Asymptotically optimal t-design curves on the Grassmann manifold

Martin Ehler
Fakultät für Mathematik
Universität Wien
Vienna, Austria
martin.ehler@univie.ac.at

Clemens Karner
Fakultät für Mathematik
Universität Wien
Zentrum für Medizinische Physik
und Biomedizinische Technik
Medizinische Universität Wien
Vienna, Austria
clemens.karner@meduniwien.ac.at

Pedro R. López-Gómez
Departamento de Matemáticas,
Estadística y Computación
Universidad de Cantabria
Santander, Spain
lopezpr@unican.es

***Abstract*—Similarly to well-known spherical t-design points, we investigate t-design curves on the Grassmannian manifold $\mathrm{Gr}_{2,4}$. Such curves have the special property that the line integral along them integrates polynomials of degree t exactly. We derive asymptotic bounds on the length of such curves based on a parametrization of $\mathrm{Gr}_{2,4}$ as $(\mathbb{S}^2 \times \mathbb{S}^2)/\pm 1$.**

***Index Terms*—Spherical designs, t-design curves, Grassmannian manifold**

I. Introduction

A *t-design point configuration* on the d-sphere $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ is a finite set of points $X = \{x_1, \ldots, x_n\}$, with $n \in \mathbb{N}$, such that

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \int_{\mathbb{S}^d} f \quad \text{for all } f \in \Pi_t,$$

where Π_t is the set of polynomials in $d+1$ variables of degree at most t, and $d, t \in \mathbb{N}$.

Spherical t-design points have been an ongoing topic of research, resulting in a wide variety of publications, including [20], [27], [28], [30], [37], [4], [5], [17], [18]. Further existence results on more general compact Riemannian manifolds are contained in [16], [15], [11].

The concept of spherical designs was originally introduced in 1977 in the fundamental paper [9] by Delsarte, Goethals, and Seidel. If we denote by $N(d,t)$ the minimal number of points on a spherical t-design on $\mathbb{S}^d$ for each $d, t \in \mathbb{N}$, Delsarte, Goethals, and Seidel also proved that a spherical t-design in d dimensions satisfies the lower bound $N(d,t) \geq c_d t^d$. Here and throughout this text, we use c_d and C_d to denote sufficiently small and sufficiently large positive constants, respectively, depending only on d. Several years later, in 1984, Seymour and Zaslavsky [29] proved the existence of t-design points on $\mathbb{S}^d$ for all $d, t \in \mathbb{N}$. A key open question was the existence of a tight upper bound on the number of points. The first results in this regard were presented in 1991 by Wagner and Volkmann [36], who showed that $N(d,t) \leq C_d t^{C_d d^4}$, and in 1992 by Bajnok [3], who obtained the upper bound $N(d,t) \leq C_d t^{C_d d^3}$. Shortly afterwards, in 1993, Korevaar and Meyers proved in their well-known paper [22] that $N(d,t) \leq C_d t^{(d^2+d)/2}$ and stated what later became known as the Korevaar–Meyers conjecture about the existence for all $t \in \mathbb{N}$ of spherical t-designs with cardinality $n \leq C_d t^d$. Families of spherical t-designs for all $t \in \mathbb{N}$ that match this bound are called *asymptotically optimal*, meaning[1] $n \asymp t^d$. In this regard, one of the major breakthroughs was the proof by Bondarenko, Radchenko and Viazovska [4] in 2013 of the aforementioned Korevaar–Meyers conjecture on the existence of asymptotically optimal t-design configurations on $\mathbb{S}^d$, a problem that had been open for 20 years. Their result is stated in the following theorem.

Theorem I.1 ([4, Theorem 1])**.** *In $\mathbb{S}^d$, there exists a sequence of spherical t-design points $(X_t)_{t\in\mathbb{N}}$ such that the cardinalities satisfy $|X_t| \asymp t^d$.*

Recently, in 2023, the concept of t-design points was generalized to curves on the d-sphere in [14]. A *t-design curve* on $\mathbb{S}^d$ is a curve $\gamma\colon [0,1] \to \mathbb{S}^d$ of length $\ell(\gamma) = \int_0^1 \|\dot\gamma(s)\|\,\mathrm{d}s$ such that the line integral $\int_\gamma f = \int_0^1 f(\gamma(s))\|\dot\gamma(s)\|\,\mathrm{d}s$ integrates all polynomials in Π_t exactly, that is,

$$\frac{1}{\ell(\gamma)}\int_\gamma f = \int_{\mathbb{S}^d} f \quad \text{for all } f \in \Pi_t.$$

Note that we allow arbitrary self-intersections, so that we are here not quite as restrictive as in [14], where the following theorem on the minimal length of such a curve is derived.

