Generalized Average Sampling over Quasi Shift Invariant Spaces

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***Abstract*—We analyse the generalized average sampling and reconstruction for quasi shift invariant spaces with frame generators. We show that for any functions in the quasi shift invariant space can be uniquely reconstructed from its generalized average samples.**

***Index Terms*—Generalized Average Sampling, Quasi Shift Invariant Spaces.**

I. Introduction

Sampling play an important role in digital communications and signal processing, which provides theoretical basis for signal analysis. A fundamental challenge in signal processing is representing a continuous signal using its discrete samples. The sampling theorems provide the reconstruction formulas and hence such theorems become the very powerful tool in the field of signal and image processing. The famous Shannon sampling theorem states that finite energy bandlimited signals are completely characterized by their sample values [2]–[4], [6], [17]. Furthermore, Shannon gave the following reconstruction formula

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\Omega}\right) \frac{\sin(\Omega x - k\pi)}{\Omega x - k\pi}.$$

In Shannon reconstruction formula, sinc function defined by $\operatorname{sinc}(x) = \dfrac{\sin x}{x}$ is used as the kernel function and is in fact a scaling function of a multiresolution analysis used in wavelet theory. In practical situation, all the signals need not be bandlimited. For rapid communications the $sinc$ function itself is not very suitable. Many generalizations of the classical Shannon sampling theorem have been proposed. Various researchers in this field [16], [24], [33], [36] have discussed the extension of classical Shannon sampling theorem to wavelet subspaces. Studies mentioned in [2]–[4], [6], [17] have carried out the analysis of sampling procedure in shift-invariant spaces and spline spaces.

Consideration of sampling and reconstruction problem in shift invariant spaces is deemed to be a recent trend to handle sampling. As a result, modern analysis has a special role in these spaces. In continuation, wavelets, approximation theory and finding efficient prefilters in wavelet theory provide an opportunity for the usage of these kinds of sampling in shift invariant spaces. The uniform and non uniform sampling and reconstruction problem over shift invariant subspaces like wavelet subspaces, spline spaces and reproducing kernel Hilbert spaces have been analysed in [1], [2], [17], [24], [29], [33].

Average sampling is motivated by realistic needs. In [9], [17], [30], [32], the authors presented an average sampling theorem for bandlimited signals, generalizing the Shannon sampling theorem. Several authors have studied the average sampling theorem for spline subspaces with both uniform and non-uniform sampling points, as presented in [6], [30], [31]. Furthermore, uniform and non-uniform average sampling and reconstruction problems in general shift-invariant spaces are studied in [3], [7], [26], [32]. In [5], [20], [27], an average sampling theorem for reproducing kernel Hilbert spaces and finitely generated shift-invariant spaces is presented.

As a generalization of shift invariant space, we consider the quasi shift invariant space $V_X(\varphi)$ is defined as,

$$V_X(\varphi) = \left\{ \sum_{k\in\mathbb{Z}} c_k \varphi(t - x_k) : \{c_k\} \in \ell^2 \right\},$$

where $X = \{x_k : k \in \mathbb{Z}\}$ is any strictly increasing sequence of $\mathbb{R}$ and $\varphi(t) \in L^2(\mathbb{R})$. Quasi shift-invariant spaces have been studied by several authors in [37] - [40] and are widely utilized in approximation theory [40]. The authors in [38] investigated the sampling problem in quasi shift-invariant spaces generated by a function φ belonging to a class of totally positive functions of finite type. Recently, Anuj and Sivananthan [41] studied sampling and average sampling problem in quasi shift invarint spaces $V_X(\varphi)$.

As a generalization of average sampling, we consider generalized average sampling in quasi shift-invariant spaces. Averages are not taken from a signal itself but from its channeled version by some linear time invariant system, which is known as generalized average sampling. In 1977, Papoulis [28] studied, how to reconstruct a bandlimited function f from the sample values $\mathcal{L}_m f$, where $\mathcal{L}_m f$ are time invariant operators. These types of samples are known as generalized samples, also referred to as channel sampling. In [8], [19], [34], [35], the concept of generalized sampling has been extended to non-bandlimited functions.

The studies in [10]–[13], [22], [23] explores regular and irregular generalized sampling theorems over a shift-invariant space generated by a continuous Riesz generator. In particular, multichannel sampling theorems in shift-invariant spaces generated by a Riesz generator have been analyzed in [10], [13], [18], [22], [23]. As an extension of average sampling, we consider the generalized average sampling expansion (i.e., $\langle \mathcal{L}[f], u_n \rangle$, where $u_n(t)$ are averaging functions) for quasi shift invariant spaces. In this paper, we discuss the generalized average sampling problem for quasi shift invariant spaces.

