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006 **Anonymous authors**

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ABSTRACT

013 In scientific machine learning, models are routinely deployed with parameter val-
014 ues or boundary conditions far from those used in training. This paper studies
015 the learning-where-to-learn problem of designing a training data distribution that
016 minimizes average prediction error across a family of deployment regimes. A theo-
017 retical analysis shows how the training distribution shapes deployment accuracy.
018 This motivates two adaptive algorithms based on bilevel or alternating optimization
019 in the space of probability measures. Discretized implementations using parametric
020 distribution classes or nonparametric particle-based gradient flows deliver opti-
021 mized training distributions that outperform nonadaptive designs. Once trained,
022 the resulting models exhibit improved sample complexity and robustness to distri-
023 bution shift. This framework unlocks the potential of principled data acquisition
024 for learning functions and solution operators of partial differential equations.

1 INTRODUCTION

027 In scientific settings, machine learning models encounter data at deployment time that can differ
028 substantially from the data seen during training. Standard empirical risk minimization (ERM), which
029 minimizes an average loss over the fixed training distribution, may yield models that perform well
030 in-sample yet degrade under distribution shift. This challenge is acute in operator learning for partial
031 differential equations (PDEs) and infinite-dimensional inverse problems. For example, in data-driven
032 electrical impedance tomography (EIT), neural operators are trained to map electrical patterns on the
033 boundary to internal conductivities (Fan & Ying, 2020; Molinaro et al., 2023; Park et al., 2021). The
034 input is a collection of voltage and corresponding current pairs that represent Dirichlet and Neumann
035 boundary conditions, respectively. The probability distribution ν that boundary voltages are sampled
036 from is a user choice. Crucially, the choice of ν shapes how well the trained model generalizes to
037 new boundary measurement distributions. Fig. 1 illustrates this effect with a variant of the Neural
038 Inverse Operator (NIO) architecture (Molinaro et al., 2023); see SM E–F for details.

039 These observations motivate the central premise of the present work: *learning where to learn* by
040 designing an optimal training distribution ν for a prescribed family of deployment regimes. Guided
041 by theoretical and practical considerations, this paper develops a principled and computationally
042 feasible framework for optimizing ν in function regression and operator learning tasks. It establishes
043 intelligent data acquisition as a crucial component of scientific machine learning (SciML) workflows.

044 **Contributions.** Our main contributions are as follows.

- 045 (C1) *Lipschitz-based distribution shift bounds.* We develop quantitative bounds that connect
046 out-of-distribution (OOD) error to the training error, Lipschitz continuity of the model and
047 target map, and the Wasserstein mismatch between training and test distributions.
- 048 (C2) *Algorithms for training distribution optimization.* Motivated by the theory, we develop two
049 adaptive algorithms that actively optimize the training distribution for general model classes:
050 (1) a bilevel procedure that directly minimizes OOD error and (2) a computationally efficient
051 alternating scheme that minimizes an upper bound on the OOD error. We implement (1) and
052 (2) over the space of probability measures by considering (a) parametric training distribution
053 families (e.g., Gaussian processes) and (b) nonparametric particle representations of the
training distribution (e.g., that are updated with Wasserstein gradient flows).

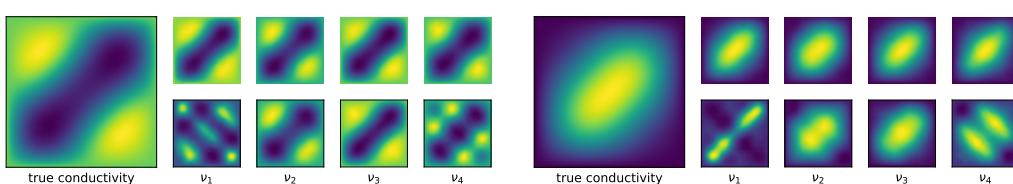


Figure 1: Two conductivity samples in EIT. The true conductivity is on the left of each panel, followed by predictions from NIO models trained on four Dirichlet boundary condition distributions $\{\nu_i\}_{i=1}^4$. Top row: in-distribution predictions; bottom row: out-of-distribution predictions. See SM E–F for details.

(C3) *Empirical performance in SciML tasks.* We perform numerical experiments involving function approximation as well as forward and inverse operator learning for a range of **elliptic, parabolic, and hyperbolic PDEs** (EIT, Darcy flow, viscous Burgers', and radiative transport). Consistent with our theory, the results show that the training data distribution strongly governs OOD deployment error. In the experiments, the two proposed adaptive algorithms efficiently optimize the training distribution, greatly reduce OOD error, and often outperform adaptive and nonadaptive baselines.

Related work. The careful selection of sampling points for regression appears frequently in numerical analysis (Barron et al., 2008; Krieg et al., 2022). Weighted least-squares regression with Christoffel sampling is one prominent example (Cohen & Migliorati, 2017; Adcock et al., 2022). One of the main findings of these works is that training on samples from the target test distribution is a suboptimal strategy in general (cf. Canatar et al. (2021)). Instead, a change of measure is derived by minimizing an upper bound on average error. These ideas have their origin in numerical integration and Monte Carlo methods (Caflisch, 1998). Some other related design strategies include greedy Bayesian herding (Huszar & Duvenaud, 2012) and kernel quadrature (Epperly & Moreno, 2023). However, quadrature-based sampling does not necessarily deliver optimal performance for regression.

In SciML for PDEs, several adaptive sampling and reweighting strategies have emerged, particularly for unsupervised methods such as physics-informed neural networks (PINNs) (Wu et al., 2023). Common approaches either reweight a fixed pool of training data samples (Chen et al., 2025a; McCleny & Braga-Neto, 2023) or sample new data points from a density proportional to the PDE residual (Gao et al., 2023; Tang et al., 2023). Similar methods based on importance sampling, reweighting, or coresnet selection have been applied to neural operators (Wang et al., 2022b; Toscano et al., 2025). Others use residual information to move samples directly (Ouyang et al., 2025) or iteratively expand training datasets (Lye et al., 2021). The current work builds on these advances by operating at the level of probability distributions. This enables a wide range of parametrizations and discretizations beyond samples only and provides extra freedom to explore the design space.

For a discussion of broader connections to other areas of machine learning and statistics, see SM A.

Outline. This paper is organized as follows. Sec. 2 provides the necessary background. Sec. 3 establishes motivating theoretical results (C1). Sec. 4 proposes two implementable data distribution selection algorithms (C2). Sec. 5 performs numerical studies (C3) and Sec. 6 gives concluding remarks. Proof and implementation details are deferred to the supplementary material (SM).

2 PRELIMINARIES

This section sets the notation, provides essential background on key mathematical concepts, and reviews the framework of operator learning. SM B contains additional reference material.

Notation. This paper works with an abstract input space $(\mathcal{U}, \langle \cdot, \cdot \rangle_{\mathcal{U}}, \|\cdot\|_{\mathcal{U}})$ and output space $(\mathcal{Y}, \langle \cdot, \cdot \rangle_{\mathcal{Y}}, \|\cdot\|_{\mathcal{Y}})$, both assumed to be real separable Hilbert spaces. We often drop the subscripts on norms and inner products when the meaning is clear from the context. Unless otherwise stated, \mathcal{U} or \mathcal{Y} could be infinite-dimensional. We denote the Lipschitz constant of a map $F: \mathcal{U} \rightarrow \mathcal{Y}$ by $\text{Lip}(F) := \sup_{u \neq u'} \|F(u) - F(u')\|_{\mathcal{Y}} / \|u - u'\|_{\mathcal{U}}$. For any element ζ of a metric space, the Dirac probability measure assigning unit mass to ζ is written as δ_{ζ} . For $p \in [1, \infty]$, (Bochner) $L_{\mu}^p(\mathcal{U}; \mathcal{Y})$ spaces are defined in the usual way. We say that a probability measure $\mu \in \mathcal{P}(\mathcal{U})$ belongs to $\mathcal{P}_p(\mathcal{U})$ if $m_p(\mu) < \infty$, where $m_p: \mathcal{P}(\mathcal{U}) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ denotes the uncentered p -th moment defined by

108 $\mu \mapsto m_p(\mu) := \mathbb{E}_{u \sim \mu} \|u\|_{\mathcal{U}}^p$. We often work with Gaussian measures $\mathcal{N}(m, \mathcal{C}) \in \mathcal{P}_2(\mathcal{U})$, where
109 $m \in \mathcal{U}$ is the mean and $\mathcal{C}: \mathcal{U} \rightarrow \mathcal{U}$ is the symmetric trace-class positive semidefinite covariance
110 operator; see Dashti & Stuart (2017, App. A.3) for details in the infinite-dimensional setting.