Theorem I.2 ([14, Theorem 1.1])**.** *Assume that a piecewise smooth, closed curve $\gamma\colon [0,1] \to \mathbb{S}^d$ satisfies*

$$\frac{1}{\ell(\gamma)}\int_\gamma f = \int_{\mathbb{S}^d} f \quad \textit{for all } f \in \Pi_t.$$

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[1] The notation $f(t) \asymp g(t)$ means that $f(t) = O(g(t))$ and $g(t) = O(f(t))$, that is, f and g are asymptotically of the same order.

Then, its length is bounded from below by

$$\ell(\gamma) \geq c_d t^{d-1},$$

with some constant $c_d > 0$ that may depend on the dimension d but is independent of t and γ.

Collections of spherical t-design curves are presented in [14], [12]. An analogue of the Korevaar–Meyers conjecture for the case of t-design curves asks for the existence of asymptotically optimal t-design curves on $\mathbb{S}^d$, that is, a sequence $(\gamma_t)_{t\in\mathbb{N}}$ of t-design curves whose length grows at most as t^{d-1}. The case of the 2-sphere was proven in [14] and that of the 3-sphere in [24]. In this paper, we employ the result on the 2-sphere, stated in the following theorem.

Theorem I.3 ([14, Theorem 1.2])**.** *In $\mathbb{S}^2$, there exists a sequence of t-design curves $(\gamma_t)_{t\in\mathbb{N}}$ with length $\ell(\gamma_t) \asymp t$.*

The concept of t-design curves is significant not only in the case of the sphere but also in the more general setting of compact Riemannian manifold; see [19], [33], [32], [26] for applications of this notion in mobile sampling. Mobile sampling allows for the approximation of a function based on values along a curve. This offers the advantage of requiring only a single sensor moving along a curve for data acquisition, compared to several sensors for point-based (e.g. t-design points) sampling. In this paper, instead of the unit sphere $\mathbb{S}^2$, we are especially interested in the concept of t-design curves on the Grassmannian manifold, which encode orthogonal projections. Such projections can be used to represent data in lower dimensions with a wide variety of applications, including visualization of high-dimensional data sets [25], [34], [7], classification of clusters [7], linear regression [31], linear programming [35], scattering theory [23] and learning frameworks [8]. There are currently no results on t-design curves on the Grassmann manifold.

The main goal of this paper is to prove the corresponding version of Theorem I.3 for the case of the Grassmannian $\mathrm{Gr}_{2,4}$, which is closely related to $\mathbb{S}^2$ and will be properly introduced in the next section.

II. The Grassmann manifold

A. Basic definitions

The (real, unoriented) Grassmann manifold $\mathrm{Gr}_{m,d}$ is defined as the set of m-dimensional subspaces of $\mathbb{R}^d$. The Grassmannian $\mathrm{Gr}_{m,d}$ is a homogeneous space under the action of the orthogonal group $\mathrm{O}(d)$ and we have the following representation of $\mathrm{Gr}_{m,d}$ as a homogeneous space:

$$\mathrm{Gr}_{m,d} = \mathrm{O}(d)/\big(\mathrm{O}(m) \times \mathrm{O}(d-m)\big).$$

In this work, we will instead use the following identification of $\mathrm{Gr}_{m,d}$ with the set of orthogonal projectors on $\mathbb{R}^d$ of rank m:

$$\mathrm{Gr}_{m,d} = \{P \in \mathbb{R}^{d\times d} \,:\, P^\mathsf{T} = P,\ P^2 = P,\ \mathrm{rank}(P) = m\}.$$

The Grassmannian $\mathrm{Gr}_{m,d}$ is a compact, connected Riemannian manifold of dimension $m(d-m)$. We will denote the unique $\mathrm{O}(d)$-invariant probability measure on $\mathrm{Gr}_{m,d}$ induced by the Haar measure on $\mathrm{O}(d)$ as σ and we will refer to it as the *uniform measure* on $\mathrm{Gr}_{m,d}$. From now on, we focus on the Grassmannian $\mathrm{Gr}_{2,4}$.

