On the (5,3)-Grassmannian and the p-Frame Potentials

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***Abstract*—This paper establishes that the minimizing configuration for the p-frame potentials with 5 points in 3 dimensions is not the (5,3)-Grassmannian frame for any finite p, and shows that no configuration can minimize the p-frame potentials for N=5 in d=3 for all sufficiently large p. To prove this, a family of 5-point configurations parameterized by the value of p is explicitly constructed, and it is shown to be a smooth interpolation between the known solution at p=2 and the (5,3)-Grassmannian frame, evolving under a set of coupled differential equations.**

***Index Terms*—p-frame potentials, frame force, tight frames, Grassmannian frames, coherence, equiangular tight frames, optimization**

I. Introduction

For any finite collection $X = \{x_k\}_{k=1}^N$ of N-many unit-norm vectors in $\mathbb{R}^d$ and any $p \in (0, \infty]$, the *p-frame potential* of X is defined as

$$\mathrm{FP}_{p,N,d}(X) := \begin{cases} \sum_{k=1}^N \sum_{\ell \neq k}^N |\langle x_k, x_\ell \rangle|^p & p < \infty \\ \max_{k \neq \ell} |\langle x_k, x_\ell \rangle| & p = \infty. \end{cases} \tag{1}$$

Note that the p-frame potential of a configuration $\{x_k\}$ is invariant under multiplication by an orthogonal matrix U, that is, $\mathrm{FP}_{p,N,d}(\{x_k\}) = \mathrm{FP}_{p,N,d}(\{Ux_k\})$. Similarly, permuting the order of the elements of a collection leaves the potential unchanged.

The compactness of the sphere S^{d-1} and continuity of the p-frame potential guarantee that the infimum of the potential across all such collections X is in fact achieved. The properties of the optimal configurations, that is, the configurations that minimize this potential for a given p, N, and d, are of particular interest. Given the invariance properties mentioned above, the minimizing configurations are defined up to any composition of orthogonal transformations and reordering of its elements.

A *finite frame* in $\mathbb{R}^d$ is any finite collection of vectors whose span is $\mathbb{R}^d$. For all $N \geq d$, the minimizing configuration for any $p \in (0, \infty]$ must be a frame, as was shown in [1, Prop. 2.1]. The case of $p = 2$ was originally studied by Benedetto and Fickus [2] under the name "frame potential." They proved that any local minimizer of the frame potential is also a global minimizer, and furthermore is a *finite unite norm tight frame* (FUNTF). A FUNTF is a collection of N-many unit-norm vectors $\{x_k\}_{k=1}^N \subset S^{d-1}$ such that

$$\frac{N}{d} \|x\|^2 = \sum_{k=1}^N |\langle x, x_k \rangle|^2 \quad \text{for all } x \in \mathbb{R}^d. \tag{2}$$

See [2, Th. 3.1] for details on the constant $\frac{N}{d}$. As FUNTFs exist for all $N \geq d$, the problem of characterizing the minimizing configurations of the p-frame potential when $p = 2$ is completely solved. For a FUNTF $\{x_k\}$, if some constant $C \geq 0$ satisfies $|\langle x_k, x_\ell \rangle| = C$ for all $x_k, x_\ell \in X$ where $k \neq \ell$, then the FUNTF is called an *equiangular tight frame* (ETF). It is well-known that ETFs cannot exist if $N < d$ or if $N > \frac{d(d+1)}{2}$ [3, Th. 12.2]. For a given N and d, if an ETF exists, it minimizes the p-frame potential for all $p \geq 2$ [4], including $p = \infty$ [5].

When $p = \infty$, the quantity $\mathrm{FP}_{\infty,N,d}(X)$ is referred to as the *coherence* of X, and is sometimes denoted $c(X)$. The minimizing configurations for $N \geq d$ are called *Grassmannian* frames [5]. It is shown in [1, Prop. 2.2] that if a sequence of configurations $\{X^{(p)}\}_p$, where each configuration $X^{(p)}$ minimizes the p-frame potential, has a cluster point X, then X minimizes the coherence $c(X)$ across all configurations.

This paper focuses on dimension $d = 3$, where several methodologies have been used to find the minimizing configurations of $\mathrm{FP}_{p,N,3}$, so much so that we feel it is necessary to organize and collect the results in one place. As the case corresponding to $p = 2$ is completely known, Table I summarizes the known minimizing configurations of the p-frame potentials for $p \neq 2$ in dimension $d = 3$ for $N \leq 12$.