II. Preliminaries

In numerous practical scenarios, the sampled values do not precisely correspond to the actual values of f at the sample points. Hence, we consider the local averages of f near x_k is

$$\langle f, u_k \rangle = \int f(x) u_k(x) dx,$$

with respect to the collection of averaging functions $u_k(x)$, $k \in \mathbb{Z}$. It is assumed that the averaging functions $u_k(x)$ are nonnegative and $u_k \in L^2(\mathbb{R})$, $\text{supp}(u_k) \subset [x_k - a, x_k + b]$ ($a, b \geq 0$ and $a + b > 0$), and $\int_{\mathbb{R}} u_k(x) dx = 1$.

Let $\mathcal{L}[\cdot]$ be a linear time invariant system with an impulse response $l(t)$ such that

(i) $l(t) = \delta(t + a)$, $a \in \mathbb{R}$ or

(ii) $l(t) \in L^2(\mathbb{R})$ or

Let $\psi(t) := \mathcal{L}[\varphi](t) = (\varphi \star l)(t)$.

Definition 2.1: Let $X = \{x_k : k \in \mathbb{Z}\}$ be a strictly increasing sequence of $\mathbb{R}$. A quasi shift-invariant space is defined as a space of functions on $\mathbb{R}$, represented in the form:

$$V_X(\varphi) = \left\{ f(t) = \sum_{k \in \mathbb{Z}} c_k \varphi(t - x_k) : \{c_k\} \in \ell^2(\mathbb{Z}) \right\},$$

where $\varphi \in L^2(\mathbb{R})$ satisfies Riesz basis condition.

If $X = \mathbb{Z}$, then quasi shift invariant space $V_X(\varphi)$ is called a shift invariant space V.

Definition 2.2: The Wiener amalgam space $W(L^p)$ for $p \in [1, \infty]$, consists of all measurable functions f such that

$$||f||_{W(L^p)} := \left(\sum_{k \in \mathbb{Z}} \sup_{x \in [0,1]} |f(x + k)|^p \right)^{\frac{1}{p}} < \infty.$$

Proposition 2.1 ([21], p. 184): If $f(x)$ is differentiable on $[a, b]$, f, $f^{'} \in L^2[a, b]$, and $f(a)f(b) = 0$, then

$$\int_a^b |f(x)|^2 dx \leq \frac{4}{\pi^2}(b - a)^2 \int_a^b |f^{'}(x)|^2 dx.$$

Proposition 2.2 ([25], p. 303-304): Let f be an integrable function on $[a, b]$ and let $F(x) = \int_a^x f(t) dt$, $|F(x)| \leq M(x - a)$ for $a < x \leq b$ ($M > 0$); furthermore, let g be a nonnegative, nonincreasing and integrable function. Then

$$\left| \int_a^b f(x) g(x) dx \right| \leq M \int_a^b g(x) dx.$$

Lemma 2.1 ([41], Lemma 2.1): Let X be a translation set and $\varphi \in L^2(\mathbb{R})$ such that $\{\varphi(. - x_k) : k \in \mathbb{Z}\}$ be a frame for a closed subspace V of $L^2(\mathbb{R})$ with bounds C and D. If φ is a continuous function such that $\sum_{k \in \mathbb{Z}} |\varphi(x - x_k)|^2 \leq A < \infty$, then for any frame $\{f_k(x) : k \in \mathbb{Z}\}$ of V, $\sum_{k \in \mathbb{Z}} |f_k(x)|^2$ is bounded on $\mathbb{R}$.

Now we derive the main results of this paper.

III. Generalized Average Sampling for quasi shift invariant space

Theorem 3.1: Let $\varphi \in W(C, L^1)$ such that $\hat{\varphi}(x) \geq 0$ for every $x \in \mathbb{R}$ and X is a translation set such that $\{\varphi(. - x_k) : k \in \mathbb{Z}\}$ be a frame for $V_X(\varphi)$ with bounds C_1 and C_2. Let us assume φ is locally absolutely continuous and $\varphi^{'} \in L^2(\mathbb{R})$, such that

$$\max \left(\sup_k \sum_{j \in \mathbb{Z}} |\psi^{'}(x_k + t - x_j)|, \sup_j \sum_{k \in \mathbb{Z}} |\psi^{'}(x_k + t - x_j)| \right) \leq K$$