111 **Background.** We review several key concepts that frequently appear in this work.

113 The *pushforward operator* maps a measure from one measurable space to another via a measurable
114 map. Specifically, given measure spaces (X, \mathcal{B}_X, μ) and (Y, \mathcal{B}_Y, ν) and a measurable map $T: X \rightarrow$
115 Y , the pushforward operator $T_{\#}$ assigns to μ a measure $T_{\#}\mu$ on Y defined by $T_{\#}\mu(B) = \mu(T^{-1}(B))$
116 for all $B \in \mathcal{B}_Y$, where $T^{-1}(B) = \{x \in X \mid T(x) \in B\}$ is the pre-image of B . Intuitively, $T_{\#}$
117 redistributes the measure μ on X to a measure $T_{\#}\mu$ on Y following the structure of the map T .

118 **Definition 2.1** (p -Wasserstein distance). Consider probability measures μ and ν on a metric
119 space (X, d) with finite p -th moments, where $p \in [1, \infty)$. Define the set of all couplings
120 $\Gamma(\mu, \nu) := \left\{ \gamma \in \mathcal{P}(X \times X) \mid \pi_1^{\#}\gamma = \mu, \pi_2^{\#}\gamma = \nu \right\}$, where $\pi^1(x, y) = x$ and $\pi^2(x, y) = y$. The
121 p -Wasserstein distance between μ and ν is $W_p(\mu, \nu) := (\inf_{\gamma \in \Gamma(\mu, \nu)} \int_{X \times X} d(x, y)^p \gamma(dx, dy))^{1/p}$.

123 The cases of $p = 1$ and $p = 2$ are often considered, corresponding to the W_1 and W_2 metrics (Villani,
124 2021). The case when μ and ν are both Gaussian and X is a Hilbert space is also used frequently.

125 We will use the notion of random measure (Kallenberg, 2017) to quantify OOD error in Secs. 3–4.

126 **Definition 2.2** (random measure). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let (M, \mathcal{B}) be a measurable
127 space. A *random measure* on M is a map $\Xi: \Omega \rightarrow \mathcal{P}(M)$, $\Xi(\omega) := \mu_{\omega}$, such that for every
128 $A \in \mathcal{B}$, the function $\omega \mapsto \mu_{\omega}(A)$ is a random variable, i.e., measurable from (Ω, \mathcal{F}) to $(\mathbb{R}_{\geq 0}, \mathcal{B}(\mathbb{R}))$.

130 **Operator learning.** Operator learning provides a powerful framework for approximating mappings
131 $\mathcal{G}^*: \mathcal{U} \rightarrow \mathcal{Y}$ between function spaces. It plays an increasingly pivotal role for accelerating forward
132 and inverse problems in the computational sciences (Boullé & Townsend, 2024; Kovachki et al.,
133 2023; 2024b; Morel et al., 2025; Nelsen & Stuart, 2024; Subedi & Tewari, 2025). As in traditional
134 regression, the learned operator $\mathcal{G}_{\hat{\theta}}$ is parametrized over a set Θ and minimizes an *empirical risk*

$$135 \quad \hat{\theta} \in \arg \min_{\theta \in \Theta} \frac{1}{N} \sum_{n=1}^N \|\mathcal{G}^*(u_n) - \mathcal{G}_{\theta}(u_n)\|_{\mathcal{Y}}^2. \quad (2.1)$$

138 The Hilbert norm in (2.1) and discretization-invariant architectures (e.g., DeepONet (Lu et al., 2021),
139 Fourier Neural Operator (FNO) (Li et al., 2021), NIO (Molinaro et al., 2023)) are points of departure
140 from classical function approximation. In (2.1), $\{u_n\} \sim \nu^{\otimes N}$, where ν is the training distribution
141 over input functions. The choice and properties of ν are crucial. An inadequately selected ν may
142 result in $\mathcal{G}_{\hat{\theta}}(u)$ being a poor approximation to $\mathcal{G}^*(u)$ for $u \sim \nu'$, where $\nu' \neq \nu$ (Adcock et al., 2022;
143 Boullé et al., 2023; de Hoop et al., 2023; Li et al., 2024; Musekamp et al., 2025; Pickering et al.,
144 2022; Subedi & Tewari, 2024). The present work aims to improve accuracy under distribution shift in
145 both function approximation and operator learning settings.

146 3 LIPSCHITZ THEORY FOR OUT-OF-DISTRIBUTION ERROR BOUNDS

148 This section studies how loss functions behave under distribution shifts and the implications of such
149 shifts on average-case OOD accuracy. The assumed Lipschitz continuity of the target map and the
150 elements in the approximation class play a central role in the analysis. **Although our theoretical
151 contributions build upon established tools, we are not aware of any existing work that presents these
152 results in the precise form developed here.**

153 **Distribution shift inequalities.** In the infinite data limit, the functional in (2.1) naturally converges
154 to the so-called *expected risk*, which serves as a notion of model accuracy in this work. The following
155 result forms the foundation for our forthcoming data selection algorithms in Sec. 4.

156 **Proposition 3.1** (distribution shift error). *Let \mathcal{G}_1 and \mathcal{G}_2 both be Lipschitz continuous maps from
157 Hilbert space \mathcal{U} to Hilbert space \mathcal{Y} . For any $\nu \in \mathcal{P}_1(\mathcal{U})$ and $\nu' \in \mathcal{P}_1(\mathcal{U})$, it holds that*

$$159 \quad \mathbb{E}_{u' \sim \nu'} \|\mathcal{G}_1(u') - \mathcal{G}_2(u')\|_{\mathcal{Y}} \leq \mathbb{E}_{u \sim \nu} \|\mathcal{G}_1(u) - \mathcal{G}_2(u)\|_{\mathcal{Y}} + (\text{Lip}(\mathcal{G}_1) + \text{Lip}(\mathcal{G}_2)) W_1(\nu, \nu'). \quad (3.1)$$

160 Moreover, for any $\mu \in \mathcal{P}_2(\mathcal{U})$ and $\mu' \in \mathcal{P}_2(\mathcal{U})$, it holds that

$$161 \quad \mathbb{E}_{u' \sim \mu'} \|\mathcal{G}_1(u') - \mathcal{G}_2(u')\|_{\mathcal{Y}}^2 \leq \mathbb{E}_{u \sim \mu} \|\mathcal{G}_1(u) - \mathcal{G}_2(u)\|_{\mathcal{Y}}^2 + c(\mathcal{G}_1, \mathcal{G}_2, \mu, \mu') W_2(\mu, \mu'), \quad (3.2)$$

162 where $c(\mathcal{G}_1, \mathcal{G}_2, \mu, \mu')$ equals
 163

$$164 (\text{Lip}(\mathcal{G}_1) + \text{Lip}(\mathcal{G}_2)) \sqrt{4(\text{Lip}(\mathcal{G}_1) + \text{Lip}(\mathcal{G}_2))^2 [\mathbf{m}_2(\mu) + \mathbf{m}_2(\mu')] + 16 [\|\mathcal{G}_1(0)\|_{\mathcal{Y}}^2 + \|\mathcal{G}_2(0)\|_{\mathcal{Y}}^2]}. \\ 165$$

166 See **SM C** for the proof. Provided that the maps \mathcal{G}_1 and \mathcal{G}_2 are Lipschitz, Prop. 3.1 shows that the
 167 functional $\mu \mapsto \mathbb{E}_{u \sim \mu} \|\mathcal{G}_1(u) - \mathcal{G}_2(u)\|$ is globally Lipschitz continuous in the W_1 metric. Similarly,
 168 the functional $\mu \mapsto \mathbb{E}_{u \sim \mu} \|\mathcal{G}_1(u) - \mathcal{G}_2(u)\|^2$ is *locally* Lipschitz continuous in the W_2 metric; in
 169 particular, it is Lipschitz on sets with uniformly bounded second moments. We remark that the
 170 Lipschitz map hypotheses in Prop. 3.1 are invoked for simplicity. Hölder continuity is also compatible
 171 with Wasserstein distances, and the preceding result can be adapted to hold under such regularity.

