On the Interaction of Noise, Compression, and Adaptivity under (L_0, L_1)-Smoothness: An SDE Approach

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Abstract

Using stochastic differential equation (SDE) approximations, we study the dynamics of Distributed SGD, Distributed Compressed SGD, and Distributed SignSGD under (L_0, L_1)-smoothness and flexible noise assumptions. Our analysis provides insights – which we validate through simulation – into the intricate interactions between batch noise, stochastic gradient compression, and adaptivity in this modern theoretical setup. For instance, we show that *adaptive* methods such as Distributed SignSGD can successfully converge under standard assumptions on the learning rate scheduler, even under heavy-tailed noise. On the contrary, Distributed (Compressed) SGD with pre-scheduled decaying learning rate fails to achieve convergence, unless such a schedule also accounts for an inverse dependency on the gradient norm – de facto falling back into an adaptive method.

1. Introduction

Understanding the dynamics of stochastic optimization algorithms is crucial, especially in distributed machine learning settings where batch noise, compression, and adaptivity significantly impact convergence and generalization. Despite extensive studies in the literature, the interplay among these three aspects under the general condition of (L_0, L_1)-smoothness remains underexplored.

Contributions. Our key contributions include:

- Establishing convergence bounds for Distributed SGD (DSGD), Distributed Compressed SGD (DCSGD), and Distributed SignSGD (DSignSGD) under the (L_0, L_1)-smoothness condition;
- Showcasing how normalizing the update step of D(C)SGD naturally emerges as a condition for convergence, thus confirming the superiority of adaptive methods for ill-conditioned loss landscapes, especially for pathological batch noise or when unbiased compression is used;
- Highlighting that an *adaptive* method such as DSignSGD converges even under heavy-tailed noise with standard assumptions on the learning rate scheduler.

2. Related work

SDE Approximations and Applications. In [22], a rigorous theoretical framework was introduced to derive SDEs that faithfully model the stochastic behavior intrinsic to optimization algorithms widely employed in machine learning. Since then, such SDE-based formulations have found application across several domains, including *stochastic optimal control* for tuning stepsizes [22, 23] and batch sizes [46]. Notably, SDEs have been instrumental in analyzing *convergence bounds* and *stationary distributions* [5, 6, 8], *scaling laws* [7, 8, 15], *implicit regularization* effects [5, 37], and *implicit preconditioning* [26, 43].

Interplay of noise, compression, and adaptivity under (L_0, L_1)-smoothness Previous research has extensively studied the effect of batch noise, compression, and adaptivity on the convergence of optimizers. Batch noise significantly influences stochastic gradient algorithms, affecting their convergence speed and stability [8, 19, 36, 44]. Noise characteristics such as heavy-tailed distributions have been shown to profoundly impact the optimization trajectories, necessitating robust algorithmic strategies [12, 35]. Compression methods, including unbiased techniques such as sparsification and quantization [1, 28, 38] and biased approaches like SignSGD [2, 3], are critical for reducing communication overhead in distributed training. These compression techniques come with theoretical guarantees under various smoothness assumptions [1, 7, 11, 28]. Adaptive methods such as SignSGD normalize gradient elements to cope effectively with large or heavy-tailed gradient noise, thus demonstrating improved empirical robustness [7, 8, 18, 33].

However, most of the aforementioned works rely on restrictive assumptions such as L-smoothness, i.e., the L-Lipschitz continuity of the gradient. To relax this condition, Zhang et al. [44] introduces and empirically validates the (L_0, L_1)-*smoothness* assumption, which allows the norm of the Hessian to be bounded by an affine function of the gradient norm, thereby significantly expanding the class of admissible problems. Various (stochastic) first-order methods have been analyzed under (L_0, L_1)-smoothness, including Clip-SGD and its variants [13, 17, 30, 39, 44, 45], Normalized SGD and its variants [4, 14, 47], SignSGD [9], AdaGrad [10, 42], Adam [21, 41], and SGD [20]. In the context of compressed communication, Khirirat et al. [16] proposed and analyzed a momentum-based variant of normalized EF21-SGD [31] under the assumption of bounded noise variance.

To the best of our knowledge, no study has jointly considered all these aspects, namely, batch noise, communication compression, and adaptivity, under the (L_0, L_1)-smoothness condition. In particular, we consider flexible noise assumptions ranging from bounded to unbounded variance, and even encompassing heavy-tailed noise. Our work closes this gap by providing a comprehensive analysis of their interplay within a unified theoretical framework.

3. Preliminaries

Distributed Setup. Let us consider the problem of minimizing an objective function expressed as an average of N functions: $\min_{x\in\mathbb{R}^d} \left[f(x) \coloneqq \frac{1}{N}\sum_{i=1}^N f_i(x)\right]$, where each $f_i : \mathbb{R}^d \to \mathbb{R}$ is lower bounded and twice continuously differentiable, and represents the loss over the local data of the i-th agent. In our stochastic setup, each agent only has access to gradient estimates: let n_i be the number of datapoints accessible to agent i; at a given $x \in \mathbb{R}^d$, agent i estimates $\nabla f_i(x)$ using a batch of data $\gamma_i \subseteq \{1, \ldots, n_i\}$, sampled uniformly with replacement and uncorrelated from the previously

sampled batches. Given the sampling properties above, this estimate, which we denote by $\nabla f_{i,\gamma_i}(x)$, can be modeled as a perturbation of the global gradient: $\nabla f_{i,\gamma_i}(x) = \nabla f(x) + Z_i(x)$.

Noise assumptions. We assume the sampling process and agent configurations are such that, for all $x \in \mathbb{R}^d$ and each agent pair (i, j) with $i \neq j$, $Z_i(x)$ is independent of $Z_j(x)$. Regarding assumptions on the noise structure, we always assume that at each $x \in \mathbb{R}^d$, $Z_i(x)$ is absolutely continuous and with coordinate-wise symmetric distribution. If we discuss the setting $Z_i(x) \in L^1(\mathbb{R}^d)$, then we assume $\mathbb{E}[Z_i(x)] = 0$. Last, if $Z_i(x) \in L^2(\mathbb{R}^d)$, we denote $\Sigma_i(x) := Cov(Z_i(x))$.

Next, we define our two structural assumptions. The first one strictly concerns the global landscape; the second concerns how global landscape features affect the noise distribution of each agent.

Definition 1 ([44]) *f is (L_0, L_1)-smooth ($L_0, L_1 \geq 0$) if, $\forall x \in \mathbb{R}^d$, $\left\|\nabla^2 f(x)\right\| \leq L_0 + L_1 \|\nabla f(x)\|$.*

Definition 2 (Mod. of the assumptions from [34, 40]) *The gradient noise for agent i has $(\sigma_{0,i}^2, \sigma_{1,i}^2)$-variance if $\|\Sigma_i(x)\|_\infty \leq \sigma_{0,i}^2 + \sigma_{1,i}^2 \|\nabla f(x)\|_2^2$. If $\sigma_{1,i} = 0$, the noise has bounded variance.*

SDE approximations. The following definition formalizes the idea that an SDE can be a "reliable surrogate" to model an optimizer. It is drawn from the field of numerical analysis of SDEs (see [27]) and it quantifies the disparity between the discrete and the continuous processes.

Definition 3 *A continuous-time stochastic process $(X_t)_{t\in[0,T]}$ is an order α weak approximation of a discrete stochastic process $(x_k)_{k=0}^{\lfloor T/\eta \rfloor}$ if for every polynomial growth function g, there exists a positive constant C, independent of η, such that $\max_{k=0,\dots,\lfloor T/\eta \rfloor} \left|\mathbb{E}g\left(x_k\right) - \mathbb{E}g\left(X_{k\eta}\right)\right| \leq C\eta^\alpha$.*

Optimizers and SDEs. We study: 1) DSGD defined as $x_{k+1} = x_k - \frac{\eta}{N}\sum_{i=1}^N \nabla f_{i,\gamma_i}(x_k)$ and whose SDE is defined in Eq. 27 (see Thm. 3.2 in [7]); 2) DCSGD defined as $x_{k+1} = x_k - \frac{\eta}{N}\sum_{i=1}^N \mathcal{C}_{\xi_i}\left(\nabla f_{i,\gamma_i}(x_k)\right)$, where the stochastic compressors $\mathcal{C}_{\xi_i}$ are independent for different i and satisfy (i) $\mathbb{E}_{\xi_i}\left[\mathcal{C}_{\xi_i}(x)\right] = x$ and (ii) $\mathbb{E}_{\xi_i}\left[\|\mathcal{C}_{\xi_i}(x) - x\|_2^2\right] \leq \omega_i \|x\|_2^2$ for some compression rates $\omega_i \geq 0$: Its SDE is defined in Eq. 70 (see Thm. 3.6 in [7]); 3) DSignSGD defined as $x_{k+1} = x_k - \frac{\eta}{N}\sum_{i=1}^N \operatorname{sign}(\nabla f_{i,\gamma_i}(x_k))$ and whose SDE is in Eq. 94 (see Thm. 3.10 in [7]).

