Application of frames in non-uniform discrete dynamical system

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***Abstract*—Aldroubi et al., in [11], studied source recovery problems in a discrete dynamical system for the stable recovery of source terms with the help of frames. Inspired by the work of Aldroubi et al., we introduce discrete dynamical system indexing over a non-uniform discrete set arising from spectral pairs, which is not necessarily a group. We call it *non-uniform discrete dynamical system* (NUDDS, in short). Using the techniques given in [11], we analyze the stability of the source term of the NUDDS in terms of frames.**

***Index Terms*—Sampling theory; forcing; frames; reconstruction.**

I. Introduction

A countable collection $\{\varphi_k\}_{k\in\mathbb{I}}$ of members of an infinite-dimensional Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist finite positive scalars L_o, $U_o \in (0,\infty)$ such that

$$L_o\|\varphi\|^2 \leq \sum_{k\in\mathbb{I}} |\langle \varphi,\varphi_k\rangle|^2 \leq U_o\|\varphi\|^2 \text{ for all } \varphi\in\mathcal{H}. \tag{1}$$

Scalars L_o and U_o, are called the *lower frame bound* and *upper frame bound*, respectively, of $\{\varphi_k\}_{k\in\mathbb{I}}$. If only the upper inequality holds in (1), then we say that $\{\varphi_k\}_{k\in\mathbb{I}}$ is a *Bessel sequence* with Bessel bound U_o. The operator $T:\ell^2(\mathbb{I})\to\mathcal{H}$ defined as $T(\{c_k\}_{k\in\mathbb{I}} = \sum_{k\in\mathbb{I}} c_k\varphi_k,\ \{c_k\}_{k\in\mathbb{I}}\in\ell^2(\mathbb{I})$ is called the *pre-frame operator* or the *synthesis operator*. The Hilbert-adjoint operator T^* of T is called the *analysis operator*, given by $T^*:\mathcal{H}\to\ell^2(\mathbb{I}),\quad T^*\varphi=\{\langle\varphi,\varphi_k\rangle\}_{k\in\mathbb{I}},\ \varphi\in\mathcal{H}$. The composition $TT^*=\Theta\colon\mathcal{H}\to\mathcal{H}$, defined by $\Theta\colon\varphi\to\sum_{k\in\mathbb{I}}\langle\varphi,\varphi_k\rangle\varphi_k$ is called the *frame operator* which is bounded, linear and invertible on $\mathcal{H}$. This gives the *reconstruction formula* for each vector $\varphi\in\mathcal{H}$: $\varphi=\Theta\Theta^{-1}\varphi=\sum_{k\in\mathbb{I}}\langle\Theta^{-1}\varphi,\varphi_k\rangle\varphi_k$. For fundamental properties of frames with different structures and their applications in pure mathematics and engineering science, we refer to texts [17]–[19], [22], [26].

Dynamical sampling, initiated by Aldroubi et al. [1], [2], has revolutionized our understanding in recovering signals from spatially and temporally evolving measurements. Recent advancements extend these concepts to challenges in dynamic environments, such as medical diagnostics and artificial intelligence. Researchers have developed frameworks for stable recovery of signals and source terms in separable Hilbert spaces, leveraging Bessel systems, spectral pairs [15], and non-uniform multiresolution analysis [16]. Current studies explore stability conditions for discrete dynamical systems and recovery of source terms in both finite and infinite-dimensional settings, offering insights for applications like environmental monitoring and numerical methods for partial differential equations. Dynamical sampling problems can be stratified in three types depending on the specific variable of interest. These types are the *space-time trade-off* [1], [4], [5], [7], [12], [14], *system identification* [3], [6], [13], [24] *source recovery problems*, see [8]–[11] and many references therein.