172 **Average-case performance.** OOD performance may be quantified in several ways. This paper
 173 chooses *average-case accuracy* with respect to the law $\mathbb{Q} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ of a random probability
 174 measure $\nu' \sim \mathbb{Q}$. For a ground truth map \mathcal{G}^* and any hypothesis \mathcal{G} , we strive for

$$175 \mathcal{E}_{\mathbb{Q}}(\mathcal{G}) := \mathbb{E}_{\nu' \sim \mathbb{Q}} \mathbb{E}_{u' \sim \nu'} \|\mathcal{G}^*(u') - \mathcal{G}(u')\|_{\mathcal{Y}}^2 \quad (3.3) \\ 176$$

177 to be small. An equivalent viewpoint in (D.2) from **SM D.1** rewrites (3.3) as a single expectation
 178 against a mixture. Other approaches to quantify OOD error are possible, such as worst-case analysis.
 179 **We choose average-case accuracy because it is more natural for learning from finite samples and**
 180 **because the linearity of (3.3) with respect to \mathbb{Q} is useful for algorithm development.**

181 The following corollary is useful. It is a consequence of Prop. 3.1 and the Cauchy–Schwarz inequality.

182 **Corollary 3.2** (basic inequality). *For any $\mu \in \mathcal{P}_2(\mathcal{U})$ and maps \mathcal{G}^* and \mathcal{G} , it holds that*

$$183 \mathcal{E}_{\mathbb{Q}}(\mathcal{G}) \leq \mathbb{E}_{u \sim \mu} \|\mathcal{G}^*(u) - \mathcal{G}(u)\|_{\mathcal{Y}}^2 + \sqrt{\mathbb{E}_{\nu' \sim \mathbb{Q}} c^2(\mathcal{G}^*, \mathcal{G}, \mu, \nu')} \sqrt{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\mu, \nu')}. \quad (3.4) \\ 184$$

185 The upper bound (3.4) on the average-case OOD error $\mathcal{E}_{\mathbb{Q}}$ is intuitive. We view the arbitrary
 186 probability measure μ as the training data distribution that we are free to select. The first term on
 187 the right-hand side of (3.4) then represents the in-distribution generalization error of \mathcal{G} . On the
 188 other hand, the second term can be considered a regularizer that penalizes how far μ is from \mathbb{Q} on
 189 average with respect to the W_2 metric. The factor involving the quantity c further penalizes the size
 190 of moments of μ and \mathbb{Q} as well as the size of the Lipschitz constants of \mathcal{G}^* and \mathcal{G} .

191 We instantiate Cor. 3.2 in the concrete context of Gaussian mixtures in **SM C**, Ex. C.1.

192 Although the preceding OOD results hold in a quite general setting, they do not distinguish between
 193 an arbitrary model and a trained model obtained by ERM (2.1). Sharpening the bounds for a trained
 194 model would require probabilistic estimates on the model’s Lipschitz constant and in-distribution
 195 error. These are challenging theoretical tasks. Instead, we turn our attention to implementable
 196 algorithms inspired by Cor. 3.2 that are designed to reduce OOD error in practice.

200 4 PRACTICAL ALGORITHMS FOR TRAINING DISTRIBUTION DESIGN

201 Let $\mathcal{G}^* : \mathcal{U} \rightarrow \mathcal{Y}$ be the target mapping and \mathbb{Q} be a probability measure over $\mathcal{P}_2(\mathcal{U})$. We assume
 202 query access to an oracle that returns the label $\mathcal{G}^*(u) \in \mathcal{Y}$ when presented with an input $u \in \mathcal{U}$. The
 203 idealized infinite data training distribution selection problem that we wish to address is

$$204 \inf \left\{ \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu)}) \mid \nu \in \mathcal{P}_2(\mathcal{U}) \text{ and } \widehat{\mathcal{G}}^{(\nu)} \in \arg \min_{\mathcal{G} \in \mathcal{H}} \mathbb{E}_{u \sim \nu} \|\mathcal{G}^*(u) - \mathcal{G}(u)\|_{\mathcal{Y}}^2 \right\}, \quad (4.1) \\ 205$$

206 where $\mathcal{E}_{\mathbb{Q}}$ is the average-case OOD accuracy functional (3.3) and \mathcal{H} is the model hypothesis class.

207 Eqn. (4.1) is a bilevel optimization problem. The inner minimization problem seeks the best-fit model
 208 $\widehat{\mathcal{G}}^{(\nu)}$ for a fixed training distribution ν . The outer problem minimizes $\mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu)})$ over ν in the space
 209 of probability measures. The map $\nu \mapsto \widehat{\mathcal{G}}^{(\nu)}$ can be highly nonlinear, which poses a challenge to
 210 solve (4.1). To devise practical solution methods, this section considers the bilevel optimization
 211 problem (4.1) in Sec. 4.1 and a computationally tractable relaxation of (4.1) in Sec. 4.2.

212 **Remark 4.1** (finite data). It is necessary to use finite data to estimate the optimal training distribution
 213 (**SM D.1**). We thus assume sample access to the chosen feasible set of training distributions, e.g.,
 214 i.i.d. sampling or interacting particles. Moreover, we only require sample access to samples from \mathbb{Q} .

216 4.1 EXACT BILEVEL FORMULATION
217

218 This subsection focuses on directly solving the bilevel optimization problem (4.1) to identify the
219 optimal training distribution. Bilevel optimization has been extensively studied in the literature; see,
220 for example, the work of Colson et al. (2007); Golub & Pereyra (2003); Van Leeuwen & Aravkin
221 (2021). For the infinite-dimensional problem (4.1), our approach begins by specifying a Hilbert
222 space \mathcal{H} . Using its linear structure, we calculate the gradient of the Lagrangian associated with (4.1)
223 with respect to the variable ν . This gradient computation leverages the *adjoint state method* and the
224 envelope theorem (Afriat, 1971). In what follows, we specialize this general framework to the case
225 where \mathcal{H} is a reproducing kernel Hilbert space (RKHS) with scalar output space $\mathcal{Y} = \mathbb{R}$. The resulting
226 algorithm can be efficiently implemented using representer theorems for kernel methods (Micchelli &
227 Pontil, 2005). Additionally, other linear classes of functions can be incorporated into this framework
228 with minimal effort. Extending the framework to nonlinear, vector-valued function spaces \mathcal{H} , such as
229 neural network or neural operator spaces, is an exciting direction for future research.

