New Properties of Clifford Prolate Spheroidal Wave Functions

Hamed Baghal Ghaffari
School of Information and Phys Sci
University of Newcastle
Australia
hamed.baghalghaffari@newcastle.edu.au

Jeffrey A. Hogan
School of Information and Phys Sci
University of Newcastle
Australia
jeff.hogan@newcastle.edu.au

Joseph D. Lakey
Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003-8001
USA
jlakey@nmsu.edu

***Abstract*—We review a recent construction of Clifford prolate spheroidal wave functions (CPSWFs) – a multidimensional multi-channel generalization of one-dimensional PSWFs. We introduce new properties of the corresponding eigenvalues such as decay and spectral accumulation.**

I. Introduction

The first version of higher dimensional prolate spheroidal wave functions has been studied in [1]. The mentioned prolates have been defined as $\psi_{n,k}(r,\theta) = R_{n,k}(r)H_k(\theta)$ where the $R_{n,k}(r)$ and $H_k(\theta)$ are radial functions and spherical harmonics, respectively. A modified version of the radial part, $\varphi_{n,k}(r) = \sqrt{r}R_{n,k}(r)$ is the eigenfunction of the following differential operator

$$M_c(u)(t) = (1-t^2)\frac{d^2u}{dt} - 2t\frac{du}{dt} + (\frac{\frac{1}{4}-N^2}{t^2} - c^2t^2)u = 0. \tag{1}$$

The operator has a singularity at the origin, causing instabilities. The prolates are also the eigenfunctions of the Fourier transformation truncated to the unit disc, i.e.,

$$\mathcal{F}_c(\psi_{n,k})(x) = \int_{B(1)} e^{ic\langle x,y\rangle}\psi_{n,k}(y)\,dy. \tag{2}$$

In recent years there have been some attempts to improve the numerical computation of the prolates [2], [3]. There are also some other versions of the higher dimensional prolates that have been developed in the Clifford analysis [4], [5]. In [6] new Clifford-type prolate spheroidal wave functions (CPSWFs) have been developed with different properties. Here we present some new properties for CPSWFs.

II. Clifford Analysis

Let $\{e_1,\ldots,e_m\}$ be the standard basis for m-dimensional euclidean space $\mathbb{R}^m$. We declare the non-commutative multiplication in the Clifford algebra $\mathbb{R}_m$ by the rules

$$\begin{aligned} e_j^2 &= -1 \quad j=1,\cdots,m \\ e_ie_j &= -e_je_i \quad i\neq j. \end{aligned}$$

A canonical base for $\mathbb{R}_m$ is obtained by considering for any ordered set $A=\{j_1,j_2,\cdots,j_h\}\subset\{1,\cdots,m\}=M$, the element $e_A = e_{j_1}e_{j_2}\cdots e_{j_h}$. For example, each $\lambda\in\mathbb{R}_2$, may be written as $\lambda = \lambda_0+\lambda_1e_1+\lambda_2e_2+\lambda_{12}e_1e_2$, where $\lambda_i\in\mathbb{R}$. The conjugation $\overline{\lambda}$ of $\lambda=\sum_A\lambda_Ae_A\in\mathbb{R}_m$ is given by $\overline{\lambda} = \sum_A\lambda_A\overline{e_A}$ where $\overline{e_j}=-e_j$ and $\overline{\alpha\beta}=\overline{\beta}\,\overline{\alpha}$ for all $\alpha,\beta\in\mathbb{R}_m$. Similarly $\mathbb{C}_m$ can be obtained if $\lambda_A\in\mathbb{C}$. The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebra $\mathbb{R}_m$ by identifying the point $x=(x_1,\cdots,x_m)\in\mathbb{R}^m$ with the 1-vector $x=\sum_{j=1}^m e_jx_j$. The product of two 1-vectors splits up into a scalar part and a 2-vector (also called bivector): $xy = -\langle x,y\rangle + x\wedge y$ where $\langle x,y\rangle = \sum_{j=1}^m x_jy_j$ and $x\wedge y = \sum_{i<j}e_ie_j(x_iy_j - x_jy_i)$. Note also that if x is a 1-vector, then $x^2 = -\langle x,x\rangle = -|x|^2$.