a.e. for some K and there exist a positive constant R and a constant $E > \frac{2}{\sqrt{3}}$ such that $\hat{\psi}(\xi) \geq R$ for all $\xi \in \left[-\frac{E}{2r}, \frac{E}{2r}\right]$, where $r = \inf_{k \in \mathbb{Z}}(x_{k+1} - x_k) > 0$. Let $\{u_k(x) : k \in \mathbb{Z}\}$ be a sequence of averaging functions such that $supp(u_k) \subset [x_k - a, x_k + b]$, and $\delta := \max\{a, b\}$. If the real sequence $\{x_k\}$ satisfy $0 < \alpha \leq x_{k+1} - x_k \leq \beta < 1$ for some constants α and β and $\sqrt{\delta(a + b)} < \frac{B_1}{K}$, then there is a frame $\{S_k(x) : k \in \mathbb{Z}\}$ for V_0 such that for any $f \in V_X(\varphi)$,

$$f(x) = \sum_{k \in \mathbb{Z}} \langle \mathcal{L}[f], u_k \rangle S_k(x), \tag{III.1}$$

where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$.

Proof 3.1: For any $f \in V_X(\varphi)$ can be written as

$$f(t) = \sum_{k \in \mathbb{Z}} c_k \varphi(t - x_k)$$

Hence

$$\begin{aligned} \mathcal{L}[f](t) &= \sum_{k \in \mathbb{Z}} c_k \mathcal{L}[\varphi](t - x_k) \\ &= \sum_{k \in \mathbb{Z}} c_k \psi(t - x_k), \end{aligned}$$

where $\psi(t) := \mathcal{L}[\varphi](t) = (\varphi \star l)(t)$.

As $supp(u_k(t)) \subset [x_k - a, x_k + b]$ and $\int_{x_k - a}^{x_k + b} u_k(t) dt = 1$,

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} |\langle \mathcal{L}[f], u_k \rangle - \mathcal{L}[f](x_k)|^2 \\ &= \sum_{k \in \mathbb{Z}} \left| \int_{x_k - a}^{x_k + b} (\mathcal{L}[f](t) - \mathcal{L}[f](x_k)) u_k(t) dt \right|^2 && \text{(III.2)} \\ &\leq \sum_{k \in \mathbb{Z}} \int_{x_k - a}^{x_k + b} |\mathcal{L}[f](t) - \mathcal{L}[f](x_k)|^2 u_k(t) dt && \text{(III.3)} \\ &= \sum_{k \in \mathbb{Z}} \int_{-a}^{b} |\mathcal{L}[f](x_k + t) - \mathcal{L}[f](x_k)|^2 dt. && \text{(III.4)} \end{aligned}$$

Now

$$\begin{aligned}
& |\mathcal{L}[f](x_k+t)-\mathcal{L}[f](x_k)| \\
= \;& \left|\int_0^t \mathcal{L}[f^{'}](x_k+s)ds\right| \\
\leq \;& \sqrt{t}\left(\int_0^t \left|\mathcal{L}[f^{'}](x_k+s)\right|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence we get for any $-a \leq t \leq b$,

$$\begin{aligned}
& |\mathcal{L}[f](x_k+t)-\mathcal{L}[f](x_k)|^2 \\
\leq \;& \delta \int_{-a}^{b} \left|\mathcal{L}[f^{'}](x_k+s)\right|^2 ds. \qquad \text{(III.5)}
\end{aligned}$$

From equations III.4 and III.5, we obtain

$$\begin{aligned}
& \sum_{k\in\mathbb{Z}} |\langle \mathcal{L}[f], u_k\rangle - \mathcal{L}[f](x_k)|^2 \\
\leq \;& \sum_{k\in\mathbb{Z}} \delta \int_{-a}^{b} \left|\mathcal{L}[f^{'}](x_k+s)\right|^2 ds \\
= \;& \delta \int_{-a}^{b} \sum_{k\in\mathbb{Z}} \left|\sum_{j\in\mathbb{Z}} c_n \psi^{'}(x_k+s-x_j)\right|^2 ds \\
\leq \;& \delta \int_{-a}^{b} \sum_{k\in\mathbb{Z}} \left[\sum_{j\in\mathbb{Z}} |c_n|^2 \left|\psi^{'}(x_k+s-x_j)\right| \right. \\
& \left. \sum_{j\in\mathbb{Z}} \left|\psi^{'}(x_k+s-x_j)\right|\right] ds \\
= \;& K^2 ||c||^2 \delta [a+b].
\end{aligned}$$

