On Generalized Sampling Series

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***Abstract*—In the present paper, we analyse the approximation behaviour of generalized sampling series and Kantorovich sampling series. First, we obtain the approximation error in-terms of the modulus of continuity for the generalized sampling series for functions in $C(\mathbb{R}^d)$. Further, for the class of log-Hölderian functions, the order of uniform norm convergence is discussed. Furthermore, for functions in $C(\mathbb{R}^d)$ and log-Hölderian functions, we provide the approximation error for the Kantorovich sampling series.**

***Index Terms*—Generalized Sampling Series, Kantorovich Sampling Series, Degree of Approximation, Modulus of Smoothness, Log-Hölderian Class.**

I. Introduction

The celebrated Shannon sampling theorem is one of the fundamental theorem in Fourier analysis. This theorem can be used to convert the analog signals into discrete sequence of samples without loosing any information. If f is band-limited to $[-\pi w, \pi w]$, $w > 0$, then f can be completely reconstructed by

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) \frac{\sin \pi(wt-k)}{\pi(wt-k)}, \quad (t \in \mathbb{R}).$$

The above sampling reconstruction formula has lot of applications in signal and image processing, information theory, communication theory etc. The approximation results of the above sampling series were studied by several researchers. In particular, $L^p(\mathbb{R})$ convergence for the above sampling series was studied by Rahman and Vértesi in [17]. Further, the rate of convergence in-terms of averaged modulus of smoothness for the above series in $L^p(\mathbb{R})$-norm for $1 < p < \infty$ for non-smooth signals were analyzed in [5]. The approximation of multivariate signals with respect $L^p(\mathbb{R}^n)$-norm for $1 < p < \infty$ were analyzed in [3]. Burinska, Runovski and Schmeisser generalized the above sampling series in [6]. The generalized sampling series of f with respect to the kernel $\mathcal{F}\psi \in L_1(\mathbb{R}^d)$ is given by

$$S_\sigma(f; \mathbf{x}) = (2\pi)^{-d} \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{\sigma}\right) \mathcal{F}\psi(\sigma\mathbf{x} - k), \quad (\mathbf{x} \in \mathbb{R}^d)$$

where $\mathcal{F}$ denotes the Fourier transform and ψ is compactly supported functions satisfying some more additional conditions. We note that if $(2\pi)^{-d}\mathcal{F}\psi(\mathbf{x}) = \sin c(\mathbf{x})$, then above generalized sampling series reduces to the classical sampling reconstruction formula. We further see that if $(2\pi)^{-d}\mathcal{F}\psi(\mathbf{x}) = \psi(\mathbf{x})$, then the above sampling series reduces to the sampling series considered by Butzer and his collaborators, see [8]–[12].

Let f be a continuous function defined on $\mathbb{R}^d$ and let $\sigma > 0$. Then we consider the following generalized sampling series of f with respect to the kernel K_σ^ψ by

$$S_\sigma(f; \mathbf{x}) = (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{\sigma}\right) K_\sigma^\psi \left(\mathbf{x} - \frac{k}{\sigma}\right), \ (\mathbf{x} \in \mathbb{R}^d)$$

where

$$K_\sigma^\psi(\mathbf{x}) = \mathcal{F}\psi\left(\frac{\cdot}{\sigma}\right)(\mathbf{x}),$$

such that ψ is continuous on $\mathbb{R}^d$, centrally symmetric, have a compact support, satisfying $\psi(0) = 1$ and $\mathcal{F}\psi \in L_1$. The uniform and point-wise approximation theorems and the approximation results with respect to L_p metric for the above generalized sampling series S_σ were discussed in [6]. The quality of approximation by generalized sampling series in-terms of suitable K-functional were analyzed in [7] with respect to L_p metric. In this paper, we estimate the error in the approximation by generalized sampling series and Kantorovich sampling series. In order to do this, we need certain assumptions on the kernel K_σ^ψ.