Definition 1. *Let $f:\mathbb{R}^m\to\mathbb{R}_m$ be defined and continuously differentiable in an open region Ω of $\mathbb{R}^m$. The Dirac operator ∂_x is defined on such functions by*

$$\partial_x f = \sum_{j=1}^m e_j\partial_{x_j}f.$$

We also allow the Dirac operator to act of the right in the sense that $f\partial_x = \sum_{j=1}^m\partial_{x_j}fe_j$. f is said to be left (resp. right) monogenic on Ω if $\partial_xf=0$ (resp. $f\partial_x=0$) on Ω. If f is left-and right monogenic, we say f is monogenic.

Definition 2. *A left (resp. right) monogenic homogeneous polynomial P_k of degree k ($k\geq 0$) in $\mathbb{R}^m$ is called a left (resp. right) solid inner spherical monogenic of order k. The set of all left (resp. right) solid inner spherical monogenics of order k will be denoted by $M_l^+(k)$, respectively $M_r^+(k)$.*

For the proof, the reader is referred to [7].

The $\mathbb{R}_m$-valued inner product of the functions $f,g:\mathbb{R}^m\to\mathbb{R}_m$ is given by $\langle f,g\rangle = \int_{\mathbb{R}^m}\overline{f(x)}g(x)\,dx$, where dx is Lebesgue measure on $\mathbb{R}^m$. The associated norm is given by $\|f\|^2 = [\langle f,f\rangle]_0$. The unitary right Clifford-module of Clifford algebra-valued measurable functions on $\mathbb{R}^m$ for which $\|f\|^2<\infty$ is a right Hilbert Clifford-module which we denote by $L_2(\mathbb{R}^m,\mathbb{C}_m)$. The multi-dimensional Fourier transform $\mathcal{F}$ is given by

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}\exp(-i\langle x,\xi\rangle)f(x)dV(x) \tag{3}$$

for $f\in L^1(\mathbb{R}^m,\mathbb{C}_m)$ and may be extended unitarily to $L^2(\mathbb{R}^m,\mathbb{C}_m)$.

Proof. For the proof see [7]. □

Theorem 1. *(Rodrigues' Formula) The Clifford Gegenbauer polynomials (CGPs) $C_n^\alpha(P_k)(x)$ are defined by*

$$C_n^\alpha(Y_k)(x) = (1-|x|^2)^{-\alpha}\partial_x^n((1-|x|^2)^{\alpha+n}Y_k(x)) \quad (4)$$

where $Y_k(x)$ monogenic homogenous polynomial given in Definition 2. The CGPs also satisfies in the following Clifford differential equation

$$\begin{aligned}(1-|x|^2)^{-\alpha}\partial_x((1-|x|^2)^{\alpha+1}\partial_x C_{n,m}^\alpha(Y_k)(x))\\ = C(\alpha,n,m,k)C_{n,m}^\alpha(Y_k)(x),\end{aligned}$$

where

$$C(\alpha,n,m,k) = \begin{cases} n(2\alpha+n+m+2k) & \text{if } n \text{ is even,}\\ (2\alpha+n+1)(n+m+2k-1) & \text{if } n \text{ is odd.}\end{cases}$$

