Model-Free, Regret-Optimal Best Policy Identification in Online CMDPs

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Abstract

This paper considers the best policy identification (BPI) problem in online Constrained Markov Decision Processes (CMDPs). We are interested in algorithms that are model-free, have low regret, and identify an optimal policy with a high probability. Existing model-free algorithms for online CMDPs with sublinear regret and constraint violation do not provide any convergence guarantee to an optimal policy and provide only average performance guarantees when a policy is uniformly sampled at random from all previously used policies. In this paper, we develop a new algorithm, named Pruning-Refinement-Identification (PRI), based on a fundamental structural property of CMDPs we discover, called *limited stochasticity*. The property says for a CMDP with N constraints, there exists an optimal policy with *at most* N stochastic decisions.

The proposed algorithm first identifies at which step and in which state a stochastic decision has to be taken and then fine-tunes the distributions of these stochastic decisions. PRI achieves trio objectives: (i) PRI is a model-free algorithm; and (ii) it outputs a near-optimal policy with a high probability at the end of learning; and (iii) in the tabular setting, PRI guarantees $\tilde{\mathcal{O}}(\sqrt{K})$[1] regret and constraint violation, which significantly improves the best existing regret bound $\tilde{\mathcal{O}}(K^{\frac{4}{5}})$ under a model-free algorithm, where K is the total number of episodes.

1. Introduction

Learning in Constrained Markov Decision Processes (CMDPs) [1] has become an active research topic recently. Existing solutions include both model-based [2–6, 8, 10] and model-free algorithms [7, 12, 13]. This paper focuses on model-free approaches for CMDPs due to their computation and memory efficiency. A fundamental drawback of existing model-free algorithms for online CMDPs is that they provide only average performance guarantees for a policy uniformly sampled at random from *all* previously used policies during learning, so they fail to identify an optimal or a near-optimal policy.[2] Therefore, a natural question arises:

Is it possible to identify an optimal or a near-optimal policy in online CMDPs with the model-free approach with optimal regret?

1. **Notation:** $f(n) = \tilde{\mathcal{O}}(g(n))$ denotes $f(n) = \mathcal{O}(g(n)\log^k n)$ with $k > 0$. The same applies to $\tilde{\Omega}$.
2. In this paper, a policy is a mapping from a state at a given step to an action distribution, without any other additional input information. An algorithm that uses multiple policies, e.g. randomly sampling one policy from many policies, is explicitly called a *mixed* policy.

There are two key challenges to answering this question: (i) CMDP problems are typically represented as Linear Programming (LP) problems, resulting in stochastic optimal policies. Model-free online CMDP algorithms often employ the primal-dual approach, utilizing Lagrange multipliers to balance reward maximization and constraint violation. However, these methods yield "greedy policies" for fixed Lagrange multipliers, which aren't necessarily optimal. Consequently, model-free algorithms such as Triple-Q [11] offer performance guarantees only in terms of averages over various greedy policies determined by different Lagrange multipliers, failing to converge to a single policy. (ii) The best-known regret bound of model-free algorithms for episodic, online CMDPs is $\tilde{\mathcal{O}}(K^{\frac{4}{5}})$ [12]. It is also known that model-based algorithms can achieve a smaller and order-wise tight regret $\tilde{\mathcal{O}}(\sqrt{K})$ [6]. The open question is whether a model-free algorithm can reach $\tilde{\mathcal{O}}(\sqrt{K})$ regret in online CMDPs?

This paper tackles both challenges, providing affirmative responses to both questions. We introduce a novel algorithm, Pruning-Refinement-Identification (PRI). The main contributions of this paper include:

- PRI is the first model-free PAC RL algorithm for CMDPs, achieving optimal regret and minimal constraint violation.
- PRI outputs a near-optimal policy with a high probability at the end. The learned policy has $\tilde{\mathcal{O}}(1/\sqrt{K})$ optimality gap with probability $1 - \tilde{\mathcal{O}}(1/\sqrt{K})$.
- In the tabular setting, PRI guarantees $\tilde{\mathcal{O}}(\sqrt{K})$ regret and constraint violation, which significantly improves the best existing regret bound $\tilde{\mathcal{O}}(K^{\frac{4}{5}})$ under a mode-free algorithm, where K is the total number of episodes. Unlike existing regret bounds, the dominating term in terms of K in the regret bound does not depend on the sizes of the state space and action space.

2. Related Work

Model-based and Model-free algorithms for online CMDPs. As mentioned in the introduction, most existing results on online CMDPs consider regret minimization. For example, [2, 6, 10] proposed model-based algorithms for episodic tabular CMDPs. [3, 8] proposed efficient algorithms with zero or bounded constraint violation. For model-free algorithms, [13] developed Triple-Q that achieves sublinear regret and zero constraint violation in episodic tabular CMDPs. Similar results have been established for linear CMDPs [5, 7] and infinite-horizon average CMDPs [4, 12]. However, these existing model-free algorithms for online CMDPs does not converge to an optimal or a near-optimal policy. Very recently, [9] considered BPI for online CMDPs. They formulated the CMDP problem as a min-max game and the proposed algorithm converges to a near-optimal policy at the last iteration with optimistic mirror descent. However, the paper does not provide any regret guarantee when learning the near-optimal policy. Table 1 summarizes the recent results on online, episodic CMDPs.

3. Problem Formulation

We consider an episodic CMDP, denoted by $(\mathcal{S}, \mathcal{A}, H, \mathbb{P}, r, g^n, n \in [N])$, where $\mathcal{S}$ is the state space ($|\mathcal{S}| = S$), $\mathcal{A}$ is the action space ($|\mathcal{A}| = A$), $\{r_h\}_{h=1}^H, \{g_h^n\}_{h=1}^H, n \in [N]$ are reward, n-th utility functions, and $\mathbb{P} = \{\mathbb{P}_h(\cdot|x, a)\}_{h=1}^H$ are the transition kernels. For simplicity, we assume that in each

Table 1: The Exploration-Exploitation Tradeoff in Episodic CMDPs.

	Algorithm	Regret	Constraint Violation	BPI?
Model-based	OPDOP [5]	$\tilde{\mathcal{O}}(H^3\sqrt{S^2AK})$	$\tilde{\mathcal{O}}(H^3\sqrt{S^2AK})$	No
	OptDual-CMDP [6]	$\tilde{\mathcal{O}}(H^2\sqrt{S^3AK})$	$\tilde{\mathcal{O}}(H^2\sqrt{S^3AK})$	No
	OptPrimalDual-CMDP [6]	$\tilde{\mathcal{O}}(H^2\sqrt{S^3AK})$	$\tilde{\mathcal{O}}(H^2\sqrt{S^3AK})$	No
	CONRL [2]	$\tilde{\mathcal{O}}(H^3\sqrt{S^3A^2K})$	$\tilde{\mathcal{O}}(H^3\sqrt{S^3A^2K})$	No
	OptPess-LP [8]	$\tilde{\mathcal{O}}(H^3\sqrt{S^3AK})$	0	No
	OptPess–PrimalDual [8]	$\tilde{\mathcal{O}}(H^3\sqrt{S^3AK})$	$\mathcal{O}(1)$	No
	OPSRL[3]	$\tilde{\mathcal{O}}(\sqrt{S^4H^7AK})$	0	No
Model-free	Triple-Q[12]	$\tilde{\mathcal{O}}(\frac{1}{\delta}H^4S^{\frac{1}{2}}A^{\frac{1}{2}}K^{\frac{4}{5}})$	0	No
	PRI	$\tilde{\mathcal{O}}(\sqrt{H^2K})$	$\tilde{\mathcal{O}}(\sqrt{H^2K})$	Yes

episode, the agent starts from the same initial state $x_1 = x_{ini}$. It is straightforward to generalize the results to the case when the initial state is sampled with a given distribution but the notation becomes cumbersome. We also assume that $r_h : \mathcal{S} \times \mathcal{A} \to [0, 1]$ and $g_h^n : \mathcal{S} \times \mathcal{A} \to [0, 1]$ are deterministic for notation simplicity. Our results can be easily generalized to random reward/utility signals.

During each episode, the agent interacts with the environment as follows: at each step h, the agent takes action a_h after observing state x_h, receives reward $r_h(x_h, a_h)$ and N utility values $g_h^n(x_h, a_h)$ ($n \in [N]$), and then observes a new state (x_{h+1}), which evolves by following the transition kernel $\mathbb{P}_h(\cdot|x_h, a_h)$. The episode terminates after H steps.

Given a stochastic policy π, which is a collection of H functions $\{\pi_h : \mathcal{S} \times \mathcal{A} \to [0, 1]\}_{h=1}^H$, the agent takes action a with probability $\pi_h(a|x)$ when being in state x at step h . The reward value function of policy π, denoted by $V_h^\pi(x)$, is the expected total reward when starting from an arbitrary state x at step h to the end of the episode: $V_h^\pi(x) = \mathbb{E}_\pi\left[\sum_{i=h}^H r_i(x_i, a_i)\middle| x_h = x\right]$, where the expectation is taken with respect to the policy π and randomness from the transition kernels. Accordingly, the reward Q-function, denoted by $Q_h^\pi(x, a)$, is the expected total reward when the agent starts from an arbitrary action-action pair (x, a) at step h and follows policy π to the end of the episode: $Q_h^\pi(x, a) = r_h(x, a) + \mathbb{E}_\pi\left[\sum_{i=h+1}^H r_i(x_i, a_i)\middle| x_h = x, a_h = a\right]$.