Importantly, extensive experimental validation [6–8, 25, 29] shows that the SDEs do track their respective optimizers accurately on a variety of architectures, e.g., MLPs, ResNets, and ViTs.

4. Theoretical Results

Recall that, in the continuous-time setup, the dynamics of the iterates is modeled by a stochastic process X_t solution to an SDE model. In this setting, the learning rate is a scalar factor in the SDE influencing both its drift and its diffusion. To decouple adaptivity from scheduling, we *parametrize our learning rate as a product*: $\eta\eta_t$. To ensure convergence, we **always** assume η_t satisfying the Robbins and Monro [32] conditions: For $\phi_t^i = \int_0^t (\eta_s)^i ds$, we require $\phi_t^1 \overset{t\to\infty}{\to} \infty$, $\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0$.

4.1. Overview

Under (L_0, L_1)-smoothness, our insights concern the structure of η for convergence, where $\eta\eta_t$ is the actual learning rate and η_t is a predetermined scheduler: See Fig. 1 for empirical validation.

- Thm. 4 shows that the dynamics of the DSGD model can converge to a first-order stationary point in expectation even when $\exists i$ s.t. $\sigma_{1,i}^2 > 0$, yet the learning rate η_t is required to scale inversely to the gradient norm – i.e. needs to be adaptive;
- Thm. 5 operates in the compressed unbiased gradient setting. The insights are similar to Thm. 4 yet assume bounded variance for pedagogical purposes only: Thm. 6 covers the more general $(\sigma_{0,i}^2, \sigma_{1,i}^2)$-variance case;
- Thm. 7 shows that the DSignSGD model does not require adaptive learning rate to converge: Not even when the expectation of the batch noise is **unbounded** – The intuition is that DSignSGD is already normalized.

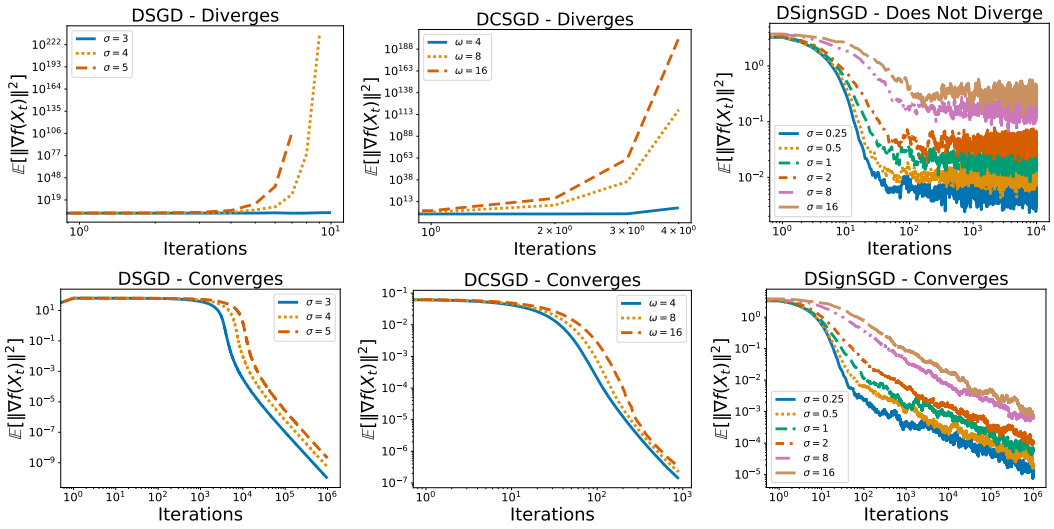

Figure 1: We optimize $f(x) = \frac{x^4}{4}$ with batch noise of variance $\sigma^2\|\nabla f(x)\|_2^2$ for different values of σ: As per Thm. 4, DSGD diverges faster and faster for larger values of σ if normalization **is not employed** (Top-Left) but always converges if it **is employed** (Bottom-Left); We optimize $f(x) = \frac{\sum_{j=1}^{1000}(x_j)^4}{4}$ with batch noise of variance $\sigma^2\|\nabla f(x)\|_2^2$ and use *Random Sparsification* for different compression rates ω: As per Thm. 5, DCSGD diverges faster and faster for larger values of ω if normalization **is not employed** but always converges if it **is employed** (Bottom-Center); We optimize $f(x) = \frac{x^4}{4}$ with batch noise of **unbounded expected value** and for different *scale parameters* σ: As per Thm. 7, DSignSGD does not converge to 0 *without* a proper learning rate scheduler (Top-Right), but does converge *with* (Bottom-Right).

4.2. Results

We state the SDE models directly in the appendix and indicate the setting with blue color.

Theorem 4 ***(DSGD, unbounded variance)*** *Let f be (L_0, L_1)-smooth, and each agent have $(\sigma_{0,i}^2, \sigma_{1,i}^2)$-variance. Define $\overline{\sigma_0^2} := \frac{1}{N}\sum_{i=1}^N \sigma_{0,i}^2$ and $\overline{\sigma_1^2} := \frac{1}{N}\sum_{i=1}^N \sigma_{1,i}^2$. For an arbitrary $\epsilon \in (0,1)$, assume*

$$\eta\eta_t < \frac{2\epsilon}{(L_0 + L_1\mathbb{E}[\|\nabla f(X_t)\|])\left(1 + \frac{d\overline{\sigma_1^2}}{N}\right) + \frac{d}{N}\overline{\sigma_0^2}L_1}. \tag{1}$$

Then, for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \frac{\eta\phi_t^2}{2N}(L_0 + L_1)d\overline{\sigma_0^2}\right) \overset{t\to\infty}{\to} 0. \tag{2}$$

Intuition: This result showcases the crucial role of the regularity of the loss landscape as well as its interaction with the gradient noise structure. Even in the noiseless setup, normalizing the update step naturally emerges as a condition to ensure convergence. Additionally: i) $L_1\overline{\sigma_1^2} > 0$ requires stronger adaptivity; ii) $\overline{\sigma_0} = \overline{\sigma_1} = 0$ recovers the standard stepsize schedule derived under L-smoothness, i.e. $\eta\eta_t < \frac{2}{L_0}$.

Theorem 5 ***(DCSGD, unbiased compression, bounded variance)*** *Let f be (L_0, L_1)-smooth and each agent i have bounded variance σ_i^2, $\overline{\sigma^2} := \frac{1}{N}\sum_{i=1}^N \sigma_i^2$, and $\overline{\sigma^2\omega} := \frac{1}{N}\sum_{i=1}^N \sigma_i^2\omega_i$. For arbitrary $\epsilon \in (0,1)$, assume*

$$\eta\eta_t < \frac{2\epsilon}{\left(L_0 + L_1\mathbb{E}\left[\|\nabla f(X_t)\|_2\right]\right)\left(1 + \frac{\overline{\omega}}{N}\right) + \frac{d\left(\overline{\sigma^2}+\overline{\sigma^2\omega}\right)L_1}{N}}. \tag{3}$$

Then, for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2\frac{\eta(L_0+L_1)d}{2N}\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)\right) \overset{t\to\infty}{\to} 0. \tag{4}$$

Intuition: This result showcases the crucial role of the regularity of the loss landscape and its interaction with gradient compression: i) Compressing the gradients, i.e. $\overline{\omega} > 0$, requires stronger adaptivity; ii) One can draw a parallel between the normalization requirement for DSGD prescribed in Eq. 1 and that of DCSGD in Eq. 3 — DCSGD with bounded variance σ^2 and compression rate ω is essentially equivalent to DSGD with (σ_0^2, σ_1^2)-variance where $\sigma_0^2 = d(\sigma^2 + \omega\sigma^2)$ and $\sigma_1^2 = \frac{\omega}{d}$.

One can generalize this result to cover the potentially unbounded variance setting.

Theorem 6 ***(DCSGD, unbiased compression, unbounded variance)*** *Let f be (L_0, L_1)-smooth, and each agent have $(\sigma_{0,i}^2, \sigma_{1,i}^2)$-variance. Define $\overline{\sigma_0^2} := \frac{1}{N}\sum_{i=1}^N \sigma_{0,i}^2$, $\overline{\sigma_1^2} := \frac{1}{N}\sum_{i=1}^N \sigma_{1,i}^2$, $\overline{\sigma_0^2\omega} := \frac{1}{N}\sum_{i=1}^N \sigma_{i,0}^2\omega_i$, and $\overline{\sigma_1^2\omega} := \frac{1}{N}\sum_{i=1}^N \sigma_{i,1}^2\omega_i$. For an arbitrary $\epsilon \in (0,1)$, assume*

$$\eta\eta_t < \frac{2\epsilon}{\left(L_0 + L_1\mathbb{E}\left[\|\nabla f(X_t)\|_2\right]\right)\left(1 + \frac{\overline{\omega}+d(\overline{\sigma_1^2\omega}+\overline{\sigma_1^2})}{N}\right) + \frac{L_1 d\left(\overline{\sigma_0^2}+\overline{\sigma_0^2\omega}\right)}{N}}. \tag{5}$$

Then, for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{(1-\epsilon)\phi_t^1}\left(f(X_0) - f(X_*) + \phi_t^2\frac{\eta(L_0+L_1)d(\overline{\sigma_0^2} + \overline{\sigma_0^2\omega})}{2N}\right) \overset{t\to\infty}{\to} 0. \tag{6}$$

Intuition: This result showcases the crucial role of the regularity of the loss landscape, the gradient noise structure, and the compression scheme: If $L_1(\overline{\omega} + d(\overline{\sigma_1^2\omega} + \overline{\sigma_1^2})) > 0$, stronger adaptivity is required.