Proof. For the proof see [7]. □

From ([7] p. 294) we have that

$$C_{2n,m}^\alpha(Y_k)(x) = A_{n,\alpha}P_n^{(\alpha,k+\frac{m}{2}-1)}(2|x|^2-1)Y_k(x), \quad (5)$$

where $A_{n,\alpha} = (-1)^n 2^{2n}(\alpha+n+1)_n(n)!$, and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial. The odd version is also given in [7]. Let $f(x) = C_{2N,m}^\alpha(Y_k^j)(x) = P_{N,k,m}^\alpha(|x|^2)Y_k^j(x)$, to obtain

$$\begin{aligned}&(1-s^2)Q_N''(s) + [(k+\frac{m}{2}-1-\alpha)\\ &- s(k+\frac{m}{2}+\alpha+1)]Q_N'(s) = -\frac{C(\alpha,2N,m,k)}{4}Q_N(s),\end{aligned} \quad (6)$$

in which $Q_N(s) = \alpha P_{N,k,m}^\alpha(2|x|^2-1)$ where α is any constant number. We can see the equation (6) is a Sturm-Liouville differential equation. In fact, (6) is a Jacobi differential equation.

III. Computations of the CPSWFs

Definition 3. *Let $c \geq 0$ and $\alpha > -1$. The Clifford operator $\mathcal{L}_c$ acting on $C^2(B(1),\mathbb{R}_m) \subset L_2(B(1),(1-|x|^2)^\alpha)$ is defined as follows*

$$\mathcal{L}_c f(x) = \partial_x((1-|x|^2)\partial_x f(x)) + 4\pi^2c^2|x|^2 f(x). \quad (7)$$

We call the eigenfunctions of (7) CPSWFs, $\psi_{n,m}^{k,c}(x)$.

Theorem 2. *Let $n=2N$. The CPSWFs also are of two parts, i.e., $\psi_{2N,m}^{k,c,i}(x) = P_{N,m}^{k,c}(|x|^2)Y_k^i(x)$. Similarly for $n=2N+1$, we have that $\psi_{2N+1,m}^{k,c,i}(x) = Q_{N,m}^{k,c}(|x|^2)xY_k^i(x)$.*

For the proof see [6].

Lemma 3. *By considering $|x|^2 = \frac{s+1}{2}$ we can see that*

$$\begin{aligned}&(1-s^2)\frac{d^2}{ds^2}\tilde{P}_{N,m}^{k,c}(s) + [(k+\frac{m}{2}-1) - s(k+\frac{m}{2}+1)]\\ &\frac{d}{ds}\tilde{P}_{N,m}^{k,c}(s) - \pi^2c^2\frac{s+1}{2}\tilde{P}_{N,m}^{k,c}(s) = -\frac{\chi_{2N,m}^{k,c}}{4}\tilde{P}_{N,m}^{k,c}(s),\end{aligned} \quad (8)$$

where $\tilde{P}_{N,m}^{k,c}(s) = P_{N,m}^{k,c}(2|x|^2-1)$ becomes a Sturm-Liouville differential equation after multiplying $y(s) = (1+s)^{k+\frac{m}{2}-1}$.

Proof. For the proof see [6]. □

Equation (8) may be written as

$$\begin{aligned}\tilde{T}_c\tilde{P}_{N,m}^{k,c}(s) = (1-s^2)\frac{d^2}{ds^2}\tilde{P}_{N,m}^{k,c}(s) + [(k+\frac{m}{2}-1)\\ -s(k+\frac{m}{2}+1)]\frac{d}{ds}\tilde{P}_{N,m}^{k,c}(s) - \pi^2c^2\frac{s+1}{2}\tilde{P}_{N,m}^{k,c}(s).\end{aligned} \quad (9)$$

Since by Sturm Liouville theory $\tilde{P}_{N,m}^{k,c}(s)$ is normalized in the weighted space $L^2([-1,1],(1+s)^{k+\frac{m}{2}-1})$ so $\tilde{P}_{N,m}^{k,c}(s) = \sum_{n=0} a_{n,k,m}\bar{P}_n^{(0,k+\frac{m}{2}-1)}(s)$ as $\bar{P}_n^{(0,k+\frac{m}{2}-1)}(s)$ are orthonormal in the same space. So

