graph  $G(\theta')$  (update  $r'_i$  for  $\forall i \in f(A)$ ).
- (5) Repeat step (2) to step (4) until all agents leave the matching market.

Output:  $\pi$

later). Finally,  $A$  requires no further updates and each agent in  $A$  is assigned to another agent in  $A$  or remains single.

After agents in  $A$  get matched and leave the market, the *sharing* process will connect their remaining neighbors, resulting in an updated graph where unmatched neighbors of  $A$  become neighbors of each other. We use  $f(A) = \{i | \langle j, i \rangle \in E(\theta'), j \in A\}$  to denote all the neighbors of  $A$ . For all agents  $i \in f(A)$ , we update  $r'_i$  and  $G(\theta')$ , so the next round will run on this updated graph.

The dynamic alliance method is crucial for reconciling the sharing process with other desired properties. To see how it works, we introduce a new concept called *closed alliance*. A *closed alliance* is an alliance  $A$  that satisfies for all  $i \in A$ , any unmatched neighbors that  $i$  prefers to  $\pi_i$  are also involved in  $A$ . Our mechanism employs the dynamic alliance method to identify several closed alliances. In each round, let  $N_{out}$  be the set of agents who have left the market and  $N_r = N \setminus N_{out}$  be the remaining agents. We next prove that every round of *dynamic alliance* identifies a closed alliance in  $N_r$ .

**PROPOSITION 3.** *In each round, the dynamic alliance method always finds a closed alliance in  $N_r$  for any given report.*

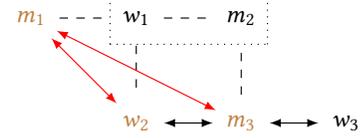
**PROOF.** We need to show that  $\forall i \in A, \forall j \in r'_i \cap N_r$  such that  $j \succ_i \pi_i$  belongs to  $A$ . In the men-propose version, we consider the following two cases: (1) If  $i \in M$ ,  $i$  has proposed to all agents  $j \succ_i \pi_i$  where  $j \in r'_i \cap N_r$ . Hence,  $j$  must have joined  $A$  by the proposal and enforcement operation. (2) If  $i \in W$ ,  $i$  receives a proposal from agent  $k$  or remains single. Therefore, all unmatched neighbors

of  $i$  who  $i$  ranks higher than  $\pi_i$  must have been in  $A$  due to the refusal and enforcement operation. Combining these results, we have demonstrated that in each round, the dynamic alliance method will identify a closed alliance in  $N_r$  for any given report.  $\square$

The dynamic alliance method ensures all the potential blocking pairs within distance of 1 are involved in the same alliance (the feature of closed alliance), and the propose-and-reject process within the alliance assembles DA where no blocking pairs will be formed.

**COROLLARY 4.** *There are no  $i \in M, j \in W$  in a closed alliance, and they form a blocking pair within distance of 1 on the updated graph (on which the alliance is constructed).*

By design, the sharing process does not affect the matched agents and can coexist with other desired properties. To have more agents benefit from the sharing process, a natural idea is to involve fewer agents in the dynamic alliance. However, the closed alliance requirement cannot be relaxed anymore because if agent  $i$  does not involve all his/her more preferred neighbors in the alliance, they may form a blocking pair, violating the Stable-1.



**Figure 5: The sharing process when DDA runs for the example in Figure 4. In round 1, alliance consists of  $\{w_1, m_2\}$  and leaves the matching market. The  $f(\{w_1, m_2\})$  are  $\{m_1, w_2, m_3\}$ . The graph is updated by connecting  $m_1$  with  $w_2$  and  $m_3$ .**

In Figure 5, we use the same example shown in Figure 4 to illustrate how DDA executes. The initial agent set is  $\{w_1\}$ , and we initialize the order as  $\mathcal{P} = (w_1, m_1, m_2, w_2, m_3, w_3)$ . First,  $w_1$  is the source of the dynamic alliance, and she adds  $m_2$  ( $w_1$  prefers  $m_2$  to single and  $m_1$  is unacceptable) into the alliance  $A$ . Then,  $m_2$  proposes to  $w_1$  because  $m_2$  has no other woman neighbors. So,  $A = \{w_1, m_2\}$  and it is a closed alliance and they are matched with each other. Next, DDA removes  $\{m_2, w_1\}$  from the market and connects their remaining neighbors  $f(A) = \{m_1, w_2, m_3\}$  shown by the red arrows in Figure 5. Now,  $m_1$  is the agent with the minimum order remaining in the market, and he will propose to his favorite neighbor  $w_2$  and add  $w_2$  into the new alliance. Then,  $w_2$  accepts the proposal, and she adds  $m_3$  to join the matching because  $m_3 \succ_{w_2} m_1$ . After joining in the alliance,  $m_3$  will propose to his favorite neighbor  $w_3$  and add  $w_3$  to the alliance. Finally,  $w_3$  accepts  $m_3$  and the alliance does not update anymore. Hence, in this round,  $A = \{m_1, w_2, m_3, w_3\}$  and the allocation is  $((m_1, w_2), (m_3, w_3))$ , so DDA removes  $\{m_1, w_2, m_3, w_3\}$  from the market. Given that no agent remains in the matching, the allocation result of DDA mechanism is  $((m_1, w_2), (m_2, w_1), (m_3, w_3))$ , the same allocation as DA.