B. Parametrization of $\mathrm{Gr}_{2,4}$ as $(\mathbb{S}^2 \times \mathbb{S}^2)/\pm 1$

In this work we focus on the Grassmannian $\mathrm{Gr}_{2,4}$, which is the simplest Grassmannian that is algebraically different from a projective space. It is well-known that $\mathbb{S}^2 \times \mathbb{S}^2$ is a double covering of $\mathrm{Gr}_{2,4}$. Specifically, there is an isometric one-to-one mapping $\mathcal{P}\colon (\mathbb{S}^2 \times \mathbb{S}^2)/\pm 1 \to \mathrm{Gr}_{2,4}$ given by

$$\begin{aligned}\mathcal{P}(-x,-y) &= \mathcal{P}(x,y) \\ &:= \frac{1}{2}\begin{pmatrix} 1 + x^\mathsf{T} y & -(x\times y)^\mathsf{T} \\ -x \times y & xy^\mathsf{T} + yx^\mathsf{T} + (1 - x^\mathsf{T} y) I_3 \end{pmatrix},\end{aligned} \tag{1}$$

where I_3 is the 3×3 identity matrix and A^T denotes the transpose of A; see [10, Section 7] for details. Note that here we are assuming that the measures on all the spaces are normalized so that the total volume is 1.

Recall that the eigenvalues and eigenfunctions of the Laplace–Beltrami operator on $\mathrm{Gr}_{2,4}$ are naturally indexed by integer partitions $\pi = (\pi_1, \pi_2) \in \mathbb{N}^2$, with $\pi_1 \geq \pi_2 \geq 0$, of *degree* $|\pi| := \pi_1 + \pi_2$; see [21, Theorem 13.2] or [2, Section 3] for details. Then, we have the following decomposition of the space $L^2(\mathrm{Gr}_{2,4})$ of square-integrable functions on $\mathrm{Gr}_{2,4}$ as the direct sum of the eigenspaces of the Laplace-Beltrami operator:

$$L^2(\mathrm{Gr}_{2,4}) = \overline{\bigoplus_{\pi_1 \geq \pi_2 \geq 0} H_\pi},$$

where H_π is the eigenspace associated with the eigenvalue λ_π.

A consequence of the identification of $\mathrm{Gr}_{2,4}$ with $(\mathbb{S}^2 \times \mathbb{S}^2)/\pm 1$ given by (1) is that, given an orthonormal basis $\{\varphi_{\pi,l}\}_{1\leq l \leq \dim(H_\pi)}$ of the eigenspace H_π, we can express the eigenfunctions $\varphi_{\pi,l}$ as a tensor product of spherical harmonics on $\mathbb{S}^2$. More specifically, consider the functions $Y^{j,k}_{l,m}\colon \mathrm{Gr}_{2,4} \to \mathbb{C}$ given by

$$Y^{j,k}_{l,m}(\mathcal{P}(x,y)) := (Y^j_l \otimes Y^k_m)(x,y) = Y^j_l(x) Y^k_m(y), \tag{2}$$

where Y^j_l denotes a spherical harmonic of degree l. These functions are well-defined for $l + m \in 2\mathbb{N}$, the latter taking into account the ambiguity in (1).

Theorem II.1 ([10, Theorem 7.2])**.** *Let H_π be the eigenspace of the Laplace–Beltrami operator associated with the partition $\pi = (\pi_1, \pi_2)$. Let $l_\pi = \pi_1 + \pi_2$ and $m_\pi = \pi_1 - \pi_2$. Then,*

$$\begin{aligned}H_\pi = \mathrm{span}\{Y^{j,k}_{l_\pi, m_\pi}, Y^{k,j}_{m_\pi, l_\pi} \,:\, -l_\pi \leq j \leq l_\pi, \\ -m_\pi \leq k \leq m_\pi\}.\end{aligned}$$

Next, we define the concept of t-designs on the Grassmannian.

III. Main result

While the concept of t-design points on the Grassmannian has been explored in [1], [2], there are no results about t-design curves on the Grassmann manifold so far. Here we focus on the Grassmannian $\mathrm{Gr}_{2,4}$.

