Frames by Iterations and Invariant Subspaces

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***Abstract*—This paper presents a characterization of systems of iterations that generate frames of abstract separable Hilbert spaces. The characterization is achieved through a correspondence with a canonical system of iterations that form Parseval frames of certain subspaces of the space of vector-valued functions $L^2(\mathbb{T},\mathcal{K})$, where $\mathcal{K}$ is a Hardy space with multiplicity. These subspaces possess the property of being invariant under two shift operators with multiplicity.**

Furthermore, we provide a clear description of the subspaces generated by these canonical systems of iterations.

I. INTRODUCTION

In recent years, there has been extensive research on systems of iterations that generate frames. One of the driving factors behind this research is the application of such systems to the Dynamical Sampling (DS) problem [5], [6], [7], which involves the reconstruction of a signal from its time-space samples. For applications of DS see [8], [9], [10], [12], [19].

The DS problem asks whether it is possible to reconstruct an element f of a separable Hilbert space $\mathcal{H}$ in a stable manner from a set of samples $\{\langle T^j f, v_i\rangle : i \in I, j \in J\}$, where $T : \mathcal{H} \to \mathcal{H}$ is a linear and bounded operator, I and J are at most countable index sets, and $\{v_i\}_{i\in I} \subset \mathcal{H}$ is a set of vectors. It can be shown that answering this question is equivalent to finding conditions on T, I, J and $\{v_i\}_{i\in I} \subset \mathcal{H}$ such that the set $\left\{T^{*j} v_i : i \in I, j \in J\right\}$ forms a frame of $\mathcal{H}$ [5].

This motivates the problem of characterizing systems of iterations by a single operator acting on a set of vectors that generate frames of Hilbert spaces. For example, in [5], this problem was completely solved for the finite-dimensional case. The infinite-dimensional case for normal operators and several generators was also solved. See [4], [5], [11], [16], [17]. There, the authors provide a characterization of those frames using tools from complex analysis, making the first connection with Hardy spaces and interpolation sets.

Later on, for general operators, the authors in [18] found that the orbit of one vector by a single operator in a separable Hilbert space forms a frame if and only if the operator is similar to the compression of the shift in a model space of the Hardy space in the open unit disk. The isomorphism that gives the similarity is given by the restriction of the synthesis operator to the model space. This was later extended in [17] to the case of several generators, which involves Hardy spaces with multiplicity. These results are valid for the two cases: when the iteration runs over $\mathbb{N}$ (unilateral case) and over $\mathbb{Z}$ (bilateral case). However, the bilateral case only applies when the iterated operator is invertible.

In the case of shift-invariant spaces (SIS) (i.e. spaces invariant under translation along $\mathbb{Z}$) in $L^2(\mathbb{R})$, the functions $\Phi = \{\phi_i\}_{i\in I}$ that give a frame by translations have been completely characterized some years ago, by several authors, see for example [14], [20]. In [1], the authors address the problem of the existence of a second linear and bounded operator L, that acts on the SIS and commutes with translations (known in the literature as shift-preserving operator). They ask whether the operator L can be used to form a frame if the translations of the functions in Φ do not suffice. That is, they look for conditions on L and Φ such that the system of translations of the iterations of L on Φ forms a frame of the SIS. Sufficient and necessary conditions were found for the case when L is normal and the SIS is finitely generated.

The latter question was extended in [2] to the general case of an abstract separable Hilbert space $\mathcal{H}$, considering the system of orbits of a set of vectors $\{v_i\}_{i\in I} \subset \mathcal{H}$ by iterations of two bounded commuting operators T and L acting on $\mathcal{H}$ (see (1)). Here, T is an invertible operator but not necessarily unitary. A characterization of such systems that form a frame of $\mathcal{H}$ was obtained with different techniques to those in [1].

The primary approach used in [3] involves examining the kernel of the synthesis operator in a suitable space. This kernel can be viewed as a subspace of $L^2(\mathbb{T}, H^2_{\ell^2(I)})$, and it is demonstrated that it reduces the bilateral shift while being invariant under the local action of the unilateral shift (as defined in III.2). This observation raised the question of identifying the subspaces of $L^2(\mathbb{T}, H^2_{\mathcal{K}})$ with similar invariant properties, where $\mathcal{K}$ is a separable Hilbert space. This question was answered in [3].

The characterization of invariant subspaces of the unilateral shift operator in the Hardy space H^2 was initially achieved by Beurling [13] and later generalized by Lax [23] and Halmos [21] to include Hardy spaces with multiplicity. Helson and Lowdenslager [22] subsequently characterized the subspaces of $L^2(\mathbb{T},\mathcal{K})$ which reduce the bilateral shift. The characterization presented in [3] follows the same framework as these influential theorems.

