NEW RESULTS FOR OPERATOR MICHELSON CONTRAST

Anonymous authors
Paper under double-blind review

ABSTRACT

We consider the generalization of the Michelson contrast for positive operators of countably decomposable W^*-algebras and prove its properties. In addition, we study how the inequalities characterizing traces interplay with the Michelson contrasts of operator variables.

Also, we developed a torch code for simulation modeling code to Monte-Carlo type I von Neumann algebras.

Let $\mathcal{A}$ denote some Banach $*$-algebras; then, $\mathcal{A}^{\text{sa}}, \mathcal{A}^{+}$ are its self-adjoint and positive parts, respectively. $\mathcal{A}^*$ is the conjugate space of continuous linear functionals. If $\mathcal{A}$ is von Neumann algebra, then $\mathcal{A}_*$ denotes its predual space. Additionally, $\mathcal{A}_*^+$, $\mathcal{A}^{*+}$ are the positive cones in $\mathcal{A}_*$ and $\mathcal{A}^*$, respectively. Tr denotes the canonical trace of $\mathbb{M}_n(\mathbb{C})$. By $C(H)$ and $B(H)$ we denote the ideal of compact operators and the algebra of bounded operators, respectively.

1 PRELIMINARIES

From the work (16) we know.

Let $\mathcal{A} = B(H)$, then the center $\mathfrak{C}(B(H))$ of $B(H)$ is equal to $\mathbb{C}\mathbf{1}$. Let us consider the function

$$\Delta(x) = \inf_{A \in \mathbb{R}^+} \left\{ \left\| \mathbf{1} - \frac{x}{A} \right\| \right\} \text{ for } x \in B(H)^+,$$

which illustrates how far the element x is from the central elements. If $x = \mathbf{1}$, then $\Delta(\mathbf{1}) = 0$ ($A = 1$) and $\Delta(\mathbf{0}) = 1$.

Proposition 1 *Let x be positive operator ($x \in B(H)^+$), then $\Delta(x) \leq 1$.*

Proposition 2 *Let x be positive non-invertible (singular) operator, then*

$$\Delta(x) = 1.$$

Corollary 1 *Let x be positive compact operator, then $\Delta(x) = 1$.*

Theorem 1 *Let x be invertible positive operator ($x \in B(H)^+$), with the inverse x^{-1}, then*

$$\Delta(x) = \frac{\|x\|\|x^{-1}\| - 1}{\|x\|\|x^{-1}\| + 1}. \tag{1}$$

Corollary 2 *Let $x \in B(H)^+$ be invertible element with the inverse element $x^{-1} \in B(H)$, then $\Delta(x) < 1$.*

Corollary 3 *Let the sequence x_n from $B(H)^+$ that converges to element $\mathbf{0} \neq x \in B(H)^+$ in terms of norm,*

$$\lim_n \Delta(x_n) = \Delta(x),$$

i.e. $\Delta : (B(H)^+ \setminus \{\mathbf{0}\}, \|\cdot\|) \mapsto [0, 1]$ is a continuous function.

Corollary 4 *If the sequence of operators is converging to a non-singular (invertible) operator, then the sequence contains not more than a finite quantity of non-invertible operators.*

Corollary 5 *For any x in $B(H)^+$ the following properties*

1.
$$\Delta(x) = \frac{\sup(\sigma(x)) - \inf(\sigma(x))}{\sup(\sigma(x)) + \inf(\sigma(x))};$$

2.
$$\Delta(x) = \frac{\sup\sigma(x)}{\sup\sigma(x) + \inf\sigma(x)} \left\| 1 - \frac{x}{\|x\|} \right\|.$$

hold.

The first equality states that the Δ is indeed the Michelson contrast.

Theorem 2 *Let $x, y \in B(H)^+$, then $\Delta(x+y) \leq \max\{\Delta(x), \Delta(y)\}$.*

2 Computational experiments

We conducted additional computational experiments on the inequalities violations for the higher dimensions compared to (16).

2.1 Gardner's inequality inspired simulation

In 1979 L.T. Gardner showed the inequality $|\varphi(X)| \leq \varphi(|X|)$ characterizes traces in C^*-algebras among all functionals, i.e.

