Nonparametric Estimation of Local Bandwidth from Level Crossings Sampling

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***Abstract*—We study the problem of a local bandwidth recovery for nonstationary stochastic signals when the measured information is given in terms of level crossings. We propose a kernel estimate of the local bandwidth from samples generated from level crossings of stochastic signals being the time-warped version of stationary Gaussian processes. The positivity and bandlimited nature of the local intensity is captured by the properly selected class of bandlimited and positive kernel functions. The asymptotic properties of the estimator are derived.**

I. Introduction

In the conventional signal processing system the sampling rate corresponding to the Nyquist frequency is kept fixed when the signal is processed. Such an approach is justified in relation to signals whose spectral properties do not evolve in time. However, there are numerous classes of signals whose local spectral content is strongly varying and the concept of the global bandwidth is not sufficient. This fact has motivated a number of researchers [1]–[4] to utilize time-varying local properties of the signal and adapt the sampling rate to the changing frequency content. This, however, requires the knowledge of the local bandwidth to control time-varying sampling rate. The approach which can be used for this purpose is the level-crossing sampling as the mean rate of level crossings is higher when the signal varies quickly and lower when it changes more slowly.

In this paper we observe that the local bandwidth can be directly related to the concept of a local intensity function characterizing a counting process of level crossings. This allows us to develop the nonparametric kernel estimate for the local bandwidth recovery, see [5], [6] for the related preliminary results. The celebrated Rice theory [7], [8] for average number of level crossings of stochastic processes is utilized to establish the link between the local bandwidth and the level crossings counting process. We consider a class of locally stationary processes being the time-warping deformation version of stationary Gaussian processes [9]. The positivity and bandlimitness of the local bandwidth requires the suitable correction of our estimate. This is achieved by the proper choice of bandlimited and positive kernel functions. The asymptotic properties of the proposed estimate are established based on the theory of the local martingale characterization of counting processes.

M. Pawlak, D. Rzepka and M. Miśkowicz were supported by the Polish National Center of Science under Grant DEC-2017/27/B/ST7/03082.

The remainder of the paper is organized as follows: Section 2 introduces the examined class of nonstationary signals. In Section 3 the problem of estimating the time deformation model from level crossings is formulated. We also present our basic mathematic tools utilizing the Rice level crossing theory and the local martingale characterization of counting processes. Section 4 defines the kernel method for the local bandwidth estimation. Also in the same section we present the asymptotic theory of the proposed estimate. The detailed proofs and simulations studies will be presented elsewhere. We shall denote by $\mathbf{1}(A)$ the indicator function of the set A.

II. Nonstationary Stochastic Signals: The Time Deformation Model

Many processes encountered in biomedical and communications systems arise from nonstationary phenomenon. The nonstationary nature of a signal can be manifested by the local variability or periodically correlated structure. The latter case is represented by a class of cyclostationary random signals [10], [11]. A nonstationary stochastic signal $X(t)$ often reveals a local stationarity that can be defined in various ways. In [12] the local stationarity is characterized by the Hölder condition imposed on the autocorrelation function. Nonstationary processes with a locally time varying spectral representation were examined in [13], [14]. Yet another approach defines the local stationarity based on modeling the signal $X(t)$ as a certain deformation of the stationary process $Z(t)$ [15], [16], [9]. A rich class of nonstationary signals can be obtained by imposing amplitude and phase deformation mappings. Hence, the following deformation model can be considered

$$X(t) = a(t)Z(\theta(t)), \tag{1}$$

where $Z(t)$ is a zero-mean stationary stochastic process with finite variance μ_0 and covariance function $R(\tau)$. The function $a(t)$ defines the amplitude variation, whereas $\theta(t)$ is the time warping deformation. The nonstationarity of $X(t)$ results from the time-varying behavior of the variance of $X(t)$, i.e., $Var[X(t)] = a^2(t)\mu_0$. The further essential contribution to the signal nonstationarity is provided by the warp function $\theta(t)$. In fact, we have

$$R_X(t,s) = a(t)a(s)R(\theta(s) - \theta(t)). \tag{2}$$

The important case of (1) is the time deformation model

$$X(t) = Z(\theta(t)). \tag{3}$$

This model plays an important role in a number of real-world applications such as the Doppler effect and also is an inherent part of nonstationary signal processing, e.g., in speech analysis to model local expansion (or compression) of time [13], [9], [17]. This is the deformation model that will be examined in this paper.

