DYNAMIC SYSTEM RECONSTRUCTION FROM MULTI-VARIATE TIME SERIES VIA MULTILINEAR MAP AND TIME DELAY EMBEDDING

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ABSTRACT

This paper examines the properties of dynamic system reconstruction using time delay embedding and multilinear (tensor) algebra. The tensor dynamical system model for time series prevents the loss of higher-order information. The key idea is to use the tensor as a multilinear map from set phase spaces to one subspace. Due to the simplicity of the linear approach and linear dependencies between components, the results show that the method in several cases allows for a better reconstruction of the original attractor from an incomplete set of variables. A computational experiment was carried out on Lorenz attractor and measurements of the accelerometer of a mobile device with three classes of human movements.

1 INTRODUCTION

The role of multilinear algebra has been explored for the reconstruction of linear and nonlinear dynamics. For example, the segmentation problem for human activity recognition of a quasiperiodic time series with the study of the dynamic is solved in (Motrenko & Strijov, 2015; Ignatov & Strijov, 2016; Grabovoy & Strijov, 2020). This paper studies the dimensionality reduction method for time series with multilinear (tensor) algebra approaches (Kruppa, 2017; Chen et al., 2019; Chen, 2022). The resulting tensor dynamic system helps in various applications. The key idea is to use the tensor as a multilinear map from set phase spaces to one subspace. Usual separated methods and graph models cause a loss of higher-order information (Wolf et al., 2016) due to separate use of each multivariate time series.

In this (Kliková & Raidl, 2011) paper, the delay method is used to construct a phase space. The dimension of the phase space is the length of a vector with previous values in time. The resulting vector is a point in the phase space. As the dimension of the phase space increases, the distances between the points of the trajectory tend to the constant value. The proper dimension is significantly less than the dimension of the original phase space shown in (Motrenko & Strijov, 2015; Grabovoy & Strijov, 2020; Usmanova et al., 2020). That makes distances uninformative and unstable due to the curse of dimensionality (Powell, 2007). Also, it assumes that a more stable and robust model is possible in the subspace than in the original one. To prevents the curse of dimensionality various method is used. The most common method for such analysis is the principal component method (PCA). This is a linear method. To extend it, it is proposed to use the tensor method for characterizing the state of the multidimensional data.

The key contributions of the paper: the application of the previously proposed dynamic system model has been expanded in addition to the time delay embedding, and the computational experiment explores walking and squats (Ignatov & Strijov, 2016). The experiment was performed on data obtained from a mobile device's accelerometer (Malekzadeh et al., 2019). The main conclusions about the convergence and theoretical validity of the approach are the same as the conclusions in (Chen, 2022).

The paper is organized into three sections. In section 2, multilinear dynamical systems with time delay embedding are introduced. If the dependencies between variables are linear, the multilinear map method can effectively reconstruct an attractor of the dynamical system. In section 2.2, a tensor preliminaries review includes notations and various tensor products. In section 3 conclusion results with numerical examples are presented. Section 4 draws some conclusions and plans for future work.

2 Multilinear dynamical system

2.1 Time delay embedding

First described in (Packard et al., 1980), time delay embedding allows the augmentation of a scalar time series s_t into a higher dimension through the construction of delay vector $\boldsymbol{s}_t$ given as $\boldsymbol{s}_t = [s_t, s_{t-\tau}, ..., s_{t-(n-1)\tau}]$, where the embedding parameters τ is delay lag and n is embedding dimension. For this article let τ is always equal to 1. According to Taken's theorem, only one variable with time delays reconstructs a dynamical system. This augmentation with previous measures is also called the trajectory matrix. Thus trajectory matrix $\boldsymbol{S}$ of a time series $\boldsymbol{s}$ is defined as

$$\boldsymbol{S} = \begin{bmatrix} s_1 & \dots & s_n \\ s_2 & \dots & s_{n+1} \\ \vdots & \ddots & \vdots \\ s_k & \dots & s_N \end{bmatrix}^{\mathsf{T}} = [\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_k], \quad k = N - n + 1, \tag{1}$$

where n is the width of the window, N is the lengths of the time series$\boldsymbol{s}$.