Similarly, we can define the N utility value functions as $W_h^{\pi,n}(x) = \mathbb{E}_\pi\left[\sum_{i=h}^H g_i^n(x_i, a_i)\middle| x_h = x\right]$ and utility Q-functions as $C_h^{\pi,n}(x, a) = g_h^n(x, a) + \mathbb{E}_\pi\left[\sum_{i=h+1}^H g_i^n(x_i, a_i)\middle| x_h = x, a_h = a\right]$. The objective of the CMDP is to find an optimal policy that maximizes the expected total reward while making sure the $n-$th expected total utility is no less than ρ^n for all $n \in [N]$:

$$\pi^* \in \arg\max_\pi V_1^\pi(x_{ini}) \quad \text{s.t.} \quad W_1^{\pi,n}(x_{ini}) \geq \rho^n \quad \forall n \in [N]. \tag{1}$$

To avoid triviality, we assume $\rho^n \in [0, H]$. For simplicity, we use V_1^π to represent $V_1^\pi(x_{ini})$ and $W_1^{\pi,n}$ to represent $W_1^{\pi,n}(x_{ini})$.

We evaluate an online RL algorithm for CMDP using regret and constraint violation over K episodes: $\text{Regret}(K) = KV_1^{\pi^*}(x_{ini}) - \mathbb{E}\left[\sum_{k=1}^K V_1^{\pi_k}(x_{ini})\right]$ and $\text{Violation}^n(K) = K\rho^n - \mathbb{E}\left[\sum_{k=1}^K W_1^{\pi_k,n}(x_{ini})\right]$, where π_k is the policy used in episode k.

4. PRI (Pruning-Refinement-Identification)

Before formally introducing our algorithm, we first present two structural properties of the optimal solution to the CMDP problem (1). These properties have been overlooked in the literature but serve as the foundation of our proposed algorithm. Consider a CMDP problem with N constraints. It is well-known that the problem can be formulated as a linear programming (LP) problem [1]:

$$\max_{\{q_h(x,a)\}} \sum_{h,x,a} q_h(x,a) r_h(x,a) \tag{2}$$

$$\text{s.t.:} \sum_{h,x,a} q_h(x,a) g_h^{(n)}(x,a) \geq \rho^n \quad \forall n \in [N] \tag{3}$$

$$\sum_a q_{h+1}(x,a) = \sum_{x',a'} \mathbb{P}_h(x|x',a') q_h(x',a') \quad \forall x \in \mathcal{S}, h \in [H] \tag{4}$$

$$\sum_a q_1(x_{ini},a) = 1, \sum_a q_1(x,a) = 0, \ x \neq x_{ini} \tag{5}$$

$$q_h(x,a) \geq 0, \tag{6}$$

where $q_h(x,a)$ denotes the probability that state-action pair (x,a) is visited at step h, called the occupancy measure. Each feasible solution $\{q_h(x,a)\}_{h,x,a}$ to the problem leads to a corresponding Markov policy: $\pi_h(a|x) = \frac{q_h(x,a)}{\sum_a q_h(x,a)}$.

In this paper, we call probability distribution $\pi_h(\cdot|x)$ *decision* at state x at step h. So a policy consists of $S \times H$ decisions. A decision $\pi_h(\cdot|x)$ is called *greedy* if $\pi_h(a|x) = 1$ for some $a \in \mathcal{A}$ and stochastic otherwise.

Lemma 1 (Limited Stochasticity) *If $q^* = \{q_h^*(x,a)\}_{h,x,a}$ is an optimal solution to the CMDP problem (2)-(6) and is an extreme point, then there are at most $HS + N$ nonzero values in q^*. This implies that the optimal policy derived from q^* includes at most N stochastic decisions.*

The detailed proof can be found in Appendix D. The following corollary, which is a well-known result, is a direct consequence of the lemma.

Corollary 1 *For unconstrained MDP problems, one of the optimal policies is a greedy policy.*

Proof One of the optimal solutions to the LP is an extreme point. Since $N = 0$, all decisions from an optimal policy must be greedy according to Lemma 1. ■

Given an occupancy measure q and its induced policy π, we define $\mathcal{D}_{h,x}(q) = \{a : q_h(x,a) > 0\}$, which is the set of actions that will be taken with a nonzero probability in state x at step h under the policy π induced by q. Note that if $\pi_h(\cdot|x)$ is a greedy decision, then $|\mathcal{D}_{h,x}(q)| = 1$; and if $\pi(\cdot|x)$ is greedy, then $|\mathcal{D}_{h,x}(q)| > 1$. Let $M_q = \prod_{h,x} |\mathcal{D}_{h,x}(q)|$, and let π^m represent the mth greedy policy $(m = 1, \cdots, M_q)$ constructed from $\otimes_{h,x}\mathcal{D}_{h,x}(q)$ such that $\pi_h^m(a|x) = 1$ only if $a \in \mathcal{D}_{h,x}(q)$. Note that a greedy policy is a policy under which all decisions are greedy. Next, we will show that a Markov policy is equivalent to a mixed policy of many greedy policies in the following lemma, whose proof can be found in Appendix D.2.

Lemma 2 (Decomposition) *Given any Markov policy π and its corresponding occupancy measure q, there exists a set of M greedy policies and a probability distribution $\{a_m\}_{m=1,\cdots,M}$ such that the mixed policy, which selects a greedy policy π^m at the start of an episode with probability a_m and subsequently follows it, has the same occupancy measure q as the original policy π.*

We will first consider the case where *the LP has a unique optimal solution*. Leveraging these two observations from Lemma 1 and 2, we propose a novel three-phase algorithm (Algorithm 1), including policy pruning, policy refinement, and policy identification, called PRI. If the LP has more than one solution, we will introduce a multi-solution pruning algorithm to the policy pruning phase of PRI to resolve the issue. The algorithm and the analysis can be found in Appendix B.

The algorithm is presented in Algorithm 1, which includes $\sqrt{K}+2K$ episodes, $\sqrt{K}$ episodes for pruning, K episodes for refinement and K episodes for identification. In the first phase (policy pruning), we run Triple-Q for $\sqrt{K}$ episodes, we denote $\{\pi_{k,h}\}_{h=1}^{H}$ as the policy used by Triple-Q in the kth episode, and it is a greedy policy. For fixed (h,x,a) in the policy pruning phase, we use $\tilde{N}_h(x,a)$ to count the number of episodes in which the greedy policy we follow is $\pi_h(a|x)=1$, which is the number of greedy policies (among the $\sqrt{K}$ greedy policies) that would take action a if the agent visits state x at step h. Because of the sub-linear regret and zero violation guaranteed by Triple-Q, we expect that $\frac{N_h(x,a)}{\sqrt{K}}$ is close to zero if $\pi_h^*(a|x)=0$ and is a non-negligible positive value if otherwise. Therefore, with a high probability, $\tilde{\mathcal{D}}_{h,x}=\mathcal{D}_{h,x}(q^*)$, where $\tilde{\mathcal{D}}_{h,x}$ is gradually updated in Algorithm 1 (Lines 8-10).

At each round of the second phase (policy refinement), the following optimization is solved.

$$\begin{aligned}\textbf{Decomposition-Opt: } &\max_{\{a_m\}_{m=1}^{M}} \sum_{m=1}^{M} a_m \bar{V}_1^{\pi^m} \\ \text{s.t.: } &\left|\sum_{m=1}^{M} a_m \bar{W}_1^{\pi^m,n}-\rho^n\right| \leq \sqrt{\frac{H^2\log(t\epsilon' K)}{\epsilon' t\sqrt{K}}} \quad \forall n, \\ &\sum_m a_m = 1, a_m \geq \epsilon' \quad \forall m.\end{aligned} \tag{7}$$

After the first phase, PRI obtains M greedy policies. In the second phase, PRI learns the weights $\{a_m\}$ so that a mixed policy that chooses policy π^m with probability a_m is statistically identical to the optimal policy. This is achieved by learning the reward and utility value functions of the greedy policies and then solving an approximated version of the CMDP (Decompsotion-Opt (7)). After learning sufficiently accurate $\{a_m\}$ in the second phase, PRI learns the occupancy measure under the mixed policy defined by $\{a_m\}$ and constructs a Markov policy $\tilde{\pi}$ based on the learned occupancy measure.

An informal statement of the main results is presented below. The formal statements of the theorems and the proofs will be presented in the appendix.

Main Results: With a high probability, PRI yields policy $\tilde{\pi}$ such that

- $\{(h,x,a):\tilde{\pi}_h(a|x)>0\}=\{(h,x,a):\pi_h^*(a|x)>0\}$,
- PRI guarantees $\mathcal{O}(\sqrt{K})$ regret and constraint violation over the $\sqrt{K}+2K$. episodes, and
- $|\tilde{\pi}_h(a|x)-\pi_h^*(a|x)|=\mathcal{O}(1/\sqrt{K})$ for all (h,x,a), and $\tilde{\pi}_h(a|x)=\pi_h^*(a|x)$ if $\pi_h^*(a|x)\in\{0,1\}$.

Algorithm 1: PRI

[Phase 1: Policy Pruning] **for** $t = 1, \cdots, logK$ **do**

Initialize $\tilde{N}_h(x,a) = 0$ for all h, x and a. Reinitialize all parameters in Triple-Q

for $k = 1, \cdots, \sqrt{K}$ **do**

For all (h,x,a), $\tilde{N}_h(x,a) \leftarrow \tilde{N}_h(x,a) + \pi_{k,h}(a|x)$, execute Triple-Q for one episode.

end

$\tilde{\mathcal{D}}^{(t)} = \varnothing$

for *all* (h,x,a) **do**

$\tilde{\mathcal{D}}^{(t)}(h,x) \leftarrow \tilde{\mathcal{D}}(h,x) \bigcup \{a\}$ if $\frac{\tilde{N}_h(x,a)}{\sqrt{K}} \geq \frac{\epsilon}{2}$.

end

end

Obtain $\tilde{\mathcal{D}}$ by majority vote on $\tilde{\mathcal{D}}^{(t)}$. Obtain M greedy policies from $\tilde{\mathcal{D}}$ where $M = \prod_{h,x} |\tilde{\mathcal{D}}(h,x)|$.