230 **RKHS.** Let $\mathcal{H} := (\mathcal{H}_\kappa, \langle \cdot, \cdot \rangle_\kappa, \|\cdot\|_\kappa)$ be a separable RKHS of real-valued functions on \mathcal{U} with
231 bounded continuous reproducing kernel $\kappa: \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$. For fixed $\nu \in \mathcal{P}_2(\mathcal{U})$, let $\iota_\nu: \mathcal{H} \hookrightarrow L_\nu^2$
232 be the canonical inclusion map (which is compact under the assumptions on κ). The adjoint of the
233 inclusion map is the kernel integral operator $\iota_\nu^*: L_\nu^2 \rightarrow \mathcal{H}$ defined by $\iota_\nu^* h = \int_{\mathcal{U}} \kappa(\cdot, u) h(u) \nu(du)$.
234 We also define the self-adjoint operator $\mathcal{K}_\nu := \iota_\nu^* \iota_\nu: \mathcal{H} \rightarrow \mathcal{H}$, which has the same action as ι_ν^* .

235 **Method of adjoints.** Assume that $\mathcal{G}^* \in L_\nu^2 \equiv L_\nu^2(\mathcal{U}; \mathbb{R})$ for every $\nu \in \mathcal{P}_2(\mathcal{U})$ under consideration;
236 this is true, for example, if \mathcal{G}^* is uniformly bounded on \mathcal{U} . Define the data misfit $\Psi: \mathcal{H} \rightarrow \mathbb{R}$ by
237 $\Psi(\mathcal{G}) := \frac{1}{2} \|\iota_\nu \mathcal{G} - \mathcal{G}^*\|_{L_\nu^2}^2$; this is an abstract way to write the convex inner objective in (4.1) (rescaled
238 by 1/2). By Lem. D.1, the inner solution $\widehat{\mathcal{G}}^{(\nu)}$ solves the equation $\mathcal{K}_\nu \widehat{\mathcal{G}}^{(\nu)} = \iota_\nu^* \mathcal{G}^*$ in \mathcal{H} . This fact
239 allows us to define the Lagrangian $\mathcal{L}: \mathcal{P}_2(\mathcal{H}) \times \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ for (4.1) by

$$240 \quad \mathcal{L}(\nu, \mathcal{G}, \lambda) = \frac{1}{2} \mathbb{E}_{\nu' \sim \mathbb{Q}} \|\iota_{\nu'} \mathcal{G} - \mathcal{G}^*\|_{L_{\nu'}^2}^2 + \langle (\mathcal{K}_\nu \mathcal{G} - \iota_\nu^* \mathcal{G}^*), \lambda \rangle_\kappa. \quad (4.2)$$

242 Differentiating \mathcal{L} with respect to $\mathcal{G} \in \mathcal{H}$ leads to the adjoint equation.

243 **Lemma 4.2** (adjoint state equation). *At a critical point of the map $\mathcal{G} \mapsto \mathcal{L}(\nu, \mathcal{G}, \lambda)$ for fixed (ν, \mathcal{G}) ,
244 the adjoint state $\lambda = \lambda^{(\nu)}(\mathcal{G}) \in \mathcal{H}$ satisfies the compact operator equation*

$$246 \quad \mathcal{K}_\nu \lambda^{(\nu)}(\mathcal{G}) = \mathbb{E}_{\nu' \sim \mathbb{Q}} \mathbb{E}_{u' \sim \nu'} [\kappa(\cdot, u') (\mathcal{G}^*(u') - \mathcal{G}(u'))]. \quad (4.3)$$

248 The preceding results enable us to differentiate \mathcal{L} with respect to ν along the constraints of (4.1).

249 **Proposition 4.3** (derivative: infinite-dimensional case). *Define $\mathcal{J}: \mathcal{P}_2(\mathcal{U}) \rightarrow \mathbb{R}$ by $\mathcal{J}(\nu) :=$
250 $\mathcal{L}(\nu, \widehat{\mathcal{G}}^{(\nu)}, \lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)}))$. Then, in the sense of duality, $(D\mathcal{J})(\nu) = (\widehat{\mathcal{G}}^{(\nu)} - \mathcal{G}^*) \lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)})$ for each ν .*

252 **SM D.2** provides the proofs of Lem. 4.2 and Prop. 4.3 and [implementation details](#).

253 **Remark 4.4** (Wasserstein gradient flow). A nonparametric approach to solve the bilevel problem (4.1)
254 is to derive the Wasserstein gradient flow and its Lagrangian equivalence $\dot{u} = -\nabla(D\mathcal{J})(\text{Law}(u))(u)$,
255 where $D\mathcal{J}$ is as in Prop. 4.3. This leads to a curve of measures $t \mapsto \text{Law}(u(t))$ converging to a
256 steady-state probability distribution. It is important to note that this dynamic requires gradients of the
257 true map \mathcal{G}^* , which may not be available or could be too expensive to compute even if available. To
258 implement this mean-field evolution, we can discretize it with interacting particles; see **SM F.5**.

259 **Parametrized training distributions.** Although Prop. 4.3 provides the desired derivative of the
260 bilevel problem (4.1), it does not specify how to use the derivative to compute the optimal ν . This
261 issue is nontrivial because $\nu \in \mathcal{P}_2(\mathcal{U})$ belongs to a nonlinear metric space. One approach assumes
262 a parametrization $\nu = \nu_\vartheta$, where $\vartheta \in \mathcal{V}$ for some separable Hilbert parameter space \mathcal{V} . Typically
263 $\mathcal{V} \subseteq \mathbb{R}^P$ for some $P \in \mathbb{N}$. We then use the chain rule to apply Euclidean gradient-based optimization
264 tools to solve (4.1). To this end, define $\mathbb{J}: \mathcal{V} \rightarrow \mathbb{R}$ by $\vartheta \mapsto \mathbb{J}(\vartheta) := \mathcal{J}(\nu_\vartheta)$, where \mathcal{J} is as in Prop. 4.3.
265 The following theorem computes the gradient $\nabla \mathbb{J}: \mathcal{V} \rightarrow \mathcal{V}$.

266 **Theorem 4.5** (gradient: parametric case). *Suppose there exists a dominating measure μ_0 such that
267 for every $\vartheta \in \mathcal{V}$, ν_ϑ has density p_ϑ with respect to μ_0 . If $(u, \vartheta) \mapsto p_\vartheta(u)$ is sufficiently regular, then*

$$269 \quad \nabla \mathbb{J}(\vartheta) = \int_{\mathcal{U}} (\widehat{\mathcal{G}}^{(\nu_\vartheta)}(u) - \mathcal{G}^*(u)) (\lambda^{(\nu_\vartheta)}(\widehat{\mathcal{G}}^{(\nu_\vartheta)}))(u) (\nabla_\vartheta \log p_\vartheta(u)) \nu_\vartheta(du). \quad (4.4)$$

270 **Algorithm 1** Training Data Design via Gradient Descent on a Parametrized Bilevel Objective
271

```

272 1: Initialize: Parameter  $\vartheta^{(0)}$ , step sizes  $\{t_k\}$ 
273 2: for  $k = 0, 1, 2, \dots$  do
274 3:   Gradient step: With  $\nabla J$  as in (4.4), update the training distribution's parameters via
275   
$$\vartheta^{(k+1)} = \vartheta^{(k)} - t_k \nabla J(\vartheta^{(k)})$$

276 4:   if stopping criterion is met then
277   5:     Return  $\vartheta^{(k+1)}$  then break
278
279

```

280 In Thm. 4.5, μ_0 is typically the Lebesgue measure if $\mathcal{U} \subseteq \mathbb{R}^d$. **SM D.2** contains the proof.

281 **Example 4.6** (Gaussian parametrization). Consider the Gaussian model $\nu_\vartheta := \mathcal{N}(m_\vartheta, \mathcal{C}_\vartheta)$ with μ_0
282 being the Lebesgue measure on $\mathcal{U} = \mathbb{R}^d$. Explicit calculations with Gaussian densities show that

283
$$\nabla_\vartheta \log p_\vartheta(u) = -\nabla_\vartheta \Phi(u, \vartheta), \text{ where } \Phi(u, \vartheta) := \frac{1}{2} \|\mathcal{C}_\vartheta^{-1/2}(u - m_\vartheta)\|_2^2 + \frac{1}{2} \log \det(\mathcal{C}_\vartheta). \quad (4.5)$$

285 The parameters ϑ of ν_ϑ can now be updated using any gradient-based optimization method. For
286 instance, a simple gradient flow in Hilbert space is given by $\dot{\vartheta} = -\nabla J(\vartheta)$.