There are two primary results that we prove in this paper; in Section II we show that the $(5,3)$-Grassmannian frame is not a minimizing configuration of $\mathrm{FP}_{p,5,3}$ for any finite p and that there are no configurations that minimize $\mathrm{FP}_{p,5,3}$ for all sufficiently large p. This situation is notably in contrast with the optimal configurations of $\mathrm{FP}_{p,5,2}$ [1, Th. 3.7] and $\mathrm{FP}_{p,4,3}$ and $\mathrm{FP}_{p,6,3}$ (see Table I). In Section III, we show that each configuration that we constructed to prove the primary results is a local minimizer of $\mathrm{FP}_{p,5,3}$, at least within machine precision. Furthermore, the family of configurations solves a set of coupled differential equations which can physically be interpreted as the evolution of a known minimizing configuration at $p = 2$ under the force induced by $\mathrm{FP}_{p,5,3}$ as p increases to ∞, which smoothly interpolates between the known optimal configurations at $p = 2$ and $p = \infty$.

TABLE I
MINIMIZING CONFIGURATIONS OF $\mathrm{FP}_{p,N,3}$

N	p	Shape	From
1,2,3	$(0, \infty]$	(Subset of) orthonormal basis	
4	$(2, \infty]$	Tetrahedron	[4], [5]
5	∞	Any 5 non-antipodal vertices of regular icosahedron	[5]
6	$(0, 2)$	Two copies of orthonormal basis	[6]
6	$(2, \infty]$	Non-antipodal vertices of regular icosahedron	[4], [5]
9	$(0, 2)$	Three copies of orthonormal basis	[6]
10	4	Non-antipodal vertices of regular dodecahedron	[1, Prop. 1.1] [7]
12	$(0, 2)$	Four copies of orthonormal basis	[6]
12	$(2, 4]$	Regular icosahedron	[8]

II. LARGE-p BEHAVIOR OF $\mathrm{FP}_{p,5,3}$

Let $N = 5$ and $d = 3$. As shown in Table I, there are no finite values of p (other than $p = 2$) where the minimizing configurations of $\mathrm{FP}_{p,5,3}$ are characterized. Denote by X_∞ a collection of 5 non-antipodal vertices of the regular icosahedron, for example, five equally-spaced points on the circle of intersection of the plane $z = \frac{1}{\sqrt{5}}$ with S^2. For the known Grassmannian frames corresponding to $N = 4$ and $N = 6$ in Table I, the configurations also minimize $\mathrm{FP}_{p,N,3}$ for all p greater than some lower bound. One may wonder if such a lower bound on p exists for the equiangular (but not tight) frame given by X_∞. We show that such a result is impossible by explicitly constructing a configuration with a lower p-frame potential for all finite p.

Proposition 1: The Grassmannian frame X_∞ for $N = 5$ is not a minimizing configuration for the p-frame potential for any finite p.

Proof: Let $0 < p < \infty$ be arbitrary, and let X^t be the set of 5 equally-spaced points on the intersection of S^2 with the plane $z = t$ where $0 \le t \le 1$. An explicit construction is as follows:

$$X^t = \{x_k\}_{k=1}^5, \text{ where } x_k = \begin{pmatrix} \cos\left(\frac{2k\pi}{5}\right)\sqrt{1-t^2} \\ \sin\left(\frac{2k\pi}{5}\right)\sqrt{1-t^2} \\ t \end{pmatrix}.$$

Notice that when $t = \frac{1}{\sqrt{5}}$, this is exactly X_∞ (again, up to multiplication by an orthogonal matrix and reordering of its elements), and when $t = \frac{1}{\sqrt{3}}$, this is a tight frame and thus a minimizer of the $p = 2$ frame potential. For any $1 \le k, \ell \le 5$:

$$\begin{aligned} |\langle x_k, x_\ell \rangle|^p = \Bigg| &(1-t^2)\left(\cos\left(\frac{2\pi k}{5}\right)\cos\left(\frac{2\pi \ell}{5}\right)\right. \\ &\left.+ \sin\left(\frac{2\pi k}{5}\right)\sin\left(\frac{2\pi \ell}{5}\right)\right) + t^2\Bigg|^p \\ = &\left|(1-t^2)\cos\left(\frac{2\pi}{5}(k-\ell)\right) + t^2\right|^p. \end{aligned}$$

Thus for a fixed ℓ,

$$\sum_{k \ne \ell} |\langle x_k, x_\ell \rangle|^p = \sum_{k=1}^{4} \left|(1-t^2)\cos\left(\frac{2\pi k}{5}\right) + t^2\right|^p,$$

and therefore

$$\begin{aligned} \mathrm{FP}_{p,5,3}(X^t) &= 5\sum_{k=1}^{4} \left|(1-t^2)\cos\left(\frac{2\pi k}{5}\right) + t^2\right|^p \\ &= 5\sum_{k=1}^{4} \left|\cos\left(\frac{2\pi k}{5}\right) + \left(1 - \cos\left(\frac{2\pi k}{5}\right)\right)t^2\right|^p \\ &= 10\left|\frac{\sqrt{5}-1}{4} + \frac{5-\sqrt{5}}{4}t^2\right|^p \\ &\quad + 10\left|\frac{-\sqrt{5}-1}{4} + \frac{5+\sqrt{5}}{4}t^2\right|^p. \end{aligned}$$