DSignSGD, structured noise, unbounded expected value. To provide tight results for the convergence of DSignSGD under **unbounded second and even first moments**, we additionally assume structured (heavy-tailed) noise following a student-t distribution: $\nabla f_{\gamma_i}(x) = \nabla f(x) + \sqrt{\Sigma_i}Z_i$ s.t. $Z_i \sim t_\nu(0, I_d)$, ν are the d.o.f, and *scale matrices*[1] $\Sigma_i = \mathrm{diag}(\sigma_{1,i}^2, \cdots, \sigma_{d,i}^2)$. Note that if $\nu = 1$, the **expected value** of Z_i is **unbounded**, thus modeling much more pathological noise than simple (σ_0^2, σ_1^2)-variance.

Theorem 7 *Let f be (L_0, L_1)-smooth, $\Sigma_i \leq \sigma_{max,i}^2$, $\sigma_{\mathcal{H},1}$ be the harmonic mean of $\{\sigma_{max,i}\}$, $M_\nu > 0$ and $\ell_\nu > 0$ constants, and $K := \left(\frac{L_1}{2N} + \frac{(L_0+L_1)\sigma_{\mathcal{H},1}^{-1}M_\nu}{\sqrt{d}}\right)$. Then, for a scheduler $\eta\eta_t < \frac{\ell_\nu K^{-1}}{\sigma_{\mathcal{H},1}d}$ and a random time $\tilde{t}$ with distribution $\frac{\eta_t\ell_\nu\sigma_{\mathcal{H},1}^{-1} - \eta_t^2 K}{\phi_t^1\ell_\nu\sigma_{\mathcal{H},1}^{-1} - \phi_t^2 K}$, we have that*

$$\mathbb{E}\|\nabla f\left(X_{\tilde{t}}\right)\|_2^2 \leq \frac{1}{\phi_t^1\ell_\nu\sigma_{\mathcal{H},1}^{-1} - \phi_t^2 K}\left(f(X_0) - f(X_*) + \phi_t^2\eta(L_0+L_1)d\left(\frac{1}{2N} + \frac{M_\nu}{\sigma_{\mathcal{H},1}\sqrt{d}}\right)\right) \overset{t\to\infty}{\to} 0. \tag{7}$$

1. These are *not* covariance matrices, but we use the same notation to facilitate comparability.

5. Conclusion

In this paper, we provided the first application of SDEs to (L_0, L_1)-smooth problems, deriving the first convergence guarantees for DSGD, DCSGD, and DSignSGD under such a condition as we coupled it with flexible batch noise assumptions. Importantly, we show that some sort of adaptivity is beneficial to ensure the convergence of stochastic optimizers. On one hand, an adaptive method such as DSignSGD converges even under heavy-tailed noise of **unbounded** expected value. On the other hand, for DCSGD normalizing the updates emerges naturally as a strategy to ensure convergence, and even more so if either the compression rate $\overline{\omega}$ or the $\overline{\sigma_1^2}$ is positive. These findings prompt us to include the study of Normalized SGD under heavy-tailed noise in future work. Our final message is that the success of adaptive methods in Deep Learning has to be partially credited to the fact that their updates are, to some extent, normalized, thus actively countering the destabilizing effects of ill-conditioned landscapes even under large and possibly heavy-tailed noise.

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Appendix A. Theoretical Results

A.1. Distributed SGD

A.1.1. FIRST ORDER SDE

The following is the first-order SDE model of DSGD (see Theorem 3.2 in [7]). Let us consider the stochastic process $X_t \in \mathbb{R}^d$ defined as the solution of

$$dX_t = -\nabla f(X_t)dt + \sqrt{\frac{\eta}{N}}\sqrt{\hat{\Sigma}(X_t)}dW_t, \tag{8}$$

where $\hat{\Sigma}(x) \coloneqq \frac{1}{N}\sum_{i=1}^N \Sigma_i(x)$ is the average of the covariance matrices of the N agents.

Theorem 8 *Let f be (L_0, L_1)-smooth, $\|\Sigma_i(x)\|_\infty < \sigma_{0,i}^2 + \sigma_{1,i}^2\|\nabla f(x)\|_2^2$, the learning rate scheduler η_t s.t. $\phi_t^i = \int_0^t (\eta_s)^i ds$, $\phi_t^1 \overset{t\to\infty}{\to} \infty$, $\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0$, $\overline{\sigma_0^2} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_{0,i}^2$, and $\overline{\sigma_1^2} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_{1,i}^2$. Then, for $0 < \epsilon < 1$,*

$$\eta\eta_t < \frac{2N\epsilon}{d\left(\overline{\sigma_1^2}L_0 + \overline{\sigma_0^2}L_1 + L_1\overline{\sigma_1^2}\mathbb{E}\left[\|\nabla f(X_t)\|_2\right]\right)}, \tag{9}$$

and for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2\frac{\eta d(L_0 + L_1)(\overline{\sigma_0^2} + \overline{\sigma_1^2})}{2N}\right) \overset{t\to\infty}{\to} 0. \tag{10}$$

Proof Using Itô's Lemma and using a learning rate scheduler η_t during the derivation of the SDE, we have

$$d(f(X_t) - f(X_*)) = -\eta_t\|\nabla f(X_t)\|_2^2 dt + \mathcal{O}(\text{Noise}) + (\eta_t)^2\frac{\eta}{2N}\text{Tr}(\nabla^2 f(X_t)\tilde{\Sigma}(X_t))dt \tag{11}$$

$$\leq -\eta_t\|\nabla f(X_t)\|_2^2 dt + \mathcal{O}(\text{Noise}) \tag{12}$$

$$+ (\eta_t)^2\frac{\eta(\overline{\sigma_0^2} + \overline{\sigma_1^2}\|\nabla f(X_t)\|_2^2)d(L_0 + L_1\|\nabla f(X_t)\|)}{2N}dt. \tag{13}$$

Phase 1: If $\|\nabla f(X_t)\| \leq 1$, then the proof and conditions are the same as the L-smoothness case. Let us observe that since $\int_0^t \frac{\eta_s}{\phi_t^1}ds = 1$, the function $s \mapsto \frac{\eta_s}{\phi_t^1}$ defines a probability distribution and let $\tilde{t}$ have that distribution. Then, by integrating over time and by the Law of the Unconscious Statistician, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\tilde{t}})\|_2^2\right] = \frac{1}{\phi_t^1}\int_0^t \|\nabla f(X_s)\|_2^2\eta_s ds, \tag{14}$$

meaning that

$$\mathbb{E}\left[\|\nabla f(X_{\tilde{t}})\|_2^2\right] \leq \frac{f(X_0) - f(X_*)}{\phi_t^1} + \frac{\eta(L_0 + L_1)(\overline{\sigma_0^2} + \overline{\sigma_1^2})d}{2N}\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0. \tag{15}$$

Phase 2**:** If $\|\nabla f(X_t)\| > 1$, we have

$$d(f(X_t) - f(X_*)) = -\eta_t \|\nabla f(X_t)\|_2^2 dt + \mathcal{O}(\text{Noise}) + (\eta_t)^2 \frac{\eta}{2N} \text{Tr}(\nabla^2 f(X_t)\tilde{\Sigma}(X_t))dt \tag{16}$$

$$\leq -\eta_t \|\nabla f(X_t)\|_2^2 dt + \mathcal{O}(\text{Noise}) \tag{17}$$

$$+ (\eta_t)^2 \frac{\eta(\overline{\sigma_0^2} + \overline{\sigma_1^2}\|\nabla f(X_t)\|_2^2) d(L_0 + L_1\|\nabla f(X_t)\|)}{2N} dt \tag{18}$$

$$= -\eta_t \|\nabla f(X_t)\|_2^2 \left(1 - \frac{\eta_t \eta d}{2N}\left(\overline{\sigma_1^2} L_0 + \overline{\sigma_0^2} L_1 + L_1 \overline{\sigma_1^2}\|\nabla f(X_t)\|_2\right)\right) dt \tag{19}$$

$$+ (\eta_t)^2 \frac{\eta \overline{\sigma_0^2} d L_0}{2N} dt \tag{20}$$

Therefore, for $0 < \epsilon < 1$ we have that if

$$\eta\eta_t < \frac{2N\epsilon}{d\left(\overline{\sigma_1^2} L_0 + \overline{\sigma_0^2} L_1 + L_1 \overline{\sigma_1^2}\|\nabla f(X_t)\|_2\right)}, \tag{21}$$

and therefore that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2 \frac{\eta L_0 d\overline{\sigma^2}}{2N}\right) \overset{t\to\infty}{\to} 0, \tag{22}$$

where $\hat{t}$, is a random time with distribution $\frac{\eta_{\hat{t}}}{\phi_t^1}$.