The natural question concerns the inverse statistical inference problem of recovering the deformation transformation $\theta(t)$ from the observed nonstationary signal $X(t)$. Hence, one would like to estimate $\theta(t)$ from the single realization $\{X(t), 0 \leq t \leq T\}$ for some finite T. This inverse warping problem was thoroughly studied in [16], [9]. The detailed statistical theory for estimating $\theta(t)$ from a densely observed single realization of $X(t)$ was given in [9]. The consistent estimate of the derivative of $\theta(t)$ was proposed assuming that the input signal $Z(t)$ is a stationary Gaussian process and that the deformation function $\theta(t)$ meets some strong smoothing conditions. In fact, the weak convergence of an estimate of $(\log(\theta^{(1)}(t)))^{(1)}$ was established.

In this paper we address the analogous inverse estimation problem regarding the model in (3). Nevertheless our statistical inference is not based on the direct observation of $X(t)$ but on the event driven samples obtained from level crossings of $X(t)$. As a result we obtain a consistent estimate of $\theta^{(1)}(t)$ assuming merely that $\theta(t)$ is the differentiable, positive and non-decreasing function. Hence, $\theta(t)$ can be represented as

$$\theta(t) = \int_{-\infty}^{t} \Omega(s) ds, \tag{4}$$

where $\Omega(t) = \theta^{(1)}(t)$ is a positive function often interpreted as a local bandwidth. Thus, our theory provides consistent nonparametric estimates of the local bandwidth. As we have already mentioned, the concept of local bandwidth plays essential role in asynchronous signal processing where irregular sampling is performed according to the time-varying shape of the bandwidth [18] , [19], [1], [2], [4], [6], [5].

III. Estimating Local Bandwidth from Level Crossings

At the given level u the level crossings measurements can be represented by the counting process $N_u^X(t)$ that is the number of $u-$level crossings of the signal $X(t)$ over the interval $[0, t]$. The goal of this paper is to recover the local bandwidth $\Omega(t)$ representing the model in (3) from the level counting process $N_u^X(t)$ over the interval $[0, T]$. The theory of level crossings for stationary stochastic processes is well established and has originated from the celebrated Rice formula for the average number of level crossings in stationary Gaussian processes [20], [7], [8]. In this paper we use the extended Rice formula for nonstationary processes and this gives us the link between the average number of level crossings and the local crossing intensity function. In the context of the model in (3) the latter is directly related to the local bandwidth $\Omega(t)$.

Our estimation method relies on the counting process $\{N_u^X(t), 0 \leq t \leq T\}$. In this context no asymptotic convergence is possible in the classical setting (when the sample size tends to infinity) since the intensity estimation problem for counting processes does not fall into the large-sample - smaller distance between sample points framework. In order to obtain the proper asymptotic we must increase the number of points falling into the interval $[0, T]$. To do so, we let the mean bandwidth of the input signal $Z(t)$ to increase without bound. If $Z(t)$ is bandlimited this is effectively equivalent to the fact that the absolute bandwidth of $Z(t)$ tends to infinity. The asymptotic with respect to T is not appropriate since, as it was remarked in [21], [22], this will only add new observations for $t > T$ but not everywhere.

Fig. 1 illustrates our problem of estimating the local bandwidth $\Omega(t)$ from counting process $N_u^X(t)$.

In the following sections we summarize fundamental results on level crossings for both stationary and nonstationary processes. In particular we present the version of the Rice formula for the time deformation process in (3).

A. Stationary Signals and Level Crossings

Let us begin with the simplest case of the stationary Gaussian process $Z(t)$ that is assumed to be zero mean with smooth trajectories. The latter is formalized by the requirement that the first two spectral moments of $Z(t)$ exist or equivalently that $\mu_0 = \text{var}[Z(t)]$ and $\mu_2 = \text{var}[Z^{(1)}(t)]$ exist. As it was already defined let $N_u^Z(t)$ denote the number of times the process $Z(t)$ crosses a fixed level u over the interval $[0, t]$.

The following celebrated Rice formula [20], [7], [8] gives the average value of $N_u^Z(t)$.

$$\mathbb{E}[N_u^Z(t)] = \frac{t}{\pi} \gamma e^{-u^2/2\mu_0}, \tag{5}$$

where

$$\gamma = \sqrt{\mu_2/\mu_0} \tag{6}$$

is the so-called *mean bandwidth* of the process $Z(t)$. Hence, the stationary Gaussian process is characterized by the constant intensity $\lambda_u(t) = \frac{1}{\pi}\gamma e^{-u^2/2\mu_0}$ at the level u. It is of great interest to evaluate the formula in (5) for bandlimited processes. This is illustrated in the following example.