We denote by $\mathrm{Pol}_t(\mathrm{Gr}_{2,4})$ the collection of classical polynomials of degree at most $t \in \mathbb{N}$ on the Grassmannian $\mathrm{Gr}_{2,4}$:

$$\mathrm{Pol}_t(\mathrm{Gr}_{2,4}) = \{f\big|_{\mathrm{Gr}_{2,4}} \,:\, f \in \mathbb{R}[X]_t\},$$

where $\mathbb{R}[X]_t$ is the collection of multivariate polynomials of degree at most t with 16 variables arranged as a matrix $X \in \mathbb{R}^{4\times 4}$. It is well known (see, for example, [6, Section 4.2]) that $\mathrm{Pol}_t(\mathrm{Gr}_{2,4})$ can be identified with the following direct sum of the eigenspaces of the Laplace–Beltrami operator:

$$\mathrm{Pol}_t(\mathrm{Gr}_{2,4}) = \bigoplus_{|\pi|\le t} H_\pi. \tag{3}$$

We finally define the notion of t-design curves on the Grassmannian.

Definition III.1. For $t \in \mathbb{N}$, we say that a continuous, closed curve $\gamma\colon [0,1] \to \mathrm{Gr}_{2,4}$ is a *t-design curve* on $\mathrm{Gr}_{2,4}$ if

$$\frac{1}{\ell(\gamma)} \int_\gamma f = \int_{\mathrm{Gr}_{2,4}} f \quad \text{for all } f \in \mathrm{Pol}_t(\mathrm{Gr}_{2,4}).$$

The lower bound on the length of t-design curves in Theorem I.2 ([14, Theorem 1.1]) is originally proved for the sphere $\mathbb{S}^d$, but it was already mentioned in [14] that the proof directly translates to d-dimensional compact Riemannian manifolds (with the appropriate concept of t-designs). Since the Grassmannian $\mathrm{Gr}_{2,4}$ is 4-dimensional, we therefore note that every sequence $(\gamma_t)_{t\in\mathbb{N}}$ of t-design curves on $\mathrm{Gr}_{2,4}$ satisfies

$$\ell(\gamma_t) \ge c_d t^3\,.$$

Thus, we are looking for sequences of t-design curves that match this lower bound.

Our main result is the following theorem, which establishes the existence of asymptotically optimal t-design curves on $\mathrm{Gr}_{2,4}$; it can be seen as the analogue of Theorem I.3 for the case of the Grassmann manifold $\mathrm{Gr}_{2,4}$. It should be noted that the idea behind our construction is similar to the one in the proof of [13, Theorem 10].

Theorem III.2. *There exists a sequence $(\Gamma_t)_{t\in\mathbb{N}}$ of t-design curves on $\mathrm{Gr}_{2,4}$ with length $\ell(\Gamma_t) \asymp t^3$.*

Proof. For every $t > 0$, let $X = \{x_1,\dots,x_n\} \subset \mathbb{S}^2$ be a t-design with $n \asymp t^2$ points, and let γ be a t-design curve on $\mathbb{S}^2$ satisfying $\ell(\gamma) \asymp t$. Without loss of generality, we can assume that γ contains two antipodal points, since otherwise we could simply concatenate γ with a copy of itself conveniently rotated so that the resulting curve has two antipodal points. Then, there exists a rotation $O_{ij} \in \mathrm{O}(3)$ such that x_i and x_j lie on $\gamma^{(i,j)} := O_{ij}\gamma$. Consider the curve Γ_t on $(\mathbb{S}^2\times\mathbb{S}^2)/\pm 1$ given by the following set of curve-point pairs on $\mathbb{S}^2\times\mathbb{S}^2$:

$$\Gamma_t = \left\{ \begin{matrix} (\gamma^{(1,2)}, x_1), & (x_1, \gamma^{(1,2)}), \\ (\gamma^{(1,3)}, x_2), & (x_2, \gamma^{(1,3)}), \\ (\gamma^{(2,4)}, x_3), & (x_3, \gamma^{(2,4)}), \\ \vdots & \vdots \\ (\gamma^{(n-2,n)}, x_{n-1}), & (x_{n-1}, \gamma^{(n-2,n)}), \\ (\gamma^{(n-1,n)}, x_n), & (x_n, \gamma^{(n-1,n)}) \end{matrix} \right\}$$

that is,

$$\Gamma_t = \left\{(\gamma^{(1,2)}, x_1), (x_1, \gamma^{(1,2)}), (\gamma^{(n-1,n)}, x_n), (x_n, \gamma^{(n-1,n)})\right\} \cup \bigcup_{i=2}^{n-1} \{(\gamma^{(i-1,i+1)}, x_i)\} \cup \bigcup_{i=2}^{n-1} \{(x_i, \gamma^{(i-1,i+1)})\}.$$