### 5.3 Properties of the DDA Mechanism

In this section, we will prove that DDA mechanism is IC4M(W), wICI4W(M), Stable-1, wPE4M(W)-L, and local optimal Stable-1 for men (women).

To begin with, we introduce three lemmas: the first one demonstrates that order  $\mathcal{P}$  is independent of agents' reports; and the second one claims that every agent's report cannot influence the allocation before the round he/she leaves; and the third lemma shows that the allocation within every closed alliance is the same as executing DA with incomplete preference.

LEMMA 5.  $\forall i \in N$ , order of  $i$  in  $\mathcal{P}$  is independent of the  $i$ 's report.

LEMMA 6. In each round of DDA, for each  $i \in N_r$ ,  $i$ 's report has no impact on the allocation of  $N_{out}$ .

LEMMA 7. Given any report profile, in each round of DDA,  $\forall i \in A$ , the allocation  $\pi_i$  under DDA is the same as executing DA in  $N_r$  with a restricted preference list where only neighbors are acceptable.

PROOF. Without loss of generality, we suppose agent  $i$  is on the propose side, i.e.,  $i \in M$ , and its report profile is  $\theta'_i$ . Let the allocation given by DA be  $\alpha$ . Since DA gives the propose side optimal allocation, we have  $\alpha_i \succeq_i \pi_i$ . If  $\alpha_i \neq \pi_i$ , it can only be  $\alpha_i \succ'_i \pi_i$ . When executing DA in  $N_r$ , we assumed only neighbors are acceptable in every  $i$ 's preference, so it is clear that  $\alpha_i \in r'_i$ . In DDA,  $\forall j \succ'_i \pi_i$ ,  $j \in r'_i$  must have been added to  $i$ 's alliance  $A$ . If  $j = \alpha_i \succ'_i \pi_i$ ,  $j$  must have been paired with another  $i' \in M \cap A$  in DDA, and we have  $\alpha_{i'} \succ'_{i'} \pi_{i'}$ . Similarly, we can find a subset  $A' \subseteq M \cap A$  whose allocation is worse than that in  $\alpha$ , and their partners set  $\pi_{A'} = \{j | j = \pi_i, \forall i \in A'\}$  is the same as  $\alpha_{A'} = \{j' | j' = \alpha_i, \forall i \in A'\}$ . In men-propose DDA, agents in  $A'$  must have proposed to their partners in  $\alpha$  because the man side will first propose to a more preferred agent. At that time, every man and woman had a proposal (including single), and DDA will pair agents in  $A'$  up in this way. Thus,  $\pi_i = \alpha_i$  for all  $i \in A$ .  $\square$

THEOREM 5. For any ordering  $\mathcal{P}$ , men (women)-propose DDA satisfies IC4M(W).

PROOF. First, we show that every agent  $i \in M$  reports preference truthfully. For  $\forall i \in M$ , we know  $i$ 's report does not affect  $N_{out}$ 's allocation from Lemma 6. Assume  $\pi_i((\succ_i, r'_i), \theta'_{-i}) = w_i$  and  $i$  belongs to closed alliance  $A$ . When  $i$  misreports, assume  $\pi'_i((\succ'_i, r'_i), \theta'_{-i}) = w'_i$  and the closed alliance involving  $i$  is  $A'$ . Due to lemma 7, we know that  $w_i = \alpha_i((\succ_i, r'_i), \theta'_{-i})$  and  $w'_i = \alpha_i((\succ'_i, r'_i), \theta'_{-i})$ . Since DA ensures truthful preference report, agents cannot misreport their preferences for neighbors to improve matching results. Also, the preference list beyond neighbors will not be considered by DDA because those agents are not in the alliance. So, agent  $i$  can not make his allocation better by misreporting preference, and  $w'_i \succ_i w_i$  is impossible. Thus,  $\forall i \in M$  will report preference truthfully.

