Dynamics of Riemann‘s Zeta Function on the critical line.

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Prime numbers are the starry arch of the mathematics.

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Abstract The dynamics of the Riemann‘s Zeta function on the critical line while t rises are studied in this paper. Stated the reason why with big values of t start occurring exceptions from the Green‘s Law and the Lemer‘s phenomenon is explained. It‘s shown that almost all non-trivial zeros lie on the critical line and how to calculate the approximate values of these zeros using known prime numbers.

Inroduction

1. For complex numbers zeta function $s = \sigma + it$ with $\sigma > 1$ is defined as sum of series:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1}$$

L. Euler showed that for real $s > 1$ takes place the identity:

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}, \tag{2}$$

where composition is taken for all prime numbers. The first one to study zeta function for complex means was B. Riemann, who indicated the set of important characteristics of this function [1].He proved that this function can be analytically extended for the whole complex pole, where it has one simple pole with residue which equals 1 in point $s = 1$. Riemann also showed (although in different ways) that this function complies with functional equation.

$$\zeta(s) = \chi(s)\zeta(1 - s), \tag{3}$$

where

$$\chi(s) = (2\pi)^s \pi^{-1} \sin \frac{\pi s}{2} \Gamma(1 - s) = \pi^{s - \frac{1}{2}} \Gamma\left(\frac{1 - s}{2}\right) \left(\Gamma\left(\frac{s}{2}\right)\right)^{-1}.$$

Zeros of a function when $\sigma < 0$ will be values $s = -2, -4, ...$ (trivial zeros). Part of the plane $0 \leq \sigma \leq 1$ is called the critical strip and the line $\sigma = \frac{1}{2}$ – is the critical line. Zeta function has infinite quantity of complex zeros in the critical strip. They are situated symmetrically around straight lines $t = 0$ and $\sigma = \frac{1}{2}$. These zeros are called non-trivial. One of the main areas of focus in the theory of zeta functions is related to studying non-trivial zeros.

2. If $N(T)$ – is the amount of zeros in rectangle $0 \leq \sigma \leq 1$, $0 \leq t \leq T$, then asymptotic equality (Riemann-von Mangoldt function) exists.

$$N(T) = \frac{T}{2\pi} ln \frac{T}{2\pi} - \frac{T}{2\pi} + O(lnT) \tag{4}$$

This formula is the logical corollary of the formula

$$N(T) = \frac{1}{\pi}\theta(T) + 1 + \frac{1}{\pi} arg\zeta\left(\frac{1}{2} + iT\right), \tag{5}$$

where through and $\theta(T)$ и $arg\zeta\left(\frac{1}{2} + iT\right)$ increments of continuous branches of the function $arg\left\{\pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\right\}$ and $arg\zeta(s)$ along polygonal path with tops at points $2; 2 + iT; \frac{1}{2} + iT$ [2] are denoted. Stirling‘s formula for gamma function leads to the equality

$$\theta(T) = \frac{T}{2} ln \frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + \Delta(T), \tag{6}$$

where $\Delta(T) = O(T^{-1})$. Let $S(T) = \frac{1}{\pi} arg\zeta\left(\frac{1}{2} + iT\right)$, $S(T) = \frac{1}{2}\left(S(T-0) + S(T+0)\right)$, if $T = \gamma_k$ – is ordinate of the zero $\zeta(s)$. Then from (5) follows that

$$S(T) = N(T) - \frac{1}{\pi}\theta(T) - 1 \tag{7}$$

Function $S(T)$ is called the argument of Riemann‘s zeta function on the critical line. This function is a piecewise smooth function and is strongly oscillatory. In case of indefinite growth of T valuations take place, which belong to J. J. Littlewood [2]:

$$S(T) = O(lnT), \int_0^T S(t)dt = O(lnT). \tag{8}$$

The set of important characteristics of this function was created by G. Bor, E. Landau, J. J. Littlewood, A. Selberg, K. M. Tsung and M. Korolev. In particular Bor and Landau proved that function $S(T)$) is unbounded above and below.