$$\tilde{T}_c(\sum_{n=0} a_{n,k,m}\bar{P}_n^{(0,k+\frac{m}{2}-1)}(s)) = \sum_{n=0} a_{n,k,m}\tilde{T}_c(\bar{P}_n^{(0,k+\frac{m}{2}-1)}(s)),$$

where $\bar{P}_n^{(0,k+\frac{m}{2}-1)}(s) = \sqrt{\frac{(2n+k+\frac{m}{2})}{2^{k+\frac{m}{2}}}}P_n^{(0,k+\frac{m}{2}-1)}(s)$. Now by the use of the following identity for non-normalized Jacobi polynomials

$$\begin{aligned}&(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)(2n+\alpha+\beta)(s+1)P_n^{(\alpha,\beta)}(s)\\ &= 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)} +\\ &\quad [(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)(2n+\alpha+\beta)-\\ &\quad (2n+\alpha+\beta+1)(\alpha^2-\beta^2)]P_n^{(\alpha,\beta)}(s)+\\ &\quad 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(s),\end{aligned}$$

and from (6) we have that

$$\begin{aligned}&\tilde{T}_c(\tilde{P}_{N,m}^{k,c}(s)) = \sum_{n=0} a_{n,k,m}\tilde{T}_c(\bar{P}_n^{(0,k+\frac{m}{2}-1)}(s))\\ &= \sum_{n=0} a_{n,k,m}[\alpha_{n,k,m}\bar{P}_{n-1}^{(0,k+\frac{m}{2}-1)}(s) + \beta_{n,k,m}\bar{P}_n^{(0,k+\frac{m}{2}-1)}(s)\\ &+ \gamma_{n,k,m}\bar{P}_{n+1}^{(0,k+\frac{m}{2}-1)}(s)] = \sum_{n=0}[a_{n+1,k,m}\alpha_{n+1,k,m}\\ &+ a_{n,k,m}\beta_{n,k,m} + a_{n-1,k,m}\gamma_{n-1,k,m}]\bar{P}_n^{(0,k+\frac{m}{2}-1)}(s)\\ &= -\frac{\chi_{2N,m}^{k,c}}{4}\tilde{P}_{N,m}^{k,c}(s).\end{aligned}$$

Therefore,

$$\begin{aligned}a_{n+1,k,m}\alpha_{n+1,k,m} + a_{n,k,m}(\beta_{n,k,m} + \frac{\chi_{2N,m}^{k,c}}{4})\\ + a_{n-1,k,m}\gamma_{n-1,k,m} = 0\end{aligned}$$

where

$$\begin{cases}\alpha_{n+1,k,m} &= \frac{-\pi^2c^2(n+1)(n+k+\frac{m}{2})}{(2n+k+\frac{m}{2}+1)\sqrt{(2n+k+\frac{m}{2})(2n+k+\frac{m}{2}+2)}},\\[2ex] \beta_{n,k,m} &= [-n(n+k+\frac{m}{2})\\ &\quad -\frac{\pi^2c^2}{2}(1+\frac{(k+\frac{m}{2}-1)^2}{(2n+k+\frac{m}{2}-1)(2n+k+\frac{m}{2}+1)})],\\[2ex] \gamma_{n-1,k,m} &= \frac{-\pi^2c^2\ n\ (n+k+\frac{m}{2}-1)}{(2n+k+\frac{m}{2}-1)\sqrt{(2n+k+\frac{m}{2}-2)(2n+k+\frac{m}{2})}}.\end{cases}$$

This recurrence formula holds for all $n, N \geq 0$ so the problem reduces to finding the eigenvectors $a_{n,k,m}$ and associated eigenvalues $\chi_{2N,m}^{k,c}$ of the doubly-infinite matrix $M_{k,m}^e$ with the following entries:

$$M_{k,m}^e(i,j) = \begin{cases} \gamma_{i-1,k,m}, & \text{if } i \geq 1,\ j = i-1, \\ \beta_{i,k,m}, & \text{if } i = j \geq 0, \\ \alpha_{i+1,k,m}, & \text{if } i \geq 0,\ j = i+1, \\ 0 & else. \end{cases}$$

The matrix is symmetric as $\gamma_{i,k,m} = \alpha_{i+1,k,m}$. The odd CPSWFs may be computed similarly. So claculation of the CPSWFs reduces to calculations of its radial parts.