[Phase 2: Policy Refinement] **if** $M = 1$ **then**

Output the greedy policy.

end

else

Set $\hat{V}_1^{\pi^m} = 0, \hat{W}_1^{\pi^m,n} = 0$, and $a_m = \frac{1}{M}$ for all n and m.

end

for *round* $t = 1, \cdots, \sqrt{K}$ **do**

for $m = 1, \cdots, M$ **do**

for $k = 1, \cdots, a_m\sqrt{K}$ **do**

Execute greedy policy π^m for one episode.

if $k \leq \epsilon'\sqrt{K}$ **then**

Set $\hat{V}_1^{\pi^m} \leftarrow \hat{V}_1^{\pi^m} + V_{k,1}^{\pi^m}$ and $\hat{W}_1^{\pi^m,n} \leftarrow \hat{W}_1^{\pi^m,n} + W_{k,1}^{\pi^m,n}$ for all n, where $V_{k,1}^{\pi^m}$ and $W_{k,1}^{\pi^m,n}$ are the total reward and utility of type n received in the kth episode.

end

end

Set $\bar{V}_1^{\pi^m} = \frac{\hat{V}_1^{\pi^m}}{t\epsilon'\sqrt{K}}$ and $\bar{W}_1^{\pi^m,n} = \frac{\hat{W}_1^{\pi^m,n}}{t\epsilon'\sqrt{K}}$ for all n. Update $\{a_m\}$ by solving Decomposition-Opt (7).

end

end

[Phase 3: Policy Identification] Initialize $N_h(x,a) = 0$ for all h, x and a.

for $t = 1, \cdots, \sqrt{K}$ **do**

for $m = 1, \cdots, M$ **do**

for $k = 1, \cdots, a_m\sqrt{K}$ **do**

for $h = 1, \cdots, H$ **do**

Take action a_h given by policy π^m, i.e. $\pi^m(a_h|x_h) = 1$, $N_h(x_h,a_h) \leftarrow N_h(x_h,a_h) + 1$.

end

end

end

end

For all (h,x,a), set $\tilde{\pi}_h(a|x) = \frac{N_h(x,a)}{\sum_{\tilde{a}\in\mathcal{A}} N_h(\tilde{a},x)}$, output policy $\tilde{\pi}$.

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Appendix A. Main Results

In this section, we provide our main results assuming that the LP associated with the CMDP problem has a unique solution. This assumption can be relaxed and the results can be found in Section B. Letπ^* be the unique optimal policy and $\{q_h^{\pi^*}(x,a)\}$ is the corresponding occupancy measure. Furthermore, let $\{\pi^m\}$ $(m = 1, \cdots, M)$ be the set of greedy policies associated with the optimal policy as defined in Lemma 2, and $\{a_m^*\}$ the associated weights. We also make the following additional assumptions.

Assumption 1 *The ϵ and ϵ' used in PRI satisfy $q_h^{\pi^*}(x,a) \geq \epsilon$ for any (h,x,a) such that $\pi_h^*(a|x) > 0$, and $\min_m a_m^* \geq \epsilon' > 0$.*

Assumption 2 *There exist two positive constants c_v and c_w such that given a feasible occupancy measure q^π to the LP and the corresponding reward value function and utility value function V^π and $W^{\pi,n}$, we have either $V_1^{\pi^*} - V_1^\pi \geq c_v ||q^{\pi^*} - q^\pi||_1$ or for some $n \in [N]$, $W_1^{\pi^*,n} - W_1^{\pi,n} \geq c_w ||q^{\pi^*} - q^\pi||_1$, where $\|\cdot\|_1$ is the $L1$-norm.*

Recall a feasible occupancy measure defines a unique Markov policy. The assumption above states that when a policy's occupancy measure is different from that of the unique optimal policy, then either the reward value function or one of the utility reward functions should also be different from that under the optimal policy.

Assumption 3 *Under any greedy policy π, for all x and h, we have*

$$\Pr(x_h = x) = \sum_{x',a'} q_{h-1}^\pi(x',a')\mathbb{P}_h(x|x',a') > p_{\min}.$$

This assumption above says all states should be visited with a non-negligible probability under any greedy policy. It is worth noting that this assumption can be removed if we apply the extension version of PRI, which is stated in section B. To prove our main result, we first recall the regret and constraint violation guaranteed under Triple-Q [12] in the following lemma.

Lemma 3 *For sufficiently large K, over K episodes, Triple-Q guarantees $\tilde{\mathcal{O}}(K^{0.8})$ regret and zero constraint violation, and furthermore,*

$$\Pr\left(K\rho^n - \sum_{k=1}^{K} W_1^{\pi_k,n} \leq 0\right) = 1 - \mathcal{O}\left(\frac{1}{K^2}\right). \tag{8}$$

In the following theorem, we show that PRI can correctly classify stochastic and greedy decisions with a high probability after the pruning phase.

Theorem 4 (Pruning) *Let $\mathcal{D}^* = \{(h,x,a) : \pi_h^*(a|x) > 0\}$ and $\tilde{\mathcal{D}} = \left\{(h,x,a) : \frac{\tilde{N}_h(x,a)}{\sqrt{K}} \geq \frac{\epsilon}{2}\right\}$. Under Assumptions 1 and 3, after policy pruning, we have*

$$\Pr\left(\tilde{\mathcal{D}}_{h,x} = \mathcal{D}_{h,x}(q^*),\ \forall (h,x)\right) = 1 - \tilde{\mathcal{O}}\left(K^{-0.1}\right). \tag{9}$$

The detailed proof is deferred to Appendix E. Note that since the pruning phase includes $\sqrt{K}$ episodes, the regret and constraint violation are both bounded by $H\sqrt{K}$.

The following theorem shows that the regret and constraint violation during the refinement phase are both $\tilde{\mathcal{O}}(\sqrt{K})$. Note that $\tilde{\mathcal{O}}(\sqrt{K})$ regret and constraint violation imply that the learned mixed policy is close to optimal. The proof can be found in Appendix F.

Theorem 5 (Refinement) *Assume $\tilde{\mathcal{D}} = \mathcal{D}^*$ after policy pruning. Under Assumption 1 to 3, with probability $1 - \tilde{\mathcal{O}}(\frac{1}{\sqrt{K}})$, the regret and constraint violation during the policy refinement phase are both $\tilde{\mathcal{O}}(H\sqrt{K})$.*

The refinement phase learns a near optimal mixed policy, which is a combination of M greedy policies for $M \leq 2^N$. In the following theorem we show that the identification phase is to find a single near-optimal policy by using the occupancy measure of the mixed policy. The proof can be found in Appendix G.

Theorem 6 (Identification) *Assume $\tilde{\mathcal{D}} = \mathcal{D}^*$ after policy pruning. Under Assumption 1 to 3, with probability $1 - \tilde{\mathcal{O}}(\frac{1}{K})$, the regret and constraint violation during the policy identification phase are both $\mathcal{O}(\sqrt{K})$. Furthermore, $|\tilde{\pi}_h(a|x) - \pi_h^*(a|x)| = \mathcal{O}(\frac{1}{\sqrt{K}})$ if $0 < \pi_h^*(a|x) < 1$ and $\tilde{\pi}_h(a|x) = \pi_h^*(a|x)|$ if $\pi_h^*(a|x) \in \{0, 1\}$.*

By summarizing the results from the three theorems above, we have the regret and the constraint violation over the $\sqrt{K} + 2K$ episodes are $\tilde{\mathcal{O}}(H\sqrt{K})$ with probability $1 - \tilde{\mathcal{O}}(\frac{1}{K^{0.1}})$. Consider the regret, the pruning phase includes $\sqrt{K}$ episodes, resulting in at most $H\sqrt{K}$ regret. Theorem 5 and Theorem 6 show that the regret in the refinement and identification phases are both $\tilde{\mathcal{O}}(H\sqrt{K})$. Note that the order-wise bounds are independent of S and A, unlike those in the literature. However, there is an implicit dependence on S and A as the results hold only when K is sufficiently large and how large K needs to be depends on S and A.

Appendix B. Extension to CMDPs with Multiple Optimal Solutions

In this section, we consider the case where the optimal policy is not unique so the LP has multiple optimal solutions. Here, the RL agent's objective is to learn one of these optimal policies. According to Lemma 1, an optimal solution associated with an extreme point of the LP involves no more than $HS + N$ stochastic decisions. Additionally, any optimal policy can be viewed as a combination of the optimal policies associated with the extreme points. We define the set of optimal policies as Π^* and the subset associated with extreme points as $\Pi^{*,e}$. We expand our assumptions to the case of multiple solutions as follows.

Assumption 4 *The ϵ used PRI satisfies* $\min_{(h,x,a):\pi_h(a|x)\neq 0} q_h^\pi(x, a) \geq \epsilon \quad \forall \pi \in \Pi^{*,e}$.

Assumption 5 *Given any occupancy measure q' and the induced Markov policy π', there exists an optimal policy $\pi^* \in \Pi^*$ such that $V_1^{\pi^*} - V_1^{\pi'} \geq c_v||q^{\pi^*} - q'||_1$ or for some $n, W_1^{\pi^*,n} - W_1^{\pi',n} \geq c_w||q^{\pi^*} - q'||_1$, where c_v and c_w are two positive constants.*

Note that if π' is an optimal policy, then the assumption holds trivially with $\pi^* = \pi'$. Under Assumptions 3-5, the following theorem shows that a unique optimal policy is identified after Multi-Solution Pruning. The algorithm result in at most $H^2SAK^{0.25}$ regret and constraint violation.