Let us consider the bi-infinite matrix with entries $\Psi := \psi_{jk} = \psi(x_j - x_k)$ for $j,k \in \mathbb{Z}$. Since

$$\sum_{k\in\mathbb{Z}} |\psi_{jk}| = \sum_{k\in\mathbb{Z}} |\psi(x_j - x_k)| \leq \left(1+\lfloor\frac{1}{r}\rfloor\right) ||\psi||_{W(C,L^1)} < \infty$$

for any $j \in \mathbb{Z}$. Therefore

$$||\Psi c||_{l^2(\mathbb{Z})} \leq B_2 ||c||_{l^2(\mathbb{Z})} \; for all c = (c_k) \in l^2(\mathbb{Z}),$$

where $B_2 = \left(1+\lfloor\frac{1}{r}\rfloor\right)||\psi||_{W(C,L^1)} > 0$. Let us consider

$$\begin{aligned}
& |\langle \Psi c, c\rangle| \\
= \;& \sum_{j,k\in\mathbb{Z}} \bar{c_j} c_k \psi(x_j - x_k) \\
= \;& \sum_{j,k\in\mathbb{Z}} \bar{c_j} c_k \int_{\mathbb{R}} \hat{\psi}(\xi) e^{-2\pi i (x_j - x_k)\xi} d\xi \\
= \;& \int_{\mathbb{R}} \hat{\psi}(\xi) \sum_{j,k\in\mathbb{Z}} \bar{c_j} c_k e^{-2\pi i (x_j - x_k)\xi} d\xi \\
= \;& \int_{\mathbb{R}} \hat{\psi}(\xi) \left|\sum_{j\in\mathbb{Z}} c_j e^{-2\pi i x_j \xi}\right|^2 d\xi \\
\geq \;& \int_{|\xi|\leq \frac{E}{2r}} \hat{\psi}(\xi) \left|\sum_{j\in\mathbb{Z}} c_j e^{-2\pi i x_j \xi}\right|^2 d\xi \\
\geq \;& R \int_{|\xi|\leq \frac{E}{2r}} \left(1-\frac{2r|\xi|}{E}\right) \left|\sum_{j\in\mathbb{Z}} c_j e^{-2\pi i x_j \xi}\right|^2 d\xi \\
\geq \;& \frac{R}{2\pi} \int_{|\xi|\leq \frac{E}{2r}} \left(1-\frac{2r|\xi|}{E}\right) \sum_{j,k\in\mathbb{Z}} \bar{c_j} c_k e^{-i(x_j - x_k)\xi} d\xi \\
= \;& \frac{R}{2\pi} \sum_{j,k\in\mathbb{Z}} \bar{c_j} c_k (2) \int_0^{\frac{E}{2r}} \left(1-\frac{2r|\xi|}{E}\right) \cos(x_j - x_k)\xi d\xi \\
= \;& \frac{RE}{\pi r} \sum_{j,k\in\mathbb{Z}} \bar{c_j} c_k \left(sinc\left(\frac{(x_j - x_k)E}{2r}\right)\right)^2 \\
= \;& \frac{RE}{\pi r} \left[\sum_{j\in\mathbb{Z}} |c_j|^2 + \sum_{j\in\mathbb{Z}} \sum_{k\neq j} \bar{c_j} c_k \left(sinc\left(\frac{(x_j - x_k)E}{2r}\right)\right)^2\right] \\
\geq \;& \frac{RE}{\pi r} \left[|||c||^2_{l^2(\mathbb{Z})} - \sum_{j\in\mathbb{Z}} \sum_{k\neq j} \bar{c_j} c_k \left(sinc\left(\frac{(x_j - x_k)E}{2r}\right)\right)^2\right] \\
\geq \;& \frac{RE}{\pi r} \left[|||c||^2_{l^2(\mathbb{Z})} - \left(\sum_{j\in\mathbb{Z}} |c_j|^2 \sum_{k\neq j} \left(sinc\left(\frac{(x_j - x_k)E}{2r}\right)\right)^2\right)^{\frac{1}{2}} . \right. \\
& \left. \left(\sum_{k\in\mathbb{Z}} \sum_{j\neq k} |c_k|^2 \left(sinc\left(\frac{(x_j - x_k)E}{2r}\right)\right)^2\right)^{\frac{1}{2}}\right] \\
\geq \;& \frac{RE}{\pi r} \left[|||c||^2_{l^2(\mathbb{Z})} - \left(\sum_{j\in\mathbb{Z}} |c_j|^2\right)^{\frac{1}{2}} \cdot \frac{4}{3E^2} \left(\sum_{k\in\mathbb{Z}} |c_k|^2\right)^{\frac{1}{2}}\right] \\
= \;& \frac{RE}{\pi r} \left[|||c||^2_{l^2(\mathbb{Z})} - \frac{4}{3E^2} ||c||^2_{l^2(\mathbb{Z})}\right] \\
= \;& \frac{RE}{\pi r} \left[1 - \frac{4}{3E^2}\right] |||c||^2_{l^2(\mathbb{Z})}
\end{aligned}$$