In view of Lemma 3.4 [6], the kernel K_σ^ψ satisfies the following conditions:

(K1) $(2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} K_\sigma^\psi \left(\mathbf{x} - \frac{k}{\sigma}\right) = 1$

(K2) $M_0(\psi) = \sup_{\mathbf{x}\in\mathbb{R}^d} (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left| K_\sigma^\psi \left(\mathbf{x} - \frac{k}{\sigma}\right)\right| < \infty.$

Further, we assume that K_σ^ψ satisfies the following condition:

(K3) For some $\beta > 0$, we have

$$M_\beta(\psi) = \sup_{\mathbf{x}\in\mathbb{R}^d} (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left| K_\sigma^\psi \left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \left\| \mathbf{x} - \frac{k}{\sigma}\right\|_2^\beta < \infty,$$

where $\|\cdot\|_2$ denotes the usual Euclidean norm on $\mathbb{R}^d$.

Remark 1. *It is easy to see that if $M_\beta(\psi) < \infty$, for some $\beta > 0$, then we get $M_\nu(\psi) < \infty$, for every $0 \le \nu \le \beta$.*

Now, we shall recall the definition of modulus of continuity for $f \in C(\mathbb{R}^d)$.

Definition 1. *For $f \in C(\mathbb{R}^d)$, the modulus of continuity is defined by*

$$\omega(f,\delta) = \sup_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{R}^d \\ \|\mathbf{x}-\mathbf{y}\|_2 \le \delta}} |f(\mathbf{x}) - f(\mathbf{y})|, \quad \delta > 0.$$

$w(f,\delta)$ satisfies the following properties:

(i) $\omega(f,\delta) \to 0$, as $\delta \to 0$.

(ii) $\omega(f,\lambda\delta) \le (\lambda+1)\omega(f,\delta)$, for every $\lambda > 0$.

(iii) $|f(\mathbf{x}) - f(\mathbf{y})| \le \omega(f,\delta)\left(1 + \dfrac{\|\mathbf{x}-\mathbf{y}\|_2}{\delta}\right)$.

Next, we define the Log-Hölderian class.

Definition 2. *Let $0 < \alpha \le 1$. The log-Hölderian function of order α is defined by*

$$\begin{aligned} L_\alpha \;\; := \;\; & \{f : \mathbb{R}^d \to \mathbb{R} : \;\; \exists M > 0 \;\; s.t. \\ & |f(\mathbf{x}) - f(\mathbf{y})| \le M\|\mathbf{x}-\mathbf{y}\|_2^\alpha, \;\; \mathbf{x},\mathbf{y} \in \mathbb{R}^d\}. \end{aligned}$$

II. Degree of approximation

In this section, we obtain the error estimates of the generalized sampling series S_σ.

Theorem 1. *Let $f \in C(\mathbb{R}^d)$, $\sigma > 0$ and $\mathbf{x} \in \mathbb{R}^d$. Suppose that $M_\beta(\psi) < \infty$, for some $\beta \ge 1$. Then, we have*

$$|(S_\sigma f)(\mathbf{x}) - f(\mathbf{x})| \le M_0(\psi)\;\omega(f,\delta) + \frac{2K}{\sigma\delta}\;M_1(\psi),$$

where $K > 0$ is a constant.

Proof. Let $\mathbf{x} \in \mathbb{R}^d$. By the definition of the sampling operators S_σ and by the condition (K1), we have

$$\begin{aligned} & |(S_\sigma f)(\mathbf{x}) - f(\mathbf{x})| \\ & = \left|(2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{\sigma}\right) K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right) - f(\mathbf{x})\right| \\ & = \left|(2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{\sigma}\right) K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right) - \right. \\ & \qquad \left. (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} f(\mathbf{x})\; K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \\ & \le (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left|f\left(\frac{k}{\sigma}\right) - f(\mathbf{x})\right| \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \end{aligned}$$

$$\begin{aligned} & = (2\pi\sigma)^{-d} \left(\sum_{\substack{k\in\mathbb{Z}^d \\ \|\frac{k}{\sigma}-\mathbf{x}\|_2 \le \delta}} + \sum_{\substack{k\in\mathbb{Z}^d \\ \|\frac{k}{\sigma}-\mathbf{x}\|_2 > \delta}} \right) \left|f\left(\frac{k}{\sigma}\right) - f(\mathbf{x})\right| \\ & \qquad \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \\ & := I_1 + I_2. \end{aligned}$$