Theorem 7 *Under Assumption 4 and 5, with probability* $1 - \mathcal{O}(1/K^{0.02})$, *for sufficiently large* K, *multi-solution pruning outputs a unique optimal policy with at most* N *stochastic decisions. The regret and constraint violation during multi-solution pruning are bounded by* $H^2SAK^{0.25}$ *with probability one.*

More discussions and the detailed proof are deferred to Appendix H.1 due to the page limit. We note that adding this multi-solution pruning to PRI only increases the regret and constraint violation by $HSAK^{0.25}$ which is order-wise smaller than $\tilde{\mathcal{O}}(H\sqrt{K})$ in terms of K. Therefore, the regret and constraint violation remains to be $\tilde{\mathcal{O}}(H\sqrt{K})$ for sufficiently large K.

Appendix C. Experiments

Synthetic CMDP

This section presents numerical evaluations of the proposed algorithm. We first evaluated our algorithm for a synthetic CMDP with a single constraint. The transition kernels, rewards, and utilities are chosen such that the problem has a unique optimal solution and satisfies Assumption 3. The objective is to maximize the cumulative reward while guaranteeing that the cumulative utility is at least 2. Comparison between Triple-Q and PRI can be found in Figure 1 and 2. Experiment details can be found in the Appendix I.1.

We can observe that PRI converges significantly faster than Triple-Q. Remarkably, both regret and constraint violation level off at the beginning of policy refinement after approximately $110,000$ episodes. However, the regret of Triple-Q continues to increase sublinearly. PRI significantly outperforms Triple-Q on regret. At the end of the $1,100,000$ episodes, Triple-Q has a regret of 2.05×10^5 and constraint violation of -3.86×10^4. In contrast, the regret and constraint violation under PRI are -1.73×10^3 and 1.06×10^4, respectively. Thus, the regret is significantly lower than Triple-Q. Since the full CMDP model is given, we can obtain the optimal solution by using the linear programming approach. The cumulative reward and cumulative utility under PRI are are 1.57301, and 2.00008, which are very close to the optimal solution 1.57306 and 2.

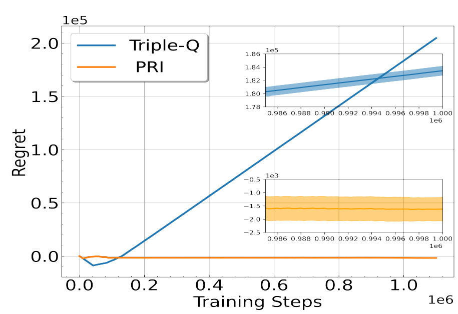

Figure 1: Regret for a synthetic CMDP with a unique solution, the shaded region represents the 95% confidence interval.

Grid-world

In our second experiment, which is a grid-world environment (refer to Appendix I.2 for details), we compared Triple-Q with PRI, and the results are shown in Figure 3 and 4. This problem has multiple optimal policies. Therefore, we used the extended PRI wth multi-solution-pruning. PRI

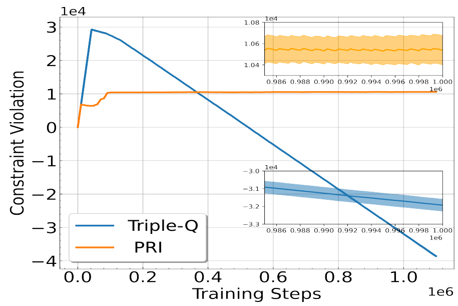

Figure 2: Constraint Violation for a synthetic CMDP with a unique solution, the shaded region represents the 95% confidence interval.

consists of $200,000$ episodes for the initial phase, followed by $200,000$ episodes for each multi-solution pruning phase. Both policy refinement and policy identification phases include $5,000,000$ episodes each. For reference, we ran Triple-Q for the same number of episodes. The outcomes concerning regret and constraint violation are visualized in Figure 3 and 4. We can observe that Triple-Q has a regret of 3.19×10^6 and a constraint violation of -5.26×10^5, whereas PRI achieves 1.54×10^5 regret and 2.98×10^3 constraint violation, indicating substantially lower regret with PRI.

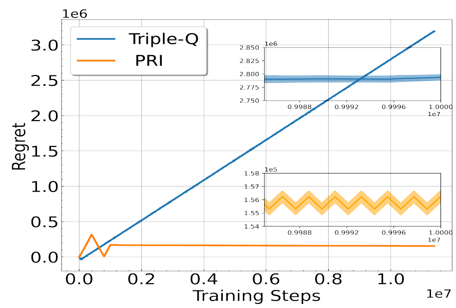

Figure 3: Regret for the grid world environment, the shaded region represents the 95% confidence interval.

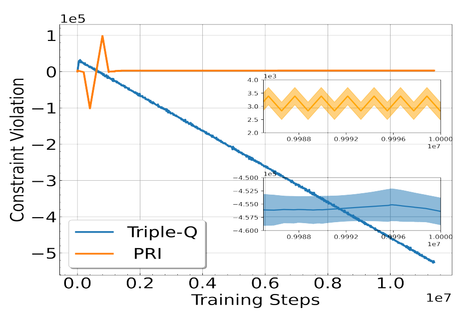

Figure 4: Constraint Violation for the grid world environment, the shaded region represents the 95% confidence interval.

Appendix D. Proofs of the Technical Lemmas

D.1. Proof of Lemma 1 (Limited Stochasticity)

Lemma 1 *If $q^* = \{q_h^*(x,a)\}_{h,x,a}$ is an optimal solution to the CMDP problem (2)-(6) and is an extreme point, then there are at most $HS + N$ nonzero values in q^*. This implies that the optimal policy derived from q^* includes at most N stochastic decisions.*

Proof The LP has HSA decision variables $\{q_h(s,a)\}$ in total. So at an extreme point, at least HSA constraints become tight. In other words, at least HSA constraints become equalities under solution q^*. Since there are only $HS + N$ constraints defined in (3)-(5), at least

$$HSA - HS - N = HS(A-1) - N$$

constraints in (6) become tight (equality) under q^*. Therefore, there are at least $HS(A-1) - N$ zeros in q^* or at most $HS + N$ nonzero values in q^*.

Now suppose the optimal policy obtained from q^* has less than $HS - N$ greedy decisions. Then q^* would have at least

$$HS - N - 1 + 2(N+1) = HS + N + 1$$

nonzero values because each greedy decision requires one nonzero $q_h(x,a)$ and each stochastic decision requires at least two nonzero $q_h(x,a)$. This leads to a contradiction. ■

D.2. Proof of Lemma 2 (Decomposition)

Lemma 2 *Given any Markov policy π and its corresponding occupancy measure q, there exists a set of M greedy policies and a probability distribution $\{a_m\}_{m=1,\cdots,M}$ such that the mixed policy, which selects a greedy policy π^m at the start of an episode with probability a_m and subsequently follows it, has the same occupancy measure q as the original policy π.*

Proof To simplify the notation, we will prove the lemma for the case where $|\mathcal{D}_{h,x}(q)| \in \{1,2\}$, i.e., any stochastic decision takes two possible actions and assume $\mathcal{A} = \{0,1\}$. The extension to the general case is trivial.

Under a Markov policy $\{\pi_h\}_{h=1}^H$, the actions are independently chosen given state x and step h. Suppose we will execute the Markov policy for K episodes. We will generate K matrices $\{B_k\}_{k=1}^K$ of size $H \times S$ such that $B_k(h,x)$ is a realization of a random variable with distribution $\pi_h(\cdot|x)$. All these values are independently generated. Now to execute policy π at episode k, at state x and step h, the agent takes action a such that $B_k(h,x) = a$. This is statistically the same as sampling an action using $\pi_h(\cdot|x)$ when reaching state x at step h.

We note that each binary matrix B_k corresponds to a greedy policy from the M_q greedy policies and vice versa. Furthermore, the binary matrix associated with greedy policy π^m is generated with probability

$$a_m = \prod_{h,x}\left(\sum_{a\in\mathcal{D}_{h,x}(q)} \pi_h(a|x)\pi_h^m(a|x)\right),$$

because

$$\sum_{a\in\mathcal{D}_{h,x}(q)} \pi_h(a|x)\pi_h^m(a|x) = \sum_{a\in\mathcal{D}_{h,x}(q)} \pi_h(a|x)\mathbb{I}(\pi_h^m(a|x)=1),$$

which is the probability that action selected by the greedy policy π^m is also selected under policy π. Therefore, if we consider a mixed policy that chooses policy π^m with probability a_m, then it is statistically the same as policy π and has the same occupancy measure q. ■

Appendix E. Proof of Theorem 4 (Pruning)

In this part, we are going to show the detailed proof of Pruning.

Theorem 1 *Let* $\mathcal{D}^* = \{(h,x,a) : \pi_h^*(a|x) > 0\}$ *and* $\tilde{\mathcal{D}} = \left\{(h,x,a) : \frac{\tilde{N}_h(x,a)}{\sqrt{K}} \geq \frac{\epsilon}{2}\right\}$. *Under Assumptions 1 to 3, after policy pruning, we have*

$$\Pr\left(\tilde{\mathcal{D}}_{h,x} = \mathcal{D}_{h,x}(q^*),\ \forall (h,x)\right) = 1 - \tilde{\mathcal{O}}\left(K^{-0.05\log K}\right).$$

Proof At the end of the first phase, i.e., $\sqrt{K}$ episodes, we consider a mixed policy $\hat{\pi}$ that selects the policy used in the kth episode, π_k, with probability $1/\sqrt{K}$. We assume that all constraints are satisfied under $\hat{\pi}$, which occurs with probability $1 - \mathcal{O}(K^{-2})$. The reward value function of the policy $\hat{\pi}$ is

$$V_1^{\hat{\pi}} = \frac{1}{\sqrt{K}} \sum_{k=1}^{\sqrt{K}} V_1^{\pi_k} \tag{10}$$

and $V^{\pi^*} - V_1^{\hat{\pi}} \geq 0$ because the constraints are satisfied under $\hat{\pi}$. Note that policy $\hat{\pi}$ is not a Markov policy. We next prove that the occupancy measure induced by policy $\hat{\pi}$ is a valid solution to the LP problem:

$$\begin{aligned}
\sum_a q_h^{\hat{\pi}}(x,a) &= \sum_a \frac{1}{\sqrt{K}} \sum_{k=1}^{\sqrt{K}} q_h^{\pi_k}(x,a) \\
&= \frac{1}{\sqrt{K}} \sum_{k=1}^{\sqrt{K}} \sum_a q_h^{\pi_k}(x,a) \\
&= \frac{1}{\sqrt{K}} \sum_{k=1}^{\sqrt{K}} \sum_{x',a'} q_{h-1}^{\pi_k}(x',a') \mathbb{P}_h(x|x',a') \\
&= \sum_{x',a'} \mathbb{P}_h(x|x',a') \frac{1}{\sqrt{K}} \sum_{k=1}^{\sqrt{K}} q_{h-1}^{\pi_k}(x',a') \\
&= \sum_{x',a'} \mathbb{P}_h(x|x',a') q_{h-1}^{\hat{\pi}}(x',a').
\end{aligned}$$

Besides, it is easy to verify that $\forall h,x,a, q_h^{\hat{\pi}}(x,a) \geq 0$. Thus the policy $\hat{\pi}$ is a valid policy for the LP problem.