In this note, we provide a survey of the results obtained in [2] (Section IV) and [3] (Section V). We refer the reader to the original articles for a more comprehensive development of the results presented here.

II. Notation and definitions

In this article, we will write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the letters $\mathcal{H}$ and $\mathcal{K}$ will denote separable complex Hilbert spaces. The set of linear and bounded operators from $\mathcal{H}$ to $\mathcal{K}$ will be denoted, as usual, by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$.

Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. We say that $\mathcal{M}$ is *invariant* under an operator $T \in \mathcal{B}(\mathcal{H})$ if $T(\mathcal{M}) \subseteq \mathcal{M}$. In the case when $\mathcal{M}$ is invariant under T and also under its adjoint operator T^*, it is said that $\mathcal{M}$ *reduces* T (or that $\mathcal{M}$ is a reducing subspace for T).

III. Spaces of measurable vector-valued functions

In this section we give an introduction to the basic theory of vector-valued functions that we need in the following sections.

Definition III.1. A vector-valued function $f : \mathbb{T} \to \mathcal{K}$ is said to be measurable if for each $x \in \mathcal{K}$, the complex-valued function $\lambda \mapsto \langle f(\lambda), x\rangle_{\mathcal{K}}$ is measurable on $\mathbb{T}$.

Denote by $L^2(\mathbb{T}, \mathcal{K})$ the Hilbert space of all measurable vector-valued functions $f : \mathbb{T} \to \mathcal{K}$ such that $\int_{\mathbb{T}} \|f(\lambda)\|^2_{\mathcal{K}}\, d\lambda$ is finite, equipped with the inner product

$$\langle f, g\rangle = \int_{\mathbb{T}} \langle f(\lambda), g(\lambda)\rangle_{\mathcal{K}}\, d\lambda.$$

Observe that any $v \in \mathcal{K}$ induces a vector-valued function, namely the function that is constantly v over $\mathbb{T}$. Thus, $\mathcal{K}$ can be seen as a subspace of $L^2(\mathbb{T}, \mathcal{K})$.

If we choose an orthonormal basis $\mathcal{B} = \{\varepsilon_i\}_{i\in I}$ of $\mathcal{K}$, then we have the following expansion of a function $f \in L^2(\mathbb{T}, \mathcal{K})$:

$$f(\lambda) = \sum_{i\in I} \langle f(\lambda), \varepsilon_i\rangle_{\mathcal{K}}\, \varepsilon_i, \quad \text{a.e } \lambda \in \mathbb{T}.$$

We define the i-th coordinate function of f with respect to the basis $\mathcal{B}$ as $f_i := \langle f(\cdot), \varepsilon_i\rangle_{\mathcal{K}}$. It can be easily seen that $f_i \in L^2(\mathbb{T})$ for all $i \in I$.

The *Hardy space with multiplicity* is denoted by $H^2_{\mathcal{K}}$ and it can be defined as the space of all vector-valued functions $f \in L^2(\mathbb{T}, \mathcal{K})$ such that their coordinates functions $\{f_i\}_{i\in I}$ (with respect to any orthonormal basis $\mathcal{B}$ of $\mathcal{K}$) belong to the scalar Hardy space H^2, where

$$H^2 := \left\{ f \in L^2(\mathbb{T}) : \int_{\mathbb{T}} f(z) z^{-n}\, dz = 0 \text{ for } n < 0 \right\}.$$

Acting on these spaces we can consider three fundamental shift operators.

Definition III.2. Let $\mathcal{K}$ be a separable Hilbert space.

1) The *bilateral shift with multiplicity* is the operator $U : L^2(\mathbb{T}, \mathcal{K}) \to L^2(\mathbb{T}, \mathcal{K})$, defined by
$$Uf(\lambda) = \lambda f(\lambda), \quad \text{a.e } \lambda \in \mathbb{T} \text{ and } f \in L^2(\mathbb{T}, \mathcal{K}).$$
2) The *unilateral shift with multiplicity* is the operator $S : H^2_{\mathcal{K}} \to H^2_{\mathcal{K}}$ given by the restriction of U to $H^2_{\mathcal{K}}$.
3) The *pointwise shift* is the operator defined by $\widehat{S} : L^2(\mathbb{T}, H^2_{\mathcal{K}}) \to L^2(\mathbb{T}, H^2_{\mathcal{K}})$,
$$\widehat{S}f(\lambda) = S(f(\lambda)), \quad \text{a.e } \lambda \in \mathbb{T} \text{ and } f \in L^2(\mathbb{T}, H^2_{\mathcal{K}}).$$

Observe that with these operators one can construct orthonormal bases of $L^2(\mathbb{T}, \mathcal{K})$, $H^2_{\mathcal{K}}$ and $L^2(\mathbb{T}, H^2_{\mathcal{K}})$, formed by the iterations over an orthonormal basis of the underlying space $\mathcal{K}$ (see, for instance, [2]).