Example 1. Let us consider the stationary bandlimited Gaussian process with $R(\tau) = \frac{\sin(\omega_0 \tau)}{\omega_0 \tau)}$ and the corresponding spectral density $S(\omega) = \frac{1}{2\omega_0}\mathbf{1}(|\omega| \leq \omega_0)$. Then, $\mu_0 = 1$ and $\mu_2 = \frac{\omega_0^2}{3}$ yielding the mean mean bandwidth

$$\gamma = \frac{\omega_0}{\sqrt{3}}. \tag{7}$$

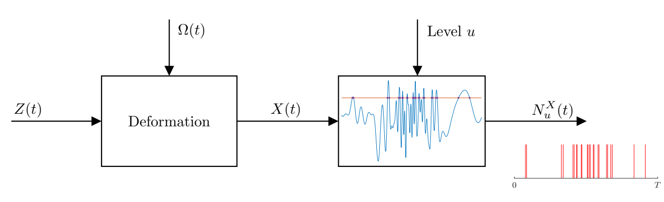

Fig. 1. The process of recovery of the local bandwidth $\Omega(t)$ from the level-crossing counting process $N_u^X(t)$.

Hence, the Rice formula in (5) reads as follows

$$\mathbb{E}[N_u^Z(t)] = \frac{t}{\pi}\frac{\omega_0}{\sqrt{3}}e^{-u^2/2\mu_0}. \quad (8)$$

This reveals that the average number of level crossings over a finite time interval is increasing with the bandwidth ω_0.
If $u = 0$ (zero crossings) the Rice formula gives the average sampling rate equal to $\tau_R = \pi\sqrt{3}/\omega_0$. Tthe Shannon sampling theory suggests the average sampling rate $\tau_S = \pi/\omega_0$. As $\tau_S < \tau_R$ this clearly reveals the advantage of level-crossing sampling over the classical time-domain scheme.

B. Nonstationary Signals and Level Crossings

The level-crossing rate for nonstationary stochastic processes is characterized by the local crossings intensity function $\lambda_u(t)$ at the level u. In fact, the the average number of level crossings of the smooth nonstationary stochastic process $X(t)$ over the interval (s,t) is given by

$$\mathbb{E}[N_u^X((s,t))] = \int_s^t \lambda_u(\tau)d\tau. \quad (9)$$

It is known [7] (Chapter 8), [23] (Chapter 3.10) that for a wide range of nonstationary random processes the local crossings intensity $\lambda_u(t)$ is given by the following formula

$$\lambda_u(t) = \int_{-\infty}^{\infty} |v| f_{X,X^{(1)}}(u,v;t)dv, \quad (10)$$

where $f_{X,X^{(1)}}(u,v;t)$ is the joint density of $(X(t), X^{(1)}(t))$ that depends on t due to the non-stationarity of the process $X(t)$. The explicit form of $\lambda_u(t)$ in (10) is difficult to obtain as the joint density function of $(X(t), X^{(1)}(t))$ can have a complex form. Nevertheless, specializing the result in (10) to the time deformation model in (3) leads to the following result.
Lemma 1. Let $X(t)$ be the time deformation process in (3), where $\theta(t)$ is the positive, differentiable and non-decreasing function with $\theta^{(1)}(t) = \Omega(t)$. Let $Z(t)$ be the zero mean stationary Gaussian process with finite spectral moments μ_0, μ_2. Then, we have

$$\lambda_u(t) = \Omega(t)\mathbb{E}[N_u^Z(1)], \quad (11)$$

where $\mathbb{E}[N_u^Z(1)]$ is defined in (5) .
The result in (11) yields the following formula for the average value of level crossings for the nonstationary process in (3)

$$\mathbb{E}[N_u^X(t)] = \alpha_u \int_0^t \Omega(s)ds, \quad (12)$$

where $\alpha_u = \frac{1}{\pi}\gamma e^{-u^2/2\mu_0}$ and γ is the mean bandwidth of $Z(t)$. Thus, the intensity function is of the multiplicative form, i.e., we have

$$\lambda_u(t) = \alpha_u \Omega(t). \quad (13)$$

If $Z(t)$ is bandlimited (with the bandwidth ω_0) α_u is proportional to ω_0. The class of counting processes for which the local intensity function has this form is called the multiplicative intensity model that was thoroughly examined in [24]. The following example illustrates the identity in (13).