Recall that $\mathcal{D}_{h,x}(q) = \{a : q_h(x,a) > 0\}$. We have $\tilde{\mathcal{D}}_{h,x}(q) = \mathcal{D}_{h,x}(q^*),\ \forall (h,x)$ is equivalent to $\mathcal{D}^* = \tilde{\mathcal{D}}$.

We further define event

$$\mathcal{E} = \left\{ \exists (h, x, a) \in \mathcal{D}^*, \frac{\tilde{N}_h(x, a)}{\sqrt{K}} < \frac{\epsilon}{2} \right\},$$

i.e., the event that at the end of the pruning phase, the algorithm eliminates an action used by the optimal policy. Note that π_k's are greedy policies ($\pi_{k,h}(a|x) \in \{0, 1\}$). Therefore, we have

$$\tilde{N}_h(x, a) = \sum_{k=1}^{\sqrt{K}} \pi_{k,h}(a|x).$$

Thus, assuming this event $\mathcal{E}$ occurs, we can obtain

$$\begin{aligned}
& q_h^{\hat{\pi}}(x, a) \\
=& \frac{1}{\sqrt{K}} \sum_{k=1}^{\sqrt{K}} q_h^{\pi_k}(x, a) \\
=& \frac{1}{\sqrt{K}} \sum_{k=1}^{\sqrt{K}} \left(\sum_{x', a'} q_{h-1}^{\pi_k}(x', a') \mathbb{P}_h(x|x', a') \right) \pi_{k,h}(a|x) \\
\leq& \frac{1}{\sqrt{K}} \sum_{k=1}^{\sqrt{K}} \pi_{k,h}(a|x) \\
=& \frac{1}{\sqrt{K}} \tilde{N}_h(x, a) \\
<& \frac{\epsilon}{2}.
\end{aligned}$$

According to Assumption 1, we have

$$q_h^{\pi^*}(x, a) \geq \epsilon \quad \forall (h, x, a) \in \mathcal{D}^*, \tag{11}$$

which implies $||q^{\hat{\pi}} - q^*||_1 \geq \frac{\epsilon}{2}$. According to Assumption 2, we have either

$$V_1^{\pi^*} - V_1^{\hat{\pi}} \geq c_v \frac{\epsilon}{2} \quad \text{(Case 1)} \tag{12}$$

or

$$W_1^{\pi^*, n} - W_1^{\hat{\pi}, n} \geq c_w \frac{\epsilon}{2} \text{ for some } n \quad \text{(Case 2)}. \tag{13}$$

Therefore, we have

$$\begin{aligned}
\Pr(\mathcal{E}) \leq & \Pr\left(\sqrt{K} \left(V_1^{\pi^*} - V^{\hat{\pi}} \right) \geq c_v \frac{\epsilon}{2} \sqrt{K} \right) \\
& + \Pr\left(\exists n \in [N], \sqrt{K} \left(W_1^{\pi^*, n} - W_1^{\hat{\pi}, n} \right) \geq c_w \frac{\epsilon}{2} \sqrt{K} \right).
\end{aligned}$$

Based on Lemma 3's result on regret and the Markov inequality, we have

$$\Pr\left(\sqrt{K}\left(V_1^{\pi^*}-V^{\hat{\pi}}\right)\geq c_v\frac{\epsilon}{2}\sqrt{K}\right)\leq\frac{c_1\sqrt{K}^{0.8}}{c_v\frac{1}{2}\epsilon\sqrt{K}}=\frac{2c_1}{c_v\epsilon K^{0.1}}.$$

From Lemma 3's result on constraint violation, the high probability bound, we have

$$\begin{aligned}&\Pr\left(\exists n\in[N],\sqrt{K}\left(W_1^{\pi^*,n}-W_1^{\hat{\pi},n}\right)\geq c_w\frac{\epsilon}{2}\sqrt{K}\right)\\ \leq&\Pr\left(\exists n\in[N],\sqrt{K}\left(\rho^n-W_1^{\hat{\pi},n}\right)\geq c_w\frac{\epsilon}{2}\sqrt{K}\right)\\ =&\mathcal{O}\left(\frac{1}{K}\right).\end{aligned}$$

Thus for sufficiently large $\sqrt{K}$, $\Pr\left(\mathcal{E}\right)=\mathcal{O}\left(K^{-0.1}\right)$, i.e., with probability $1-\mathcal{O}\left(K^{-0.1}\right)$, we have

$$\mathcal{D}^*\subseteq\tilde{\mathcal{D}}.$$

Now define event $\mathcal{E}'=\left\{\exists(h,x,a)\notin\mathcal{D}^*,\frac{N_h(x,a)}{\sqrt{K}}\geq\frac{\epsilon}{2}\right\}$. Similar to (11) and based on Assumption 3, we can obtain

$$q_h^{\hat{\pi}}(x,a)=\frac{1}{\sqrt{K}}\sum_{k=1}^{\sqrt{K}}q_h^{\pi_k}(x,a)\geq p_{\min}\frac{1}{\sqrt{K}}\tilde{N}_h(x,a)\geq\frac{\epsilon p_{\min}}{2}. \tag{14}$$

Since $q_h^{\pi^*}(x,a)=0$ for $(h,x,a)\notin\mathcal{D}^*$, $||q^{\hat{\pi}}-q^{\pi^*}||_1\geq\frac{\epsilon p_{\min}}{2}$. Similar to the analysis on $\mathcal{E}$, we obtain

$$\Pr(\mathcal{E}')=\mathcal{O}\left(K^{-0.1}\right). \tag{15}$$

In other words, with probability $1-\mathcal{O}\left(K^{-0.1}\right)$, Denote the number of rounds that we classify wrongly as X. When we run Triple-Q for $\log K$ rounds, by Chernoff bound, we have

$$\Pr(X\geq 0.5\log K)\leq e^{0.5\log K-K^{-0.1}\log K-0.05(\log K)^2+0.5\log 2\log K} \tag{16}$$

In other words, considering that K is sufficiently large, with probability $1-\mathcal{O}\left(K^{-0.05\log K}\right)$, we have

$$\tilde{\mathcal{D}}\subseteq\mathcal{D}^*,$$

which completes the proof. ■

Appendix F. Proof of Theorem 5 (Refinement)

Theorem 2 *Assume $\tilde{\mathcal{D}}=\mathcal{D}^*$ after policy pruning. Under Assumptions 1 and 3, with probability $1-\tilde{\mathcal{O}}\left(\frac{1}{\sqrt{K}}\right)$, the regret and constraint violation during the policy refinement phase are both $\mathcal{O}\left(H\sqrt{K}\right)$.*

Proof Recall that in Lemma 2, we have shown that there exists a mixed policy of M greedy policies defined by $\mathcal{D}^*_{h,x}$ that has the same occupancy measure as that under the optimal policy, and $\{a^*_m\}$ are the associated weights.

Recall that the policy refinement consists of $\sqrt{K}$ rounds. Let $\{a_{t,m}\}_{m=1,\cdots,M}$ be the weights used in round t. Then in the tth round, greedy policy π^m is used for $a_{t,m}\sqrt{K}$ episodes, where $a_{t,m}$ is the optimal solution to Decomposition-Opt (7).

First, we will bound the estimation errors of the reward and utility value functions. Recall that PRI uses $\epsilon'\sqrt{K}$ episodes in each round to estimate the reward value function and the utility value functions instead of all episodes because $\{a_{t,m}\}$ are random variables correlated with the estimated value functions from the previous round. At the beginning of round t, we have $(t-1)\epsilon'\sqrt{K}$ samples from the previous round. Indexing the samples by k', we have

$$\bar{W}_1^{\pi^m,n} = \frac{\sum_{k'=1}^{(t-1)\epsilon'\sqrt{K}} W_{k',1}^{\pi^m,n}}{(t-1)\epsilon'\sqrt{K}}. \tag{17}$$

Define

$$\delta W_1^{\pi^m,n} = \bar{W}_1^{\pi^m,n} - W_1^{\pi^m,n} \tag{18}$$

$$= \frac{\sum_{k'=1}^{(t-1)\epsilon'\sqrt{K}} \left(W_{k',1}^{\pi^m,n} - W_1^{\pi^m,n}\right)}{(t-1)\epsilon'\sqrt{K}}. \tag{19}$$