By taking a worst-case scenario approach, we merge these two bounds into a single one:

$$d(f(X_t) - f(X_*)) \leq -\eta_t \|\nabla f(X_t)\|_2^2 \left(1 - \frac{\eta_t \eta d}{2N}\left(\overline{\sigma_1^2} L_0 + \overline{\sigma_0^2} L_1 + L_1 \overline{\sigma_1^2}\|\nabla f(X_t)\|_2\right)\right) dt \tag{23}$$

$$+ (\eta_t)^2 \frac{\eta d(L_0 + L_1)(\overline{\sigma_0^2} + \overline{\sigma_1^2})}{2N} dt, \tag{24}$$

and, therefore, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2 \frac{\eta d(L_0 + L_1)(\overline{\sigma_0^2} + \overline{\sigma_1^2})}{2N}\right) \overset{t\to\infty}{\to} 0, \tag{25}$$

where $\hat{t}$, is a random time with distribution $\frac{\eta_{\hat{t}}}{\phi_t^1}$.

Finally, for practical reasons, we leverage the distributed setting to tighten the requirements on the learning rate scheduler to make it experimentally viable, and rather require

$$\eta\eta_t < \frac{2N\epsilon}{d\left(\overline{\sigma_1^2} L_0 + \overline{\sigma_0^2} L_1 + L_1 \overline{\sigma_1^2}\mathbb{E}\left[\|\nabla f(X_t)\|_2\right)\right]}. \tag{26}$$

■

A.1.2. SECOND ORDER SDE

The following is the second-order SDE model of DSGD and is a straightforward generalization of Theorem 3.2 in [7]. Let us consider the stochastic process $X_t \in \mathbb{R}^d$ defined as the solution of

$$dX_t = -\nabla f(X_t)dt - \frac{\eta}{2}\nabla^2 f(X_t)\nabla f(X_t)dt + \sqrt{\frac{\eta}{N}}\sqrt{\hat{\Sigma}(X_t)}dW_t, \tag{27}$$

where $\hat{\Sigma}(x) \coloneqq \frac{1}{N}\sum_{i=1}^N \Sigma_i(x)$ is the average of the covariance matrices of the N agents.

Theorem 9 *Let f be (L_0, L_1)-smooth, $\|\Sigma_i(x)\|_\infty < \sigma_{0,i}^2 + \sigma_{1,i}^2\|\nabla f(x)\|_2^2$, the learning rate scheduler η_t s.t. $\phi_t^i = \int_0^t (\eta_s)^i ds$, $\phi_t^1 \overset{t\to\infty}{\to} \infty$, $\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0$, $\overline{\sigma_0^2} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_{0,i}^2$, and $\overline{\sigma_1^2} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_{1,i}^2$. Then, for $0 < \epsilon < 1$,*

$$\eta\eta_t < \frac{2\epsilon}{L_0 + L_1\mathbb{E}\left[\|\nabla f(X_t)\|\right] + \frac{d}{N}\left(\overline{\sigma_1^2}L_0 + \overline{\sigma_0^2}L_1 + L_1\overline{\sigma_1^2}\mathbb{E}\left[\|\nabla f(X_t)\|\right]\right)}, \tag{28}$$

and for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \frac{\eta\phi_t^2}{2N}(L_0 + L_1)d\overline{\sigma_0^2}\right) \overset{t\to\infty}{\to} 0. \tag{29}$$

Proof Using Itô's Lemma and using a learning rate scheduler η_t during the derivation of the SDE, we have

$$\begin{aligned} d(f(X_t) - f(X_*)) &= -\eta_t\|\nabla f(X_t)\|_2^2 dt - \frac{\eta\eta_t^2}{2}\left(\nabla f(X_t)\right)^\top \nabla^2 f(X_t)\nabla f(X_t)dt && (30)\\ &\quad + \mathcal{O}(\text{Noise}) + (\eta_t)^2\frac{\eta}{2N}\text{Tr}(\nabla^2 f(X_t)\tilde{\Sigma}(X_t))dt && (31)\\ &\leq -\eta_t\|\nabla f(X_t)\|_2^2 dt + \frac{\eta\eta_t^2}{2}(L_0 + L_1\|\nabla f(X_t)\|)\|\nabla f(X_t)\|^2 dt && (32)\\ &\quad + \mathcal{O}(\text{Noise}) + (\eta_t)^2\frac{\eta(\overline{\sigma_0^2} + \overline{\sigma_1^2}\|\nabla f(X_t)\|_2^2)d(L_0 + L_1\|\nabla f(X_t)\|)}{2N}dt. && (33)\end{aligned}$$

Phase 1**:** If $\|\nabla f(X_t)\| \leq 1$,

$$\|\nabla f(X_t)\|_2^2\left(\eta_t - \frac{\eta\eta_t^2}{2}(L_0 + L_1\|\nabla f(X_t)\|_2)\left(1 + \frac{d\overline{\sigma_1^2}}{N}\right)\right)dt \leq -d(f(X_t) - f(X_*)) + \frac{\eta\eta_t^2}{2N}.(L_0 + L_1)d\overline{\sigma_0^2}dt \tag{34}$$

Therefore, for $\epsilon \in (0, 1)$, we have that

$$\eta\eta_t < \frac{2\epsilon}{(L_0 + L_1\|\nabla f(X_t)\|_2)\left(1 + \frac{d\overline{\sigma_1^2}}{N}\right)} < \frac{2}{(L_0 + L_1)\left(1 + \frac{d\overline{\sigma_1^2}}{N}\right)} \tag{35}$$

meaning that

$$\mathbb{E}\left[\|\nabla f(X_{\tilde{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1 - \phi_t^2\frac{\eta}{2}(L_0 + L_1)\left(1 + \frac{d\overline{\sigma_1^2}}{N}\right)}\left(f(X_0) - f(X_*) + \frac{\eta\phi_t^2}{2N}(L_0 + L_1)d\overline{\sigma_0^2}\right) \overset{t\to\infty}{\to} 0. \tag{36}$$

Phase 2**:** If $\|\nabla f(X_t)\| > 1$, we have

$$d(f(X_t) - f(X_*)) = -\eta_t \|\nabla f(X_t)\|_2^2 dt + \mathcal{O}(\text{Noise}) + (\eta_t)^2 \frac{\eta}{2N} \text{Tr}(\nabla^2 f(X_t)\tilde{\Sigma}(X_t))dt \tag{37}$$

$$\leq -\eta_t \|\nabla f(X_t)\|_2^2 dt + \frac{\eta\eta_t^2}{2}(L_0 + L_1\|\nabla f(X_t)\|)\|\nabla f(X_t)\|^2 dt \tag{38}$$

$$+ \mathcal{O}(\text{Noise}) + (\eta_t)^2 \frac{\eta(\overline{\sigma_0^2} + \overline{\sigma_1^2}\|\nabla f(X_t)\|_2^2)d(L_0 + L_1\|\nabla f(X_t)\|)}{2N} dt \tag{39}$$

$$= -\eta_t \|\nabla f(X_t)\|_2^2 \left(1 - \frac{\eta_t \eta}{2}\left(L_0 + L_1\|\nabla f(X_t)\| + \frac{d}{N}\left(\overline{\sigma_1^2}L_0 + \overline{\sigma_0^2}L_1 + L_1\overline{\sigma_1^2}\|\nabla f(X_t)\|_2\right)\right)\right) dt \tag{40}$$

$$+ (\eta_t)^2 \frac{\eta\overline{\sigma_0^2}dL_0}{2N} dt \tag{41}$$

Therefore, for $0 < \epsilon < 1$ we have that if

$$\eta\eta_t < \frac{2\epsilon}{L_0 + L_1\|\nabla f(X_t)\| + \frac{d}{N}\left(\overline{\sigma_1^2}L_0 + \overline{\sigma_0^2}L_1 + L_1\overline{\sigma_1^2}\|\nabla f(X_t)\|_2\right)}, \tag{42}$$

and therefore that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2 \frac{\eta L_0 d\overline{\sigma^2}}{2N}\right) \overset{t\to\infty}{\to} 0, \tag{43}$$

where $\hat{t}$, is a random time with distribution $\frac{\eta_{\hat{t}}}{\phi_t^1}$.