Example 2. Let us consider the bandlimited Gaussian signal $Z(t)$ examined in Example 1. Then, by (11) and (8) we obtain

$$\lambda_u(t) = \Omega(t)\frac{\omega_0}{\pi\sqrt{3}}e^{-u^2/2\mu_0}. \quad (14)$$

C. Level Crossings Counting Process

The level crossings counting process $N_u^X(t)$ can be examined by techniques developed in the general theory of counting processes [24], [25]. The process $N_u^X(t)$ can be written as

$$N_u^X(t) = \sum_i \mathbf{1}(t_i \leq t), \quad (15)$$

where $\{t_i\}$ are ordered time points where level crossing events are taking place. The cumulative intensity $\int_0^t \lambda_u(\tau)d\tau$ is just the average $\mathbb{E}[N_u^X(t)]$ given in (12). This reveals that the process

$$M_u^X(t) = N_u^X(t) - \int_0^t \lambda_u(\tau)d\tau \quad (16)$$

is the residual process for which we have $\mathbb{E}[M_u^X(t)] = 0$. It turns out [25] that the process $M_u^X(t)$ forms the local martingale with the increment $dM_u^X(t)$ satisfying the following formula

$$dM_u^X(t) = dN_u^X(t) - \lambda_u(t)dt. \quad (17)$$

Hence, the increment of the counting process $dN_u^X(t)$ can be written in the signal plus noise form $dN_u^X(t) = \lambda_u(t)dt + dM_u^X(t)$, where the increment $dM_u^X(t)$ is the zero mean martingale. The martingale process reveals jump points at the level crossing points of the counting process $N_u^X(t)$. The probabilistic properties of $dM_u^X(t)$ are crucial for the asymptotic results presented in this paper. The local behavior of the noise process $M_u^X(t)$ with $u = 0$ (zero crossings) is illustrated in Fig.2.

IV. Nonparametric Local Bandwidth Estimation

Owing to (15) we note that the formal derivative of $N_u^X(t)$ is the sum of delta functions defined at the event points $\{t_i\}$. This and (12) suggest the following naive estimate of $\lambda_u(t)$

$$\widetilde{\lambda}_u(t) = \sum_{t_k \leq T} \delta(t - t_k),$$

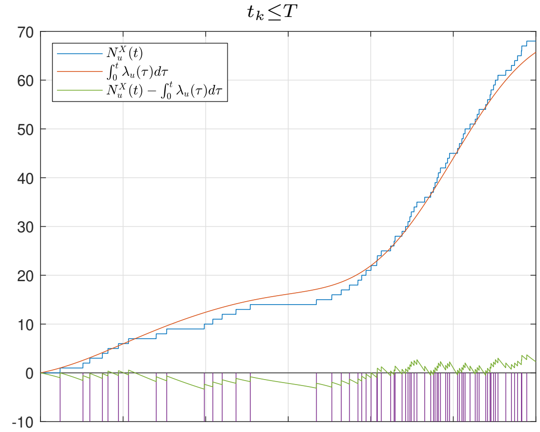

Fig. 2. The zero crossings counting process $N_u^X(t)$, the cumulative intensity $\int_0^t \lambda_u(\tau)d\tau$ and the corresponding residual process $M_u^X(t) = N_u^X(t) - \int_0^t \lambda_u(\tau)d\tau$. The vertical lines show the time positions $\{t_i\}$ of the zero crossings.

where $\delta(t)$ is the delta function. Clearly, this is the impractical and inconsistent estimate of $\lambda_u(t)$ since it includes infinite spikes yielding the estimate with the unbounded variance. A consistent estimate of $\lambda_u(t)$ can be obtained by a proper smoothing of the naive estimate., i.e., by taking the convolution of the naive estimate with the locally tuned kernel function, i..e, $\widetilde{\lambda}_u(t) * b_W(t)$, where $b_W(t) = Wb(Wt)$. This leads to the following kernel estimate

$$\widehat{\lambda}_u(t) = \sum_{t_k \leq T} b_W(t - t_k), \tag{18}$$

where $b(t)$ is the positive kernel function. The smoothing parameter W controls the bias-variance tradeoff of $\widehat{\lambda}_u(t)$. Hence, small W results in large bias and small variance. On the other hand, large W gives the opposite effect.

The problem of estimating the local intensity function has been examined by a number of authors, see [21], [25] and the references cited therein. These contributions have considered the classical setting when data are generated according a certain point process. Furthermore, the employed kernel functions belong to a class of compact supported density functions. In event based systems the underlying signals are often bandlimited and one is required to consider a class of band-limited intensity functions that have bandwidth not larger than W. By selecting $b(t)$ in (18) as the bandlimited positive function with the unit bandwidth the estimate $\widehat{\lambda}_u(t)$ becomes the positive bandlimited function with the bandwidth W. We denote the class of bandlimited positive functions with bandwidth W as $BL_+(W)$.