Second, we show that every agent  $i \in M$  reports neighbors truthfully. From Lemma 6 and Lemma 5, we know that agent  $i$  cannot report to determine who left before his round, and  $i$  also cannot control his order in  $\mathcal{P}$ . Thus,  $i$  cannot misreport his neighbor set to switch to a different alliance. Suppose when  $i$  reports  $(\succ'_i, r_i)$  his allocation is  $w_i$  and the alliance involving  $i$  is  $A$ , and when  $i$  reports  $(\succ'_i, r'_i)$  his allocation is  $w'_i$  and the alliance involving  $i$  is  $A'$ . Since other agents' neighbors are independent of  $i$ 's report, the difference between  $A'$  and  $A$  is caused by  $i$  not including some of its neighbors in the dynamic alliance process. Thus, it is identical to reporting these neighbors as unacceptable, i.e., misreporting preference. However, we have proved that preference misreport cannot improve agent  $i$ 's matching under a given neighbor set report. Hence, we

can conclude that agent  $i$  cannot misreport neighbor set to improve his matching.

Hence, men (women)-propose DDA satisfies IC4M(W).  $\square$

THEOREM 6. For any order  $\mathcal{P}$ , men (women)-propose DDA satisfies ICI4W(M).

PROOF. For any  $i \in W$ , assume  $\pi_i((\succ'_i, r_i), \theta'_{-i}) = w_i$  and  $i$  belongs to closed alliance  $A$ . Under the same reported preference  $\succ'_i$ , assume  $i$  misreports neighbor set and his allocation is  $\pi'_i((\succ'_i, r'_i), \theta'_{-i}) = w'_i$ , and  $i$  now belongs to closed alliance set  $A'$ . We construct a new preference report  $\succ''_i$  where  $i$  reports all neighbors in  $r_i \setminus r'_i$  as unacceptable. If agent  $i$  reports  $(\succ''_i, r_i)$ , we have  $A' = A''$  because if  $i$  can unqualify these agents, only  $i$  can add them into the alliance. Thus, we have  $\pi_i((\succ''_i, r_i), \theta'_{-i}) \succeq_i \pi_i((\succ'_i, r'_i), \theta'_{-i})$ , which means DDA is ICI4W(M).  $\square$

THEOREM 7. For any ordering  $\mathcal{P}$ , men (women)-propose DDA satisfies Stable-1.

PROOF. For each closed alliance  $A$ , by its definition, we can know that  $\forall i \in A$  and  $\forall j \in N_r \setminus A$  cannot form a blocking pair within distance of 1 on the updated graph. Due to Corollary 4, there is also no blocking pair within distance of 1 in  $A$ . Thus, when we iterate over all closed alliances, there will be no blocking pairs between any two agents within distance of 1, i.e., DDA satisfies Stable-1.  $\square$

THEOREM 8. For any ordering  $\mathcal{P}$ , men (women)-propose DDA mechanism satisfies local optimal Stable-1 and wPE4M(W)-L for every report profile  $\theta$ .

PROOF. Suppose there exists a Stable-1 matching  $\pi'$  for a certain profile where a bipartite fully-connected coalition  $C$  satisfies (1)  $\pi'_i \succeq_i \pi_i$  for all  $i \in M \cap C$ , with at least one strictly better off, (2)  $\pi'_i$  coincides with  $\pi_i$  outside  $C$ . Assume the agents in  $C$  belong to closed alliances  $A_1, A_2, \dots, A_k$  when running DDA.

Consider any  $j \in [k]$ . Since  $C$  is a bipartite fully-connected coalition,  $C \cap A_j$  is also a BFC in  $A_j$ . In alliance  $A_j$ , given Lemma 7, we can simply consider the allocation under DA with a restricted preference list. From previous work [12], we know that DA is man-optimal among all Stable-1 (Stable-1 is identical to Stable in BFC) matchings on  $A_j$ . Thus, no reassignment within  $A_j \cap C$  can weakly improve every man and strictly improve at least one man without violating stability. Additionally, since each  $A_j$  is a closed alliance, no reassignment across different alliances can improve any man. Hence,  $\pi'_i$  cannot exist, and  $\pi_i$  under DDA is man-local optimal Stable-1 allocation. Due to Theorem 3, DDA also satisfies wPE4M(W)-L.  $\square$

## 6 CONCLUSION

In this paper, we model the stable marriage problem on a social network setting and our goal is to incentivize all the participants (on both sides) to invite their neighbors to join in and have a satisfying allocation in this enlarged market. We redefine traditional stability and efficiency notions in this new model and demonstrate that they cannot coexist with incentives for invitation. Furthermore, we identify the tightest stability, Stable-1, and efficiency, wPE4M(W)-L and local optimal Stable-1, compatible with the new IC, i.e., IC4M(W) and ICI4W(M). Finally, we propose the Dynamic Deferred Acceptance mechanism to satisfy these desirable properties.