For the $sinc$ series, we applied the following estimates, since $|x_j - x_k| \geq r|j-k|$. We take

$$\sum_{k\neq j}\left(sinc\left(\frac{(x_j-x_k)E}{2r}\right)\right)^2 \leq \frac{4}{E^2\pi^2}\sum_{k\neq j}\frac{1}{(j-k)^2} = \frac{8}{E^2\pi^2}\sum_{n=1}\frac{1}{n^2}=\frac{4}{3E^2}.$$

Therefore

$$B_1||c||_{l^2(\mathbb{Z})}\leq ||\Psi c||_{l^2(\mathbb{Z})}\ forall c=(c_k)\in l^2(\mathbb{Z}),$$

where $B_1=\frac{RE}{\pi r}\left[1-\frac{4}{3E^2}\right]$. Hence, we can write

$$B_1||c||_{l^2(\mathbb{Z})}\leq ||\Psi c||_{l^2(\mathbb{Z})}=\sum_{k\in\mathbb{Z}}|\mathcal{L}[f](x_k)|^2\leq B_2||c||_{l^2(\mathbb{Z})}.$$

As a consequence of proposition 3.1 in [42], we get $c\in N(T)^{\perp}$. For $c\in N(T)^{\perp}$, there are two positive constants C_1 and C_2 such that

$$\frac{1}{C_2}||f||_2^2\leq ||\mathbf{c}||_2^2\leq \frac{1}{C_1}||f||_2^2.$$

Hence,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|\langle\mathcal{L}[f],u_k\rangle|^2 &\geq \left(\left[\sum_{k\in\mathbb{Z}}|\mathcal{L}[f](x_k)|^2\right]^{\frac{1}{2}}-\right.\\ &\left.\left[\sum_{k\in\mathbb{Z}}|\langle\mathcal{L}[f],u_k\rangle-\mathcal{L}[f](x_k)|^2\right]^{\frac{1}{2}}\right)^2\\ &= \left(B_1-K\sqrt{\delta[a+b]}\right)^2||c||_2^2\\ &\geq \frac{1}{C_2}\left(B_1-K\sqrt{\delta[a+b]}\right)^2||f||_2^2.\end{aligned}$$

Applying similar arguments we get

$$\sum_{k\in\mathbb{Z}}|\langle\mathcal{L}[f],u_k\rangle|^2\leq\frac{1}{C_1}\left(B_1+K\sqrt{\delta[a+b]}\right)^2||f||_2^2.$$

Further

$$\langle\mathcal{L}[f],u_k\rangle=\left\langle f,\bar{\tilde{l}}\star u_k\right\rangle=\left\langle f,\mathcal{P}(\bar{\tilde{l}}\star u_k)\right\rangle,\ f\in V_X(\varphi),$$

where $\tilde{l}(t):=l(-t)$ and $\mathcal{P}$ is the orthogonal projection of $L^2(\mathbb{R})$ onto $V_X(\varphi)$. Therefore $\{g_k(t):=\mathcal{P}(\bar{\tilde{l}}\star u_k):\ n\in\mathbb{Z}\}$ is a frame for $V_X(\varphi)$. Let $\{S_k:k\in\mathbb{Z}\}$ be the dual frame of $\{g_k:k\in\mathbb{Z}\}$. Then for any $f\in V_X(\varphi)$,

$$f(x)=\sum_{k\in\mathbb{Z}}\langle\mathcal{L}[f],u_k\rangle S_k(x), \tag{III.6}$$

By Lemma 2.1, $\sum_{k\in\mathbb{Z}}|S_k(x)|^2$ is bounded on $\mathbb{R}$. Therefore the above series III.6 is uniformly convergent on $\mathbb{R}$. □

The previous theorem can be significantly enhanced, as the generalized average sampling theorem applies to a broader range of quasi shift invariant spaces.