By taking a worst-case scenario approach, we merge these two bounds into a single one:

$$d(f(X_t) - f(X_*)) \leq -\eta_t \|\nabla f(X_t)\|_2^2 \left(1 - \frac{\eta_t \eta}{2}\left(L_0 + L_1\|\nabla f(X_t)\| + \frac{d}{N}\left(\overline{\sigma_1^2}L_0 + \overline{\sigma_0^2}L_1 + L_1\overline{\sigma_1^2}\|\nabla f(X_t)\|_2\right)\right)\right) dt \tag{44}$$

$$+ (\eta_t)^2 \frac{\eta}{2N}(L_0 + L_1)d\overline{\sigma_0^2}dt, \tag{45}$$

and, therefore, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \frac{\eta\phi_t^2}{2N}(L_0 + L_1)d\overline{\sigma_0^2}\right) \overset{t\to\infty}{\to} 0, \tag{46}$$

where $\hat{t}$, is a random time with distribution $\frac{\eta_{\hat{t}}}{\phi_t^1}$.

Finally, for practical reasons, we leverage the distributed setting to tighten the requirements on the learning rate scheduler to make it experimentally viable, and rather require

$$\eta\eta_t < \frac{2\epsilon}{L_0 + L_1\mathbb{E}\left[\|\nabla f(X_t)\|\right] + \frac{d}{N}\left(\overline{\sigma_1^2}L_0 + \overline{\sigma_0^2}L_1 + L_1\overline{\sigma_1^2}\mathbb{E}\left[\|\nabla f(X_t)\|\right]\right)}. \tag{47}$$

■

A.2. Distributed Compressed SGD with Unbiased Compression

A.2.1. FIRST ORDER SDE

The following is the first-order SDE model of DCSGD (see Theorem 3.6 in [7]). Let us consider the stochastic process $X_t \in \mathbb{R}^d$ defined as the solution of

$$dX_t = -\nabla f(X_t)dt + \sqrt{\frac{\eta}{N}}\sqrt{\tilde{\Sigma}(X_t)}dW_t, \tag{48}$$

where for $\Phi_{\xi_i,\gamma_i}(x) := \mathcal{C}_{\xi_i}\left(\nabla f_{\gamma_i}(x)\right) - \nabla f_{\gamma_i}(x)$

$$\tilde{\Sigma}(x) = \frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}_{\xi_i\gamma_i}\left[\Phi_{\xi_i,\gamma_i}(x)\Phi_{\xi_i,\gamma_i}(x)^\top\right] + \Sigma_i(x)\right). \tag{49}$$

Theorem 10 *Let f be (L_0, L_1)-smooth, the learning rate scheduler η_t such that $\phi_t^i = \int_0^t (\eta_s)^i ds$, $\phi_t^1 \overset{t\to\infty}{\to} \infty$, $\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0$, and $\overline{\sigma^2\omega} := \frac{1}{N}\sum_{i=1}^N \sigma_i^2\omega_i$. Then, for $0 < \epsilon < 1$,*

$$\eta\eta_t < \frac{2N\epsilon}{\overline{\omega}L_0 + \left(\overline{\sigma^2}d + d\overline{\sigma^2\omega}\right)L_1 + \overline{\omega}L_1\mathbb{E}\left[\|\nabla f(X_t)\|_2\right]}, \tag{50}$$

and for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \le \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2\frac{\eta(L_0+L_1)d\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)}{2N}\right) \overset{t\to\infty}{\to} 0. \tag{51}$$

Proof Since it holds that

$$\mathbb{E}_{\xi_i,\gamma_i}\|(\mathcal{C}_{\xi_i}\left(\nabla f_{\gamma_i}(x)\right) - \nabla f(x))\|_2^2 \le \omega_i\|\nabla f(x)\|_2^2 + d\sigma_i^2(\omega_i+1),$$

we have that

$$d(f(X_t) - f(X_*)) = -\eta_t\|\nabla f(X_t)\|_2^2 dt + \mathcal{O}(\text{Noise}) \tag{52}$$

$$+ (\eta_t)^2\frac{\eta(L_0 + L_1\|\nabla f(X_t)\|_2)}{2N}\left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}_{\xi_i,\gamma_i}\|(\mathcal{C}_{\xi_i}\left(\nabla f_{\gamma_i}(x)\right) - \nabla f(x))\|_2^2\right)dt \tag{53}$$

$$\le -\eta_t\|\nabla f(X_t)\|_2^2 dt + \mathcal{O}(\text{Noise}) \tag{54}$$

$$+ (\eta_t)^2\frac{\eta(L_0 + L_1\|\nabla f(X_t)\|_2)}{2N}\left(\overline{\omega}\|\nabla f(X_t)\|_2^2 + \overline{\sigma^2}d + d\overline{\sigma^2\omega}\right)dt \tag{55}$$

Phase 1: If $\|\nabla f(X_t)\|_2 \le 1$, then we have that

$$\mathbb{E}\left[\|\nabla f(X_t)\|_2^2\right]\left(\eta_t - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}(\eta_t)^2\right)dt \le -d(f(X_t) - f(X_*)) \tag{56}$$

$$+ (\eta_t)^2\frac{\eta(L_0+L_1)d}{2N}\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)dt. \tag{57}$$

Let us now observe that since $\int_0^t \frac{\eta_s - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}\eta_s^2}{\phi_t^1 - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}\phi_t^2} ds = 1$, the function $s \mapsto \frac{\eta_s - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}\eta_s^2}{\phi_t^1 - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}\phi_t^2}$ defines a probability distribution and let $\tilde{t}$ have that distribution. Then by integrating over time and by the Law of the Unconscious Statistician, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\tilde{t}})\|_2^2\right] = \frac{1}{\phi_t^1 - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}\phi_t^2} \int_0^t \|\nabla f(X_s)\|_2^2 \left(\eta_s - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}\eta_s^2\right) ds, \tag{58}$$

meaning that

$$\mathbb{E}\left[\|\nabla f(X_{\tilde{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1 - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}\phi_t^2} \left(f(X_0) - f(X_*) + \phi_t^2 \frac{\eta(L_0+L_1)d}{2N}\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)\right) \overset{t\to\infty}{\to} 0, \tag{59}$$

where $\tilde{t}$, is a random time with distribution $\frac{\eta_{\tilde{t}} - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}(\eta_{\tilde{t}})^2}{\phi_t^1 - \frac{\eta(L_0+L_1)\overline{\omega}}{2N}\phi_t^2}$.

Phase 2: If $\|\nabla f(X_t)\|_2 > 1$, we have that

$$\begin{aligned}
d(f(X_t) - f(X_*)) &\leq -\eta_t \|\nabla f(X_t)\|_2^2 dt + \mathcal{O}(\text{Noise}) && (60)\\
&\quad + (\eta_t)^2 \frac{\eta(L_0 + L_1\|\nabla f(X_t)\|_2)}{2N}\left(\overline{\omega}\|\nabla f(X_t)\|_2^2 + \overline{\sigma^2}d + d\overline{\sigma^2\omega}\right) dt && (61)\\
&\leq -\eta_t \|\nabla f(X_t)\|_2^2 \left(1 - \frac{\eta_t\eta}{2N}\left(\overline{\omega}L_0 + d\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)L_1 + \overline{\omega}L_1\|\nabla f(X_t)\|_2\right)\right) dt && (62)\\
&\quad + \eta_t^2 \frac{\eta L_0 d}{2N}\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right) dt. && (63)
\end{aligned}$$

Therefore, for $0 < \epsilon < 1$ we have that if

$$\eta\eta_t < \frac{2N\epsilon}{\overline{\omega}L_0 + d\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)L_1 + \overline{\omega}L_1\|\nabla f(X_t)\|_2}, \tag{64}$$

then,

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2 \frac{\eta L_0 d}{2N}\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)\right) \overset{t\to\infty}{\to} 0, \tag{65}$$

where $\hat{t}$, is a random time with distribution $\frac{\eta_{\hat{t}}}{\phi_t^1}$. Finally, for practical reasons, we leverage the distributed setting to tighten the requirements on the learning rate scheduler to make it experimentally viable, and rather require

$$\eta\eta_t < \frac{2N\epsilon}{\overline{\omega}L_0 + \left(\overline{\sigma^2}d + d\overline{\sigma^2\omega}\right)L_1 + \overline{\omega}L_1\mathbb{E}\left[\|\nabla f(X_t)\|_2\right]}, \tag{66}$$

By taking a worst-case scenario approach, we merge these two bounds into a single one and have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2 \frac{\eta(L_0+L_1)d\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)}{2N}\right) \overset{t\to\infty}{\to} 0, \tag{67}$$

where $\hat{t}$, is a random time with distribution $\frac{\eta_{\hat{t}}}{\phi_t^1}$. ■

Finally, one can generalize this result to cover the (σ_0^2, σ_1^2)-Variance.