The first term in absolute values is always positive. The second term in absolute values is negative when $t \le \frac{1}{5^{1/4}}$, so

$$\begin{aligned} \mathrm{FP}_{p,5,3}(X^t) = 10&\left(\frac{\sqrt{5}-1}{4} + \frac{5-\sqrt{5}}{4}t^2\right)^p \\ &+ 10\left(\frac{\sqrt{5}+1}{4} - \frac{5+\sqrt{5}}{4}t^2\right)^p. \end{aligned}$$

This is a continuous function of t, whose value at $t = \frac{1}{\sqrt{5}}$ agrees with $\mathrm{FP}_{p,5,3}(X_\infty)$. For $p > 0$, the t-derivative of the potential, given by $\frac{d}{dt}\left[\mathrm{FP}_{p,5,3}(X^t)\right]$, is

$$\begin{aligned} = 10pt\Bigg(&\left(\frac{\sqrt{5}-1}{4} + \frac{5-\sqrt{5}}{4}t^2\right)^{p-1}\frac{5-\sqrt{5}}{2} \\ -&\left(\frac{\sqrt{5}+1}{4} - \frac{5+\sqrt{5}}{4}t^2\right)^{p-1}\frac{5+\sqrt{5}}{2}\Bigg). \end{aligned} \tag{3}$$

At $t = \frac{1}{\sqrt{5}}$, we have

$$\begin{aligned} &\frac{10p}{\sqrt{5}}\left(\left(\frac{1}{\sqrt{5}}\right)^{p-1}\frac{5-\sqrt{5}}{2} - \left(\frac{1}{\sqrt{5}}\right)^{p-1}\frac{5+\sqrt{5}}{2}\right) \\ &= -\frac{10p}{5^{\frac{p-1}{2}}} < 0. \end{aligned}$$

Thus there exists an $\epsilon > 0$ such that for all $t \in \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} + \epsilon\right)$, we have $\mathrm{FP}_{p,5,3}(X^t) < \mathrm{FP}_{p,5,3}(X_\infty)$. □

Corollary 1: Let X be any configuration of five points on S^2. There exists a p_0 such that X is not a minimizer of $\mathrm{FP}_{p,5,3}$ for all $p > p_0$. In other words, there are no stable minimizing configurations as p increases.

Proof: If for all $k \ne \ell$, the points $x_k, x_\ell \in X$ satisfy $|\langle x_k, x_\ell \rangle| \le \frac{1}{\sqrt{5}}$, then $X = X_\infty$ as shown in [5]. In this case, we have shown above that X is not a minimizer for all sufficiently large p.

Otherwise, some $k \neq \ell$ and $\epsilon > 0$ satisfy $|\langle x_k, x_\ell \rangle| = \frac{1}{\sqrt{5}} + \epsilon$. For all $p > \frac{\ln 10}{\ln(1+\epsilon\sqrt{5})}$,

$$\begin{aligned} \text{FP}_{p,5,3}(X) &\geq 2\,|\langle x_k, x_\ell\rangle|^p = 2\left|\frac{1}{\sqrt{5}} + \epsilon\right|^p \\ &> 20\left(\frac{1}{\sqrt{5}}\right)^p = \text{FP}_{p,5,3}(X_\infty), \end{aligned}$$

and therefore X is not a minimizer for sufficiently large p. □

In the language of [1], another way to interpret Corollary 1 is that there does not exist a configuration of points and a $p_0 > 0$ such that the configuration is *universally optimal* for the class of potentials $\text{FP}_{p,5,3}$ for all $p > p_0$.

III. Concluding Remarks

We have shown that the minimizer of $\text{FP}_{p,5,3}$ is not equivalent to X_∞ for all $p > 0$ by exhibiting a configuration $X^{1/\sqrt{5}+\epsilon_p}$ with a lower potential. We now analyze this family of collections by solving for which t minimizes $\text{FP}_{p,5,3}(X^t)$ as a function of p. In doing so, we show that such a configuration is a local minimizer of $\text{FP}_{p,5,3}$ across *all* configurations. We then compare these solutions, parameterized by p, to the known optimal configurations for $p = 2$ and $p = \infty$, and show that this family of solutions smoothly interpolates between these known solutions. More interestingly, it is exactly the evolution as p increases to ∞ of the optimal configuration at $p = 2$ under the changing force $\text{FP}_{p,5,3}$, which approaches X_∞ in the limit.