Since $W_{k',1}^{\pi^m,n} \in [0,H]$ are i.i.d. random variables, by the Azuma-Hoeffding inequality, we have

$$\Pr\left(\left|\delta W_1^{\pi^m,n}\right| \leq \sqrt{\frac{2H^2\log\left((t-1)\epsilon'\sqrt{K}\right)}{\epsilon'(t-1)\sqrt{K}}}\right) \tag{20}$$

$$\geq 1 - \frac{2}{(t-1)\epsilon'\sqrt{K}}. \tag{21}$$

Similarly, defining

$$\delta V_1^{\pi^m} = \bar{V}_1^{\pi^m} - V_1^{\pi^m} = \frac{\sum_{k'=1}^{(t-1)\epsilon' K^\alpha} \left(V_{k',1}^{\pi^m} - V_1^{\pi^m}\right)}{(t-1)\epsilon'\sqrt{K}},$$

we have

$$\Pr\left(\left|\delta V_1^{\pi^m}\right| \leq \sqrt{\frac{2H^2\log\left((t-1)\epsilon'\sqrt{K}\right)}{\epsilon'(t-1)\sqrt{K}}}\right) \tag{22}$$

$$\geq 1 - \frac{2}{(t-1)\epsilon'\sqrt{K}}. \tag{23}$$

Therefore, with probability at least $1 - \frac{2M}{(t-1)\epsilon'\sqrt{K}}$,

$$\begin{aligned}
\left|\sum_{m=1}^{M} a_m^* \bar{W}_1^{\pi_m,n} - \rho^n\right| &= \left|\sum_{m=1}^{M} a_m^* \left(W_1^{\pi_m,n} + \delta W_1^{\pi_m,n}\right) - \rho^n\right| \\
&\leq \sum_{m=1}^{M} a_m^* \left|\delta W_1^{\pi_m,n}\right| \\
&\leq \sqrt{\frac{2H^2 \log\left((t-1)\epsilon'\sqrt{K}\right)}{\epsilon'(t-1)\sqrt{K}}}.
\end{aligned}$$

In other words, $\{a_m^*\}$ is a feasible solution to Decomposition-Opt (7) with a high probability, which implies that Decomposition-Opt (7) has a solution with a high probability.

We now consider $\{a_{t,m}\}$ and the regret and constraint violation in round t. If $\{a_m^*\}$ is a feasible solution to Decomposition-Opt (7), then

$$\begin{aligned}
&\sum_{m=1}^{M} a_{t,m}\sqrt{K} V_1^{\pi^m} \\
=&\sum_{m=1}^{M} a_{t,m}\sqrt{K}\left(\bar{V}_1^{\pi^m} - \delta V_1^{\pi^m}\right) \\
\geq&\sqrt{K}\left(\sum_{m=1}^{M} a_{t,m}\bar{V}_1^{\pi^m}\right) - \sqrt{K}\max_m \left|\delta V_1^{\pi^m}\right| \\
\geq_{(a)}&\sqrt{K}\left(\sum_{m=1}^{M} a_m^*\left(V_1^{\pi^m} + \delta V_1^{\pi^m}\right)\right) - \sqrt{K}\max_m \left|\delta V_1^{\pi^m}\right| \\
\geq&\sqrt{K}\left(\sum_{m=1}^{M} a_m^* V_1^{\pi^m}\right) - 2\sqrt{K}\max_m \left|\delta V_1^{\pi^m}\right| \\
=&\sqrt{K} V_1^{\pi^*} - 2\sqrt{K}\max_m \left|\delta V_1^{\pi^m}\right| \\
\geq&\sqrt{K}\left(V_1^{\pi^*} - 2\sqrt{\frac{2H^2 \log\left((t-1)\epsilon'\sqrt{K}\right)}{\epsilon'(t-1)\sqrt{K}}}\right),
\end{aligned}$$

where (a) holds because $\{a_{t,m}\}_m$ is the optimal solution to Decomposition-Opt (7). In other words, with a high probability, the regret is bounded by

$$2\sqrt{\frac{2\sqrt{K}H^2 \log\left((t-1)\sqrt{K}\right)}{\epsilon'(t-1)}}. \tag{24}$$

Thus, with probability

$$\prod_{t=2}^{\sqrt{K}}\left(1 - \frac{2}{(t-1)\epsilon'\sqrt{K}}\right) \geq 1 - \frac{2\log K}{\epsilon'\sqrt{K}} \tag{25}$$

regret in round t is bounded by (24) for all t. Therefore, the regret during policy refinement is bounded by

$$
\begin{aligned}
&2H\sqrt{K} + \sum_{t=3}^{\sqrt{K}} 2\sqrt{\frac{2\sqrt{K}H^2 \log\left((t-1)\sqrt{K}\right)}{\epsilon'(t-1)}} \\
\leq &2H\sqrt{K} + 2\sqrt{\frac{2KH^2 \log K}{\epsilon'}} \sum_{t=3}^{\sqrt{K}} \sqrt{\frac{1}{(t-1)\sqrt{K}}} \\
\leq &2H\sqrt{K} + 2\sqrt{\frac{2\sqrt{K}H^2 \log K}{\epsilon'}} \int_{t=1}^{\sqrt{K}} \sqrt{\frac{1}{t}} dt \\
\leq &2H\sqrt{K} + 2\sqrt{\frac{2KH^2 \log K}{\epsilon'}} \\
= &\mathcal{O}(H\sqrt{K}).
\end{aligned}
$$

The analysis is the same for the constraint violation. ■

Appendix G. Proof of Theorem 6 (Identification)

Theorem 3 *Assume $\tilde{\mathcal{D}} = \mathcal{D}^*$ after policy pruning. Under Assumptions 1 and 3, with probability $1 - \tilde{\mathcal{O}}\left(\frac{1}{K}\right)$, the regret and constraint violation during the policy identification phase are both $\mathcal{O}\left(\sqrt{K}\right)$. Furthermore, $|\tilde{\pi}_h(a|x) - \pi^*_h(a|x)| = \mathcal{O}\left(\frac{1}{\sqrt{K}}\right)$ if $0 < \pi^*_h(a|x) < 1$ and $\tilde{\pi}_h(a|x) = \pi^*_h(a|x)|$ if $\pi^*_h(a|x) \in \{0, 1\}$.*

Proof Consider the $\{a_m\}$ obtained at the end of the refinement phase, and the mixed policy $\hat{\pi}$ defined by $\{a_m\}$. According to the proof of Theorem 5, we have with probability $1 - \mathcal{O}(K^{-1})$,

$$
\begin{aligned}
V_1^{\hat{\pi}} &= \sum_{m=1}^{M} a_m V_1^{\pi^m} \\
&\geq \left(V_1^{\pi^*} - 2\sqrt{\frac{H^2 \log\left((t-1)\epsilon' K\right)}{\epsilon' K}} \right) \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
W_1^{\hat{\pi},n} &= \sum_{m=1}^{M} a_m W_1^{\pi^m,n} \\
&\geq \left(W_1^{\pi^*,n} - 2\sqrt{\frac{H^2 \log\left((t-1)\epsilon' K\right)}{\epsilon' K}} \right) \quad \forall n. \qquad (27)
\end{aligned}
$$

Therefore, the regret and constraint violation during the identification phase, which includes K episodes, are both $\mathcal{O}(H\sqrt{K})$.

When both (26) and (27) hold, under Assumption 2, we have

$$
\|q^{\hat{\pi}} - q^{\pi^*}\|_1 = \mathcal{O}(1/\sqrt{K}).
$$

For any (h, x, a) such that $0 < \pi_h^*(x|a) < 1$,

$$\mathbb{E}\left[\sum_{k=1}^{a_m K} \mathbb{I}(x_{k,h} = x, a_{k,h} = a)\right] = a_m q_h^{\pi^m}(x, a)K,$$

which implies that

$$\begin{aligned}&\Pr\left(\left|\sum_{k=1}^{a_m K} \mathbb{I}_{(x_{k,h}=x,a_{k,h}=a)} - a_m q_h^{\pi^m}(x, a)K\right| \leq \sqrt{K \log K}\right)\\&= 1 - \mathcal{O}\left(\frac{1}{K}\right)\end{aligned}$$

according to the Azuma-Hoeffding inequality. Define event

$$\Phi = \left\{\left|\frac{\sum_{k=1}^{K} \mathbb{I}_{(x_{k,h}=x,a_{k,h}=a)}}{K} - \sum_m a_m q_h^{\pi^m}(x, a)\right| \leq M\sqrt{\frac{\log K}{K}}\right\}. \tag{28}$$

We have

$$\Pr(\Phi) = 1 - \mathcal{O}\left(\frac{1}{K}\right).$$

Define $\tilde{q}_h(x, a) = \frac{N_h(x,a)}{K}$, which is the empirical occupant measure under policy $\hat{\pi}$. Note that

$$N_h(x, a) = \sum_{k=1}^{K} \mathbb{I}(x_{k,h} = x, a_{k,h} = a)$$

and

$$q_h^{\hat{\pi}}(x, a) = \sum_m a_m q_h^{\pi^m}(x, a).$$

Therefore, we have with probability $1 - \mathcal{O}(1/K)$,

$$\|\tilde{q} - q^{\pi^*}\|_1 = \mathcal{O}(1/\sqrt{K}),$$

which implies that

$$\|\tilde{\pi} - \pi^*\|_1 = \mathcal{O}(1/\sqrt{K}).$$

Furthermore, since $\tilde{\mathcal{D}} = \mathcal{D}^*$, we immediately have $\tilde{\pi}_h(a|x) = \pi_h^*(a|x)|$ if $\pi_h^*(a|x) \in \{0, 1\}$. ■

Appendix H. Extension to CMDPs with multiple optimal solutions

When the optimal solution to the CMDP is not unique, or the RL agent does not know whether the CMDP has a unique solution or not, the agent adds Multi-Solution Pruning after the pruning phase in PRI to keep one and only one optimal policy belonging to $\Pi^{*,e}$. Recall that after the pruning phase, the action space for state x and step h, denoted by $\mathcal{A}_{h,x}$, is limited to $\mathcal{A}_{h,x} = \tilde{D}_{h,x}$. The key idea of the multi-solution pruning algorithm is to evaluate each stochastic decision (h', x') such that

$|\mathcal{A}_{h',x'}| > 1$. The algorithm first decides whether some of the actions in $\mathcal{A}_{h',x'}$ can be removed, e.g., a', while retaining at least one optimal policy with the following action space:

$$\otimes_{(h,x)\neq(h',x')}\mathcal{A}_{h,x} \otimes (\mathcal{A}_{h',x'} \setminus \{a'\}).$$

This is done by running Triple-Q with the above action space for $K^{0.25}$ episodes. If the regret is small, then with a high probability, at least one of the optimal policies is retained so we can remove action a' from $\mathcal{A}_{h',x'}$. The detailed algorithm is presented in Algorithm 2.