IV. Eigenvalues of CPSWFs

Theorem 4. *Let the real constants $\chi_{n,m}^{k,0}$, $\chi_{n,m}^{k,c}$, $\alpha_{n,k}$, $\beta_{n,k}$, $\gamma_{n,k}$ be given by*

$$\begin{aligned} L_0 \bar{C}_{n,m}^0(Y_k^i) &= \chi_{n,m}^{k,0} \bar{C}_{n,m}^{k,0}(Y_k^i), \\ L_c \psi_{n,m}^{k,c,i} &= \chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i}, \\ x^2 \bar{C}_{n,m}^0(Y_k^i) &= \alpha_{n,k,m} \bar{C}_{n+2,m}^0(Y_k^i) + \beta_{n,k,m} \bar{C}_{n,m}^0(Y_k^i) \\ &\quad + \gamma_{n,k,m} \bar{C}_{n-2,m}^0(Y_k^i). \end{aligned}$$

Then the asymptotic behaviours of $\chi_{n,m}^{k,c}$ and $\psi_{n,m}^{k,c}$ are as follows:

$$\chi_{n,m}^{k,c} = \chi_{n,m}^{k,0} - 4\pi^2 c^2 \beta_{n,k,m} + O(c^4). \tag{10}$$

and

$$\begin{aligned} \psi_{n,m}^{k,c,i} &= \bar{C}_{n,m}^0(Y_k^i) - 4\pi^2 c^2 \left(\frac{\alpha_{n,k,m}}{\chi_{n,m}^{k,0} - \chi_{n+2,m}^{k,0}} \bar{C}_{n+2,m}^0(Y_k^i) \right. \\ &\quad \left. + \frac{\gamma_{n,k,m}}{\chi_{n,m}^{k,0} - \chi_{n-2,m}^{k,0}} \bar{C}_{n-2,m}^0(Y_k^i) \right) + O(c^4). \end{aligned} \tag{11}$$

Proof. We assume an asymptotic expansion of the form

$$\psi_{n,m}^{k,c,i}(x) = \bar{C}_{n,m}^0(Y_k^i)(x) + c^2 f(x) + O(c^4). \tag{12}$$

Since $\|\psi_{n,m}^{k,c,i}\|_2 = \|\bar{C}_{n,m}^0(Y_k^i)\|_2 = 1$, we have $\langle C_n^0(Y_k^i), f \rangle = 0$. Therefore,

$$\begin{aligned} &\chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} = L_c \bar{C}_{n,m}^0(Y_k^i) + c^2 L_c f + O(c^4) \\ &= L_0 \bar{C}_{n,m}(Y_k^i) + 4\pi^2 c^2 |x|^2 \bar{C}_{n,m}^0(Y_k^i) + c^2 L_0 f + O(c^4) \\ &= [\chi_{n,m}^{k,0} - 4\pi^2 c^2 \beta_{n,k}] \bar{C}_{n,m}^0(Y_k^i) - 4\pi^2 c^2 [\alpha_{n,k} \bar{C}_{n+2,m}^0(Y_k^i) \\ &+ \gamma_{n,k} \bar{C}_{n-2,m}^0(Y_k^i)] + c^2 L_0 f + O(c^4). \end{aligned} \tag{13}$$

On the other hand,

$$\chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} = \chi_{n,m}^{k,c} [\bar{C}_{n,m}^0(Y_k^i) + c^2 f] + O(c^4). \tag{14}$$