It should be mentioned that the analytic qualities of this function is completely described through function analytic $\theta(T)$, and the only characteristic connected with non-trivial zeros $\rho_k = \frac{1}{2} + i\gamma_k$ is that in points $T = \gamma_k$ function $S(T)$ has function jumps of a size which equals to the multiplicity of zero ρ_k. Formula (7) doesn‘t describe why jumps take place in the mentioned points. The aim of further contemplation is in researching this fact. Further it will also be proved that almost all non-trivial zeros lie on the critical line, and the reason why "Gram's law" doesn't work will be shown, and Lemer's phenomenon will be explained.

Results

3. Let $f(s)$ – be an analytic expression with the help of which it is possible to find values of zeta function in the right strip $0 \leq \sigma \leq \frac{1}{2}$ (this can be, for example, one of the variants of approximate functional equation [2; 3]. Then using (3) values of the function in points of the nearby left strip $0 \leq \sigma \leq \frac{1}{2}$ it should be found $\zeta(s) = \chi(s)\zeta(1-s)$. Zeta function is continuous in all points of the critical line, that is why with fixed t we will have: t

$$\zeta\left(\frac{1}{2}+it\right) = \frac{1}{2}\left(\zeta\left(\frac{1}{2}+it+0\right)+\zeta\left(\frac{1}{2}+it-0\right)\right) = \tag{9}$$

$$= \frac{1}{2}\left(f\left(\frac{1}{2}+it\right)+\chi\left(\frac{1}{2}+it\right)f\left(\frac{1}{2}-it\right)\right).$$

It's known [2; 3], that $\chi\left(\frac{1}{2}+it\right) = e^{-i2\theta(t)}$, where $\theta(t)$ is represented like (6). Then formula (3) shall become

$$\zeta\left(\frac{1}{2}+it\right) = k(t)e^{i\varphi(t)}\left(1+e^{-2i(\theta(t)+\varphi(t))}\right), \tag{10}$$

$$k(t) = \frac{1}{2}\left|f\left(\frac{1}{2}+it\right)\right| = \frac{1}{2}\left|f\left(\frac{1}{2}-it\right)\right|, \varphi(t) = argf\left(\frac{1}{2}+it\right),$$

Let's use partial case of approximate functional equation. Then according to Riemann-Siegel formula we have

$$\zeta\left(\frac{1}{2}+it\right) = e^{-i\theta(t)}Z(t) = \sum_{n\leq x}\frac{n^{-it}}{\sqrt{n}} + e^{-2i\theta(t)}\sum_{n\leq x}\frac{n^{it}}{\sqrt{n}} + e^{-i\theta(t)}R(t)$$

(here $x = \sqrt{\frac{t}{2\pi}}$; $R(t) = O\left(t^{-\frac{1}{4}}\right)$ [2; 3]).

Taking in account that $\zeta\left(\frac{1}{2}+it\right) = \frac{1}{2}\left(\zeta\left(\frac{1}{2}+it\right)+\chi\left(\frac{1}{2}+it\right)\zeta\left(\frac{1}{2}-it\right)\right)$, we get:

$$\zeta\left(\frac{1}{2}+it\right) = g(t) + e^{-2i\theta(t)}g(-t), \tag{11}$$

where

$$g(t) = \sum_{n\leq x}\frac{n^{-it}}{\sqrt{n}} + e^{-i\theta(t)}r(t), r(t) = \frac{1}{2}R(t) \tag{12}$$

Hence, in formula (10) we can take $k(t) = |g(t)|, \varphi(t) = argg(t)$.