By construction, Γ_t is indeed connected in $(\mathbb{S}^2\times\mathbb{S}^2)/\pm 1$ and hence induces a connected curve $\mathcal{P}\circ\Gamma_t$ on $\mathrm{Gr}_{2,4}$. Now we must check that it is a t-design curve of length $\ell(\mathcal{P}\circ\Gamma_t) \asymp t^3$ on $\mathrm{Gr}_{2,4}$.

According to (3), to prove that $\mathcal{P}\circ\Gamma_t$ is a t-design curve on $\mathrm{Gr}_{2,4}$, it suffices to check the integration conditions for the eigenfunctions $Y^{j,k}_{l_\pi,m_\pi}, Y^{k,j}_{m_\pi,l_\pi}$ in Theorem II.1. We will perform the computations explicitly for the eigenfunctions of the form $Y^{j,k}_{l_\pi,m_\pi}$; the other case is handled analogously. The isometric parametrization via $\mathcal{P}$ yields

$$\frac{1}{\ell(\mathcal{P}\circ\Gamma_t)} \int_{\mathcal{P}\circ\Gamma_t} Y^{j,k}_{l_\pi,m_\pi} = \frac{1}{\ell(\Gamma_t)} \int_{\Gamma_t} Y^{j,k}_{l_\pi,m_\pi} \circ \mathcal{P}.$$

The tensor structure of $Y^{j,k}_{l_\pi,m_\pi}$ under this parametrization (2) and the normalization of the measures involved lead to

$$\int\limits_{\mathbb{S}^2\times\mathbb{S}^2} Y^{j,k}_{l_\pi,m_\pi} \circ \mathcal{P} = \int\limits_{\mathbb{S}^2\times\mathbb{S}^2} Y^j_{l_\pi} \otimes Y^k_{m_\pi} = \int\limits_{(\mathbb{S}^2\times\mathbb{S}^2)/\pm 1} Y^j_{l_\pi} \otimes Y^k_{m_\pi}.$$

Hence, we must verify that

$$\frac{1}{\ell(\Gamma_t)} \int\limits_{\Gamma_t} Y^{j,k}_{l_\pi,m_\pi} \circ \mathcal{P} = \int\limits_{\mathbb{S}^2\times\mathbb{S}^2} Y^j_{l_\pi} \otimes Y^k_{m_\pi}.$$

We now prove that this condition is satisfied. From Equation (2), we have

$$\begin{aligned}
\int_{\Gamma_t} Y^{j,k}_{l_\pi,m_\pi} \circ \mathcal{P} &= \int_{\Gamma_t} Y^j_{l_\pi} \otimes Y^k_{m_\pi} \\
&= \int\limits_{\gamma^{(1,2)}\times\{x_1\}} Y^j_{l_\pi} \otimes Y^k_{m_\pi} + \int\limits_{\{x_1\}\times\gamma^{(1,2)}} Y^j_{l_\pi} \otimes Y^k_{m_\pi} \\
&\quad + \int\limits_{\gamma^{(n-1,n)}\times\{x_n\}} Y^j_{l_\pi} \otimes Y^k_{m_\pi} + \int\limits_{\{x_n\}\times\gamma^{(n-1,n)}} Y^j_{l_\pi} \otimes Y^k_{m_\pi} \\
&\quad + \sum_{i=2}^{n-1} \int_{\gamma^{(i-1,i+1)}\times\{x_i\}} Y^j_{l_\pi} \otimes Y^k_{m_\pi} \\
&\quad + \sum_{i=2}^{n-1} \int_{\{x_i\}\times\gamma^{(i-1,i+1)}} Y^j_{l_\pi} \otimes Y^k_{m_\pi} \\
&= Y^k_{m_\pi}(x_1) \int\limits_{\gamma^{(1,2)}} Y^j_{l_\pi} + Y^j_{l_\pi}(x_1) \int\limits_{\gamma^{(1,2)}} Y^k_{m_\pi} \\
&\quad + Y^k_{m_\pi}(x_n) \int\limits_{\gamma^{(n-1,n)}} Y^j_{l_\pi} + Y^j_{l_\pi}(x_n) \int\limits_{\gamma^{(n-1,n)}} Y^k_{m_\pi} \\
&\quad + \sum_{i=2}^{n-1} Y^k_{m_\pi}(x_i) \int\limits_{\gamma^{(i-1,i+1)}} Y^j_{l_\pi} + \sum_{i=2}^{n-1} Y^j_{l_\pi}(x_i) \int\limits_{\gamma^{(i-1,i+1)}} Y^k_{m_\pi}.
\end{aligned}$$