Theorem 11 *Let f be (L_0, L_1)-smooth, $\max(\Sigma_i(x)) < \sigma_{i,0}^2 + \sigma_{i,1}^2\|\nabla f(x)\|_2^2$, the learning rate scheduler η_t such that $\phi_t^i = \int_0^t (\eta_s)^i ds$, $\phi_t^1 \overset{t\to\infty}{\to} \infty$, $\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0$, $\overline{\sigma_0^2} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_{0,i}^2$, $\overline{\sigma_1^2} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_{1,i}^2$, $\overline{\sigma_0^2\omega} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_{i,0}^2\omega_i$, and $\overline{\sigma_1^2\omega} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_{i,1}^2\omega_i$. Then, for $0 < \epsilon < 1$,*

$$\eta\eta_t < \frac{2N\epsilon}{L_0(\overline{\omega} + d(\overline{\sigma_1^2\omega} + \overline{\sigma_1^2})) + L_1 d\left(\overline{\sigma_0^2} + \overline{\sigma_0^2\omega}\right) + L_1(\overline{\omega} + d(\overline{\sigma_1^2\omega} + \overline{\sigma_1^2}))\mathbb{E}\left[\|\nabla f(X_t)\|_2\right]}, \tag{68}$$

and for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{(1-\epsilon)\phi_t^1}\left(f(X_0) - f(X_*) + \phi_t^2 \frac{L_0(\overline{\omega} + d(\overline{\sigma_1^2\omega} + \overline{\sigma_1^2})) + L_1 d\left(\overline{\sigma_0^2} + \overline{\sigma_0^2\omega}\right)}{2N}\right) \overset{t\to\infty}{\to} 0. \tag{69}$$

A.2.2. SECOND ORDER SDE

The following is the second-order SDE model of DCSGD and is a straightforward generalization of Theorem 3.6 in [7]. Let us consider the stochastic process $X_t \in \mathbb{R}^d$ defined as the solution of

$$dX_t = -\nabla f(X_t)dt - \frac{\eta}{2}\nabla^2 f(X_t)\nabla f(X_t)dt + \sqrt{\frac{\eta}{N}}\sqrt{\tilde{\Sigma}(X_t)}dW_t, \tag{70}$$

where for $\Phi_{\xi_i,\gamma_i}(x) \coloneqq \mathcal{C}_{\xi_i}\left(\nabla f_{\gamma_i}(x)\right) - \nabla f_{\gamma_i}(x)$

$$\tilde{\Sigma}(x) = \frac{1}{N}\sum_{i=1}^N \left(\mathbb{E}_{\xi_i\gamma_i}\left[\Phi_{\xi_i,\gamma_i}(x)\Phi_{\xi_i,\gamma_i}(x)^\top\right] + \Sigma_i(x)\right). \tag{71}$$

Theorem 12 *Let f be (L_0, L_1)-smooth, the learning rate scheduler η_t such that $\phi_t^i = \int_0^t (\eta_s)^i ds$, $\phi_t^1 \overset{t\to\infty}{\to} \infty$, $\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0$, and $\overline{\sigma^2\omega} \coloneqq \frac{1}{N}\sum_{i=1}^N \sigma_i^2\omega_i$. Then, for $0 < \epsilon < 1$,*

$$\eta\eta_t < \frac{2\epsilon}{L_0 + L_1\mathbb{E}\left[\|\nabla f(X_t)\|_2\right] + \frac{\overline{\omega}L_0 + d\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)L_1 + \overline{\omega}L_1\mathbb{E}[\|\nabla f(X_t)\|_2]}{N}}, \tag{72}$$

and for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2\frac{\eta(L_0 + L_1)d}{2N}\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)\right) \overset{t\to\infty}{\to} 0. \tag{73}$$

Proof Since it holds that

$$\mathbb{E}_{\xi_i,\gamma_i}\|(\mathcal{C}_{\xi_i}\left(\nabla f_{\gamma_i}(x)\right) - \nabla f(x))\|_2^2 \leq \omega_i\|\nabla f(x)\|_2^2 + d\sigma_i^2(\omega_i + 1),$$

we have that

$$d(f(X_t) - f(X_*)) = -\eta_t\|\nabla f(X_t)\|_2^2 dt - \frac{\eta\eta_t^2}{2}\left(\nabla f(X_t)\right)^\top \nabla^2 f(X_t)\nabla f(X_t)dt + \mathcal{O}(\text{Noise}) \tag{74}$$

$$+ \frac{\eta\eta_t^2}{2}\frac{(L_0 + L_1\|\nabla f(X_t)\|_2)}{N}\left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}_{\xi_i,\gamma_i}\|(\mathcal{C}_{\xi_i}\left(\nabla f_{\gamma_i}(x)\right) - \nabla f(x))\|_2^2\right)dt \tag{75}$$

$$\leq -\eta_t\|\nabla f(X_t)\|_2^2 dt + \frac{\eta\eta_t^2}{2}(L_0 + L_1\|\nabla f(X_t)\|)\|\nabla f(X_t)\|^2 dt + \mathcal{O}(\text{Noise}) \tag{76}$$

$$+ \frac{\eta\eta_t^2}{2}\frac{(L_0 + L_1\|\nabla f(X_t)\|_2)}{N}\left(\overline{\omega}\|\nabla f(X_t)\|_2^2 + \overline{\sigma^2}d + d\overline{\sigma^2\omega}\right)dt \tag{77}$$

Phase 1: If $\|\nabla f(X_t)\|_2 \leq 1$, then we have that

$$\mathbb{E}\left[\|\nabla f(X_t)\|_2^2\right]\left(\eta_t - \frac{\eta_t^2\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)\right)dt \leq -d(f(X_t)-f(X_*)) \tag{78}$$

$$+ (\eta_t)^2\frac{\eta(L_0+L_1)d}{2N}\left(\overline{\sigma^2}+\overline{\sigma^2\omega}\right)dt. \tag{79}$$

Let us now observe that since $\int_0^t \frac{\eta_s - \frac{\eta_s^2\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)}{\phi_t^1 - \frac{\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)\phi_t^2}ds = 1$, the function $s \mapsto \frac{\eta_s - \frac{\eta_s^2\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)}{\phi_t^1 - \frac{\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)\phi_t^2}$ defines a probability distribution and let $\tilde{t}$ have that distribution. Then, by integrating over time and by the Law of the Unconscious Statistician, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\tilde{t}})\|_2^2\right] = \frac{1}{\phi_t^1 - \frac{\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)\phi_t^2}\int_0^t \|\nabla f(X_s)\|_2^2\left(\eta_s - \frac{\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)\eta_s^2\right)ds, \tag{80}$$

meaning that

$$\mathbb{E}\left[\|\nabla f(X_{\tilde{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1 - \frac{\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)\phi_t^2}\left(f(X_0)-f(X_*)+\phi_t^2\frac{\eta(L_0+L_1)d}{2N}\left(\overline{\sigma^2}+\overline{\sigma^2\omega}\right)\right) \overset{t\to\infty}{\to} 0, \tag{81}$$

where $\tilde{t}$, is a random time with distribution $\frac{\eta_{\tilde{t}} - \frac{\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)(\eta_{\tilde{t}})^2}{\phi_t^1 - \frac{\eta}{2}(L_0+L_1)\left(1+\frac{\overline{\omega}}{N}\right)\phi_t^2}$.

Phase 2: If $\|\nabla f(X_t)\|_2 > 1$, we have that

$$d(f(X_t)-f(X_*)) \leq -\eta_t\|\nabla f(X_t)\|_2^2 dt + \frac{\eta\eta_t^2}{2}(L_0+L_1\|\nabla f(X_t)\|)\|\nabla f(X_t)\|^2 dt + \mathcal{O}(\text{Noise}) \tag{82}$$

$$+ (\eta_t)^2\frac{\eta(L_0+L_1\|\nabla f(X_t)\|_2)}{2N}\left(\overline{\omega}\|\nabla f(X_t)\|_2^2 + \overline{\sigma^2}d + d\overline{\sigma^2\omega}\right)dt \tag{83}$$

$$\leq -\eta_t\|\nabla f(X_t)\|_2^2\left(1 - \frac{\eta_t\eta}{2}\left(L_0 + L_1\|\nabla f(X_t)\|_2 + \frac{\overline{\omega}L_0 + d\left(\overline{\sigma^2}+\overline{\sigma^2\omega}\right)L_1 + \overline{\omega}L_1\|\nabla f(X_t)\|_2}{N}\right)\right) \tag{84}$$

$$+ \eta_t^2\frac{\eta L_0 d}{2N}\left(\overline{\sigma^2}+\overline{\sigma^2\omega}\right). \tag{85}$$

Therefore, for $0 < \epsilon < 1$ we have that if

$$\eta\eta_t < \frac{2\epsilon}{L_0 + L_1\|\nabla f(X_t)\|_2 + \frac{\overline{\omega}L_0 + d\left(\overline{\sigma^2}+\overline{\sigma^2\omega}\right)L_1 + \overline{\omega}L_1\|\nabla f(X_t)\|_2}{N}}, \tag{86}$$

then,

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0)-f(X_*)+\phi_t^2\frac{\eta(L_0+L_1)d}{2N}\left(\overline{\sigma^2}+\overline{\sigma^2\omega}\right)\right) \overset{t\to\infty}{\to} 0, \tag{87}$$

where $\hat{t}$, is a random time with distribution $\frac{\eta_{\hat{t}}}{\phi_t^1}$. Finally, for practical reasons, we leverage the distributed setting to tighten the requirements on the learning rate scheduler to make it experimentally viable, and rather require

$$\eta\eta_t < \frac{2\epsilon}{L_0 + L_1\mathbb{E}\left[\|\nabla f(X_t)\|_2\right] + \frac{\overline{\omega}L_0 + d\left(\overline{\sigma^2}+\overline{\sigma^2\omega}\right)L_1 + \overline{\omega}L_1\mathbb{E}[\|\nabla f(X_t)\|_2]}{N}}, \tag{88}$$

By taking a worst-case scenario approach, we merge these two bounds into a single one and have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{\phi_t^1(1-\epsilon)}\left(f(X_0) - f(X_*) + \phi_t^2 \frac{\eta(L_0+L_1)d\left(\overline{\sigma^2} + \overline{\sigma^2\omega}\right)}{2N}\right) \overset{t\to\infty}{\to} 0, \tag{89}$$

where $\hat{t}$, is a random time with distribution $\frac{\eta_{\hat{t}}}{\phi_t^1}$. ■

Finally, one can generalize this result to cover the (σ_0^2, σ_1^2)-Variance.