Formula (10) explains the dynamic of zeta function on the critical line while t grows. The right side of this formula can be studied as a composition of the following moves:

1) $S_1(t) = e^{-i2(\theta(t)+\varphi(t))}$ (move of the point at the unit circle with center in (0,0) in reverse direction with speed $V(t) = 2\left(\theta'(t)+\varphi'(t)\right) \approx ln\frac{t}{2\pi}$;

2) $S_2(t) = 1 + S_1(t)$ (Displacement of the circle by 1 to the right);

3) $S_3(t) = k(t)S_2(t)$) (deformation of radius vector which creates a circle relative to the point O on the $k(t)$, i.e. compression when $0 < k(t) < 1$ and stretching when $k(t) > 1$);

4) $\zeta\left(\frac{1}{2} + it\right) = e^{-i\varphi(t)}S_3(t)$ (small oscillation of a circle at the "fixed" point O)

4. Formula (10) provides the possibility to study non-trivial zeros, which lie on the critical line. Consequently $\zeta\left(\frac{1}{2} + it\right) = 0$, when $1 + e^{-2i(\theta(t)+\varphi(t))} = 0$

From this follows that

$$\theta(t) + \varphi(t) = \frac{\pi}{2} + \pi\,(k-2)\,, k = 1, 2, ... \tag{13}$$

It has already been mentioned that

$$\theta(t) = \frac{t}{2}ln\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \Delta(t),$$

i. e. $\theta(t)$ – is the increasing function when $t > 2\pi$.

From (12) follows that:

$$g(t) = \sum_{n \le x} \frac{n^{-it}}{\sqrt{n}} + e^{-i\theta(t)}r(t) = 1 + \frac{costln2}{\sqrt{2}} + \frac{costln3}{\sqrt{3}} + ... + \frac{costlnn}{\sqrt{n}} +$$

$$+r(t)cos\theta(t) - i\left(\frac{sintln2}{\sqrt{2}} + \frac{sintln3}{\sqrt{3}} + ... + \frac{sintlnn}{\sqrt{n}} + r(t)sin\theta(t)\right) =$$

$$= u(t) - i\nu(t), n \le x = \sqrt{\frac{t}{2\pi}}; r(t) = O\left(t^{-\frac{1}{4}}\right).$$

Then

$$k(t) = \sqrt{u^2\,(t) + \nu^2(t)}, \varphi(t) = \arg g(t). \tag{14}$$

It can be shown that value of $\varphi(t)$ at certain t infinitely many times gets positive and negative values and that is why $\varphi(t)$ is the oscillatory function.

From (13) follows that $N(t) = \left[\frac{1}{\pi}\left(\theta(t) + \varphi(t)\right) + \frac{3}{2}\right]$. Then for the function $S(T)$ we get formula $S(t) = -\frac{1}{\pi}\theta(t) + \left[\frac{1}{\pi}\left(\theta(t) + \varphi(t)\right) + \frac{1}{2}\right]$, that describes the reason of the jumps of a function. From this equation we get that,

$$\varphi(t) = \pi S(t) - \pi\left(\frac{1}{2} - \left\{\frac{1}{\pi}\left(\theta(t) + \varphi(t)\right) + \frac{1}{2}\right\}\right)$$

Since $S(t) = O(\ln t)$ then $\varphi(t) = O(\ln t)$.

({a} - - fractional part of a).

Analysis of formula (13) and its graph explains why with the growth of t Gram's Law gets violated, i. e., why points γ_k go beyond $(g_{k-1}; g_k)$ (g_k – Gram's points, i. e. solutions of equation $\theta(t) = \pi(k-1)$). At a large enough t function $\theta(t)$ will surge forward and the density of points will rise as well g_k. . If at these t function $\varphi(t)$ is positive and surging forward to its maximum, then some share of points of intersection of graph $\theta(t) + \varphi(t)$ with straight lines

$\frac{\pi}{2}+\pi k$, , i. e. part of points γ_k will move in negative direction. Herewith, function $S(t)$,), which makes jumps by 1 in points γ_k, on average grows by the value approximate to $m-1$, where m – is the amount of points γ_k, which have moved to the left beyond their intervals $(g_{k-1}; g_k)$. And vice versa, when $\varphi(t)$ is rapidly decreasing, points γ_k will move in positive direction, while function $S(T)$ will be decreasing by jumps on average.