The original phase space from time delay embedding has a high dimension. Thus the principal component analysis (PCA) is often used to reduce the dimensionality of the original phase space, by transforming an initial set of variables into a smaller one that is also called a subspace.

$$\boldsymbol{X} = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{S} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_k], \quad \boldsymbol{x}_i \in \mathbb{R}^p,$$

where $\mathbf{W}$ is the transformation matrix of the PCA algorithm. The number of selected components is p, corresponding to the largest eigenvalues.

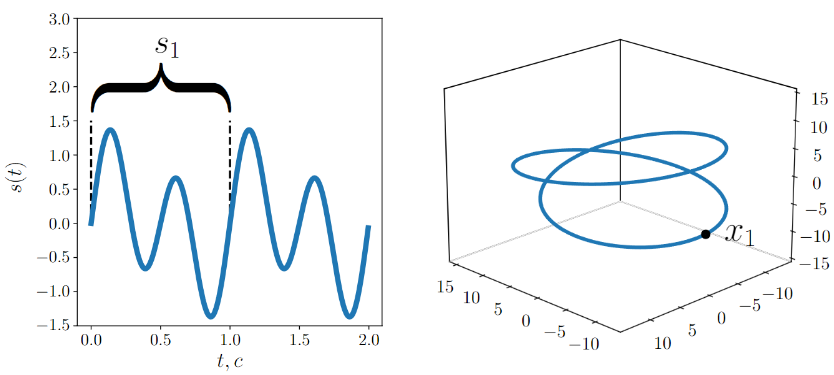

Figure 1: Segment of time series and phase trajectory with PCA in 3D.

Low-dimensional representation in phase space allows to use of more robust and simpler models and applications.

2.2 Tensor preliminaries

Tensors are multidimensional generalizations of matrices (a multidimensional array). The number of dimensions is the order of a tensor, and each dimension is called a mode. For example, a vector $\boldsymbol{v} \in \mathbb{R}^n$ has one mode, row, a matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ has two modes, rows and columns, a N-th order tensor $\mathbf{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ has N modes.

There are two tensors given in the form $\mathbf{A} \in \mathbb{R}^{I_1\times\ldots\times I_d}$ $\mathbf{B} \in \mathbb{R}^{J_1\times\ldots\times J_D}$, then $\mathbf{A} \circ \mathbf{B} \in \mathbb{R}^{I_1\times\ldots\times I_d\times J_1\times\ldots\times J_D}$ is called the outer product with elements:

$$(\mathbf{A} \circ \mathbf{B})_{i_1,\ldots,i_d,j_1,\ldots,j_D} = a_{i_1,\ldots,i_d} b_{j_1,\ldots,j_D}.$$

The n-mode multiplication of tensor $\mathbf{A} \in \mathbb{R}^{I_1\times\cdots\times I_N}$ and matrix $\boldsymbol{M} \in \mathbb{R}^{J\times I_n}$ is defined by $\mathbf{C} \in \mathbb{R}^{I_1\times\cdots\times I_{n-1}\times J\times I_{n+1}\times\cdots\times I_N}$ with the elements:

$$\mathbf{C} = \mathbf{A} \times_n^2 \boldsymbol{M} = \mathbf{A} \times_n \boldsymbol{M}, \quad c_{i_1,\ldots,i_{n-1},j,i_{n+1},\ldots,i_N} = \sum_{i_n=1}^{I_n} a_{i_1,\ldots,i_n,\ldots,i_N} m_{j,i_n}. \tag{2}$$

This n-mode multiplication $\mathbf{G}$ and matrices $\boldsymbol{M}^{(n)}$ can be extended to Tucker multiplication as

$$\mathbf{C} = \left[\mathbf{G}; \boldsymbol{M}^{(1)}, \ldots, \boldsymbol{M}^{(N)}\right] = \mathbf{G} \times_1 \boldsymbol{M}^{(1)} \times_2 \boldsymbol{M}^{(2)} \times_3 \cdots \times_N \boldsymbol{M}^{(N)}.$$