Theorem 3 ((6), Theorem 1) *The finite traces on a C^*-algebra $\mathcal{A}$ are precisely those (positive) linear functionals φ on $\mathcal{A}$ which satisfy $|\varphi(x)| \leq \varphi(|x|)$ for all $x \in \mathcal{A}$.*

If φ is a tracial functional on the C^*-algebra $\mathcal{A}$, then the Gardner exponent shows the result for all elements of $X \in \mathcal{A}$ and, conversely, if for all $X \in \mathcal{A}$ is a Gardner quality indicator and φ is a positive functional, this functional is tracial.

Let $\mathcal{M}$ be a von Neumann algebra, the normal strongly semifinite weight φ ensures that for any φ-finite projections $P \in \mathcal{M}$, the Gardner equivalent ($|\varphi(X)| \leq \varphi(|X|)$) result for all $X = PX_0P$, where $X_0 \in \mathcal{M}$, then the weight is a trace.

In case if $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$, we have that for $\varphi := \operatorname{Tr}(A\cdot)$ the inequality must be violated for some $X \in \mathbb{M}_n(\mathbb{C})$, i.r. exists $X \in \mathbb{M}(\mathbb{R})$ such that $|\operatorname{Tr}(AX)| - \operatorname{Tr}(A|X|) > 0$.

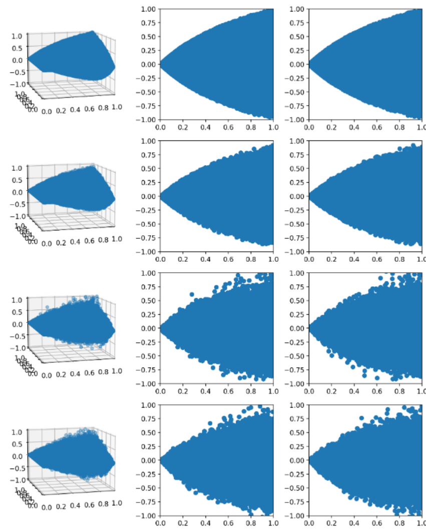

Figure 1: The scatter plots above are visualising results of simulations with $x = \Delta(X)$, $y = \Delta(Y)$, with $X \in \mathbb{M}_n(\mathbb{R}), Y \in \mathbb{M}_n(\mathbb{R}), \|X\| = \|Y\| = 1$ and $z = |\mathrm{Tr}(XY)| - \mathrm{Tr}(|X||Y|)$. The left column is a 3D scatter plot, the middle column is a plot of z vs. x and the right column is z vs. y. The rows correspond for 2, 3, 4 and 5-dimensional simulations respectively.

2.2 QUANTUM JENSEN-SHANNON DIVERGENCE

Let $X,\ Y \in \mathbb{M}_n^+(\mathbb{R})$ and $\mathrm{Tr}(X) = \mathrm{Tr}(Y) = 1$.

We call $S(X) := -\mathrm{Tr}(X \log_2(X))$ the von Neumann entropy, where $\log_2(X)$ is understood in the terms of functional calculus.

We define

$$QJSD(X,Y) := S\left(\frac{1}{2}(X+Y)\right) - \frac{1}{2}\left(S(X) + S(Y)\right)$$

following the [(29), (30),(31)].

In the following computational experiment we compare the Michelson Contrast $\Delta(XY)$ of the product of density matrices X and Y with the $\sqrt{QJSD(X,Y)}$.

We see the tendency that if we increase the dimension it seems that the following type inequalities

$$A \times \Delta(|XY|) \le \sqrt{\mathrm{QJSD}(X,Y)} \le B \times \Delta(|XY|)$$

occur. It seems logical since we know (26) where authors state the equivalence between Jensen–Shannon divergence and Michelson contrast for a continuous commutative distributions.

2.3 L_1 EQUALITY VIOLATION

From (18) we now that if $A \in \mathbb{M}_n^+(\mathbb{R})$ and $\mathrm{Tr}(|AXA|) = \mathrm{Tr}(A|X|A)$ for all $X \in \mathbb{M}_n^{sa}(\mathbb{R})$ then A is central.