Recently, Aldroubi et al., in [11], focused on the source recovery problem by characterizing frames for stable recovery of source terms in dynamical systems. Specifically, they consider the discrete dynamical system of the form

$$x_{n+1}=Ax_n+w, n\in\mathbb{N}, w\in W,$$

where $x_n\in\mathcal{H}$ is the n-th state of the system, and $\mathcal{H}$ is a separable Hilbert space. The operator $A\in\mathcal{B}(\mathcal{H}), x_0\in\mathcal{H}$ is the initial state, and W is a closed subspace of $\mathcal{H}$. Time-space sample measurements

$$D(x_0,w)=[\langle x_n,g_j\rangle]_{n\geq 0,\, j\geq 1}$$

are obtained by inner products $\langle x_n,g_j\rangle$ with vectors of a Bessel system $\{g_j\}_{j\geq1}\subset\mathcal{H}$. They provided necessary and sufficient conditions for the stable recovery of constant source terms from time-space samples in a Hilbert space. This research holds significant relevance for real-world applications such as identifying pollution sources in environmental monitoring.

On the other hand, using spectral pairs in the Lebesgue space $L^2(\mathbb{R})$, Gabardo and Nashed [16] generalized the concept of multiresolution analysis (MRA). The outcome in the generalized MRA gives non-uniform wavelets and improving numerical methods for solving partial differential equations. Unlike traditional MRA, their framework uses a spectrum rather than a group for translations, providing a characterization of non-uniform wavelets.

Definition 1. [16] Let $\Omega\subset\mathbb{R}$ be measurable and $\Lambda\subset\mathbb{R}$ a countable subset. If the collection $\{|\Omega|^{-\frac{1}{2}}e^{2\pi i\lambda\cdot}\chi_\Omega(.)\}_{\lambda\in\Lambda}$ forms complete orthonormal system for $L^2(\Omega)$, where χ_Ω is indicator function on Ω and $|\Omega|$ is Lebesgue measure of Ω, then the pair (Ω, Λ) is a spectral pair.

Example 2. [16] Let $N \in \mathbb{N}$, r be a fixed odd integer coprime with N such that $1 \leq r \leq 2N-1$, and let $\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$, and $\Omega = [0, \frac{1}{2}) \cup \left[\frac{N}{2}, \frac{N+1}{2}\right)$. Then, (Ω, Λ) is a spectral pair.

Definition 3. [23] A frame of the form $\{f_k\}_{k\in\Lambda} \subset \mathcal{H}$ for $\mathcal{H}$ is called a *non-uniform frame* for $\mathcal{H}$.

For non-uniform frames with discrete Gabor and wavelet structure, we refer to [20], [21], [25] and references therein.

Motivated by the above work, we study the stability of the source term of the non-uniform discrete dynamical system in infinite dimensional separable Hilbert spaces. More precisely, we deliberate on indexing the dynamical system and sampling vectors by sets arising from spectral pairs, which is not necessary a group but a spectrum which is based on the theory of spectral pairs [15], [16].

II. Notations

Let $N \in \mathbb{N}$ and r be a fixed odd integer co-prime with N such that $1 \leq r \leq 2N-1$. We recall a notation $\Lambda := \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$. Untill and unless specified, symbol $[2K]$ denotes the set

$$[2K] := \Big\{ -2K, -2K + \frac{r}{N}, \ldots, -2, -2 + \frac{r}{N}, \\ 0, \frac{r}{N}, 2, \ldots, 2K-2, 2K-2+\frac{r}{N}\Big\},\ K \in \mathbb{N},$$

and $|[2K]|$ denotes the cardinality of the set $[2K]$. The set $[2K]$ is only used to represent finite number of iterations. To define the notion of stable reconstruction, we need to specify the measurement spaces, where the data resides, along with an appropriate norm. This framework enables us to represent the reconstruction operator $\mathscr{R}$ as a bounded linear mapping from the data space $\mathcal{B}$ to the Hilbert space $\mathcal{H}$. The following spaces will be used in the sequel:

- $\ell^2(\Lambda) := \left\{ x = \{x_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{C} : \sum_{\lambda\in\Lambda} |x_\lambda|^2 < \infty \right\}$ is a Hilbert space with respect to the inner product defined by
$$\langle x, y\rangle = \sum_{\lambda\in\Lambda} x_\lambda \bar{y_\lambda},\ x = \{x_\lambda\}_{\lambda\in\Lambda}, y = \{y_\lambda\}_{\lambda\in\Lambda} \in \ell^2(\Lambda).$$
- $\ell^\infty(\Lambda) := \left\{ x = \{x_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{C} : \sup_{\lambda\in\lambda} |x_\lambda| < \infty \right\}$ is a Banach space endowed with the norm $\|x\|_{\ell^\infty(\Lambda)} = \sup_{\lambda\in\Lambda} |x_\lambda|$.
- $\mathbb{C}^{[2K]} := \left\{ x = (x_\lambda)_{\lambda\in[2K]} : x_\lambda \in \mathbb{C} \right\}$.

Now, we familiarize the spaces $\mathcal{B}(\ell^2(\Lambda), \mathbb{C}^{[2K]})$, $\mathcal{B}(\ell^2(\Lambda), \ell^\infty(\Lambda))$, and $\mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda))$ which are vital for our work.

Definition 4. The space $\mathcal{B}(\ell^2(\Lambda), \mathbb{C}^{[2K]})$ is the set of all infinite matrices $T = [a_{ij}]_{i\in[2K], j\in\Lambda}$ such that each row r_i of T belongs to $\ell^2(\Lambda)$. We endow $\mathcal{B}(\ell^2(\Lambda), \mathbb{C}^{[2K]})$ with the norm $\|T\|_{\ell^2(\Lambda)\to\mathbb{C}^{[2K]}} = \sum_{i\in[2K]} \|r_i\|_{\ell^2(\Lambda)}$.

The space $\mathcal{B}(\ell^2(\Lambda), \mathbb{C}^{[2K]})$ is a Banach space which is tantamount to the space of bounded linear operators from $\ell^2(\Lambda)$ to $\mathbb{C}^{[2K]}$, endowed with the operator norm.

Definition 5. The space $\mathcal{B}(\ell^2(\Lambda), \ell^\infty(\Lambda))$ is the set of all doubly infinite matrices $T = [a_{ij}]_{i,j\in\Lambda}$ such that each row r_i of T belongs to $\ell^2(\Lambda)$, and $\sup\limits_{i\in\Lambda} \|r_i\|_{\ell^2(\Lambda)}$ is finite. We endow $\mathcal{B}(\ell^2(\Lambda), \ell^\infty(\Lambda))$ with the norm $\|T\|_{\ell^2(\Lambda)\to\ell^\infty(\Lambda)} = \sup\limits_{i\in\Lambda} \|r_i\|_{\ell^2(\Lambda)}$.

The space $\mathcal{B}(\ell^2(\Lambda), \ell^\infty(\Lambda))$ is a Banach space which is tantamount to the space of bounded linear operators from $\ell^2(\Lambda)$ to $\ell^\infty(\Lambda)$, endowed with the operator norm. Now, we are ready to define the subspace $\mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda))$.

Definition 6. The space $\mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda))$ is the set of matrices $\{T = [a_{ij}] : i, j \in \Lambda\} \subset \mathcal{B}(\ell^2(\Lambda), \ell^\infty(\Lambda))$ such that there exists a $z \in \ell^2(\Lambda)$ satisfying $\lim_{|i|\to\infty} \|r_i - z\|_{\ell^2(\Lambda)} = 0$. We endow $\mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda))$ with the norm induced by $\mathcal{B}(\ell^2(\Lambda), \ell^\infty(\Lambda))$.

III. Discrete dynamical system over a non-uniform set

We begin this section with the definition of non-uniform discrete dynamical system (NUDDS, in short).