Proposition III.1. *Let $\mathcal{B} = \{\varepsilon_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{K}$. Then,*

1) *The set $\{U^k \varepsilon_i : k \in \mathbb{Z}, i \in I\}$ is an orthonormal basis of $L^2(\mathbb{T}, \mathcal{K})$.*
2) *The set $\{S^j \varepsilon_i : j \in \mathbb{N}_0, i \in I\}$ is an orthonormal basis of $H^2_{\mathcal{K}}$.*
3) *The set $\{U^k \widehat{S}^j \varepsilon_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\}$ is an orthonormal basis of $L^2(\mathbb{T}, H^2_{\mathcal{K}})$.*

IV. Frames by iterations of two operators

The main goal of this section is to review the necessary and sufficient conditions, found in [2], on two commuting operators T and L acting on a separable Hilbert space $\mathcal{H}$, and an at most countable set of vectors $\{v_i\}_{i\in I} \subset \mathcal{H}$, in order that the system of iterations

$$\left\{T^k L^j v_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\right\} \tag{1}$$

forms a frame of the corresponding space $\mathcal{H}$.

The characterization consists in establishing a correspondence between the system (1) and a *canonical system of iterations* that forms a Parseval frame of $L^2(\mathbb{T}, \mathcal{K})$, for a convenient Hilbert space $\mathcal{K}$.

Definition IV.1. For $l = 1, 2$, let $\mathcal{H}_l$ be a separable Hilbert space. Let T_l, $L_l \in \mathcal{B}(\mathcal{H})$ be such that T_l is invertible and $T_l L_l = L_l T_l$. Let $\{v_i\}_{i\in I} \subset \mathcal{H}_1$ and $\{w_i\}_{i\in I} \subset \mathcal{H}_2$ be two sets of vectors, where I is an at most countable index set.

We say that the systems of iterations

$$\left\{T_1^k L_1^j v_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\right\} \tag{2}$$

and

$$\left\{T_2^k L_2^j w_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\right\} \tag{3}$$

are *equivalent* if there exists an isomorphism $\Psi \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\Psi(v_i) = w_i$ for every $i \in I$, and the following intertwining properties hold

$$\Psi T_1 = T_2 \Psi \quad \text{and} \quad \Psi L_1 = L_2 \Psi.$$

In the case when Ψ is unitary, we say that the systems (2) and (3) are *unitarily equivalent*.

We have the following lemma.

Lemma IV.1 ([2]). *The frame property is preserved under equivalent systems of iterations. Moreover, if two equivalent systems of iterations are Parseval frames of their corresponding Hilbert space, then they are unitarily equivalent.*

Assuming that the system (1) is a Bessel sequence in $\mathcal{H}$, its associated synthesis operator can be defined by

$$C : L^2(\mathbb{T}, H^2_{\ell^2(I)}) \to \mathcal{H}, \quad Cf = \sum_{k,j,i} f^i_{k,j} T^k L^j v_i \tag{4}$$

where $f^i_{k,j} := \langle f, U^k \widehat{S}^j \delta_i \rangle$ and $\{\delta_i\}_{i \in I}$ is the canonical orthonormal basis of $\ell^2(I)$. Typically, C is defined on the space of sequences $\ell^2(\mathbb{Z} \times \mathbb{N}_0 \times I)$, but in view of item 3) of Proposition III.1, we have that the space $\ell^2(\mathbb{Z} \times \mathbb{N}_0 \times I)$ is isometrically isomorphic to the space $L^2(\mathbb{T}, H^2_{\ell^2(I)})$ via the correspondence $f \mapsto \{\langle f, U^k \widehat{S}^j \delta_i \rangle\}$.

Below, we enumerate some properties of C, which are key for establishing the main result.