288 **Implementation.** A gradient descent scheme for the bilevel optimization problem (4.1) is summa-
289 rized in Alg. 1. Alternative gradient-based optimization methods can also be explored. **We emphasize**
290 **that the derived bilevel gradients are exact; in implementation, the only source of error arises from**
291 **discretization.** Discretization details for the gradient (4.4) are provided in **SM D.2**; in particular, we
292 “empiricalize” all expectations.

293 4.2 UPPER-BOUND MINIMIZATION

295 Although Alg. 1 is implementable assuming query access to the ground truth map $u \mapsto \mathcal{G}^*(u)$, it is so
296 far restricted to linear vector spaces \mathcal{H} . This subsection relaxes problem (4.1) by instead minimizing
297 an upper bound on the objective, which is no longer a bilevel optimization problem. We then design
298 an alternating descent algorithm that applies to any hypothesis class \mathcal{H} (e.g., neural networks and
299 neural operators). Empirically, the alternating algorithm converges fast, often in a few iterations.

300 **An alternating scheme.** Recall the upper bound (3.4) from Cor. 3.2, which holds for any $\mu \in$
301 $\mathcal{P}_2(\mathcal{U})$. The particular choice $\mu := \nu$, where ν is a candidate training data distribution, delivers a
302 valid upper bound for the objective functional $\mathcal{E}_Q(\widehat{\mathcal{G}}^{(\nu)})$ in (4.1). That is,

304
$$\mathcal{E}_Q(\widehat{\mathcal{G}}^{(\nu)}) \leq \mathbb{E}_{u \sim \nu} \|\mathcal{G}^*(u) - \widehat{\mathcal{G}}^{(\nu)}(u)\|_{\mathcal{Y}}^2 + \sqrt{\mathbb{E}_{\nu' \sim Q} c^2(\mathcal{G}^*, \widehat{\mathcal{G}}^{(\nu)}, \nu, \nu')} \sqrt{\mathbb{E}_{\nu' \sim Q} W_2^2(\nu, \nu')}. \quad (4.6)$$

306 It is thus natural to approximate the solution to (4.1) by the minimizing distribution of the right-
307 hand side of (4.6). The dependence of the factor c in Prop. 3.1 on the Lipschitz constant of $\widehat{\mathcal{G}}^{(\nu)}$
308 is hard to capture in practice. One way to circumvent this difficulty is to further assume an upper
309 bound $\text{Lip}(\widehat{\mathcal{G}}^{(\nu)}) \leq R$ uniformly in ν for some constant $R > 0$. We define $c_R(\mathcal{G}^*, \nu, \nu')$ to be
310 $c(\mathcal{G}^*, \widehat{\mathcal{G}}^{(\nu)}, \nu, \nu')$ with all instances of $\text{Lip}(\widehat{\mathcal{G}}^{(\nu)})$ replaced by R . The assumption of a uniform
311 bound on the Lipschitz constant is valid for certain chosen model classes \mathcal{H} , e.g., kernels with
312 Lipschitz feature maps or Lipschitz-constrained neural networks (Gouk et al., 2021). **Although such**
313 **Lipschitz control in deep neural networks is challenging, there has been substantial recent progress on**
314 **architectures with explicit Lipschitz constraints, e.g., spectral normalization, orthogonal/Householder**
315 **layers, 1-Lipschitz activations (Miyato et al., 2018; Anil et al., 2019; Wang et al., 2020; Murari et al.,**
316 **2025).** Thus, expressive neural models with controlled Lipschitz behavior can be made practical.

317 We denote the candidate set of training distributions by $\mathcal{P} \subseteq \mathcal{P}_2(\mathcal{U})$. Finally, the upper-bound
318 minimization problem becomes

320
$$\inf_{(\nu, \mathcal{G}) \in \mathcal{P} \times \mathcal{H}} \left\{ \mathbb{E}_{u \sim \nu} \|\mathcal{G}^*(u) - \mathcal{G}(u)\|_{\mathcal{Y}}^2 + \sqrt{\mathbb{E}_{\nu' \sim Q} c_R^2(\mathcal{G}^*, \nu, \nu')} \sqrt{\mathbb{E}_{\nu' \sim Q} W_2^2(\nu, \nu')} \right\}. \quad (4.7)$$

322 Note that since only the first term of (4.7) depends on the variable \mathcal{G} , the minimizing \mathcal{G} is given by
323 $\widehat{\mathcal{G}}^{(\nu)}$, as defined previously. This observation motivates the alternating scheme outlined in Alg. 2.

Algorithm 2 Training Data Design via Alternating Model Fitting and Distribution Update Steps

- 1: **Initialize:** Training data distribution $\nu^{(0)}$, estimate of the constant $R > 0$
- 2: **for** $k = 0, 1, 2, \dots$ **do**
- 3: **Model training:** To obtain the trained model $\widehat{\mathcal{G}}^{(k)}$, solve the optimization problem

$$\widehat{\mathcal{G}}^{(k)} \leftarrow \arg \min_{\mathcal{G} \in \mathcal{H}} \mathbb{E}_{u \sim \nu^{(k)}} \|\mathcal{G}^*(u) - \mathcal{G}(u)\|_y^2$$
- 4: **Distribution update:** Update the training distribution by solving

$$\nu^{(k+1)} \leftarrow \arg \min_{\nu \in \mathbb{P}} \left\{ \mathbb{E}_{u \sim \nu} \|\mathcal{G}^*(u) - \widehat{\mathcal{G}}^{(k)}(u)\|_y^2 + \sqrt{\mathbb{E}_{\nu' \sim \mathbb{Q}} c_R^2(\mathcal{G}^*, \nu, \nu')} \sqrt{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu, \nu')} \right\}$$
- 5: **if** stopping criterion is met **then**
- 6: Return $(\widehat{\mathcal{G}}^{(k)}, \nu^{(k+1)})$ **then break**

It is important to note that not all alternating algorithms have convergence guarantees, which requires a detailed understanding of the associated fixed-point operator. In contrast, the proposed Alg. 2 comes with the following convergence guarantee.

Proposition 4.7 (monotonicity). *Under Alg. 2, the objective in (4.7) is always non-increasing.*

Proof. Line 3 ensures that the first term of the objective (4.7) does not increase, while keeping the second term constant. Next, Line 4 minimizes both terms. Iterating this argument, we deduce that the algorithm maintains a non-increasing objective value for all k . \square

Remark 4.8 (connection to Wasserstein barycenters). If the set P contains distributions with uniformly bounded second moments and the first term on the right-hand side of (4.6) is uniformly bounded over P , then the optimal value of (4.7) is bounded above by the value of another minimization problem whose solution in $\mathcal{P}_2(\mathcal{U})$ is the 2-Wasserstein barycenter of $\mathbb{Q} \in \mathcal{P}_2(\mathcal{P}_2(\mathcal{U}))$ (Backhoff-Veraguas et al., 2022, Thm 3.7, p. 444). An analogous bound is even easier to verify for the W_1 result (3.1). Thus, the \mathcal{G}^* -dependent solution of the proposed problem (4.7) over P (and hence also the solution of Eqn. (4.1)) is always *at least as good* as the barycenter, which only depends on P and \mathbb{Q} .

Implementation. To implement Alg. 2, we simply replace expectations with finite averages over empirical samples. The number of samples per iteration is a hyperparameter. Line 3 in Alg. 2 is the usual model training step, i.e., Eqn. (2.1), while the distribution update step in Line 4 can be achieved in several ways. One way involves interacting particle systems derived from (Wasserstein) gradient flows over $P \subseteq \mathcal{P}_2(\mathcal{U})$. Alternatively, if the distribution is parametrized, we can optimize over the parametrization, e.g., transport maps or Gaussian means and covariances. In this case, one can apply global optimization methods or Euclidean gradient-based solvers.