In case of tensor $\mathbf{A} \in \mathbb{R}^{I_1\times\cdots\times I_N}$ and vector $\boldsymbol{v} \in \mathbb{R}^{I_n}$ n-mode multiplication gives $\mathbf{C} \in \mathbb{R}^{I_1\times\cdots\times I_{n-1}\times I_{n+1}\times\cdots\times I_N}$:

$$\mathbf{C} = \mathbf{A} \times_n^1 \boldsymbol{v} = \mathbf{A} \bar{\times}_n \boldsymbol{v}, \quad c_{i_1,\ldots,i_{n-1},i_{n+1},\ldots,i_N} = \sum_{i_n=1}^{I_n} a_{i_1,\ldots,i_n,\ldots,i_N} v_{i_n} \tag{3}$$

Formally, mode products for a matrix (2) and a vector (3) are the same operations, but in this paper, for simplicity, only one notation (2) is used for both operations. It is implied that in the case of a matrix the second mode is used, in the case of a vector only one first mode of the vector is used.

2.3 Multilinear dynamical system for multivariate time series

This paper discusses the topic of a dynamic system, which is given by

$$\boldsymbol{v}_{t+1} = \mathbf{A} \times_1 \boldsymbol{v}_t \times_2 \boldsymbol{v}_t \times_3 \ldots \times_{k-1} \boldsymbol{v}_t, \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n\times\ldots\times n}$ is a dynamic tensor (multilinear map), and $\boldsymbol{v} \in \mathbb{R}^n$ is the state variable. It is assumed that the tensor $\mathbf{A}$ has multilinear properties in the sense of the definition of algebraic multilinearity.

Vectors of the state variables are the values of some measured quantities at time t. It is assumed that these quantities completely describe the state of the dynamic system. For example, in the case of a mathematical pendulum, these quantities are velocity and acceleration. With certain restrictions, it is possible to completely reconstruct dynamically using only these variables.

This paper proposes to construct a map into a low dimensional subspace, i.e. dimensionality reduction, instead of reconstructing the dynamics itself, as some evolution rule of a system. The evolution rule is a function that describes what future states follow from the current state of the dynamical system.

This map is used in further models for anomaly detection, classification, and signal phase extraction (in the case of periodic time series). Thus, the equation is modified as follows:

$$\boldsymbol{x}_t = \mathbf{A} \times_1 \boldsymbol{s}_t \times_2 \boldsymbol{s}_t \times_3 \ldots \times_{k-1} \boldsymbol{s}_t, \tag{5}$$

where $\boldsymbol{s}_t = [s_t, s_{t-1}, \ldots, s_{t-n}]$ is a vector from time series $\boldsymbol{s}$ with n delays, $\boldsymbol{x}_t \in \mathbb{R}^p$ is a vector with $p \ll n$ that represent system in its phase space.

In the case of (5) only univariate time series is used. It can be extended to the case of multivariate time series. In general, finding any solution to this problem for multivariate time series is challenging due to the nonlinear nature of the signal. However, in the case of linear dependencies between measurements it can be simply use in the model simultaneously.

Without loss of generality, let multivariate time series come from a triaxial accelerometer. Thus, there are three axes. Let $\boldsymbol{s}_\text{x}, \boldsymbol{s}_\text{y}, \boldsymbol{s}_\text{z}$ be the time series of acceleration along each of the axes. A signal from each axis separately restores the attractor of the dynamic system according to Taken's theorem using time delay embedding as (1). Also, there are linear maps between each variable using rotation and stretching (i.e. affine transformations)

$$\boldsymbol{S}_{\mathrm{x}} = \boldsymbol{I}^{\mathsf{T}}\boldsymbol{S}_{\mathrm{x}}, \quad \boldsymbol{S}_{\mathrm{x}} = \boldsymbol{W}_{\mathrm{y}}^{\mathsf{T}}\boldsymbol{S}_{\mathrm{y}}, \quad \boldsymbol{S}_{\mathrm{x}} = \boldsymbol{W}_{\mathrm{z}}^{\mathsf{T}}\boldsymbol{S}_{\mathrm{z}}, \tag{6}$$