Subtracting (14) fram (13) gives

$$\begin{aligned} &(\chi_{n,m}^{k,0} - 4\pi^2 c^2 \beta_{n,k} - \chi_{n,m}^{k,c}) \bar{C}_{n,m}^0(Y_k^i) + c^2 (L_0 f - \chi_{n,m}^{k,c} f) \\ &- 4\pi^2 c^2 [\alpha_{nk} \bar{C}_{n+2,m}^0(Y_k^i) + \gamma_{nk} \bar{C}_{n-2,m}^0(Y_k^i)] = O(c^4). \end{aligned} \tag{15}$$

Since $\langle L_0 f, \bar{C}_{n,m}^0(Y_k^i) \rangle = \langle f, L_0 \bar{C}_n^0(Y_k^i) \rangle = \chi_{n,m}^{k,0} \langle f, \bar{C}_{n,m}^0(Y_k^i) \rangle = 0$, taking the inner product of both sides on (15) against $\bar{C}_{n,m}^0(Y_k^i)$ gives (10).

We now aim to determine the function f in (12). Combining (10), (12) and (13) gives

$$L_0 f = \chi_{n,m}^{k,0} f - 4\pi^2 [\alpha_{n,k,m} \bar{C}_{n+2,m}^0(Y_k^i) + \gamma_{n,k,m} \bar{C}_{n-2,m}^0(Y_k^i)], \tag{16}$$

which has solutions of the form

$$\begin{aligned} f = \bar{C}_{n,m}^0(Y_k^i) A + 4\pi^2 \Bigg[&\frac{\alpha_{nk}}{\chi_{n+2,m}^{k,0} - \chi_{n,m}^{k,0}} \bar{C}_{n+2,m}^0(Y_k^i) \\ &+ \frac{\gamma_{nk}}{\chi_{n-2,m}^{k,0} - \chi_n^{k,0}} \bar{C}_{n-2,m}^0(Y_k^i) \Bigg], \end{aligned} \tag{17}$$

where A is an arbitrary Clifford constant. Substituting (17) into (12) gives

$$\begin{aligned} \psi_{n,m}^{k,c,i} &= \bar{C}_{n,m}^0(Y_k^i)(1 + Ac^2) \\ &- 4\pi^2 c^2 \left(\frac{\alpha_{n,k}}{\chi_{n,m}^{k,0} - \chi_{n+2,m}^{k,0}} \bar{C}_{n+2,m}^0(Y_k^i) \right. \\ &\left. + \frac{\gamma_{n,k}}{\chi_{n,m}^{k,0} - \chi_{n-2,m}^{k,0}} \bar{C}_{n-2,m}^0(Y_k^i) \right) + O(c^4), \end{aligned} \tag{18}$$

However, applying L_c to both sides of (18) and applying (10) gives

$$L_c \psi_{n,m}^{k,c,i} - \chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} = \bar{C}_{n,m}^0(Y_k^i) A c^2 + O(c^4),$$

from which we conclude that $A = 0$. Putting $A = 0$ in (18) gives (11). ☐

Definition 4. *We define $\mathcal{G}_c : L^2(B(1), \mathbb{C}_m) \to L^2(\mathbb{R}^m, \mathbb{C}_m)$ by*

$$\mathcal{G}_c f(x) = \chi_{B(1)}(x) \int_{B(1)} e^{2\pi i c \langle x, y \rangle} f(y)\, dy, \tag{19}$$

where $\chi_{B(1)}$ is the characteristic function of the unit ball $B(1)$ in $\mathbb{R}^m$. The adjoint $\mathcal{G}_c^*$ of $\mathcal{G}_c$ is given by

$$\mathcal{G}_c^* f(x) = \chi_{B(1)}(x) \int_{\mathbb{R}^m} e^{-2\pi i c \langle x, y \rangle} f(y)\, dy.$$

Definition 5. *The "space-limiting" operator $Q : L^2(\mathbb{R}^m, \mathbb{C}_m) \to L^2(B(1), \mathbb{C}_m)$ is given by*

$$Qf(x) = \chi_{B(1)}(x) f(x),$$

and the "bandlimiting" operator $P_c : L^2(\mathbb{R}^m, \mathbb{C}_m) \to PW_c$ is given by