To apply gradient-based optimization algorithms, we need the derivative of the objective in (4.7) with respect to ν . Up to constants, the objective takes the form $F: \mathcal{P} \subseteq \mathcal{P}_2(\mathcal{U}) \rightarrow \mathbb{R}$, where

$$\nu \mapsto F(\nu) := \mathbb{E}_{u \sim \nu} f(u) + \sqrt{1 + \mathfrak{m}_2(\nu)} \sqrt{\mathbb{E}_{\nu' \sim \mathbb{Q}} \mathbb{W}_2^2(\nu, \nu')} \quad (4.8)$$

and $f: \mathcal{Y} \rightarrow \mathbb{R}$. The next result differentiates F ; the proof is in **SM D.3**.

Proposition 4.9 (derivative: infinite-dimensional case). *Let F be as in (4.8) with f and P sufficiently regular. Then for every $\nu \in \mathsf{P}$ and $u \in \mathcal{U}$, it holds that*

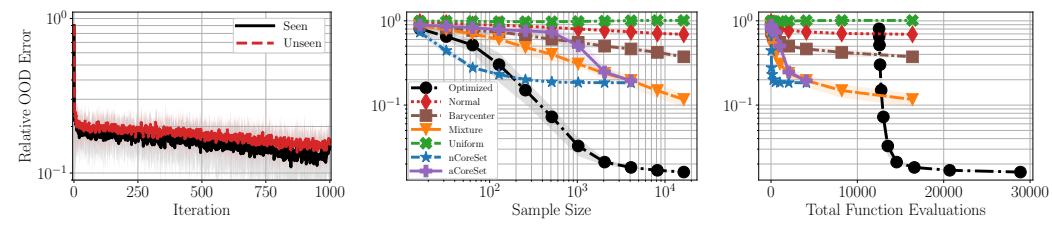
$$(DF(\nu))(u) = f(u) + \frac{1}{2} \sqrt{\frac{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu, \nu')}{1 + \mathsf{m}_2(\nu)}} \|u\|_{\mathcal{U}}^2 + \sqrt{\frac{1 + \mathsf{m}_2(\nu)}{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu, \nu')}} \mathbb{E}_{\nu' \sim \mathbb{Q}} \phi_{\nu, \nu'}(u) \quad (4.9)$$

in the sense of duality. The potential $\phi_{\nu, \nu'}: \mathcal{U} \rightarrow \mathbb{R}$ satisfies $\phi_{\nu, \nu'} = \frac{1}{2} \|\cdot\|_{\mathcal{U}}^2 - \varphi_{\nu, \nu'}$, where $\nabla \varphi_{\nu, \nu'} = T_{\nu \rightarrow \nu'}$ and $T_{\nu \rightarrow \nu'}$ is the \mathbb{W}_2 -optimal transport map from ν to ν' .

Taking the gradient of $u \mapsto DF(\nu)(u)$ in (4.9) yields the Wasserstein gradient, which can be utilized in Wasserstein gradient flow discretizations; see Rmk. 4.4. In the case of parametric distributions, combining Prop. 4.9 with the chain rule yields gradient descent schemes similar to those introduced in Sec. 4.1. **Again, the derived gradients are exact and require only discretization for their numerical evaluation.** For further details, see Cor. D.7 and Lem. D.8 and its special case (Rmk. D.9 in SM D.3).

378 Table 1: Function approximation with bilevel Alg. 1. Reported values are mean Err for kernel regressors trained
 379 with 1024 labeled data samples from the optimized ν_ϑ (ours) from Alg. 1 with 1000 iterations vs. other adaptive
 380 (aCoreSet) and nonadaptive (Normal, Barycenter, Mixture, Uniform, nCoreSet) sampling distributions. Two
 381 standard deviations from the mean Err over 10 independent runs is also reported. [Also see Fig. 10.](#)

	$g_1 (d = 2)$	$g_2 (d = 5)$	$g_2 (d = 8)$	$g_3 (d = 4)$	$g_3 (d = 5)$	$g_4 (d = 10)$
<i>Ours</i>	0.040 ± 0.010	0.169 ± 0.007	0.556 ± 0.007	0.276 ± 0.019	0.558 ± 0.009	0.025 ± 0.000
Normal	0.808 ± 0.015	0.811 ± 0.010	0.949 ± 0.003	0.937 ± 0.005	0.947 ± 0.003	0.029 ± 0.001
Barycenter	0.546 ± 0.025	0.514 ± 0.012	0.782 ± 0.005	0.771 ± 0.008	0.784 ± 0.010	0.021 ± 0.001
Mixture	0.308 ± 0.011	0.372 ± 0.015	0.712 ± 0.011	0.471 ± 0.037	0.658 ± 0.011	0.022 ± 0.001
Uniform	0.983 ± 0.001	0.971 ± 0.002	0.993 ± 0.000	0.976 ± 0.000	1.000 ± 0.000	0.121 ± 0.001
nCoreSet	0.185 ± 0.000	0.247 ± 0.002	0.622 ± 0.002	0.295 ± 0.002	0.494 ± 0.002	0.098 ± 0.012
aCoreSet	0.501 ± 0.060	0.538 ± 0.028	0.782 ± 0.018	0.793 ± 0.048	0.848 ± 0.034	0.013 ± 0.000



391 Figure 2: Alg. 1 applied to ground truth $g_1: \mathbb{R}^2 \rightarrow \mathbb{R}$. (Left) Evolution of Err over 1000 iterations ($N = 250$
 392 samples per step) of gradient descent. (Center) Err of model trained on N samples from optimized ν_ϑ (ours) vs.
 393 initial normal $\mathcal{N}(m_0, I_2)$, empirical \mathbb{Q} W_2 -barycenter, empirical \mathbb{Q} mixture, $\text{Unif}([0, 1]^2)$, and two pool-based
 394 coresets. (Right) Same as center, except incorporating the additional function evaluation cost incurred from
 395 Alg. 1. Shading represents two standard deviations away from the mean Err over 10 independent runs.
 396

5 NUMERICAL RESULTS

407 This section implements Alg. 1 (the bilevel gradient descent algorithm) for function approximation
 408 using kernel methods and Alg. 2 (the alternating minimization algorithm) for [several operator learning](#)
 409 [tasks using DeepONets](#). Comprehensive experiment details and settings are collected in [SM F](#).

410 **Bilevel gradient descent algorithm.** We instantiate a discretization of Alg. 1 on several function
 411 approximation benchmarks. The test functions $\mathcal{G}^* \in \{g_i\}_{i=1}^4$ each map \mathbb{R}^d to \mathbb{R} and are comprised
 412 of the Sobol G (g_1) and Friedmann functions (g_2, g_3), and a kernel expansion (g_4). We use the kernel
 413 framework from Sec. 4.1 with \mathcal{H} the RKHS of the squared exponential kernel with a fixed scalar
 414 bandwidth hyperparameter. Also, \mathbb{Q} is an empirical measure with 10 atoms centered at fixed Gaussian
 415 measures realized by standard normally distributed means and Wishart distributed covariance matrices.
 416 A cosine annealing scheduler determines the gradient descent step sizes. The training distribution
 417 $\nu_\vartheta := \mathcal{N}(m, \mathcal{C})$ is parametrized by the pair $\vartheta = (m, \mathcal{C})$, which we optimize using Alg. 1.