Let X^t be as before. It was demonstrated earlier that $\frac{d}{dt}\left[\text{FP}_{p,5,3}(X^t)\right]_{t=\frac{1}{\sqrt{5}}} < 0$. This function of t is minimized locally at $t = t_p$ when Equation 3 is zero and $\frac{1}{\sqrt{5}} < t_p \leq \frac{1}{5^{1/4}}$. As $p > 0$ by assumption and it is easily shown that $t_p = 0$ cannot be a minimizer, we have

$$\begin{aligned} &\left(\sqrt{5} - 1 + (5-\sqrt{5})t_p^2\right)^{p-1}(5-\sqrt{5}) \\ &= \left(\sqrt{5} + 1 - (5+\sqrt{5})t_p^2\right)^{p-1}(5+\sqrt{5}). \end{aligned}$$

This implies

$$\left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)^{\frac{1}{p-1}} = \frac{5+\sqrt{5}-(5\sqrt{5}+\sqrt{5})t_p^2}{5-\sqrt{5}+(5\sqrt{5}-\sqrt{5})t_p^2}. \qquad (4)$$

Define $a(p) := \left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)^{\frac{1}{p-1}}$, $f(t) := \sqrt{5}(1-t^2)$, and $g(t) := \sqrt{5}(1-5t^2)$. In this new shorthand, Equation 4 becomes

$$a(p) = \frac{f(t_p) + g(t_p)}{f(t_p) - g(t_p)}.$$

Define $b(p) := \frac{a(p)+1}{a(p)-1}$, so $b(p) = \frac{f(t_p)}{g(t_p)}$. Thus

$$f(t_p) - b(p)g(t_p) = 0.$$

We therefore have

$$\left(\sqrt{5} - b(p)\right) - \left(\sqrt{5} - 5b(p)\right)t_p^2 = 0,$$

and thus

$$t_p = \sqrt{\frac{\sqrt{5} - b(p)}{\sqrt{5} - 5b(p)}}.$$

In other words,

$$t_p = \sqrt{\frac{\sqrt{5} - \frac{a(p)+1}{a(p)-1}}{\sqrt{5} - 5\frac{a(p)+1}{a(p)-1}}} = \sqrt{\frac{\sqrt{5} - \frac{\left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)^{\frac{1}{p-1}}+1}{\left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)^{\frac{1}{p-1}}-1}}{\sqrt{5} - 5\frac{\left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)^{\frac{1}{p-1}}+1}{\left(\frac{5-\sqrt{5}}{5+\sqrt{5}}\right)^{\frac{1}{p-1}}-1}}}.$$

As we assumed $t_p \leq \frac{1}{5^{1/4}}$, this formula is only valid for $p \geq 1$ (by continuity it holds in the limit as $p \to 1$), it is monotone decreasing in p, and $\lim_{p\to\infty} t_p = \frac{1}{\sqrt{5}}$. As $t_2 = \frac{1}{\sqrt{3}}$, the family of configurations given by $\{X^{t_p}\}_p$ smoothly interpolates between a FUNTF at $p = 2$ and the Grassmannian frame as $p \to \infty$.

A necessary – but not sufficient – condition for a configuration X to be a minimizer of $\text{FP}_{p,5,3}$ for $p > 1$ is that it needs to be at equilibrium under the conservative force that induces the p-frame potential. In [6, Lem. 2.5], it is shown that the conservative force acting on $a, b \in S^{d-1}$ is given by $F_p = f_p(\|a-b\|)(a-b)$, where

$$f_p(x) = \begin{cases} p\left(1 - \frac{x^2}{2}\right)^{p-1} & |x| \leq \sqrt{2}, \\ -p\left(\frac{x^2}{2} - 1\right)^{p-1} & \text{otherwise.} \end{cases}$$

The effective p-frame force on each point in the configuration can be computed using a method which exactly parallels the case $p = 2$ that is discussed in [3, Ch. 6.15]. For a given $p \geq 1$, the configuration X^{t_p} can be shown within machine error to be at equilibrium under the conservative force, see [9] to download the code supports this claim.

While the effective p-frame force on each point in the configuration X^{t_p} remains (within machine precision of) 0, one can take the p-derivative of the family of conservative forces f_p to determine in what direction the conservative force would exert on each point x_k if the configuration were to remain fixed while p increased infinitesimally. Computing this for each point $x_k(p)$ in the configuration X^{t_p}, within machine precision this is exactly the direction that the p-derivative of $x_k(p)$ is pointing; see [9] to download the Mathematica code which supports this claim. In other words, denoting the effective p-frame force of a configuration X on each point $x_k \in X$ by $\text{EFF}_p(X, x_k)$, one could consider this family of configurations as a solution to the set of coupled vector differential equations $\frac{d}{dp}\text{EFF}_p(\{x_\ell(p)\}_{\ell=1}^5, x_k(p)) = \frac{d}{dp}x_k(p)$ for all $k \in \{1,2,3,4,5\}$ with boundary condition $\{x_k(2)\}_{k=1}^5 = X^{t_2}$.

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