PSolutions of equation (13) are the abscisses of graph intersection points $y = \theta(t) + \varphi(t)$ (in Cartesian coordinate system tOy) with straight lines. $y = \frac{\pi}{2} + \pi k$. By virtue of the fact that $\theta(t)$ is growing function, and $\varphi(t)$ – is the oscillatory function $\varphi(t) = O(\ln t)$, v $\theta(t) + \varphi(t)$ at $0 \le t \le T$ will intersect the abovementioned straight lines approximately as many times as graph of the function $\theta(t)$, , i. e. there will be approximately points of intersection $\frac{T}{2\pi} ln \frac{T}{2\pi} - \frac{T}{2\pi}$. That proves that almost all non-trivial zeros lay on the critical line, i.e takes place

Theorem

$$\lim_{T\to\infty} \frac{N_0(T)}{N(T)} = 1 \tag{15}$$

$N_0(T)$ is the amount of zeros on the section of the critical line $0 \le t \le T$ of the critical line. Riemann argued [1], that function $\xi\left(\frac{1}{2} + it\right)$on the gap $[0; T]$ has aproximetely $\frac{T}{2\pi} ln \frac{T}{2\pi} - \frac{T}{2\pi}$ real roots, which is equivalent to the fact that $N_0(T) \approx \frac{T}{2\pi} ln \frac{T}{2\pi} - \frac{T}{2\pi}$. Besides, the reasoning of the growth of the function $\theta(t) + \varphi(t)$ indicates that Riemann‘s hypothesis about that all trivial zeros lay on the critical line will be equal to the fact that $\theta(t) + \varphi(t)$ is the growing function for all $t > 2\pi$.

5. Let‘s study the qualities of value $k(t) = |g(t)|$ at growing t where $g(t)$ is represented by (12). In order to achieve this, first of all we study the subsum $\sum_x(s) = \sum_{n\le x} n^{-s}$. We express this sum through the multiplication by simple numbers

$p_i \le x$:

$$\sum_{n\le x} n^{-s} = \prod_{p_i^{li}\le x} \left(1 + p_i^{-s} + p_i^{-2s} + ... + p_i^{lis}\right) - \sum{}' n_j^{-s} \tag{16}$$

(in the last sum $\sum' n_j^{-s}$ additive components which we get after multiplying the composition which is given above and which are not components of the original sum occur). Then

$$\sum_{n\le x} n^{-s} = \prod_{p_i^{li}\le x} \frac{1 - p_i^{-(li+1)s}}{1 - p_i^{-s}} \left(1 - \frac{\sum'}{\prod}\right), \tag{17}$$

here $\sum' = \sum' n_j^{-s}, \prod = \prod_{p_i^{li}\le x} \left(1 + p_i^{-s} + p_i^{-2s} + ... + p_i^{lis}\right)$.

To simplify the future notes we will take

$\alpha_{p_i}^{li}(s) = \frac{1-p_i^{-(li+1)s}}{1-p_i^{-s}}, \varepsilon_1(x;s) = \frac{\sum'}{\prod}$, and then formula (17) will become

$$\sum_{n\le x} n^{-s} = \prod_{p_i^{li}\le x} \alpha_{p_i}^{li}(s)\left(1+\varepsilon_1(x;s)\right). \tag{18}$$

Now let‘s study the multipliers $\alpha_{p_i}^{li}(s)$.

$(1-p_i^{-s})^{-1} = \left(1-p_i^{-(\sigma+it)}\right)^{-1} = \left(1-p_i^{-\sigma}costlnp_i + ip_i^{-\sigma}sintlnp_i\right)^{-1}$,

Then

$$\left|\left(1-p_i^{-s}\right)^{-1}\right| = \frac{p_i^{\sigma}}{\sqrt{p_i^{2\sigma}-2p_i^{\sigma}costlnp_i+1}}. \tag{19}$$