where $\boldsymbol{S}_{\mathrm{x}}, \boldsymbol{S}_{\mathrm{y}}, \boldsymbol{S}_{\mathrm{z}}$ are trajectory matrices in initial phase space, $\boldsymbol{W}_{\mathrm{y}}^{\mathsf{T}}, \boldsymbol{W}_{\mathrm{z}}^{\mathsf{T}}$ are the transformation matrices, $\boldsymbol{I}$ is an identity matrix. Thus, the multilinear model is modified as follows:

$$\boldsymbol{x}_t = \mathbf{A} \times_1 (\boldsymbol{I}^{\mathsf{T}}\boldsymbol{s}_{\mathrm{x}\ t}) \times_2 (\boldsymbol{W}_{\mathrm{y}}^{\mathsf{T}}\boldsymbol{s}_{\mathrm{y}\ t}) \times_3 (\boldsymbol{W}_{\mathrm{z}}^{\mathsf{T}}\boldsymbol{s}_{\mathrm{z}\ t}) = \hat{\mathbf{A}} \times_1 \boldsymbol{s}_{\mathrm{x}\ t} \times_2 \boldsymbol{s}_{\mathrm{y}\ t} \times_3 \boldsymbol{s}_{\mathrm{z}\ t}, \tag{7}$$

where $\hat{\mathbf{A}} = \mathbf{A} \times_1 \boldsymbol{I}^{\mathsf{T}} \times_2 \boldsymbol{W}_{\mathrm{y}}^{\mathsf{T}} \times_3 \boldsymbol{W}_{\mathrm{z}}^{\mathsf{T}}$is modified dynamic tensor, $\boldsymbol{s}_{\mathrm{x}\ t}, \boldsymbol{s}_{\mathrm{y}\ t}, \boldsymbol{s}_{\mathrm{z}\ t}$ are state variable vectors from each axis at time t.

The tensor $\hat{\mathbf{A}}$ allows to select not only the main components, as in case of PCA for univariate time series, but filter them according to multilinear dependencies with other time series.

3 Experiment

3.1 The Lorenz system

This example uses the Lorenz attractor to analyse reconstructed phase spaces. It is defined by a system of differential equations of the form

$$\begin{cases} \dfrac{dx}{dt} = \sigma(y - x), \\ \dfrac{dy}{dt} = x(r - z) + y, \\ \dfrac{dz}{dt} = xy - bz \end{cases} \tag{8}$$

with the following parameter: $\sigma = 10$, $r = 28$, $b = 8/3$. Then the phase trajectory has the form shown in Figure 2. It is also shown reconstruction scheme and various state spaces. For comparison, an attractor is shown, that is obtained by the time delay embedding method.