Definition 7. [23] Let A be a bounded linear operator on $\mathcal{H}$, W be a closed subspace of $\mathcal{H}$ and $w \in W$ is the source term or forcing term. A system of the form

$$\begin{aligned} x_{\lambda+\frac{r}{N}} &= Ax_\lambda + w,\ \lambda \in 2\mathbb{Z}; \\ x_{\lambda+2-\frac{r}{N}} &= Ax_\lambda + w,\ \lambda \in 2\mathbb{Z}^+ + \frac{r}{N} \cup \left\{\frac{r}{N}\right\}; \\ x_{\lambda-2-\frac{r}{N}} &= Ax_\lambda + w,\ \lambda \in 2\mathbb{Z}^- + \frac{r}{N}, \end{aligned} \tag{2}$$

is called the *non-uniform discrete dynamical system* (NUDDS, in short), where $x_\lambda \in \mathcal{H}$, $\lambda \in \Lambda$, is the λ-th state of the system in $\mathcal{H}$. The terms x_0 and x_{-2} are called **initial states**.

Time-space sample measurements

$$D(x_0, x_{-2}, w) = [\langle x_\lambda, g_{\lambda'}\rangle]_{\lambda,\, \lambda'\in\Lambda}\,, \tag{3}$$

where $\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}, N \geq 1$ is an integer, and r be a fixed odd integer co-prime with N such that $1 \leq r \leq 2N-1$, are obtained via inner products $\langle x_\lambda, g_{\lambda'}\rangle$ with the vectors of a Bessel system $\{g_j\}_{j\in\Lambda} \subset \mathcal{H}$, referred to as the set of **spatial sampling vectors**. These measurements are organized in the matrix $D(x_0, x_{-2}, w)$, which is known as the **data matrix**. This matrix is also referred to as the data of the system, or alternatively as the set of time-space samples, measurements, or observations. We ruminate for the following two cases of non-uniform discrete dynamical system:

i. In the first case, the data matrix $D(x_0, x_{-2}, w) = [\langle x_\lambda, g_{\lambda'}\rangle]_{\lambda,\, \lambda'\in\Lambda}$ is obtained from finitely many iterations $|[2K]|$.
ii. In the second case, the data matrix $D(x_0, x_{-2}, w) = [\langle x_\lambda, g_{\lambda'}\rangle]_{\lambda,\, \lambda'\in\Lambda}$ is obtained from infinitely many time iterations.

For the first case, all data measurements are carried out in the space $\mathcal{B}(\ell^2(\Lambda), \mathbb{C}^{[2K]})$. In the second case of infinitely many time iterations, we utilize the space $\mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda))$ which is a closed subspace of $\mathcal{B}(\ell^2(\Lambda), \ell^\infty(\Lambda))$.

Now, we define non-uniform discrete dynamical systems that are the generalized version of (2). In this general setting, we assume that the states $x_\lambda, \lambda \in \Lambda$ are obtained via the recursive relation

$$x_\lambda = \begin{cases} \mathscr{F}_\lambda(x_0, x_{\frac{r}{N}}, x_2, x_{2+\frac{r}{N}} \cdots, x_{\lambda-2+\frac{r}{N}}, w), \ \lambda \in 2\mathbb{Z}^+; \\ \mathscr{F}_\lambda(x_0, x_{\frac{r}{N}}, x_2, x_{2+\frac{r}{N}} \cdots, x_{\lambda-\frac{r}{N}}, w), \ \lambda \in 2\mathbb{Z}^+ + \frac{r}{N} \cup \{\frac{r}{N}\}; \\ \mathscr{F}_\lambda(x_{-2}, x_{-2+\frac{r}{N}}, x_{-4}, x_{-4+\frac{r}{N}} \cdots, x_{\lambda+2+\frac{r}{N}}, w), \ \lambda \in 2\mathbb{Z}^-; \\ \mathscr{F}_\lambda(x_{-2}, x_{-2+\frac{r}{N}}, x_{-4}, x_{-4+\frac{r}{N}} \cdots, x_{\lambda-\frac{r}{N}}, w), \ \lambda \in 2\mathbb{Z}^- + \frac{r}{N}, \end{cases} \tag{4}$$

with w belongings to the closed subspace W of $\mathcal{H}$. In particular, $\mathscr{F}_\lambda$ can be a non-linear functional of its arguments.