First, we estimate I_1. Using the definition of modulus of continuity, we obtain

$$\begin{aligned} I_1 & \le (2\pi\sigma)^{-d} \sum_{\|\frac{k}{\sigma}-\mathbf{x}\|_2 \le \delta} \omega(f,\delta) \; \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \\ & \le M_0(\psi)\;\omega(f,\delta). \end{aligned}$$

Using the fact that the function is bounded on $\mathbb{R}^d$, we obtain

$$\begin{aligned} I_2 & \le 2K(2\pi\sigma)^{-d} \sum_{\|\frac{k}{\sigma}-\mathbf{x}\|_2 > \delta} \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \\ & \le 2K(2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \frac{\|k - \mathbf{x}\sigma\|_2}{\delta\sigma} \\ & \le \frac{2K}{\sigma\delta}\;M_1(\psi). \end{aligned}$$

Combining the estimates I_1 and I_2, we get

$$|(S_\sigma f)(\mathbf{x}) - f(\mathbf{x})| \le M_0(\psi)\;\omega(f,\delta) + \frac{2K}{\sigma\delta}\;M_1(\psi).$$

Thus, the proof is completed. □

Remark 2. *If we choose $\delta = \dfrac{1}{\sigma}$, we obtain*

$$|(S_\sigma f)(x) - f(x)| \le M_0(\psi)\;\omega\left(f, \frac{1}{\sigma}\right) + 2K\;M_1(\psi).$$

Theorem 2. *Let $f \in C(\mathbb{R}^d)$, $\sigma > 0$ and $\mathbf{x} \in \mathbb{R}^d$. Suppose that $M_\beta(\psi) < \infty$, for some $\beta \ge 1$. Then, we have*

$$|(S_\sigma f)(\mathbf{x}) - f(\mathbf{x})| \le C\;\omega\left(f, \frac{1}{\sigma}\right),$$

where $C = M_0(\psi) + M_1(\psi)$.

Proof. In view of the property (iii) of the modulus of smoothness, we can write

$$\begin{aligned} & |(S_\sigma f)(\mathbf{x}) - f(\mathbf{x})| \\ & \le (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left|f\left(\frac{k}{\sigma}\right) - f(\mathbf{x})\right| \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \\ & \le \omega(f,\delta) \left((2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \right. \\ & \qquad \left. \left(1 + \frac{\|\frac{k}{\sigma} - \mathbf{x}\|_2}{\delta}\right)\right) \\ & \le M_0(\psi)\omega(f,\delta) + \frac{\omega(f,\delta)}{\delta} \sum_{k\in\mathbb{Z}^d} \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \left\|\frac{k}{\sigma} - \mathbf{x}\right\|_2 \\ & = \omega(f,\delta)\left(M_0(\psi) + \frac{M_1(\psi)}{\sigma\delta}\right). \end{aligned}$$

Choosing $\delta = \dfrac{1}{\sigma}$, we obtain

$$\begin{aligned} |(S_\sigma f)(\mathbf{x}) - f(\mathbf{x})| & \le \omega\left(f, \frac{1}{\sigma}\right)(M_0(\psi) + M_1(\psi)) \\ & \le C\;\omega\left(f, \frac{1}{\sigma}\right). \end{aligned}$$

Hence, the proof is completed. □

Theorem 3. *Suppose that $M_\beta(\psi) < \infty$, for some $\beta \geq 1$. For $f \in L_\alpha$, we have*

$$\|S_\sigma f - f\|_\infty \leq \frac{KM_\alpha(\psi)}{\sigma^\alpha},$$

where $K > 0$ is a constant.