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## A ADDITIONAL ILLUSTRATION ON THEORETICAL BOUNDARIES

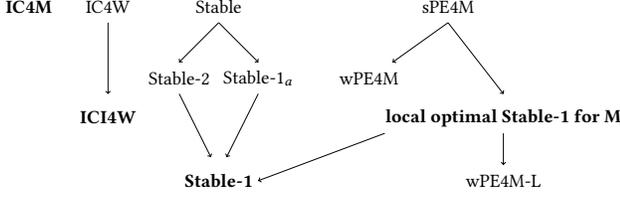


Figure 6: Relational graph of the properties where ‘W’ and ‘M’ can be reversed. An arrow from A to B represents A implies B. The bold notions are the theoretical boundaries for our model.

## B PROPERTIES OF DA

Roth [11] has proven that no mechanism can satisfy both Stable and sPE4M in the traditional stable marriage problem. Here we give a brief illustration.

Let  $M = \{m_1, m_2, m_3\}$  and  $W = \{w_1, w_2, w_3\}$  with preferences given by Table 2. Suppose agents’ social network is fully connected, so sPE4M and Stable are the same as in traditional settings.

| W     | Preferences                         | M     | Preferences                         |
|-------|-------------------------------------|-------|-------------------------------------|
| $w_1$ | $m_1 \succ m_2 \succ m_3 \succ w_1$ | $m_1$ | $w_2 \succ w_1 \succ w_3 \succ m_1$ |
| $w_2$ | $m_3 \succ m_1 \succ m_2 \succ w_2$ | $m_2$ | $w_1 \succ w_2 \succ w_3 \succ m_2$ |
| $w_3$ | $m_1 \succ m_2 \succ m_3 \succ w_3$ | $m_3$ | $w_1 \succ w_2 \succ w_3 \succ m_3$ |

Table 2: The preferences of each agent.

The allocation  $\pi_1 = ((m_1, w_1), (m_2, w_3), (m_3, w_2))$  is the only stable allocation. But considering the unstable allocation  $\pi_2 = ((m_1, w_2), (m_2, w_3), (m_3, w_1))$ ,  $m_1$  and  $m_3$  receive their first choice. So  $\pi_2$  is strictly better than  $\pi_1$  for  $m_1$  and  $m_3$ . Thus, no allocation can both satisfy sPE4M and Stable. Symmetrically, there is no mechanism satisfying sPE4W and Stable from the example in which all agents reverse their gender.

Next, we show that the men-propose version of DA can satisfy wPE4M. Assume DA gives an allocation  $\pi$ . If there exists another acceptable allocation  $\pi'$ , all men can match with a better partner. Then in DA, each man proposes to the better partner before their partner in  $\pi$ . No proposal will be rejected because each woman only receives one proposal, and the proposal is acceptable. Thus, DA gives  $\pi'$  instead of  $\pi$ , which is a contradiction.

## C PROOF OF SECTION 5.3

### C.1 Proof of Lemma 5

PROOF. Firstly, given  $\theta'_i$ , the shortest path from the initial agent set  $N_0$  to  $i$  is independent of  $r'_i$ . Moreover, for any agent  $j$  whose order is smaller than  $i$ , the shortest path from  $N_0$  to  $j$  is shorter than or equal to that to  $i$ , and  $i$  is not on that path. Thus, every agent  $i$  cannot impact its own order or the agents with a smaller order, so the order of  $i$  in  $\mathcal{P}$  is independent of  $i$ ’s report.  $\square$

### C.2 Proof of Lemma 6

PROOF. First of all, since agent  $i$  is not included in the same alliance as  $N_{out}$ ,  $i$ ’s preference is not considered when DDA generates allocations for  $N_{out}$ , so  $\succ'_i$  does not affect the allocation of  $N_{out}$ .

Next, we show that  $i$ ’s neighbor set report  $r'_i$  cannot affect the previously formed alliances. For  $\forall j \in N_{out}$ , in the round where  $j$  is involved in a dynamic alliance, there exists at least one path from the initial agent set to  $j$  that does not pass  $i$ . Because if all such paths pass  $i$ ,  $i$  must have joined in  $N_{out}$  before  $j$ . So whatever  $r'_i$  is, the alliances formed before  $i$ ’s round remain the same.  $\square$