As we have already pointed out the estimate $\widehat{\lambda}_u(t)$ cannot converge to the true $\lambda(t)$ since level-crossing points are not closely spaced. Owing to the derived multiplicative formula in (13) we can establish the asymptotic theory by allowing the factor α_u to diverge. This means that the mean bandwidth γ of the input process must increase. As a result one can define the following estimate of $\Omega(t)$

$$\widehat{\Omega}(t) = \widehat{\lambda}_u(t)/\alpha_u. \tag{19}$$

Our first asymptotic result gives the evaluation of the mean squared error of $\widehat{\Omega}(t)$.

Theorem 1. Let $\Omega(\cdot) \in BL_+(W_0)$. Let $b(\cdot) \in BL_+(1)$ such that $B_1 = \int_{\mathbb{R}} b^2(t)dt$ and $B_2 = \int_{\mathbb{R}} t^2 b(t)dt$ be finite. Then, for $t \in (0, T)$ the following asymptotic formula holds

$$\mathbb{E}[\widehat{\Omega}(t) - \Omega(t)]^2 = \frac{W}{\alpha_u} B_1 \Omega(t) + \frac{1}{4} B_2^2 (\Omega^{(2)}(t))^2 W^{-4} + o_2. \tag{20}$$

The term o_2 is smaller order than W^{-4}. The first term in (20) is the estimate variance, whereas the second one is its bias. The analogous result holds for the mean integrated squared error (MISE).

Theorem 2. Under the conditions of Theorem 1 we have

$$\mathbb{E} \int_0^T [\widehat{\Omega}(t) - \Omega(t)]^2 dt = \frac{W}{\alpha_u} B_1 \int_0^T \Omega(t)dt + \frac{1}{4} B_2^2 \int_0^T (\Omega^{(2)}(t))^2 dt W^{-4} + o_2. \tag{21}$$

It is seen that the pointwise and integrated errors tend to zero if $W = W(\alpha_u) \to \infty$ and $W(\alpha_u)/\alpha_u \to 0$ as $\alpha_u \to \infty$. Hence, α_u plays the role of 'sample size' used in the classical statistical setting. The variance-bias decomposition in (21) is minimized by the choice of W of the form

$$W_{MISE} = c\alpha_u^{1/5}, \tag{22}$$

where c depends on $\Omega(t)$ and the kernel function $b(t)$. Plugging this optimized value of W into (21) gives the minimal asymptotic version of $MISE$. Hence, we obtain

$$MISE_{opt} = c\eta^{1/5}(\Omega)\psi^{2/5}(b)\alpha_u^{-4/5}, \tag{23}$$

for some universal constant c, where

$$\eta(\Omega) = \left(\int_0^T \Omega(t)dt \right)^4 \int_0^T (\Omega^{(2)}(t))^2 dt$$

and

$$\psi(b) = \left(\int_{\mathbb{R}} b^2(t)dt \right)^2 \int_{\mathbb{R}} t^2 b(t)dt. \tag{24}$$

The careful examination of the formula in (22) allows to show that for $\Omega(\cdot) \in BL_+(W_0)$ and if the input signal $Z \in BL(\omega_0)$ then

$$W_{MISE} = cW_0^{4/5}\omega_0^{1/5}.$$

With the choice $\omega_0 = W_0$ we can obtain the desirable property that $\widehat{\Omega} \in BL_+(W_0)$. Furthermore, the optimized $MISE_{opt}$ in (23) can be further minimized with respect to the kernel choice. Ideally, one would like to minimize the functional $\psi(b)$ in (24) for $b \in BL_+(1)$. This seems to be a difficult problem and the result below gives the partial answer to this question giving the optimal kernel that minimizes the estimate bias. Hence, we wish to minimize

$$J(b) = \int_{\mathbb{R}} t^2 b(t)dt \tag{25}$$

with respect to $b \in BL_+(1)$. This variational problem has the unique solution and the optimal kernel is given by

$$b_{opt}(t) = 4\pi \frac{\cos^2(t/2)}{(\pi^2 - t^2)^2}, \tag{26}$$

with $J(b_{opt}) = \pi^2$.

V. Concluding remarks

In this paper we proposed the consistent kernel estimate of the local bandwidth that characterizes fine properties of the assumed class of nonstationary stochastic signals. Our estimation method relies on the observed level-crossings counting process at a fixed level. There are two natural extensions of the proposed approach. The first generalization concerns a larger class of nonstationary signals with a variance which is not necessarily constant [16]. The second extension may consider spatial data [26] where the theory of level sets for random fields can be utilized [27].

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