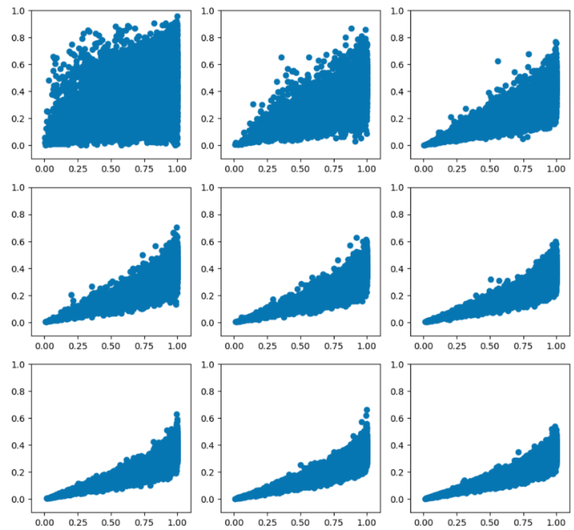

Figure 2: The scatter plots above are visualising results of simulations with $x = \Delta(|XY|)$, $y = \sqrt{QJDS(X,Y)}$, with $X,\ Y \in \mathbb{M}_n^+(\mathbb{R}), \mathrm{Tr}X = \mathrm{Tr}Y = 1$. The upper row corresponds to 2, 3 and 4-dimensional simulations, the middle row corresponds to 5, 6 and 7-dimensional simulations and the last row corresponds to 8, 9 and 10-dimensional simulations.

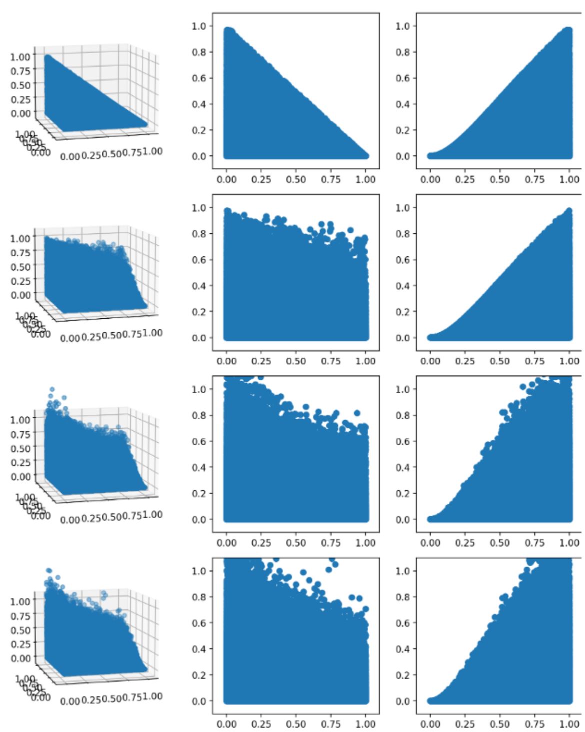

Figure 3: The scatter plots above are visualising results of simulations with $x = \Delta(|X|), y = \Delta(Y)$, with $X \in \mathbb{M}_n^{sa}(\mathbb{R}), Y \in \mathbb{M}_n^+(\mathbb{R}), \|X\| = \|Y\| = 1$ and $z = \mathrm{Tr}(Y|X|Y)| - \mathrm{Tr}(|YXY|)$. The left column is a 3D scatter plot, the middle column is a plot of z vs. x and the right column is z vs. y. The rows correspond for 2, 3, 4 and 5-dimensional simulations respectively.

3 LIMIT SIMULATION

The purely new results are obtained on the limits of sums of positive operators.

Let $X_1, \dots X_n \in \mathbb{M}_n^+(\mathbb{R})$. Consider a sequence $\Delta\left(\frac{1}{n}\sum_{k=1}^n X_k\right)$. We know, that $\Delta\left(\frac{1}{n}\sum_{k=1}^n X_k\right) = \Delta\left(\sum_{k=1}^n X_k\right)$.