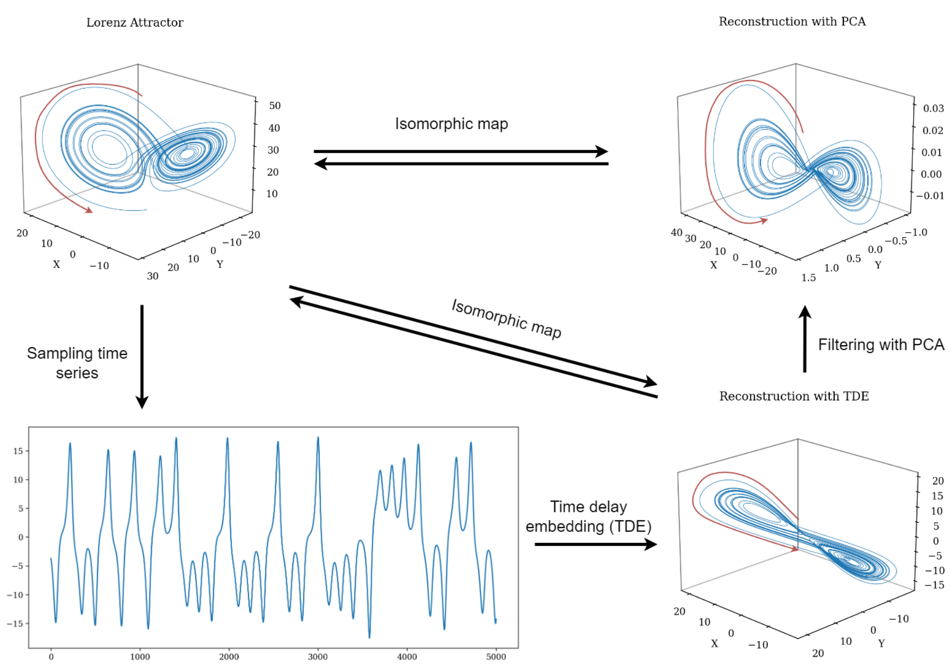

Figure 2: Schematic of the embedding process and the relationship between its reconstruction

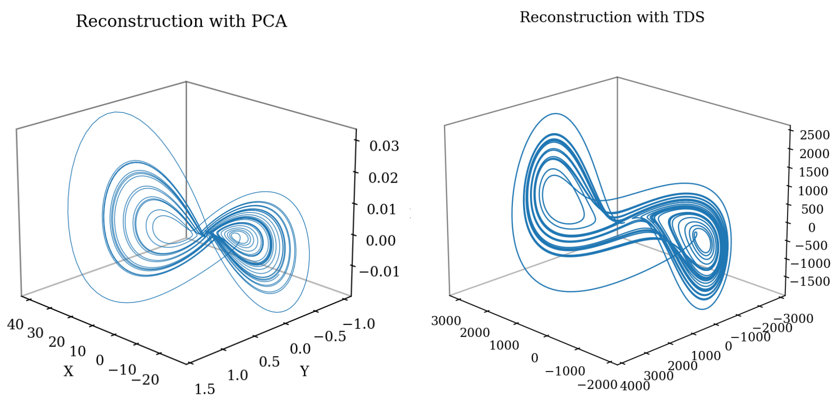

Figure 3: Phase trajectory of Lorenz linear system reconstructed with PCA (left) and tensor dynamical system (TDS) approach (right)

As shown in Figure(3), additional information in the multilinear model reconstructs the shape of the phase trajectory similar to PCA. Both methods qualitatively restore the *petals*, maintaining repeating dynamics in two different modes of the original attractor. This result is obtained due to the noise-free time series and a sufficient length of history in each of the methods.

3.2 Human movement dataset

The purpose of the computational experiment is to analyze the quality of attractor reconstruction and compare it with the PCA as a basic linear approach to real data. The experiment was performed on data obtained from the accelerometer of a mobile device (Malekzadeh et al., 2019).

This dataset includes time-series data generated by accelerometer and gyroscope sensors. It is collected with an iPhone 6s kept in the participant's front pocket using SensingKit. All data collected in 50Hz sample rate. A total of 24 participants in a various of gender, age, weight, and height performed six activities in the same environment and conditions: downstairs, upstairs, walking, jogging, sitting, and standing.

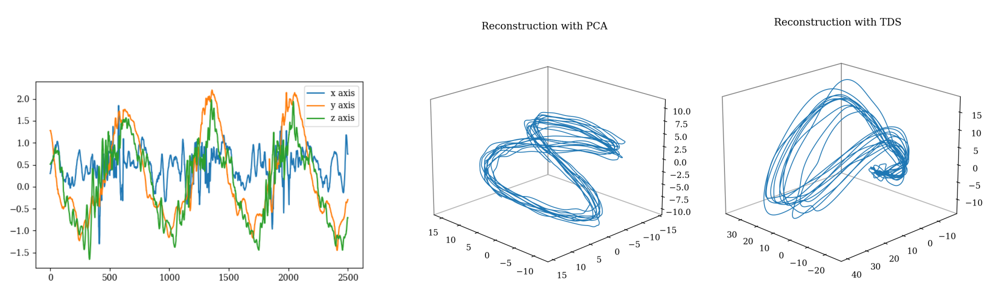

Figure 4: Time series sample (left), reconstruct phase space with PCA (center), TDS (right) of activity jogging