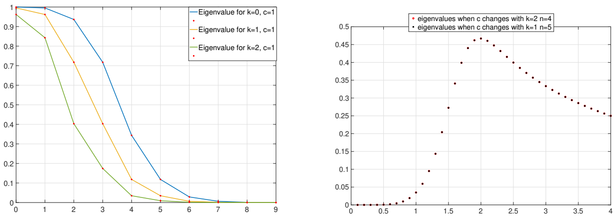
$$P_c f(x) = \int_{B(c)} \mathcal{F} f(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi.$$

Here, PW_c is the Paley-Wiener space of functions with bandlimit c, i.e.,

$$PW_c = \{ f \in L^2(\mathbb{R}^m, \mathbb{C}_m) : \hat{f}(\xi) = 0 \text{ if } |\xi| > c \}. \tag{20}$$

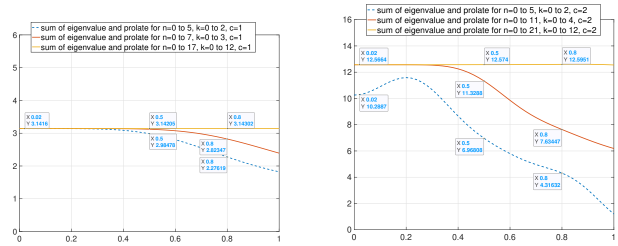
Theorem 5. *The CPSWFs $\psi_{n,m}^{k,c,i}$ are eigenfunctions of $\mathcal{G}_c$ and QP_c. In other words, $L_c \psi_{n,m}^{k,c,i}(x) = \mu_{n,m}^{k,c} \psi_{n,m}^{k,c,i}(x)$, and, $QP_c \psi_{n,m}^{k,c,i}(x) = \lambda_{n,m}^{k,c} \psi_{n,m}^{k,c,i}(x)$.*

Proof. See the [6] for the proof. ☐



(a) fig 1 (b) fig 2

Fig. 1. Behaviour of the eigenvalues $\lambda_{n,m}^{k,c}$



(a) fig 1 (b) fig 2

Fig. 2. Plotting $G(r^2)$ for $c = 1, 2$

Theorem 6. *The eigenvalues* $\{\mu_{n,m}^{k,c};\ n,k \geq 0\}$ *of* $\mathcal{G}_c$*, enjoy the following relationship*

$$\mu_{2N,m}^{k,c} = \mu_{2N+1,m}^{k-1,c}. \tag{21}$$

and for a fixed k *the eigenvalues* $\lambda_{n,m}^{k,c}$ *of* QP_c*, are non-degenerate, and*

$$\lambda_{0,m}^{k,c} > \lambda_{1,m}^{k,c} > \cdots > \lambda_{n,m}^{k,c} > \lambda_{n+1,m}^{k,c} > \cdots .$$

Proof. See [6] for the proof. □

The behaviour of the eigenvalues $\lambda_{n,m}^{k,c}$ is displayed in Figure 1.

From Theorem 5 we see that

$$\lambda_{n,m}^{k,c}\psi_{n,m}^{k,c,i}(x) = \int\limits_{B(1)} K_c(x-y)\psi_{n,m}^{k,c,i}(y)dy = QP_c\psi_{n,m}^{k,c,i}(x)$$

where $K_c(x) = c^m \int\limits_{B(1)} e^{2\pi i c\langle\omega,x\rangle}d\omega$. Then