Then, using that γ and hence $\gamma^{(i,j)}$ is a t-design curve on $\mathbb{S}^2$ with $\ell(\gamma) = \ell(\gamma^{(i,j)})$, we have

$$\begin{aligned}
&\frac{1}{\ell(\gamma)} \int_{\Gamma_t} Y^{j,k}_{l_\pi,m_\pi} \circ \mathcal{P} \\
&\quad = Y^k_{m_\pi}(x_1) \int_{\mathbb{S}^2} Y^j_{l_\pi} + Y^j_{l_\pi}(x_1) \int_{\mathbb{S}^2} Y^k_{m_\pi} \\
&\qquad + Y^k_{m_\pi}(x_n) \int_{\mathbb{S}^2} Y^j_{l_\pi} + Y^j_{l_\pi}(x_n) \int_{\mathbb{S}^2} Y^k_{m_\pi} \\
&\qquad + \sum_{i=2}^{n-1} Y^k_{m_\pi}(x_i) \int_{\mathbb{S}^2} Y^j_{l_\pi} + \sum_{i=2}^{n-1} Y^j_{l_\pi}(x_i) \int_{\mathbb{S}^2} Y^k_{m_\pi} \\
&\quad = \sum_{i=1}^{n} Y^k_{m_\pi}(x_i) \int_{\mathbb{S}^2} Y^j_{l_\pi} + \sum_{i=1}^{n} Y^j_{l_\pi}(x_i) \int_{\mathbb{S}^2} Y^k_{m_\pi}.
\end{aligned}$$

Finally, using that $\{x_1, \ldots, x_n\}$ is a spherical t-design, we obtain

$$\begin{aligned}
\frac{1}{2n\ell(\gamma)} \int_{\Gamma_t} Y^{j,k}_{l_\pi,m_\pi} \circ \mathcal{P} &= \int_{\mathbb{S}^2} Y^j_{l_\pi} \int_{\mathbb{S}^2} Y^k_{m_\pi} \\
&= \int_{\mathbb{S}^2 \times \mathbb{S}^2} Y^j_{l_\pi} \otimes Y^k_{m_\pi}.
\end{aligned}$$

Since, by construction, $n \asymp t^2$ and $\ell(\gamma) \asymp t$, we have $\ell(\Gamma_t) = 2n\ell(\gamma) \asymp t^3$, as wanted. $\square$

IV. A CONCRETE EXAMPLE

In this section we present an explicit construction of a 2-design curve on $\mathrm{Gr}_{2,4}$ using the parametrization of $\mathrm{Gr}_{2,4}$ by $(\mathbb{S}^2 \times \mathbb{S}^2)/\pm 1$ given in (1).

Example IV.1. Consider the vertices of the regular 3-dimensional simplex, which are 2-design points,

$$x_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}, \; x_2 = \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}, \; x_3 = \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \; x_4 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

on $\mathbb{S}^2$ and the 2-design curve $\gamma : [0,1] \to \mathbb{S}^2$ given by

$$\gamma(s) = \begin{pmatrix} a\cos(2\pi s) + (1-a)\cos(6\pi s) \\ a\sin(2\pi s) - (1-a)\sin(6\pi s) \\ 2\sqrt{a(1-a)}\sin(4\pi s) \end{pmatrix},$$

where $a \approx 0.7778$ [14]. Since $\gamma(\frac{1}{4}) = (0,1,0)^\top = -\gamma(\frac{3}{4})$, we have two antipodal points, and, hence, there exists a rotation $O_{ij} \in \mathrm{O}(3)$ such that x_i and x_j lie on $\gamma^{(i,j)} := O_{ij}\gamma$. These points and curves are combined to the curve Γ on $\mathrm{Gr}_{2,4}$