If the regret is large, then any optimal policy in $\otimes_{(h,x)}\mathcal{A}_{h,x}$ has to use action a' in state x' at step h'. Mulit-Solution Pruning next determines whether using a' alone is sufficient, i.e., whether an optimal policy is retained in the following action space

$$\otimes_{(h,x)\neq(h',x')}\mathcal{A}_{h,x} \otimes (\mathcal{A}_{h',x'} = \{a'\}).$$

This is again done by running Triple-Q with the above action space for $K^{0.25}$ episodes. If the regret is small, then with a high probability, one optimal policy takes a greedy decision at (h', x') with action a'; otherwise, the algorithm keeps a' in $\mathcal{A}_{h',x'}$ and moves to a different action in $\mathcal{A}_{h',x'}$. Note that we use Triple-Q for $K^{0.25}$ episodes each time, instead of $\sqrt{K}$ episodes, because it is easier to learn whether an optimal policy exists than learning the actual optimal policy.

H.1. Proof of Theorem 7 (multiple-solution pruning)

Theorem 4 *Under Assumption 4 and 5, with probability* $1 - \mathcal{O}(1/K^{0.02})$, *for sufficiently large* K, *multi-solution pruning outputs a unique optimal policy with at most* N *stochastic decisions. The regret and constraint violation during multi-solution pruning are bounded by* $H^2SAK^{0.25}$ *with probability one.*

Proof

We first consider the $K^{0.25}$ episodes after a' is removed from $\mathcal{A}_{h',x'}$. Consider the case that there still exists an optimal policy after removing the action. In this case, we will show that $v^* - \tilde{v} \leq \frac{2}{K^{0.03}}$ with a high probability. Define

$$\bar{V}_1 = \frac{1}{K^{0.25}} \sum_{k=1}^{K^{0.25}} V_1^{\pi_k},$$

where π_k is the policy used in the kth episode.

Note that

$$v^* - \tilde{v} = v^* - V_1^{\pi^*} + V_1^{\pi^*} - \bar{V}_1 + \bar{V}_1 - \tilde{v}.$$

We next bound the three terms $v^* - V_1^{\pi^*}$, $V_1^{\pi^*} - \bar{V}_1$, and $\bar{V}_1 - \tilde{v}$ individually.

Let $v_{k,1}$ be the cumulative reward received in episode k and $V_1^{\pi_k}$ be the reward value function. Note that

$$X_\tau = \sum_{k=1}^{\tau} (v_{k,1} - V_1^{\pi_k})$$

is a Martingale. By Azuma's inequality, we have

$$\Pr\left(\left|\tilde{v} - \bar{V}_1\right| \leq \sqrt{\frac{2H^2 \log K^{0.25}}{K^{0.25}}}\right) \geq 1 - \frac{1}{2K^{0.25}}. \quad (29)$$

Algorithm 2: Multi-Solution Pruning

Set v^* to be the average cumulative reward received over the $\sqrt{K}$ episodes under the policy pruning phase of PRI.
Set flag$(h,x) \leftarrow 0$ for all (h,x) such that $|\tilde{\mathcal{D}}_{h,x}| > 1$.
while $\exists(h,x)$ *such that* $|\tilde{\mathcal{D}}_{h,x}| > 1$ *and flag*$(h,x) = 0$ **do**
  Select (h',x') such that $|\tilde{\mathcal{D}}_{h',x'}| > 1$ and flag$(h',x') = 0$ with the smallest h'. Ties are broken arbitrarily.
  flag$(h',x') \leftarrow 1$
  for $a' \in \tilde{\mathcal{D}}_{h',x'}$ **do**
    Reset Triple-Q and run it for $K^{0.25}$ episodes with $\tilde{D}_{h,x}$ $((h,x) \neq (h',x'))$ and $\tilde{D}_{h',x'} \setminus \{a'\}$ as the action spaces while counting $\tilde{N}_h(x,a)$ as in policy pruning. Record the average cumulative reward $\tilde{v}$ and average cumulative utilities $\tilde{w}^n$.
    if $v^* - \tilde{v} \leq \frac{2}{K^{0.03}}$ *and* $\tilde{w}^n \geq \rho^n$ *for all* n **then**
      Update $\tilde{\mathcal{D}}_{h,x} = \left\{a : \frac{\tilde{N}_h(x,a)}{K^{0.25}} \geq \frac{\epsilon}{2}\right\}$ for all (h,x).
    end
    else
      Run Triple-Q for $K^{0.25}$ episodes with action space $\tilde{D}_{h,x}$ for $(h,x) \neq (h',x')$ and $\{a'\}$ for (h',x'). Record the average cumulative reward $\tilde{v}$
      if $v^* - \tilde{v} \leq \frac{2}{K^{0.03}} + 2\sqrt{\frac{2H^2 \log K^{0.25}}{K^{0.25}}}$ *and* $\tilde{w}^n \geq \rho^n$ *for all* n **then**
        Update $\tilde{\mathcal{D}}_{h,x} = \left\{a : \frac{\tilde{N}_h(x,a)}{K^{0.25}} \geq \frac{\epsilon}{2}\right\}$ for all (h,x).
      end
    end
  end
end

A similar argument yields that

$$\Pr\left(\left|v^* - \bar{V}_1^{\pi^*}\right| \leq \sqrt{\frac{2H^2 \log K^{0.5}}{K^{0.5}}}\right) \geq 1 - \frac{1}{2K^{0.5}}. \tag{30}$$

We next bound $V_1^{\pi^*} - \bar{V}_1$ based on Lemma 3 and the Markov inequality. First, based on the Markov inequality, we have

$$\Pr\left(V_1^{\pi^*} - \bar{V}_1 \geq K^{-0.03} \middle| V_1^{\pi^*} \geq \bar{V}_1\right) \leq \frac{\mathbb{E}\left[V_1^{\pi^*} - \bar{V}_1 \middle| V_1^{\pi^*} \geq \bar{V}_1\right]}{K^{-0.03}}.$$

Note that we have

$$\mathbb{E}\left[V_1^{\pi^*} - \bar{V}_1 \middle| V_1^{\pi^*} \geq \bar{V}_1\right] = \frac{\mathbb{E}\left[V_1^{\pi^*} - \bar{V}_1\right] - \Pr\left(V_1^{\pi^*} < \bar{V}_1\right)\mathbb{E}\left[V_1^{\pi^*} - \bar{V}_1 \middle| V_1^{\pi^*} < \bar{V}_1\right]}{\Pr\left(V_1^{\pi^*} \geq \bar{V}_1\right)} \tag{31}$$

$$\leq \frac{\frac{c_1 K^{0.2}}{K^{0.25}} + H \Pr\left(V_1^{\pi^*} < \bar{V}_1\right)}{1 - \Pr\left(V_1^{\pi^*} < \bar{V}_1\right)} \tag{32}$$

and

$$\Pr\left(V_1^{\pi^*} - \bar{V}_1 \geq K^{-0.03}\right) \tag{33}$$

$$= \Pr\left(V_1^{\pi^*} - \bar{V}_1 \geq K^{-0.03} \middle| V_1^{\pi^*} \geq \bar{V}_1\right) \Pr\left(V_1^{\pi^*} \geq \bar{V}_1\right) \tag{34}$$

$$+ \Pr\left(V_1^{\pi^*} - \bar{V}_1 \geq K^{-0.03} \middle| V_1^{\pi^*} < \bar{V}_1\right) \Pr\left(V_1^{\pi^*} < \bar{V}_1\right) \tag{35}$$

$$\leq \Pr\left(V_1^{\pi^*} - \bar{V}_1 \geq K^{-0.03} \middle| V_1^{\pi^*} \geq \bar{V}_1\right) \left(1 - \Pr\left(V_1^{\pi^*} < \bar{V}_1\right)\right) + \Pr\left(V_1^{\pi^*} < \bar{V}_1\right) \tag{36}$$

$$= K^{0.03}\left(\frac{c_1 K^{0.2}}{K^{0.25}} + H \Pr\left(V_1^{\pi^*} < \bar{V}_1\right)\right) + \Pr\left(V_1^{\pi^*} < \bar{V}_1\right). \tag{37}$$

Note that when the constraints are satisfied, we have $V_1^{\pi^*} \geq \bar{V}_1$. Therefore, according Lemma 3, $\Pr\left(V_1^{\pi^*} < \bar{V}_1\right) = \mathcal{O}(K^{-0.5})$, which implies that

$$\Pr\left(V_1^{\pi^*} - \bar{V}_1 \geq K^{-0.03}\right) = \mathcal{O}\left(K^{-0.02}\right). \tag{38}$$

Combining the inequality above with inequalities (29) and (30), we can conclude that

$$\Pr\left(v^* - \tilde{v} \leq 2K^{-0.03}\right) = 1 - \mathcal{O}\left(K^{-0.02}\right). \tag{39}$$

Based on Lemma 3's result on constraint violation, we obtain

$$\Pr\left(v^* - \tilde{v} \leq 2K^{-0.03}, \tilde{w}^n \geq \rho^n \ \forall n\right) = 1 - \mathcal{O}\left(K^{-0.02}\right), \tag{40}$$