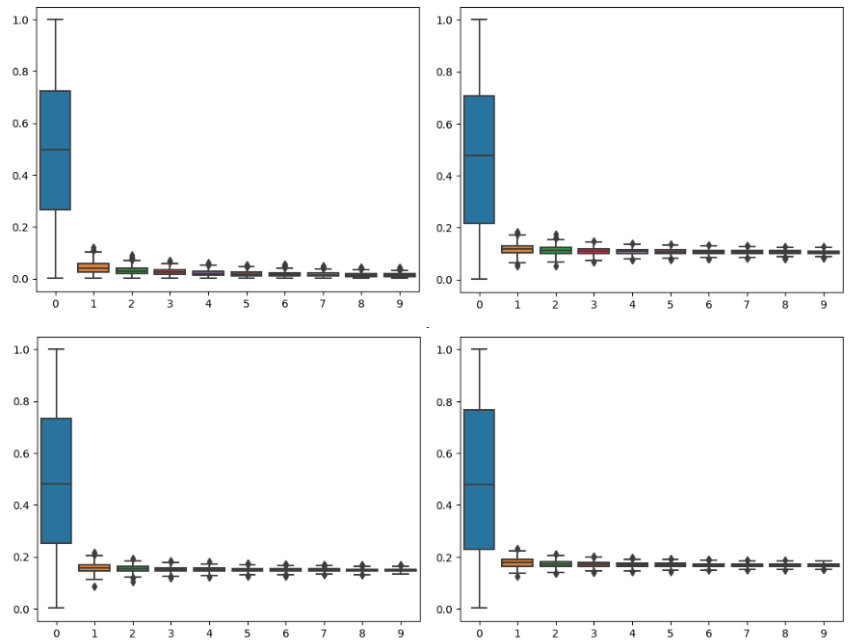

Figure 4: The upper left plot corresponds to 2-dimensional case, upper right to 3-dim, lefter bottom is 4-dim and righter bottom 5-dim.

We see that for the higher dimensions – the higher is the limit of the Michelson contrast of its sum. Yet, in any version it seems to be converging.

ACKNOWLEDGMENTS

This work is supported by Mathematical Center in Akademgorodok under agreement No.075-15-2022-281 (05.04.2022) with the Ministry of Science and Higher Education of the Russian Federation.

REFERENCES

[1] A. Bikchentaev, O.E. Tikhonov, "Characterization of the trace by monotonicity inequalities "– Linear Algebra and its Applications – 422, 1 – 274-278 (2007)

[2] A.M. Bikhentaev "Charaterization of traes on C^*-algebras: a survey", Gaspar, D. (ed.) et al., An operator theory summer. Proc. 23rd Internat. Conf. Operator Theory. Timisoara, Romania, June 29 July 4, 2010. Buharest: The Theta Foundation. Theta Series in Advanced Math. **13**, 1–12 (2012).

[3] A.M. Bikchentaev, "Commutativity of operators and characterization of traces on C*-algebras", Doklady Math. **87**, 79–82 (2013).

[4] A.M. Bikchentaev, "Commutation of projetions and haraterization of traes on von Neumann algebras. III ", Inter. J. Theor. Physis. **54** (12), 4482–4493 (2015).

[5] A.M. Bikchentaev, “Inequalities for determinants and characterization of the Sib. Math. J. **61** (2) 248–254 (2020).

[6] L.T. Gardner, “An inequality haraterizes the trae”, Canad. J. Math. **31** (6) 1322–1328. (1979).

[7] J. Dixmier, *Les C^*-algebras et leurs representations* (Gauther-Villars Editeur, Paris, 1969).

[8] D. Petz, J. Zemánek “Charaterizations of the trae”, Linear Algebra Appl. **111** 43–52 (1998).

[9] T. Sano, T. Yatsu “Charaterizations of the traial property via inequalities”, J. Inequal. Pure Appl. Math. **7** Artile 36 (2006).