$$\Gamma = \begin{Bmatrix} (\gamma^{(1,2)}, x_1), & (x_1, \gamma^{(1,2)}), \\ (\gamma^{(1,3)}, x_2), & (x_2, \gamma^{(1,3)}), \\ (\gamma^{(2,4)}, x_3), & (x_3, \gamma^{(2,4)}), \\ (\gamma^{(3,4)}, x_4), & (x_4, \gamma^{(3,4)}) \end{Bmatrix}. \tag{4}$$

A visualization of the curves and points on $\mathbb{S}^2$ is provided in Figure 1.

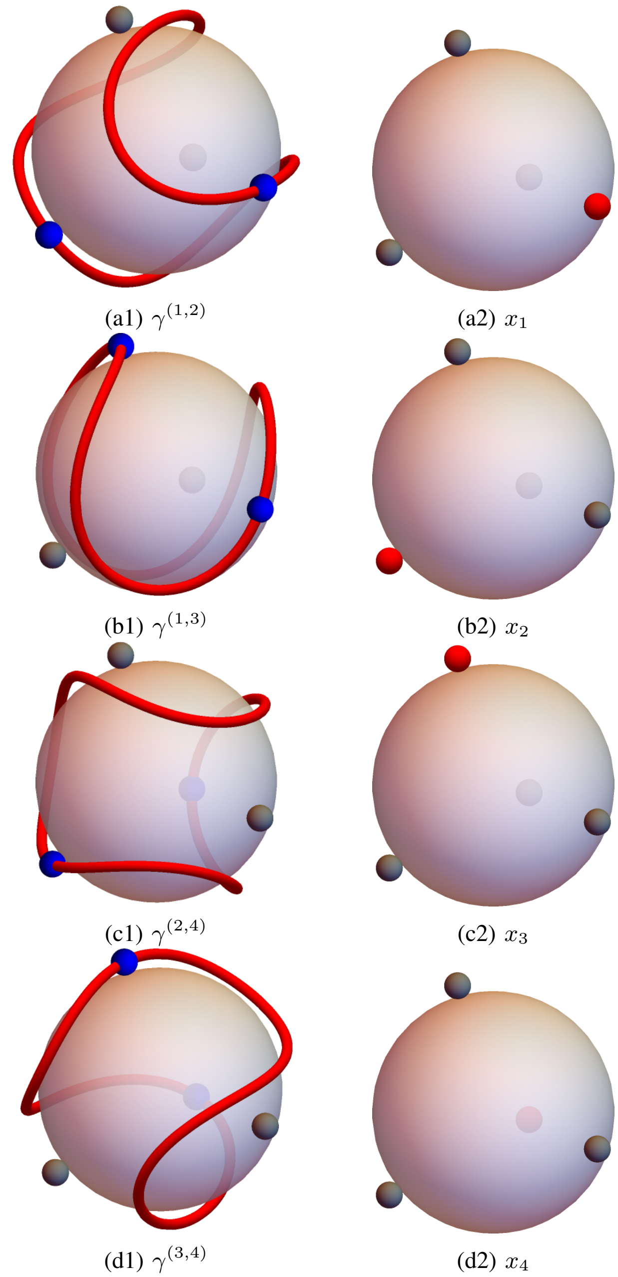

Fig. 1: Visualization of a curve on $\mathrm{Gr}_{2,4}$ using its parametrization as $(\mathbb{S}^2 \times \mathbb{S}^2)/\pm 1$. The image rows form curve-point pairs as in Equation (4). Each red 2-design curve $\gamma^{(i,j)}$, with $i, j \in \{1,2,3,4\}$ connecting the blue 2-design points x_i and x_j on the left is associated with a red 2-design point on the right.

V. CONCLUSION

In summary, we have proved the existence of asymptotically optimal t-design curves on the Grassmannian $\mathrm{Gr}_{2,4}$. The parametrization of $\mathrm{Gr}_{2,4}$ by $(\mathbb{S}^2 \times \mathbb{S}^2)/\pm 1$ was key. An open problem is the existence of asymptotically optimal t-design curves on other Grassmannians that lack such a uniquely beautiful parametrization.

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