On the other hand, if no optimal policy exists after removing a', then we have $\forall \pi \in \Pi^{*,e}$, $q_{h'}^{\pi}(x', a') \geq \epsilon$. Let π'' be an optimal policy with action spaces

$$\otimes_{(h,x)\neq(h',x')}\mathcal{A}_{h,x} \otimes (\mathcal{A}_{h',x'} \setminus \{a'\}),$$

and suppose all constraints are satisfied under π''. Note that π'' is *not* an optimal policy for the original problem. We first have

$$v^* - \tilde{v} = v^* - V_1^{\pi^*} + V_1^{\pi^*} - V_1^{\pi''} + V_1^{\pi''} - \bar{V}_1 + \bar{V}_1 - \tilde{v}$$

Based on Assumptions 4 and 5, we have

$$\begin{aligned} V_1^{\pi^*} - V_1^{\pi''} &\geq c_v ||q^{\pi^*} - q^{\pi''}||_1 \\ &\geq c_v |q_{h'}^{\pi^*}(x', a') - q_{h'}^{\pi''}(x', a')| \\ &\geq c_v \epsilon. \end{aligned}$$

This holds because all constraints are satisfied under π'' under our assumption. Similar to the first case, we have

$$\Pr\left(\left|\tilde{v} - \bar{V}_1\right| \leq \sqrt{\frac{2H^2 \log K^{0.25}}{K^{0.25}}}\right) \geq 1 - \frac{1}{2K^{0.25}}. \tag{41}$$

and

$$\Pr\left(\left|v^* - \bar{V}_1^{\pi^*}\right| \leq \sqrt{\frac{2H^2\log K^{0.5}}{K^{0.5}}}\right) \geq 1 - \frac{1}{2K^{0.5}}. \tag{42}$$

Furthermore, according to Lemma 3,

$$\Pr\left(V_1^{\pi''} - \bar{V}_1 \geq 0\right) \geq 1 - \mathcal{O}\left(\frac{1}{K^{0.5}}\right) \tag{43}$$

because the constraint is satisfied with probability $1 - \mathcal{O}\left(\frac{1}{K^{0.5}}\right)$.

Summarizing the results above, using the union bound, we conclude that with probability $1 - \mathcal{O}\left(K^{-0.02}\right)$, we have

$$v^* - \tilde{v} \geq c_v\epsilon - 2\sqrt{\frac{2H^2\log K^{0.25}}{K^{0.25}}} > 2K^{-0.03}$$

for sufficiently large K if an optimal policy is not retained. If none of the policies formed by action spaces

$$\otimes_{(h,x)\neq(h',x')}\mathcal{A}_{h,x}\otimes(\mathcal{A}_{h',x'}\setminus\{a'\})$$

can satisfy the constraints, then it can be easily shown that $\tilde{w}^n < \rho^n$ with probability $1 - \mathcal{O}(K^{-0.25})$ for some n.

If a' is deemed to be necessary, Mulit-Solution Pruning next determines whether using a' alone is sufficient, i.e., can stochastic decision (h', x') become greedy without losing optimality? The algorithm runs Triple-Q with action space $\{a'\}$ for (h', x'). If there exists an optimal policy π with $\pi_{h'}(a'|x') = 1$, then following the same analysis above, we have with probability at least $1 - \mathcal{O}\left(K^{-0.02}\right)$,

$$v^* - \tilde{v} \leq \frac{2}{K^{0.03}}.$$

Otherwise, according to the Assumption 4, $\forall \pi^* \in \Pi^{*,e}$, there exists another action $a'' \neq a'$ such that $q_{h'}^{\pi^*}(x', a'') \geq \epsilon$. Because any optimal policy can be represented as a linear combination of those policies in $\Pi^{*,e}$, we have that for any optimal policy π^*, $\sum_{a\neq a'} q_{h'}^{\pi^*}(x', a) \geq \epsilon$. Letting π'' be an optimal policy with action spaces

$$\otimes_{(h,x)\neq(h',x')}\mathcal{A}_{h,x}\otimes(\mathcal{A}_{h',x'} = \{a'\}),$$

which satisfies all constraints, we have

$$\sum_{a\neq a'} q_{h'}^{\pi^*}(x', a) - q_{h'}^{\pi''}(x', a) \geq \epsilon.$$

Thus, according to Assumption 5,

$$V_1^{\pi*} - V_1^{\pi''} \geq c_v||q^{\pi*} - q^{\pi''}||_1 \geq c_v\epsilon$$

because π'' satisfies all constraints.

The remaining analysis is identical to case when a' is removed from the action space $\mathcal{A}_{h',x'}$. If none of the policies formed by action spaces

$$\otimes_{(h,x)\neq(h',x')}\mathcal{A}_{h,x} \otimes (\mathcal{A}_{h',x'} = \{a'\})$$

can satisfy the constraints, then it can be easily shown that $\tilde{w}^n < \rho^n$ with probability $1 - \mathcal{O}(K^{-0.25})$ for some n.

After the algorithm goes through all action space $\mathcal{A}_{h,x}$, with probability $1 - \mathcal{O}(1/K^{0.02})$, we obtain action space

$$\otimes_{(h,x)}\mathcal{A}_{h,x}$$

such that none of the stochastic decision can be reduced to a greedy decision without losing optimality. Since any optimal policy can be written as a linear combination of optimal policies associated with extreme points, and any combination of two optimal policy only increases the number of stochastic decisions. Therefore, we conclude that the optimal policy induced by

$$\otimes_{(h,x)}\mathcal{A}_{h,x}$$

is an extreme point and is unique. Besides, it is easy to verify that the regret and constraint violation of multiple solution pruning are bounded by $H^2SAK^{0.25}$ because it takes at most $HSAK^{0.25}$ episodes to finish the algorithm. ■

Appendix I. Simulations

I.1. Synthetic CMDP

In the systehtic CMDP, we choose $|\mathcal{S}| = 3, |\mathcal{A}| = 3, H = 3$. The detailed parameters of the CMDP in the first experiment are shown in Table 2, 3 and 4.

We executed the pruning phase (Triple-Q) over $100,000$ episodes, followed by the refinement phase over $1,000,000$ episodes. Since the problem has only one constraint, the optimal policy has only one stochastic decision, which can be decided by evaluating the frequencies of two greedy policies. Thus, phase 3 is not necessary for this specific environment.

Table 2: Transition Kernels (the rows represent (previous state, action) and the columns represent (step, next state)).

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	0.3112981	0.35107633	0.27041442	0.42626645	0.04822746	0.14663183	0.4031534	0.19783729	0.39831431
(1,2)	0.23314339	0.32491141	0.48360071	0.24246185	0.19021328	0.43972054	0.26457139	0.21435897	0.26256243
(1,3)	0.45555851	0.32401226	0.24598487	0.3312717	0.76155926	0.41364763	0.33227521	0.58780374	0.33912326
(2,1)	0.32676574	0.35320112	0.1300059	0.35453348	0.32114495	0.40817113	0.1762648	0.30097191	0.48437535
(2,2)	0.11092341	0.28034838	0.45655888	0.23441632	0.2847394	0.235718	0.17239783	0.37273618	0.08000908
(2,3)	0.56231085	0.3664505	0.41343525	0.4110502	0.39411565	0.35611087	0.65133738	0.32629191	0.43561556

I.2. Grid World

As shown in Figure 5, the task of the agent is to go from the red grid point to the green grid point. The black grid points are the *safe* points over which the agent can move, and the yellow grid points

Table 3: Rewards (the rows represent (state, action) and the columns represent step.)

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
1	0.5507979	0.70814782	0.29090474	0.51082761	0.89294695	0.89629309	0.12558531	0.20724388	0.0514672
2	0.44080984	0.02987621	0.45683322	0.64914405	0.27848728	0.6762549	0.59086282	0.02398188	0.55885409
3	0.25925245	0.4151012	0.28352508	0.69313792	0.44045372	0.15686774	0.54464902	0.78031476	0.30636353

Table 4: Utilities (the rows represent (state, action) and the columns represent step.)

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
1	0.22195788	0.38797126	0.93638365	0.97599542	0.67238368	0.90283411	0.84575087	0.37799404	0.09221701
2	0.6534109	0.55784076	0.36156476	0.2250545	0.40651992	0.46894025	0.26923558	0.29179277	0.4576864
3	0.86053391	0.5862529	0.28348786	0.27797751	0.45462208	0.20541034	0.20137871	0.51403506	0.08722937

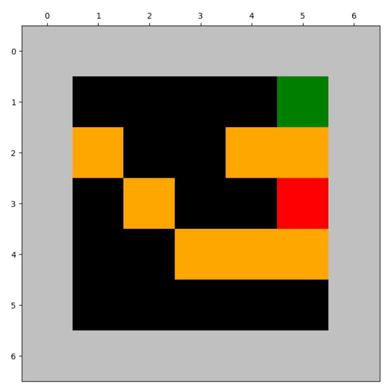

Figure 5: Grid World

are obstacles. Moving over an obstacle incurs a penalty of one. The constraint is that the agent can incur only an average cost of 0.5 or less. The agent can take six steps at maximum. The reward associated with reaching the destination is 1, and the rewards for other locations, after six steps, are the Euclidean distance from the location to the destination (normalized by the longest distance). At each grid point, the agent has five actions to choose from: up, down, left, right, and stay, except at the boundary. The goal is to maximize the reward subject to the constraint.

During the experiment, we observed that policy pruning is much more efficient than the theoretical worst case. For this specific environment, the optimal policy should have $6 \times 5 \times 5 + 1 = 155$ nonzero $\pi_h^*(a|x)$'s. After the first phase (Triple-Q), we have roughly 200 (step, state, action) triples (here "roughly" considers the difference among different trials with different random seeds), associated with stochastic decisions to check and prune. Except for the two “necessary” decisions, which are stochastic decisions in the optimal policy, for all trials, the algorithm only checked two candidate triples and eliminated the rest candidate triples in the process.