Proposition IV.1 ([2])**.** *Let C be the synthesis operator associated to the system* (1)*, given by* (4)*. Then,*

1) *If $\{\delta_i\}_{i \in I}$ is the canonical orthonormal basis of $\ell^2(I)$, then $C(\delta_i) = v_i$ for all $i \in I$. Moreover, C satisfies the following intertwining properties*

$$CU = TC, \quad CU^* = T^{-1}C \quad \text{and} \quad C\widehat{S} = LC.$$

2) *The subspace $\ker(C)^\perp \subseteq L^2(\mathbb{T}, H^2_{\ell^2(I)})$ reduces U and is invariant for $\widehat{S}^*$.*

For a closed subspace $\mathcal{N} \subseteq L^2(\mathbb{T}, H^2_{\ell^2(I)})$, we define the compression of $\widehat{S}$ to $\mathcal{N}$ as the operator $A_\mathcal{N} : \mathcal{N} \to \mathcal{N}$, given by

$$A_\mathcal{N} := P_\mathcal{N} \widehat{S}|_\mathcal{N},$$

where $P_\mathcal{N}$ is the orthogonal projection from $L^2(\mathbb{T}, H^2_{\ell^2(I)})$ onto $\mathcal{N}$.

Lemma IV.2 ([2])**.** *If $\mathcal{N} \neq \{0\}$ is a closed subspace of $L^2(\mathbb{T}, H^2_{\ell^2(I)})$ that reduces U and is invariant for $\widehat{S}^*$, then the system*

$$\left\{U^k A^j_\mathcal{N} P_\mathcal{N} \delta_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\right\} \tag{5}$$

forms a Parseval frame of $\mathcal{N}$.

From now on, any frame of the form (5) will be called a *canonical system of iterations*.

Proposition IV.2 ([2])**.** *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two closed subspaces of $L^2(\mathbb{T}, H^2_{\ell^2(I)})$ that reduce U and are invariant for $\widehat{S}^*$, and for all $i \in I$, let $\varphi^1_i = P_{\mathcal{N}_1}\delta_i$ and $\varphi^2_i = P_{\mathcal{N}_2}\delta_i$. If the canonical systems of iterations*

$$\left\{U^k A^j_{\mathcal{N}_1} \varphi^1_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\right\} \tag{6}$$

and

$$\left\{U^k A^j_{\mathcal{N}_2} \varphi^2_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\right\} \tag{7}$$

are equivalent, then $\mathcal{N}_1 = \mathcal{N}_2$ and hence $\varphi^1_i = \varphi^2_i$ for all $i \in I$.

In the following, we formulate the main result of this section which characterizes the systems of iterations that form a frame (a Parseval frame or a Riesz basis) of the corresponding Hilbert space, in terms of the equivalence relation given by Definition IV.1.

The key is to observe that if the system (1) is a frame, then the synthesis operator C is surjective. Thus, we can consider the restriction of C to the subspace $\ker(C)^\perp$, which induces by Proposition IV.1 an equivalence relation between the system (1) and the canonical system of iterations associated to $\mathcal{N} = \ker(C)^\perp$.

Theorem IV.1 ([2])**.** *The following statements hold:*

1) *The system of iterations* (1) *forms a frame of $\mathcal{H}$ if and only if it is equivalent to a canonical system of iterations.*
2) *The system of iterations* (1) *forms a Parseval frame of $\mathcal{H}$ if and only if it is unitarily equivalent to a canonical system of iterations.*
3) *The system of iterations* (1) *forms a Riesz basis of $\mathcal{H}$ if and only if it is equivalent to the canonical system of iterations corresponding to the whole space $L^2(\mathbb{T}, H^2_{\ell^2(I)})$. This means that the system* (1) *is equivalent to the orthonormal basis*

$$\left\{U^k \widehat{S}^j \delta_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\right\}.$$

Observe that by Proposition IV.2, the canonical system of iterations equivalent to (1) is unique.

Remark IV.1. Theorem IV.1 can be also stated when considering systems of iterations of T and L over $\mathbb{Z}$, i.e., systems of the form

$$\left\{T^k L^j v_i : k, j \in \mathbb{Z}, i \in I\right\}, \tag{8}$$

taking into account that L is now assumed to be invertible. To do that, it is useful to observe that $\ell^2(\mathbb{Z}^2 \times I)$ is isometrically isomorphic to $L^2(\mathbb{T}^2, \ell^2(I))$. Consider two shift operators U_1 and U_2 acting on $L^2(\mathbb{T}^2, \ell^2(I))$ defined by

$$U_i f(z_1, z_2) = z_i f(z_1, z_2), \quad i = 1, 2$$

for every $f \in L^2(\mathbb{T}^2, \ell^2(I))$ and a.e. $(z_1, z_2) \in \mathbb{T}^2$. Thus, in this case, a canonical system of iterations will be a system of the form

$$\left\{U_1|^k_\mathcal{N} U_2|^j_\mathcal{N} \varphi_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\right\} \tag{9}$$

where $\mathcal{N}$ is a closed subspace of $L^2(\mathbb{T}^2, \ell^2(I))$ that reduces U_1 and U_2 simultaneously and $\varphi_i = P_\mathcal{N}\delta_i$ for each $i \in I$. See [2] for more details.