To present certain results in the context of setting (4), we assume that the system satisfies the following properties:

i. For each $w \in W$, there is a corresponding unique pair of stationary states. More explicitly, given any $w \in W$, there is a pair of initial states $(x_0(w), x_{-2}(w))$ such that

$$x_\lambda = \frac{(x_0 + x_{-2})(w)}{2} \text{ for all } \lambda \in \Lambda.$$

ii. The correspondence between w and its unique pair of stationary states $(x_0(w), x_{-2}(w))$ is bounded. That is, the mapping $S : W \longrightarrow \mathcal{H}$ defined by

$$S(w) = \frac{(x_0 + x_{-2})(w)}{2}$$

is a bounded linear operator, and S is called as the stationary mapping operator.

iii. For any source term $w \in W$ and any arbitrary initial states $x_0, x_{-2} \in \mathcal{H}$, we have

$$\lim_{|\lambda| \to \infty} x_\lambda = S(w),$$

where the above limit is in $\|.\|_{\mathcal{H}}$.

Definition 8. [23] A non-uniform discrete dynamical system (4) satisfying the above properties $(i)-(iii)$ is denoted by the quadruple $(\mathcal{H}, W, \mathscr{F}, S)$.

Next, we define the notion of stable recovery for the non-uniform discrete dynamical system.

i. The source term $w \in W \subseteq \mathcal{H}$ is said to be stably recovered from the data matrix $D(x_0, x_{-2}, w)$ in finitely many time iterations if there exists a bounded linear operator $\mathscr{R} : \mathcal{B}(\ell^2(\Lambda), \mathbb{C}^{[2K]}) \longrightarrow \mathcal{H}$ such that $\mathscr{R}(D(x_0, x_{-2}, w)) = w$ for all $x_0, x_{-2} \in \mathcal{H}, w \in W$.

ii. The source term $w \in W \subseteq \mathcal{H}$ is said to be stably recovered from the data matrix $D(x_0, x_{-2}, w)$ in infinitely many time iterations if there exists a bounded linear operator $\mathscr{R} : \mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda)) \longrightarrow \mathcal{H}$ such that $\mathscr{R}(D(x_0, x_{-2}, w)) = w$ for all $x_0, x_{-2} \in \mathcal{H}, w \in W$.

IV. Results

In our initial results, the source w can be any element of the space $\mathcal{H}$. While this scenario is less practical compared to the more restricted case of closed subspaces, since, in reality, sources are typically confined to specific spatial regions, it offers a mathematically elegant solution to the source recovery problem.

Motivated by the work of [11, Theorem 3.1], the following result ratifies an important property of the space $\mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda))$, that it is a natural domain of the reconstruction operator $\mathscr{R}$ and the operator $\mathscr{R} : \mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda)) \longrightarrow \mathcal{H}$ is bounded.

Theorem 9. *[23, Theorem 3.10] Let $\{g_j\}_{j\in\Lambda} \subset \mathcal{H}$ be a Bessel sequence with optimal Bessel bound $\beta > 0$. Then, for each $T = [a_{ij}] \in \mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda))$, the limit*

$$\lim T\{g_j\}_{j\in\Lambda} = \lim_{|i|\to\infty} [a_{ij}]_{i,\, j\in\Lambda}\{g_j\}_{j\in\Lambda} := \lim_{|i|\to\infty} \sum_{j\in\Lambda} a_{ij} g_j$$

exists in $\mathcal{H}$. Moreover, the mapping

$$\mathscr{R} : \mathcal{B}^s(\ell^2(\Lambda), \ell^\infty(\Lambda)) \longrightarrow \mathcal{H} \text{ defined as } T \mapsto \lim T\{g_j\}_{j\in\Lambda}$$

is a well defined bounded operator whose norm is precisely $\sqrt{\beta}$.

Taking inspiration from [11, Theorem 3.2], the next theorem incorporates a necessary and sufficient condition on $\{g_j\}_{j\in\Lambda} \subset \mathcal{H}$ for the existence of reconstruction operator $\mathscr{R} : \mathcal{B}(\ell^2(\Lambda), \mathbb{C}^{[2K]}) \longrightarrow \mathcal{H}$ in finitely many iterations.