Proof. Let $f \in L_\alpha$. Then by the definition of sampling operators S_σ, we have

$$\begin{aligned}
&|(S_\sigma f)(\mathbf{x}) - f(\mathbf{x}))| \\
&= \left|(2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} f\left(\frac{k}{\sigma}\right) K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right) - f(\mathbf{x})\right| \\
&\leq (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left|f\left(\frac{k}{\sigma}\right) - f(\mathbf{x})\right| \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \\
&\leq K(2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \left\|\frac{k}{\sigma} - \mathbf{x}\right\|_2^\alpha \\
&\leq K(2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \frac{\|k - \mathbf{x}\sigma\|_2^\alpha}{\sigma^\alpha} \\
&\leq \frac{KM_\alpha(\psi)}{\sigma^\alpha}.
\end{aligned}$$

Since $M_\beta(\psi) < \infty$, for some $\beta \geq 1$, and so by the Remark 1, we have $M_\alpha(\psi) < \infty$, for $0 < \alpha \leq 1$. This completes the proof. □

III. Kantorovich Generalized Sampling Series

In this section, we consider the Kantorovich variant of the generalized sampling series S_σ. Let $\mathbf{x} \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$ and $\sigma > 0$. Then, we define the Kantorovich generalized sampling series by

$$(K_\sigma f)(\mathbf{x}) = (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left(\sigma^d \int_{I_k^\sigma} f(u)du\right) K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)$$

where $I_k^\sigma := \left[\frac{k_1}{\sigma}, \frac{k_1+1}{\sigma}\right] \times \left[\frac{k_2}{\sigma}, \frac{k_2+1}{\sigma}\right] \times \ldots \times \left[\frac{k_d}{\sigma}, \frac{k_d+1}{\sigma}\right]$ and $f : \mathbb{R}^d \to \mathbb{R}$ is a locally integrable function such that above series is converges. The Kantorovich sampling series is used to approximate the measurable, locally integrable and not-necessarily continuous functions. These operators helps to reduce the time-jitter errors, which frequently occurs in signal processing. We note that if we take $(2\pi)^{-d}\mathcal{F}\psi(\mathbf{x}) = \psi(\mathbf{x})$, then the above sampling series K_σ reduces to the classical Kantorovich sampling series which is considered by several researchers, see [1], [2], [4], [13]–[15] etc.

Remark 3. *It is easy to see that the above Kantorovich generalized sampling operators are well defined for $f \in L^\infty(\mathbb{R}^d)$. Indeed, in view of the condition (K2) we obtain*

$$\begin{aligned}
|(K_\sigma f)(\mathbf{x})| &\leq \|f\|_\infty (2\pi\sigma)^{-d} \sum_{k\in\mathbb{Z}^d} \left|K_\sigma^\psi\left(\mathbf{x} - \frac{k}{\sigma}\right)\right| \\
&\leq \|f\|_\infty M_0(\psi),
\end{aligned}$$

for every $\mathbf{x} \in \mathbb{R}^d$ and $\sigma > 0$.

Remark 4. *By the above Remark, we note that Kantorovich generalized sampling operators is a bounded linear operators maps $L^\infty(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$.*

Now we shall discuss the direct approximation results for the Kantorovich exponential sampling operators. Since the proof techniques are similar to the results discussed in Section 2, so omit the proof details.

Theorem 4. *Let $f \in C(\mathbb{R}^d)$, $\sigma > 0$ and $\mathbf{x} \in \mathbb{R}^d$. Suppose that $M_\beta(\psi) < \infty$, for some $\beta \geq 1$. Then, we have*

$$|(K_\sigma f)(\mathbf{x}) - f(\mathbf{x})| \leq C\ \omega\left(f, \frac{1}{\sigma}\right),$$

where $C = M_0(\psi) + M_1(\psi)$.