Theorem 13 *Let f be (L_0, L_1)-smooth, $\max(\Sigma_i(x)) < \sigma_{i,0}^2 + \sigma_{i,1}^2\|\nabla f(x)\|_2^2$, the learning rate scheduler η_t such that $\phi_t^i = \int_0^t (\eta_s)^i ds$, $\phi_t^1 \overset{t\to\infty}{\to} \infty$, $\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0$, $\overline{\sigma_0^2} := \frac{1}{N}\sum_{i=1}^N \sigma_{0,i}^2$, $\overline{\sigma_1^2} := \frac{1}{N}\sum_{i=1}^N \sigma_{1,i}^2$, $\overline{\sigma_0^2\omega} := \frac{1}{N}\sum_{i=1}^N \sigma_{i,0}^2\omega_i$, and $\overline{\sigma_1^2\omega} := \frac{1}{N}\sum_{i=1}^N \sigma_{i,1}^2\omega_i$. Then, for $0 < \epsilon < 1$,*

$$\eta\eta_t < \frac{2\epsilon}{L_0 + L_1\mathbb{E}\left[\|\nabla f(X_t)\|_2\right] + \frac{L_0(\overline{\omega}+d(\overline{\sigma_1^2\omega}+\overline{\sigma_1^2})) + L_1 d\left(\overline{\sigma_0^2}+\overline{\sigma_0^2\omega}\right) + L_1(\overline{\omega}+d(\overline{\sigma_1^2\omega}+\overline{\sigma_1^2}))\mathbb{E}[\|\nabla f(X_t)\|_2]}{N}}, \tag{90}$$

and for a random time $\hat{t}$ with distribution $\frac{\eta_t}{\phi_t^1}$, we have that

$$\mathbb{E}\left[\|\nabla f(X_{\hat{t}})\|_2^2\right] \leq \frac{1}{(1-\epsilon)\phi_t^1}\left(f(X_0) - f(X_*) + \phi_t^2 \frac{\eta(L_0+L_1)d(\overline{\sigma_0^2} + \overline{\sigma_0^2\omega})}{2N}\right) \overset{t\to\infty}{\to} 0. \tag{91}$$

A.3. Distributed SignSGD

A.3.1. FIRST ORDER SDE

The following is the first-order SDE model of DSignSGD (see Theorem 3.10 in [7]). Let us consider the stochastic process $X_t \in \mathbb{R}^d$ defined as the solution of

$$dX_t = -\frac{1}{N}\sum_{i=1}^N \left(1 - 2\mathbb{P}(\nabla f_{\gamma_i}(X_t) < 0)\right) dt + \sqrt{\frac{\eta}{N}}\sqrt{\overline{\Sigma}(X_t)}dW_t. \tag{92}$$

where

$$\overline{\Sigma}(X_t) := \frac{1}{N}\sum_{i=1}^N \overline{\Sigma_i}(X_t), \tag{93}$$

and $\overline{\Sigma_i}(x) = \mathbb{E}[\xi_{\gamma_i}(x)\xi_{\gamma_i}(x)^\top]$ where $\xi_{\gamma_i}(x) := \text{sign}(\nabla f_{\gamma_i}(x)) - 1 + 2\mathbb{P}(\nabla f_{\gamma_i}(x) < 0)$ the noise in the sample $\text{sign}\left(\nabla f_{\gamma_i}(x)\right)$.

Corollary 14 (Corollary C.10 in [7]) *If the stochastic gradients are $\nabla f_{\gamma_i}(x) = \nabla f(x) + \sqrt{\Sigma_i}Z_i$ such that $Z_i \sim t_\nu(0, I_d)$ does not depend on x, ν are the degrees of freedom, and scale matrices $\Sigma_i = \text{diag}(\sigma_{1,i}^2, \cdots, \sigma_{d,i}^2)$. Then, the SDE of DSignSGD is*

$$dX_t = -\frac{2}{N}\sum_{i=1}^N \Xi_\nu\left(\Sigma_i^{-\frac{1}{2}}\nabla f(X_t)\right) dt + \sqrt{\frac{\eta}{N}}\sqrt{\tilde{\Sigma}(X_t)}dW_t. \tag{94}$$

where $\Xi_\nu(x)$ is defined as $\Xi_\nu(x) := x\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)}{}_2F_1\left(\frac{1}{2}, \frac{\nu+1}{2}; \frac{3}{2}; -\frac{x^2}{\nu}\right)$, ${}_2F_1\left(a, b; c; x\right)$ is the hypergeometric function, and

$$\tilde{\Sigma}(X_t) := I_d - \frac{4}{N}\sum_{i=1}^N \left(\Xi_\nu\left(\Sigma_i^{-\frac{1}{2}}\nabla f(X_t)\right)\right)^2. \tag{95}$$

In the following, the constant ℓ_ν is defined in Proposition C.11 of [7].

Theorem 15 *Let f be (L_0, L_1)-smooth, η_t a learning rate scheduler such that $\phi_t^i = \int_0^t (\eta_s)^i ds$, $\phi_t^1 \overset{t\to\infty}{\to} \infty$, $\frac{\phi_t^2}{\phi_t^1} \overset{t\to\infty}{\to} 0$, $\Sigma_i \leq \sigma_{max,i}^2$, $\sigma_{\mathcal{H},1}$ be the harmonic mean of $\{\sigma_{max,i}\}$, and $\ell_\nu > 0$ a constant. Then, for a scheduler $\eta\eta_t < \frac{2N\ell_\nu}{\sigma_{\mathcal{H},1} d L_1}$ and a random time $\tilde{t}$ with distribution $\frac{\eta_t \ell_\nu \sigma_{\mathcal{H},1}^{-1} - \eta_t^2 \frac{\eta L_1 d}{2N}}{\phi_t^1 \ell_\nu \sigma_{\mathcal{H},1}^{-1} - \phi_t^2 \frac{\eta L_1 d}{2N}}$, we have that*

$$\mathbb{E}\|\nabla f(X_{\tilde{t}})\|_2^2 \leq \frac{1}{\phi_t^1 \ell_\nu \sigma_{\mathcal{H},1}^{-1} - \phi_t^2 \frac{\eta L_1 d}{2N}} \left(f(X_0) - f(X_*) + \frac{\eta(L_0 + L_1) d \phi_t^2}{2N} \right) \overset{t\to\infty}{\to} 0. \tag{96}$$

Proof By Ito Lemma on $f(X_t) - f(X_*)$, we have that

$$d(f(X_t) - f(X_*)) \leq -\ell_\nu \sigma_{\mathcal{H},1}^{-1} \eta_t \|\nabla f(X_t)\|_2^2 dt + \frac{\eta \eta_t^2 d}{2N} (L_0 + L_1 \|\nabla f(X_t)\|_2) dt \tag{97}$$

Phase 1: $\|\nabla f(X_t)\|_2 \leq 1$:

$$d(f(X_t) - f(X_*)) \leq -\ell_\nu \sigma_{\mathcal{H},1}^{-1} \eta_t \|\nabla f(X_t)\|_2^2 dt + \frac{\eta \eta_t^2 d}{2N} (L_0 + L_1) dt. \tag{98}$$

Phase 2: $\|\nabla f(X_t)\|_2 > 1$:

$$d(f(X_t) - f(X_*)) \leq -\ell_\nu \sigma_{\mathcal{H},1}^{-1} \eta_t \|\nabla f(X_t)\|_2^2 dt + \frac{\eta \eta_t^2 d L_1 \|\nabla f(X_t)\|_2^2}{2N} + \frac{\eta \eta_t^2 d L_0}{2N} dt. \tag{99}$$