418 Tab. 1 shows that models trained on the final iterate ν_ϑ of Alg. 1 nearly always outperforms those
 419 trained on adaptive or nonadaptive baseline distributions in terms of the root relative average OOD
 420 squared error $\text{Err} := (\mathcal{E}_\mathbb{Q}(\hat{\mathcal{G}}^{(\nu)}) / \mathcal{E}_\mathbb{Q}(0))^{1/2}$. Nonadaptive (nCoreSet) and adaptive (aCoreSet) pool-
 421 based coresets from active learning (Musekamp et al., 2025) slightly outperform ν_ϑ on the g_3 ($d = 5$)
 422 and g_4 ($d = 10$) test problems, respectively. Here, the fixed pool of size 5000 consists of the empirical
 423 samples of samples from \mathbb{Q} available to Alg. 1. Overall, the performance gap between our bilevel
 424 method and the baselines can close depending on the complexity of \mathbb{Q} or \mathcal{G}^* . Fig. 2 specializes to
 425 the $\mathcal{G}^* = g_1$ case. Although fitting the model on data sampled from the precomputed optimal ν_ϑ
 426 uniformly outperforms baseline distribution sampling for large enough N , when corrected for the
 427 “offline” cost of finding the optimal ν_ϑ , the gains are more modest (Fig. 2, right). Moreover, Fig. 2
 428 (center) shows that the nCoreSet method based on raw kernel feature maps performs better than our
 429 ν_ϑ and empirical \mathbb{Q} mixture for very small sample sizes, but eventually saturates to the mixture error
 430 level as N approaches the pool size. In contrast, the error of ν_ϑ continues to decrease.

431 **Alternating minimization algorithm.** We apply the alternating scheme Alg. 2 to several important
 432 operator learning problems from the physical sciences. Let $\Omega \subset \mathbb{R}^2$ denote a bounded domain

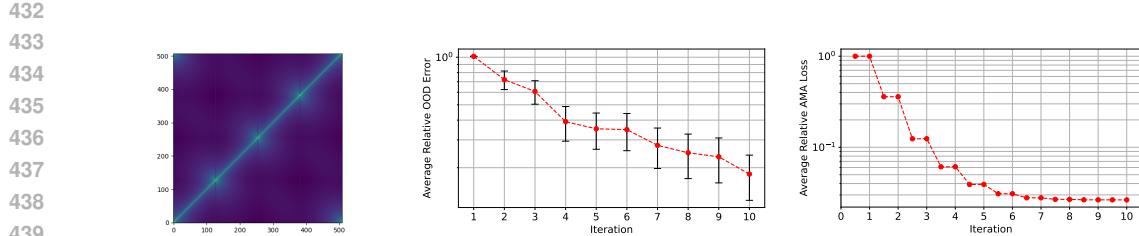


Figure 3: (Left) Matrix approximation of NtD map. (Center) After 80 independent runs, decay of the average relative OOD error of the model when trained on the optimal distribution identified at each iteration of Alg. 2; a 95% confidence interval of the true relative OOD error is provided at each iteration. (Right) Decay of average AMA loss defined in (4.7) vs. iteration relative to the same loss at initialization.

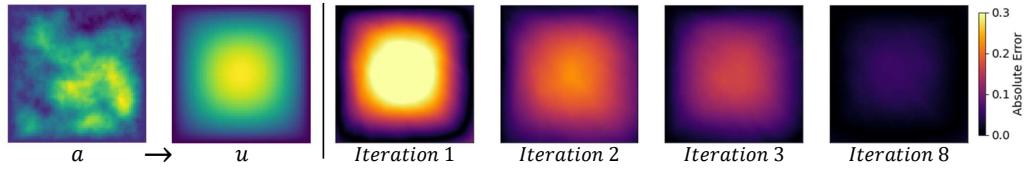


Figure 4: The first two images show the test conductivity a and true PDE solution u . The next four images show the absolute error $|u_{\text{pred}} - u|$ of the model trained on the training distribution from iterations 1, 2, 3, and 8.

with boundary $\partial\Omega$. Consider the elliptic PDE $\nabla \cdot (a(x)\nabla u(x)) = 0$, where $x \in \Omega$, $a = a(x) > 0$ is the conductivity, and $u = u(x)$ is the PDE solution. Given a mean-zero Neumann boundary condition (BC) $g = a\frac{\partial u}{\partial n}|_{\partial\Omega}$, one can obtain a unique PDE solution u and the resulting Dirichlet data $f := u|_{\partial\Omega}$. For a fixed conductivity a , we define a Neumann-to-Dirichlet (NtD) map $\Lambda_a^{-1} : g \mapsto f$, which is also used in the EIT problem (Dunlop & Stuart, 2016); see also Sec. 1. On the other hand, the map $\mathcal{G} : a \mapsto u$ with a fixed Neumann or Dirichlet BC (and nonzero source) is the Darcy flow parameter-to-solution operator, which plays a crucial role in geothermal energy extraction and carbon storage applications. Another important operator maps the scattering coefficient to the spatial-domain density in the radiative transport equation, which has key applications in optical tomography and atmospheric science. **Last, we consider the viscous Burgers' equation with small viscosity; it models steep shock-like phenomena in fluids.** Learning surrogate operators for these mappings can enable rapid simulations for efficient **inference, design, and control**.

NtD map learning. In this example, we aim to find the optimal training distribution for learning the NtD map using the alternating minimization algorithm (AMA). The domain $\Omega = (0, 1)^2$ and the conductivity a is fixed. The meta test distribution is a mixture of K components $\mathbb{Q} = \frac{1}{K} \sum_{k=1}^K \delta_{\nu'_k}$, where each ν'_k corresponds to a distribution over g , the Neumann BC. Fig. 3 illustrates the reduction in both average relative OOD error (3.3) and relative AMA loss (4.7) over 10 iterations of Alg. 2.

Darcy flow forward map learning. We use AMA to identify the optimal training distribution for learning the Darcy flow parameter-to-solution map. Here, \mathbb{Q} is as before except now each ν'_k represents a test distribution over the conductivity. Fig. 4 illustrates improved predictions of the neural operator trained on distributions obtained from Alg. 2 across various iterations.

Radiative transport operator learning. We apply a nonparametric particle-based version of AMA to find the optimal training distribution for learning the solution operator of the radiative transport equation. We vary the Knudsen number which corresponds to different physical regimes. Fig. 5 shows how the model trained on the optimal empirical measure compares to the initial model and a benchmark model trained on the empirical \mathbb{Q} mixture. When the sample size is $N = 120$ particles, particle-based AMA reduces the average relative OOD error by 88% compared to its initial value for a Knudsen number of 8.

Burgers' equation operator learning. We apply AMA to learn the operator that maps an initial condition $u(\cdot, 0)$ to the corresponding solution $u(\cdot, 1)$ of the viscous Burgers' equation. The left and center panels of Fig. 6 illustrate an example input-output pair for this operator. We compare AMA against two pool-based active learning methods: Query-by-Committee (QbC) and CoreSet (Musekamp et al.,

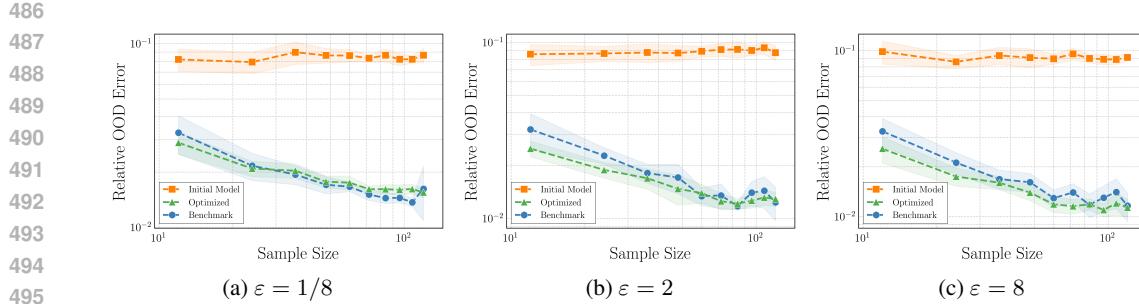


Figure 5: Relative OOD error vs. sample size N for learning the radiative transport solution operator. Results are shown for Knudsen numbers $\varepsilon \in \{1/8, 2, 8\}$. For each N , each panel displays 95% confidence intervals over 10 trials for three DeepONet models trained for 5000 epochs: the initial model, the model after particle-based AMA, and a benchmark trained the test distribution mixture $\nu_{\mathbb{Q}} := \frac{1}{3} \sum_{k=1}^3 \nu_k'$.