As shown in Figure(4), a complex phase trajectory is reconstruct with PCA. In particular, there are several intersections. Thus, different states of the system correspond to the same region of the phase space. Using a multilinear model allows to partially solve the intersection problem from one intersection to nearly zero. In comparison with Figure(5), it is clear that instead of two intersections, more complex attractor is reconstruct. This behavior occurs due to non-linear dependencies in the data between x, y, z axis of accelerometer.

In another case, in Figure(6) it is shown that the application of the tensor method allows for a better reconstruction of the structure. Thus, the appearance of a periodic component with similar behavior is clearly visible. It also allows to reduce the number of intersections to one.

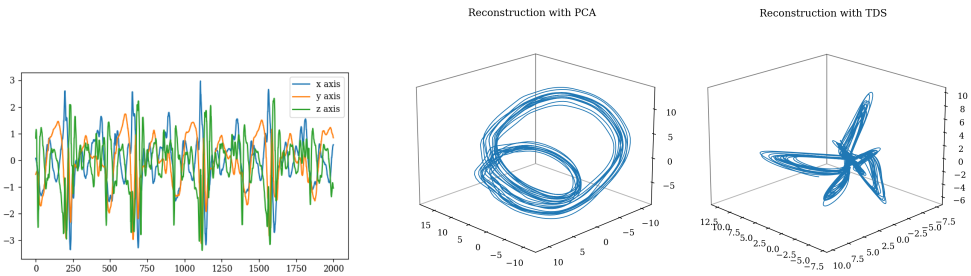

Figure 5: Time series sample (left), reconstruct phase space with PCA (center), TDS (right) of activity walk

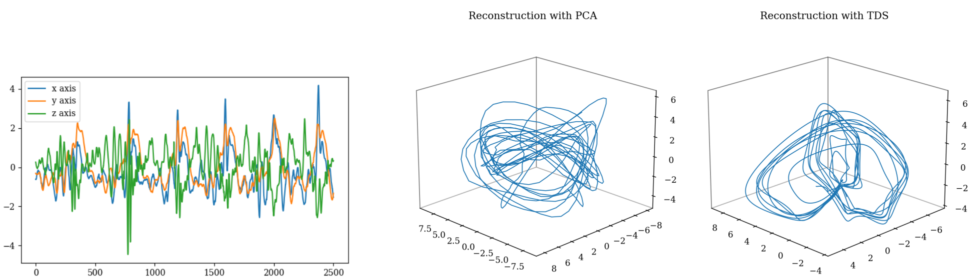

Figure 6: Time series sample (left), reconstruct phase space with PCA (center), TDS (right) of activity upstairs

Thus, on several real time series it was shown that in the case of a linear dependence, the proposed method allows to obtain more interpretable results and reduces the number of intersections. In the case of clearly nonlinear dependences, the result becomes complex.

4 Conclusion

This paper solves the problem of dimensionality reduction for the phase reconstruction of multivariate time series. The result of the work is a generalization tensor dynamical system in the case of multivariate time series. This article develops the work which investigates a tensor dynamic system with a univariate time series. Proposed method retains the required properties and reproduces the type of the original attractor with a high accuracy in linear case.

A computational experiment was performed on the Lorenz attractor and accelerometer data of human motion. Classical linear approaches and the proposed method were compared.

There are three main directions for future work. The first is to take into account nonlinear relationships through, for example, autoencoders and nonlinear activation functions. The second is to increase computational efficiency with a more complex approach which will use not all available components, but those with the highest correlation in the multivariate time series. The third is to optimize the construction of the tensor representation due to the exponential growth of the number of parameters in the case of a larger number of time series. This optimization will be important in analyzing such higher-order dynamical systems for various applications.

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