[10] O.E. Tikhonov, “Subadditivity inequalities in von Neumann algebras and haraterization of traial funtionals”, Positivity **9** (2) 259–264 (2005).

[11] A.I. Stolyarov, O.E. Tikhonov, A.N. Sherstnev, “Characterization of Normal Traces on Von Neumann Algebras by Inequalities for the Modulus”, Math. Notes. **72** 411–416 (2002).

[12] K. Cho, T. Sano, “Young’s inequality and trae”, Linear Algebra Appl. **431** 1218–1222 (2009).

[13] A.M. Bikhentaev, O.E. Tikhonov, “Charaterization of the trae by Young’s inequality”, Journal of Inequalities in Pure and Applied Mathematis. **6** (2) Artile 49 (2005).

[14] A.M. Bikhentaev, O.E. Tikhonov “Charaterization of the trae by monotonicity inequalities”, Linear Algebra Appl. **422** (1) 274–278 (2007).

[15] D. Virosztek “Connections between centrality and local monotonicity of certain J. Math. Anal. Appl. **453** (1) 221–226 (2016).

[16] S.A. Abed, I.A. Nikolaeva, A.A. Novikov, Generalisation of Michelson Contrast for Operators and Its Properties. Lobachevskii J Math 45, 3835–3848 (2024).

[17] An. An. Novikov, O.E. Tikhonov, “Charaterization of entral elements of operator algebras by inequalities”, Lobahevskii Journal of Mathematis **36** (2) 208–210 (2015).

[18] A. Novikov, “L_1-space for a positive operator affiliated with von Neumann algebra”, Positivity **21** (1) 359–375 (2017). f

[19] A. Novikov, “C^*-algebra Positive Element Invertibility Criteria in Terms of L_1-norms Equivalence and L_∞-norms Equivalence”, Lobachevskii Journal of Mathematics **40** (5) 549–552 (2019),

[20] A. Novikov, Z. Eskandarian, Z. Kholmatova “Limits of the Banach spaces associated with positive operator affiliated with von Neumann algebra, which are neither purely projective nor purely inductive”, 2020 – `https:/arxiv.org/abs/1901.09276`,

[21] A.Michelson, Studies in Optics, U. of Chicago Press, 176 p. (1927).

[22] H. Kukkonen,J. Rovamo, K. Tiippana, R. Näsänen, “Michelson contrast, RMS contrast and energy of various spatial stimuli at threshold”, Vision Research – 33(10) – 1431-1436 (1993),

[23] J. L. Prince, J. M. Links Medical Imaging Signals and Systems, Pearson, 2006, 519p.

[24] D. Oliveira, E. Giusto, B. Baheri, Q. Guan, B. Montrucchio, P. Rech “A Systematic Methodology to Compute the Quantum Vulnerability Factors for Quantum Circuits”, 2021 – `https://arxiv.org/abs/2111.07085`;

[25] A.M. Bikchentaev, S.A. Abed, “Projections and Traces on von Neumann Algebras”, Lobachevskii Journal of Mathematics – 40 (9) – 1260–1267 (2019).

[26] V. Bruni, E. Rossi and D. Vitulano, “On the Equivalence Between Jensen–Shannon Divergence and Michelson Contrast,”– IEEE Transactions on Information Theory – 58 (7) – 4278-4288 (2012)

[27] A. Haun and E. Peli “Perceived Contrast of Complex Images”– Journal of vision – 13 – 3 (2013)

[28] C. B. Wiebel, M. Singh and M. Maertens “Testing the role of Michelson contrast for the perception of surface lightness”– Journal of Vision September – 16 – 17 (2016)

[29] A.S. Holevo, “Bounds for the quantity of information transmitted by a quantum communication channel”– Probl. Inf. Transm. – 9 – 177–183 (1975)

[30] A. Majtey, P. Lamberti, D. Prato,“Jensen-Shannon divergence as a measure of distinguishability between mixed quantum states”– Physical Review A. – V.72,I.5 – 052310 – 2005.

[31] J. Briet; P. Harremoes,of classical and quantum Jensen-Shannon divergence”– Physical Review A. – V.79,I.5 – 052311 – 2009