Theorem 5. *Let $f \in L_\alpha$. Suppose that $M_\beta(\psi) < \infty$, for some $\beta \geq 1$. Then, we have*

$$|(K_\sigma f)(\mathbf{x}) - f(\mathbf{x})| \leq \frac{KM_\alpha(\psi)}{\sigma^\alpha},$$

where $K > 0$ is a constant.

IV. Examples

In this section, we consider the following examples. We construct the multivariate kernel as follows:

$$\chi(\mathbf{x}) := \prod_{i=1}^n \chi_i(x_i), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^d$, $\chi_i(x)$ denotes the uni-variate kernel.

A. Example 1

As a first example, we consider the Jackson-type kernels (see [16]) for $d = 1$ as follows: Let

$$\varphi(\zeta) = \begin{cases} 1 - |\zeta|, & \text{if } |\zeta| \leq 1 \\ 0, & \text{if } |\zeta| > 1. \end{cases}$$

The Fourier transform for φ is given by

$$\mathcal{F}\varphi(x) = \frac{4sin^2(x/2)}{x^2}.$$

Since $K_\sigma^\psi(x) = \sigma\mathcal{F}\psi(\sigma x)$, we obtain

$$K_\sigma^\psi(x) = \sigma\frac{4^N sin^{2N}(\sigma x/2)}{(\sigma x)^{2N}},$$

where $\psi = \varphi * \varphi * \ldots * \varphi$ (N times). Clearly $K_\sigma^\psi(x) = \mathcal{O}(|x|^{-2N})$, as $|x| \to \infty$, see [16]. Thus, condition (K3) is satisfied for $N \geq 2$. Now, we define the multivariate kernel as follows: Now by (1), we can write

$$K_\sigma^\psi(\mathbf{x}) := \prod_{i=1}^d K_\sigma^\psi(x_i).$$

Since $K_\sigma^\psi(x_i)$ satisfies the condition (K3), this implies that $K_\sigma^\psi(\mathbf{x})$ also satisfies (K3) for $N \geq 2$.

B. Example 2

We now consider Bochner-Riesz type kernel, see [16]. The Kernel is defined by

$$K_\sigma^\Theta(x) = \sigma \mathcal{F}\Theta(\sigma x)$$

which can be expressed as

$$K_\sigma^\Theta(x) = \sigma^{d-\eta-1/2}\frac{2^\eta}{\sqrt{2}\pi}\Gamma(\eta+1)|\sigma x|^{-\eta-1/2}J_{\eta+1/2}(|\sigma x|),$$

where $x \in \mathbb{R}, \eta > 0$ and J_λ is the Bessel function of order λ defined by

$$J_\lambda(x) = \frac{1}{\Gamma(\lambda+1/2)\Gamma(1/2)}\left(\frac{x}{2}\right)^\lambda \int_{-1}^{1} e^{ixu}(1-u^2)^{\lambda-\frac{1}{2}}du.$$

Here, Γ is the usual Euler gamma function. The function Θ is given by

$$\Theta(\zeta) = \begin{cases}(1-\zeta^2)^\eta, & \text{if } |\zeta| \leq 1\\ 0, & \text{if } |\zeta| > 1.\end{cases}$$

Clearly, we have $K_\sigma^\Theta(x) = \mathcal{O}(|x|^{-\eta-1})$, as $|x| \to \infty$, see [16]. Thus, condition (K3) is satisfied for $\eta > 1$. By the equation (1), we have

$$K_\sigma^\Theta(\mathbf{x}) := \prod_{i=1}^{d} K_\sigma^\Theta(x_i).$$

We know that $K_\sigma^\Theta(x_i)$ satisfies the condition (K3), this implies that $K_\sigma^\Theta(\mathbf{x})$ also satisfies the condition (K3) for $\eta > 1$.

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