Theorem 10. *[23, Theorem 3.14] Let $\{g_j\}_{j\in\Lambda} \subset \mathcal{H}$ be a Bessel sequence with Bessel bound $\beta > 0$. Then, for the NUDDS defined in (2), with any arbitrary initial states $x_0, x_{-2} \in \mathcal{H}$, the source term $w \in W$ can be stably recovered from the measurements $D(x_0, x_{-2}, w) = [\langle x_\lambda, g_j\rangle]_{\lambda\in[2K],\, j\in\Lambda}$ for some $1 \leq |[2K]| < \infty$ if and only if $\{g_j\}_{j\in\Lambda}$ is a frame for $\mathcal{H}$.*

In the next result, we constrain the source term to lie within a closed subspace $W \subset \mathcal{H}$. From a practical perspective, this case is particularly significant, despite being more mathematically intricate. The following theorem provides a necessary condition for the stable recovery of source term of the non-uniform discrete dynamical system (2) in finitely many iterations.

Theorem 11. *[23, Theorem 3.16] Suppose*

1) *$\{g_j\}_{j\in\Lambda}$ is a Bessel sequence in $\mathcal{H}$ with Bessel bound $\beta > 0$.*
2) *W is a closed subspace $\mathcal{H}$ and $P_W : \mathcal{H} \to \mathcal{H}$ denotes the orthogonal projection onto W.*

Then, for the NUDDS defined in (2), with any arbitrary initial states $x_0, x_{-2} \in \mathcal{H}$ and $1 \notin \sigma(A)$, $\{P_W(I - A^*)^{-1} g_j\}_{j\in\Lambda}$ is a frame for W if the source term $w \in W$ can be stably recovered from the measurements $D(x_0, x_{-2}, w) = [\langle x_\lambda, g_j\rangle]_{\lambda\in[2K],\, j\in\Lambda}$ for some $1 \leq |[2K]| < \infty$.

Remark 12. Note that the converse of the above theorem need not be true. For details, one can refer [23].

Motivated by [11, Theorem 3.4], the following theorem betokens the characterization for the stable recovery of the source term of the non-uniform discrete dynamical system (4) from the data measurements $D(x_0, x_{-2}, w)$ in infinitely many iterations.

Theorem 13. *[23, Theorem 3.19] Suppose*

1) $\{g_j\}_{j\in\Lambda} \subset \mathcal{H}$ *is a Bessel sequence with Bessel bound* $\beta > 0$.
2) W *is a closed subspace of* $\mathcal{H}$.

Then, for the NUDDS defined in (4) $(\mathcal{H}, W, \mathscr{F}, S)$ *with the assumption that* $\mathscr{F}$ *is linear, each source term* $w \in W$ *can be stably recovered from the measurements* $D(x_0, x_{-2}, w) = [\langle x_\lambda, g_j\rangle]_{\lambda,\, j\in\Lambda}$ *if and only if* $\{S^* g_j\}_{j\in\Lambda}$ *is a frame for* W.

As a fruitage of Theorem 13, we characterize stable recovery for the source term of the non-uniform discrete dynamical system (2) with the spectral radius $\rho(A) < 1$.

Theorem 14. *[23, Theorem 3.21] Suppose*

1) $\{g_j\}_{j\in\Lambda} \subset \mathcal{H}$ *is a Bessel sequence with Bessel bound* $\beta > 0$.
2) W *is a closed subspace of* $\mathcal{H}$.

Then, for the NUDDS defined in (2), *with* $\rho(A) < 1$, *each source term* $w \in W$ *can be stably recovered from the measurements* $D(x_0, x_{-2}, w) = [\langle x_\lambda, g_j\rangle]_{\lambda,\, j\in\Lambda}$ *if and only if* $\{P_W(I - A^*)^{-1} g_j\}_{j\in\Lambda}$ *is a frame for* W.

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