It‘s obvious that

$$\frac{p_i^{\sigma}}{p_i^{\sigma}+1} \le \left|\left(1-p_i^{-s}\right)^{-1}\right| \le \frac{p_i^{\sigma}}{p_i^{\sigma}-1}, \tag{20}$$

$$1-p_i^{-(li+1)s} = 1-p_i^{(-li+1)(\sigma+it)} =$$

$$1-p_i^{-(li+1)s} = cos\left(li+1\right)tlnp_i + ip_i^{-(li+1)\sigma}sin\left(li+1\right)tlnp_i.$$

$$\left|1-p_i^{-(li+1)s}\right| = \sqrt{1-\frac{2cos\left(li+1\right)tlnp_i}{p_i^{(li+1)\sigma}}+\frac{1}{p_i^{2(li+1)\sigma}}}, \tag{21}$$

$$1-\frac{1}{p_i^{(li+1)\sigma}} \le \left|1-p_i^{-(li+1)s}\right| \le 1+\frac{1}{p_i^{(li+1)\sigma}}. \tag{22}$$

Expression $\left|1-p_i^{-(li+1)s}\right|$ at large enough p_i and l_i is close to 1, that is why major oscillations $\alpha_{p_i}^{u}(s)$ are set by the first multiplier $\left|\left(1-p_i^{-s}\right)^{-1}\right|$.

Then

$$g(t) = \sum_{n\le x}\frac{n^{-it}}{\sqrt{n}} + e^{-i\theta(t)r(t)} = \sum_{n\le x}\frac{n^{-it}}{\sqrt{n}}\left(1+\frac{e^{-i\theta(t)r(t)}}{\sum}\right) =$$

$$= \prod_{p_i^{l_i}\le x}\alpha_{p_i}^{l_i}\left(\frac{1}{2}+it\right)\left(1-\varepsilon_1\left(x;\frac{1}{2}+it\right)\right)\left(1+\varepsilon_2\left(x;t\right)\right),$$

where $\varepsilon_2(x;t) = \frac{e^{-i\theta(t)}r(t)}{\sum_{n\le x} n^{-\frac{1}{2}-it}}$). Then

$$k(t) = |g(t)| = \prod_{p_i^{l_i}\le x} k_i(t)\left|1-\varepsilon_1\right|\left|1+\varepsilon_2\right|, \tag{23}$$

where

$$k_i(t) = \left|\left(1-p_i^{-\frac{1}{2}-it}\right)^{-1}\right|\left|1-p_i^{-(l_i+1)\left(\frac{1}{2}+it\right)}\right|, \varepsilon_1 = \varepsilon_1\left(x;\frac{1}{2}+it\right),$$

$$\varepsilon_2 = \varepsilon_2(x;t).$$

Formula (23) provides possibility to describe the oscillation of value $k(t)$, and make the dynamic of zeta function on the critical line more precise. Taking into account that numbers $2\pi(lnp_i)^{-1}$ create coprime periods for functions $costlnp_i$ we can set large enough values of t, when maximums of coefficients $k_i(t)$ will be almost equal and value $k(t)$ will be close to $\frac{\sqrt{2}}{\sqrt{2}-1}\frac{\sqrt{3}}{\sqrt{3}-1}...\frac{\sqrt{p_i}}{\sqrt{p_i}-1}$ (the final two multipliers $|1-\varepsilon_1|$ and $|1+\varepsilon_2|$ oscillate near 1). Then the curve $\zeta\left(\frac{1}{2}+it\right)$ will make a loop with the diameter $k(t)$. We can also set the value of t, when $k(t)$ reaches its minimum, which approximately equals to $\frac{\sqrt{2}}{\sqrt{2}+1}\frac{\sqrt{3}}{\sqrt{3}+1}...\frac{\sqrt{p_i}}{\sqrt{p_i}+1}$. The last-mentioned fact gives us a possibility to explain Lemer's effect. [3]. It is enough to make an elementary transition from function $\zeta\left(\frac{1}{2}+it\right)$ to function $Z(t)=e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right)$, which is known to have all real values for all real t to achieve this. Then loops of the curve $\zeta\left(\frac{1}{2}+it\right)$ of a very small diameter will transit into fragments of function $Z(t)$, where Lemer's effect occurs.