Theorem 3.2: Let $V_X(\varphi)$ be a closed subspace of $L^2(\mathbb{R})$ and $\{\varphi(.-x_k):k\in\mathbb{Z}\}$ be a frame for $V_X(\varphi)$. Suppose that $\{u_k:k\in\mathbb{Z}\}$ is a sequence of compactly supported averaging functions and there exist positive constants A and B such that

$$A||f||^2\leq\sum_{k\in\mathbb{Z}}|\langle\mathcal{L}[f],u_k\rangle|^2\leq B||f||^2,\ \forall f\in V_X(\varphi). \tag{III.7}$$

Then there is a frame $\{S_k:k\in\mathbb{Z}\}$ for $V_X(\varphi)$ such that

$$f(x)=\sum_{k\in\mathbb{Z}}\langle\mathcal{L}[f],u_k\rangle S_k(x),\ \forall f\in V_0, \tag{III.8}$$

where the convergence is both in $L^2(\mathbb{R})$ and uniform on $\mathbb{R}$.

Proof 3.2: For any $f\in V_X(\varphi)$, we can write

$$f(x)=\sum_{k\in\mathbb{Z}}\langle f,\varphi(.-x_k)\rangle\widetilde{\varphi}(.-x_k).$$

where $\{\widetilde{\varphi}(.-x_k):k\in\mathbb{Z}\}$ be a dual frame of $\{\varphi(.-x_k):k\in\mathbb{Z}\}$.
Hence

$$\mathcal{L}[f](x)=\sum_{n\in\mathbb{Z}}\langle\mathcal{L}[f],\varphi(.-x_k)\rangle\widetilde{\varphi}(.-x_k).$$

Let us consider, for any $n\in\mathbb{Z}$,

$$p_n=\sum_{n\in\mathbb{Z}}\langle u_n,\varphi(.-x_k)\rangle\widetilde{\varphi}(.-x_k).$$

Using u_n and φ we get $p_n\in V_X(\varphi)$.

$$\begin{aligned}\langle\mathcal{L}[f],p_n\rangle &= \left\langle\mathcal{L}[f],\sum_{k\in\mathbb{Z}}\langle u_n,\varphi(.-x_k)\rangle\widetilde{\varphi}(.-x_k)\right\rangle,\\ &= \sum_{n\in\mathbb{Z}}\langle\varphi(.-x_k),u_n\rangle\langle\mathcal{L}[f],\widetilde{\varphi}(.-x_k)\rangle,\\ &= \left\langle\sum_{k\in\mathbb{Z}}\langle\mathcal{L}[f],\widetilde{\varphi}(.-x_k)\rangle\varphi(.-x_k),u_n\right\rangle,\\ &= \langle\mathcal{L}[f],u_n\rangle.\end{aligned}$$

By inequality III.7, we obtain

$$A||f||^2\leq\sum_{n\in\mathbb{Z}}|\langle\mathcal{L}[f],p_n\rangle|^2\leq B||f||^2,\ \forall f\in V_X(\varphi). \tag{III.9}$$

Since $\langle\mathcal{L}[f],p_n\rangle=\left\langle f,\bar{\tilde{l}}\star p_n\right\rangle=\left\langle f,\mathcal{P}(\bar{\tilde{l}}\star p_n)\right\rangle,\ f\in V_X(\varphi)$, where $\tilde{l}(t):=l(-t)$ and $\mathcal{P}$ is the orthogonal projection of $L^2(\mathbb{R})$ onto $V_X(\varphi)$. Therefore $\{g_n(t):=\mathcal{P}(\bar{\tilde{l}}\star p_n):\ n\in\mathbb{Z}\}$ is a frame for $V_X(\varphi)$.

Therefore $\{u_k:k\in\mathbb{Z}\}$ is a frame for $V_X(\varphi)$. Let $\{S_k:k\in\mathbb{Z}\}$ be the dual frame of $\{p_k:k\in\mathbb{Z}\}$. Then for any $f\in V_X(\varphi)$ we can write

$$f(x)=\sum_{k\in\mathbb{Z}}\langle\mathcal{L}[f],p_k\rangle S_k(x)=\sum_{k\in\mathbb{Z}}\langle\mathcal{L}[f],u_k\rangle S_k(x).$$

By Lemma 2.1, the above sum converges uniformly on $\mathbb{R}$. □

IV. Acknowledgment

The author would like to thank the Management of SSN College of Engineering, Kalavakkam-603 110, Tamil Nadu, India.

V. References

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