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{d_{k,m}} \lambda_{n,m}^{k,c}|\psi_{n,m}^{k,c,i}(x)|^2 \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{d_{k,m}} \lambda_{n,m}^{k,c}\left[\overline{\psi_{n,m}^{k,c,i}(x)}\,\psi_{n,m}^{k,c,i}(x)\right]_0 \\
&= \left[\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{d_{k,m}} \int\limits_{B(1)} \overline{\psi_{n,m}^{k,c,i}(y)}K_c(x-y)dy\,\psi_{n,,m}^{k,c,i}(x)\right]_0 \\
&= K_c(0) = c^m|B(1)|.
\end{aligned}$$

On the other hand, if instead of summing over all $n,k \geq 0$ in the above calculation, we perform a truncated sum by restricting the values of n and k so that $0 \leq n \leq 2N+1$ and $0 \leq k \leq K$, then we have

$$\begin{aligned}
&\sum_{k=0}^{K}\sum_{n=0}^{2N+1}\sum_{i=1}^{d_{k,m}} \lambda_{n,m}^{k,c}|\psi_{n,m}^{k,c,i}(x)|^2 \\
&= \sum_{k=0}^{K}\sum_{n=0}^{N} \lambda_{2n,m}^{k,c}\left|P_{n,m}^{k,c}(|x|^2)\right|^2|x|^{2k}\frac{(k+m-2)}{|S^{m-1}|(m-2)} \\
&+ \sum_{k=0}^{K}\sum_{n=0}^{N} \lambda_{2n+1,m}^{k,c}\left|Q_{n,m}^{k,c}(|x|^2)\right|^2|x|^{2k+2}\frac{(k+m-2)}{|S^{m-1}|(m-2)} \\
&= G(|x|^2),
\end{aligned} \tag{22}$$

where we have used the reproducing kernel for monogenic functions, Theorem 3.3 in [8] . In Figures 2, we see numerical computations of the partial sums (22) demonstrating the convergence of the partial sums to the constant $c^2|B(1)|$.

V. Conclusion and Future Work

Through consideration of Clifford Gegenbauer polynomials, we developed in this paper a new method for computing multidimensional Clifford-valued PSWF's, defined as the eigenfunctions the differential operator L_c involving the Dirac operator. We then inroduced a relationship between the eigenvalues of the differential operators L_c and L_0. We defined time and frequency projections Q and P_c as CPSWFs and the eigenfunctions of QP_c. We then developed some properties for the eigenvalues of the QP_c. The spectrum accumulations property is also introduced.

Acknowledgment

JAH is supported by the Australian Research Council through Discovery Grant DP160101537. Thanks Roy. Thanks HG.

References

[1] D. Slepian, "Prolate spheroidal wave functions, fourier analysis and uncertainty—iv: extensions to many dimensions; generalized prolate spheroidal functions," *Bell System Technical Journal*, vol. 43, no. 6, pp. 3009–3057, 1964.

[2] P. Greengard and K. Serkh, "On generalized prolate spheroidal functions," *arXiv preprint arXiv:1811.02733*, 2018.

[3] R. R. Lederman, "Numerical algorithms for the computation of generalized prolate spheroidal functions," *arXiv preprint arXiv:1710.02874*, 2017.

[4] J. Morais and K. I. Kou, "Constructing prolate spheroidal quaternion wave functions on the sphere," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 14, pp. 3961–3978, 2016.

[5] K. Kou, J. Morais, and Y. Zhang, "Generalized prolate spheroidal wave functions for offset linear canonical transform in clifford analysis," *Mathematical Methods in the Applied Sciences*, vol. 36, no. 9, pp. 1028–1041, 2013.

[6] H. Baghal Ghaffari, "Higher-dimensional prolate spheroidal wave functions," Ph.D. dissertation, The University of Newcastle, 2022.

[7] R. Delanghe, F. Sommen, and V. Soucek, *Clifford algebra and spinor-valued functions: a function theory for the Dirac operator*. Springer Science & Business Media, 2012, vol. 53.

[8] H. De Bie, F. Sommen, and M. Wutzig, "Reproducing kernels for polynomial null-solutions of Dirac operators," *Constructive Approximation*, vol. 44, no. 3, pp. 339–383, 2016.