In a similar way it can be shown that

$$\varphi(t)=\sum_{p_i^{l_i}\le x}\varphi_i(t)+\beta_1(t)+\beta_2(t),$$

where

$$\varphi_i(t)==-arctg\frac{sintlnp_i}{\sqrt{p_i}-costlnp_i}+arctg\frac{sin\left(l_i+1\right)tlnp_i}{p_i^{\frac{1}{2}(l_i+1)}-cos\left(l_i+1\right)tlnp_i}$$

$$\beta_1(t)=arg\left(1-\varepsilon_1\right),\beta_2\left(t\right)=arg\left(1+\varepsilon_2\right).$$

6. If we use the above-stated arguments to study the problem which is opposite to the problem of studying the asymptotic law of prime numbers (the problem of searching for the non-trivial zeros of zeta function through the known simple numbers) then it is possible to find values of γ_k in this method.

We call the solutions of the equation $\theta(t)=\frac{\pi}{2}+\pi\left(k-2\right), k=1,2,...$ (here, $t_1=14,521...,t_2=20,655...,t_3=25,492...,t_4=25,739...,t_5=33,624...$) the Gram's zeros.

These zeros are regular in nature and they are easy to calculate. Then value of γ_k can be described as deviation of point t_k by small distances in positive and negative directions, that is why it is appropriate to take $\gamma_k=t_k+\Delta_k$ and to search for corrections Δ_k out of the conditions related to the distribution of primes and their values.

Let's similarly analyze Chebyshev's psi-function

$$\psi(x)=\sum_{p_i^k\le x}lnp_i \tag{24}$$

and using Mangoldt's formula

$$\psi(x)=x-\sum_{\rho_k}\frac{x^{\rho_k}}{\rho_k}-\frac{1}{2}ln\left(1-x^{-2}\right)-ln2\pi=$$

$$= x - \sqrt{x}\sum_{\gamma_k} \frac{cos\gamma_k lnx + 2\gamma_k sin\gamma_k lnx}{\frac{1}{4} + \gamma_k^2} - \frac{1}{2}ln\left(1 - x^{-2}\right) - ln2\pi, \qquad (25)$$

where $x\epsilon[a;b]$, $a \geq 1$ и $\rho_k = \frac{1}{2} \pm i\gamma_k$.

Then we search approximate values of function $\psi(x)$, which is described by formula (24),)) look in the form

$$\psi_n(x) = x - \sum_{k=1}^{n} \frac{x^{\delta_k}}{\delta_k} - \frac{1}{2}ln\left(1 - x^{-2}\right) - ln2\pi \qquad (26)$$

(here sum is taken the same way as in the case of (25) to group additive components with δ_k and $\bar{\delta_k}$, $\delta_k = \frac{1}{2} + i\left(t_k + \Delta_k\right)$). Unknown corrections Δ_k are found out of the condition of minimum (in any given case) disparity $\varepsilon_n(x) = \psi(x) - \psi_n(x)$,for example from the condition that

$$\int_a^b \varepsilon_n^2(x)dx = F\left(\Delta_1, \Delta_2, ..., \Delta_n\right) \to min. \qquad (27)$$

Further it‘s possible to use the necessary criterion of multivariable function minimum, to be more precise

$$\frac{\partial F}{\partial \Delta_k} = 0, k = \overline{1, n}. \qquad (28)$$

Also it is possible to use iteration process, searching for functional sequence $\psi_1(x), \psi_2(x), ...$, which is characterized by the fact that the amount of additive components in the right part of sum from equality (26) grows at each step of iteration cycle-by-cycle and herewith previously found corrections become more accurate. Δ_k.

Conclusion It is proved that on the critical line there are almost all non trivial zeros of the zeta function. The behavior of the Zeta function on this line is investigated and the Lemmer phenomenon is explained.

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