V. Reducing and invariant subspaces under two shift operators

Motivated by the characterization achieved in Theorem IV.1, the structure of the subspaces of $L^2(\mathbb{T}, H^2_\mathcal{K})$ that reduce U and are invariant for $\widehat{S}$, for some separable Hilbert space $\mathcal{K}$, was investigated in [3]. Here, we provide an overview of the results obtained.

To do this we recall some aspects on the theory of reducing subspaces for U and its connection with range functions. We remark that these are particular cases of multiplication-invariant subspaces discussed by Bownik and Ross in [15].

Definition V.1. A range function J in $\mathcal{K}$ is a mapping

$$J : \mathbb{T} \to \{\text{closed subspaces of } \mathcal{K}\}.$$

We say that J is measurable if for each $x, y \in \mathcal{K}$ the complex-valued function $\lambda \mapsto \langle P_{J(\lambda)} x, y \rangle$ is measurable, where $P_{J(\lambda)}$ denotes the orthogonal projection of $\mathcal{K}$ onto $J(\lambda)$.

We have the following characterization theorem for the subspaces of $L^2(\mathbb{T}, \mathcal{K})$ that reduce U.

Theorem V.1 ([15])**.** *A closed subspace $\mathcal{M} \subseteq L^2(\mathbb{T}, \mathcal{K})$ reduces U if and only if there exists a measurable range function J such that*

$$\mathcal{M} = \left\{ f \in L^2(\mathbb{T}, \mathcal{K}) : f(\lambda) \in J(\lambda) \text{ for a.e. } \lambda \in \mathbb{T} \right\}.$$

Identifying the range functions which are equal almost everywhere, the correspondence between reducing subspaces for U and measurable range functions is one-to-one and onto.

In order to provide a description of the closed subspaces of $L^2(\mathbb{T}, H^2_{\mathcal{K}})$ that are reducing for U and invariant for $\widehat{S}$, we need to introduce the subclass of the closed subspaces that are reducing for U and $\widehat{S}$ simultaneously.

Definition V.2. Let J be a measurable range function in $\mathcal{K}$. The *full-Hardy* subspace with *base* J is the unique closed subspace $\mathcal{W}$ of $L^2(\mathbb{T}, H^2_{\mathcal{K}})$ that reduces U whose range function is given by $\lambda \mapsto H^2_{J(\lambda)}$ for a.e. $\lambda \in \mathbb{T}$.

The full-Hardy subspaces reduce U and $\widehat{S}$, even more, they are the only subspaces with that property, as we establish below.

Theorem V.2 ([3])**.** *A subspace $\mathcal{W} \subset L^2(\mathbb{T}, H^2_{\mathcal{K}})$ is reducing for U and $\widehat{S}$ if and only if $\mathcal{W}$ is a full-Hardy subspace.*

Finally, we state the main theorem of this section. We recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry if T is an isometry in $\ker(T)^{\perp}$.

Theorem V.3 ([3])**.** *Let $\mathcal{M}$ be a closed subspace of $L^2(\mathbb{T}, H^2_{\mathcal{K}})$. The following statements are equivalent:*

1) *$\mathcal{M}$ is reducing for U and invariant for $\widehat{S}$.*
2) *There exists a full-Hardy space $\mathcal{W} \subset L^2(\mathbb{T}, H^2_{\mathcal{K}})$ and a partial isometry $\Phi : L^2(\mathbb{T}, H^2_{\mathcal{K}}) \to L^2(\mathbb{T}, H^2_{\mathcal{K}})$ with $\ker(\Phi) = \mathcal{W}^{\perp}$, that commutes with U and $\widehat{S}$, and such that $\Phi(\mathcal{W}) = \mathcal{M}$.*

The idea behind this result builds over the observation that if $\mathcal{M}$ reduces U and is invariant for $\widehat{S}$, then its associated range function J in $H^2_{\mathcal{K}}$ satisfies that $J(\lambda)$ is an invariant subspace for S, for a.e. $\lambda \in \mathbb{T}$. It may appear that one can derive the desired characterization by directly applying Beurling-Lax-Halmos (BLH) Theorem to each subspace $J(\lambda)$ and then transferring the results back to $\mathcal{M}$, a common strategy known as the fiberization technique. However, this approach is not possible due to measurability issues. Instead, the strategy is to reconstruct the isometries of the type of BLH for each $J(\lambda)$, in a measurable way with respect to $\lambda \in \mathbb{T}$.

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