By taking the worst case of these two phases, we have that

$$d(f(X_t) - f(X_*)) \leq -\ell_\nu \sigma_{\mathcal{H},1}^{-1} \eta_t \|\nabla f(X_t)\|_2^2 dt + \frac{\eta \eta_t^2 d L_1 \|\nabla f(X_t)\|_2^2}{2N} dt + \frac{\eta \eta_t^2 d}{2N} (L_0 + L_1) dt, \tag{100}$$

meaning that

$$\mathbb{E}\|\nabla f(X_{\tilde{t}})\|_2^2 \leq \frac{1}{\phi_t^1 \ell_\nu \sigma_{\mathcal{H},1}^{-1} - \phi_t^2 \frac{d\eta L_1}{2N}} \left(f(X_0) - f(X_*) + \frac{\eta(L_0 + L_1) d \phi_t^2}{2N} \right) \overset{t\to\infty}{\to} 0. \tag{101}$$

■

A.3.2. SECOND ORDER SDE

The following is the second-order SDE model of DSignSGD and is a straightforward generalization of Corollary C.10 in [7], and we observe that $\Xi_\nu'(x)$ is bounded by a positive constant M_ν.

$$dX_t = -\frac{2}{N} \sum_{i=1}^N \Xi_\nu \left(\Sigma_i^{-\frac{1}{2}} \nabla f(X_t) \right) dt - \frac{\eta}{N} \sum_{i=1}^N \Sigma_i^{-\frac{1}{2}} \nabla^2 f(X_t) \left(\Xi_\nu' \left(\Sigma_i^{-\frac{1}{2}} \nabla f(X_t) \right) \circ \Xi_\nu \left(\Sigma_i^{-\frac{1}{2}} \nabla f(X_t) \right) \right) dt$$
$$+ \sqrt{\frac{\eta}{N}} \sqrt{\tilde{\Sigma}(X_t)} dW_t. \tag{102}$$

Theorem 16 *Let f be (L_0, L_1)-smooth, $\Sigma_i \leq \sigma_{max,i}^2$, $\sigma_{\mathcal{H},1}$ be the harmonic mean of $\{\sigma_{max,i}\}$, $M_\nu > 0$ and $\ell_\nu > 0$ constants, and $K := \left(\frac{L_1}{2N} + \frac{(L_0 + L_1)\sigma_{\mathcal{H},1}^{-1} M_\nu}{\sqrt{d}} \right)$. Then, for a scheduler $\eta\eta_t < \frac{\ell_\nu K^{-1}}{\sigma_{\mathcal{H},1} d}$ and a random time $\tilde{t}$ with distribution $\frac{\eta_t \ell_\nu \sigma_{\mathcal{H},1}^{-1} - \eta_t^2 K}{\phi_t^1 \ell_\nu \sigma_{\mathcal{H},1}^{-1} - \phi_t^2 K}$, we have that*

$$\mathbb{E}\|\nabla f(X_{\tilde{t}})\|_2^2 \leq \frac{1}{\phi_t^1 \ell_\nu \sigma_{\mathcal{H},1}^{-1} - \phi_t^2 K} \left(f(X_0) - f(X_*) + \phi_t^2 \eta (L_0 + L_1) d \left(\frac{1}{2N} + \frac{M_\nu}{\sigma_{\mathcal{H},1} \sqrt{d}} \right) \right) \overset{t\to\infty}{\to} 0. \tag{103}$$

Proof By Ito Lemma on $f(X_t) - f(X_*)$, we have that

$$d(f(X_t) - f(X_*)) \leq -\ell_\nu \sigma_{\mathcal{H},1}^{-1} \eta_t \|\nabla f(X_t)\|_2^2 dt + \eta\eta_t^2 \sigma_{\mathcal{H},1}^{-1}(L_0 + L_1\|\nabla f(X_t)\|_2) M_\nu \|\nabla f(X_t)\|_1 dt \quad (104)$$

$$+ \frac{\eta\eta_t^2 d}{2N}(L_0 + L_1\|\nabla f(X_t)\|_2)dt \quad (105)$$

Phase 1: $\|\nabla f(X_t)\|_2 \leq 1$:

$$d(f(X_t) - f(X_*)) \leq -\ell_\nu \sigma_{\mathcal{H},1}^{-1} \eta_t \|\nabla f(X_t)\|_2^2 dt + \eta\eta_t^2 \sigma_{\mathcal{H},1}^{-1}(L_0 + L_1) M_\nu \sqrt{d} dt \quad (106)$$

$$+ \frac{\eta\eta_t^2 d}{2N}(L_0 + L_1)dt. \quad (107)$$

Phase 2: $\|\nabla f(X_t)\|_2 > 1$: Since $\|\nabla f(X_t)\|_1 < \sqrt{d}\|\nabla f(X_t)\|_2 < \sqrt{d}\|\nabla f(X_t)\|_2^2$, we have that

$$d(f(X_t) - f(X_*)) \leq -\ell_\nu \sigma_{\mathcal{H},1}^{-1} \eta_t \|\nabla f(X_t)\|_2^2 dt + \eta\eta_t^2 \sigma_{\mathcal{H},1}^{-1}(L_0 + L_1) M_\nu \sqrt{d} \|\nabla f(X_t)\|_2^2 dt \quad (108)$$

$$+ \frac{\eta\eta_t^2 dL_1\|\nabla f(X_t)\|_2^2}{2N} + \frac{\eta\eta_t^2 dL_0}{2N}dt. \quad (109)$$

By taking the worst case of these two phases, we have that

$$d(f(X_t) - f(X_*)) \leq -\ell_\nu \sigma_{\mathcal{H},1}^{-1} \eta_t \|\nabla f(X_t)\|_2^2 dt + \eta\eta_t^2 \sigma_{\mathcal{H},1}^{-1}(L_0 + L_1) M_\nu \sqrt{d} \|\nabla f(X_t)\|_2^2 dt \quad (110)$$

$$+ \frac{\eta\eta_t^2 dL_1\|\nabla f(X_t)\|_2^2}{2N} dt + \eta\eta_t^2 (L_0 + L_1) d\left(\frac{1}{2N} + \frac{M_\nu}{\sigma_{\mathcal{H},1}\sqrt{d}}\right) dt, \quad (111)$$

meaning that

$$\mathbb{E}\|\nabla f(X_{\hat{t}})\|_2^2 \leq \frac{1}{\phi_t^1 \ell_\nu \sigma_{\mathcal{H},1}^{-1} - \phi_t^2 d\eta\left(\frac{L_1}{2N} + \frac{(L_0+L_1)\sigma_{\mathcal{H},1}^{-1} M_\nu}{\sqrt{d}}\right)} \left(f(X_0) - f(X_*) + \phi_t^2 \eta(L_0 + L_1) d\left(\frac{1}{2N} + \frac{M_\nu}{\sigma_{\mathcal{H},1}\sqrt{d}}\right)\right) \overset{t\to\infty}{\to} 0. \quad (112)$$

■

A.4. Limitations

As noted by [24], the approximation power of SDEs can fail when the stepsize η is large or if certain conditions on ∇f and the noise covariance matrix are not met. Although these issues can be addressed by increasing the order of the weak approximation, we believe that the primary purpose of SDEs is to serve as simplification tools that enhance our intuition: We would not benefit significantly from added complexity.

Importantly, extensive experimental design empirically validated that the SDEs do track their respective optimizers precisely on a variety of architectures, including MLPs, CNNs, ResNets, and ViTs, [6–8, 29].

Appendix B. Experiments

B.1. DSGD - Figure 1 - (Left Column)

We optimize $f(x) = \frac{x^4}{4}$ as we inject gaussian noise with mean 0 and variance $\sigma^2\|\nabla f(x)\|_2^2$ on the gradient. The learning rate is $\eta = 0.01$, $\sigma \in \{3, 4, 5\}$, and we average over 1000 runs. In the top figure, we use no scheduler, while in the bottom one we use a scheduler as per Eq. 1.

B.2. DCSGD - Figure 1 - (Center Column)

We optimize $f(x) = \frac{\sum_{j=1}^{1000}(x_j)^4}{4}$ as we inject gaussian noise with mean 0 and variance $\sigma^2\|\nabla f(x)\|_2^2$ on the gradient. The learning rate is $\eta = 0.1$, $\sigma = 0.1\}$, we use *random sparsification* with $\omega \in \{4, 8, 16\}$, and we average over 1000 runs. In the top figure, we use no scheduler, while in the bottom one we use a scheduler as per Eq. 3.

B.3. DSignSGD - Figure 1 - (Right Column)

We optimize $f(x) = \frac{x^4}{4}$ as we inject student's t noise with $\nu = 1$ and scale parameters σ on the gradient. The learning rate is $\eta = 0.1$, $\sigma \in \{0.25, 0.5, 1, 2, 8, 16\}$, and we average over 10000 runs. In the top figure, we use no scheduler, while in the bottom one we use a scheduler as per Theorem 7, e.g. $\eta_t = \frac{1}{\sqrt{t+1}}$.