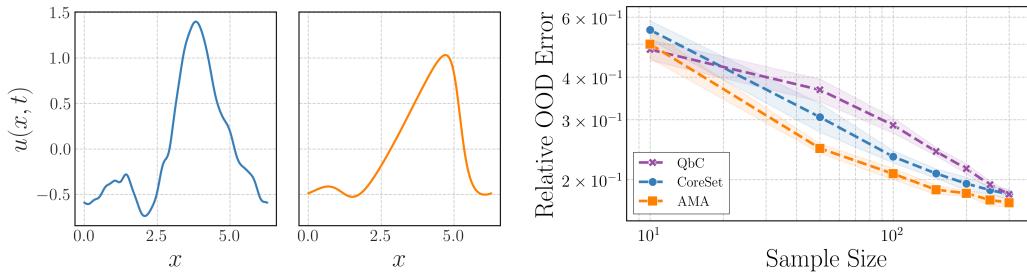


Figure 6: (Left) Example of a random initial condition $u(\cdot, 0)$ for the Burgers' equation. (Center) Corresponding solution $u(\cdot, 1)$ of the Burgers' equation at time 1 for the given initial condition. (Right) Relative OOD error vs. sample size N for learning the mapping from the initial condition $u(\cdot, 0)$ to the solution $u(\cdot, 1)$. After computing the optimal training distribution using a fixed data pool, the model labeled "AMA" is trained on N samples drawn from this distribution. This model is compared with two models trained using pool-based active learning methods, QbC and CoreSet. All models share the same DeepONet architecture and differ only in the training data received. The experiment is repeated 10 times, and the plot shows the 95% confidence interval for the expected relative OOD error.

2025). Using a fixed pool, we first compute the optimal training distribution from AMA and then draw N samples from it to train the DeepONet model. The same pool is used to obtain models trained with each active learning strategy. The right panel of Fig. 6 reports the relative OOD performance as N increases. The AMA-trained model performs comparably to the active learning baselines and, in the mid-range of sample sizes, exceeds their performance. When the sample size is small, all methods begin with randomly selected points, leading to similar performance. For large sample sizes, the training sets seem to overlap, causing the methods to converge. In the intermediate regime, however, our model consistently outperforms the other two approaches. Moreover, AMA provides the additional advantage of generating arbitrarily many high-quality training samples beyond the fixed pool, yielding not only a stronger learned operator but also a robust training distribution.

6 CONCLUSION

This paper shows how optimizing training distributions using theoretically founded bilevel and alternating optimization algorithms produces more accurate and robust models for function approximation and PDE operator learning. Numerical experiments instate the methods with parametric distribution families and nonparametric particle-based discretizations of Wasserstein gradient flows. Though promising, the proposed approach is limited by distribution family expressivity, global optimization, and line search steps. Future work will address these limitations by incorporating transport maps and importance reweighting techniques. Understanding cost versus accuracy trade-offs for sample allocation and iteration complexity of the methodology is another important future challenge.

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REPRODUCIBILITY STATEMENT

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542

We have taken several steps to ensure the reproducibility of our results. All proposed algorithms are described in detail in the main text, with pseudocode and additional implementation and discretization details provided in **SM D**, **E**, and **F**. Complete proofs of all theoretical results from the main text, together with explicit statements of assumptions, interpretations, and discussion, are included in **SM C** and **D**. The function approximation and PDE benchmarks are introduced in the main text, with full specifications of parameter ranges, boundary conditions, and discretizations in **SM E** and **F**. An anonymous numerical implementation, including code for training distribution optimization, model training, and evaluation, is provided in the supplementary material as a zip file. Hyperparameters and experimental settings are listed in **SM F** and in configuration files within the code repository. Together, these resources should enable full replication and extension of our results.

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Supplementary Material for:
 Learning where to learn:
 Training data distribution optimization for scientific machine learning

A EXPANDED OVERVIEW OF RELATED WORK

Beyond the SciML literature reviewed in Sec. 1, our work also sits alongside research on learning under distribution shift (Mallinar et al., 2024). This includes meta-learning, domain generalization and adaptation, and active learning and optimal experimental design (OED). We now explain these connections.

Meta-learning aims to learn models or priors that adapt rapidly across tasks drawn from a task distribution (Hospedales et al., 2021; Mouli et al., 2023). While related in spirit, meta-learning typically treats the data-generating distribution per task as given; in contrast, we *optimize* the training distribution itself to improve average performance across deployment regimes. Both frameworks involve a two-level structure: an inner optimization that optimizes the model over a training dataset, and an outer optimization that adjusts a higher-level quantity beyond model parameters, guided by generalization performance. In optimization-based meta-learning methods, the outer loop adjusts the model initialization or update rules, allowing for rapid adaptation across new tasks. On the other hand, the outer loop in our work optimizes the training data distribution.

Domain generalization seeks representations that transfer across multiple source domains without access to target data (Wang et al., 2022a; Deng et al., 2024; Gagnon-Audet et al., 2023; Eastwood et al., 2022; Singh et al., 2024; Wei et al., 2022). Instead, domain adaptation leverages (often unlabeled) target data to reduce source-target discrepancy (Fang et al., 2023). Both paradigms assume a fixed training set and focus on achieving robustness during training. Common benchmarks for unsupervised domain adaptation only assume access to samples from a single target domain (ν' in $\mathbb{Q} = \delta_{\nu'}$) and do not have the ability to generate new labels (Setinek et al., 2025); this setting is orthogonal to our work. Robust domain generalization methods, such as distributionally robust optimization and invariant risk minimization (Lin et al., 2022a; Rahimian & Mehrotra, 2022; Arjovsky et al., 2019; Kamath et al., 2021; Lin et al., 2022b; Zhou et al., 2022), modify objectives or constraints to ensure stability across domains given the training data. In contrast, we *design* the training distribution *before* learning: we retain standard training objectives but *actively shape* where data are drawn for improved sample complexity and robustness to distribution shift in regression problems. We do so with bilevel (or alternating) optimization in the space of probability measures, guided by theoretical bounds. In particular, the minimizing training data distribution depends not just on the chosen model architecture and the accuracy metric, but also on the target map itself.

The perspective presented in our paper naturally connects to active learning and OED. Active learning selects informative samples from a given data pool to improve label efficiency (Deng et al., 2023; Jain et al., 2025; Musekamp et al., 2025; Settles, 2009; Yang et al., 2023); however, it typically operates *reactively* over a fixed unlabeled set during training. As a result, the support of the underlying distribution remains fixed as well. We instead study the *a priori* optimization of the sampling distribution itself. Like our work, there are active learning algorithms that are adaptive to labels (Zhu et al., 2019). However, this approach is pool-based, involves joint optimization instead of bilevel optimization, applies only to classification instead of regression, and adopts a worst-case instead of average-case perspective. On the other hand, OED explicitly reasons about where to make measurements, often through A- or D-optimality or Bayesian and sequential criteria (Atkinson et al., 2007; Fedorov, 2010; Foster et al., 2019; Huan et al., 2024). The classical OED problem emphasizes parameter inference and uncertainty reduction. In contrast, we target distributions that minimize the average OOD prediction error for function and operator learning, a design principle not commonly prioritized in OED. Nevertheless, our work shares similarities with modern measure-centric OED (Hellmuth et al., 2025; Jin et al., 2024a), in which the design variable is a probability measure over the design space instead of a fixed number of sample points. In the nonlinear case, measure-based OED requires the formulation of a bilevel optimization problem (Jin et al., 2024b). The difference is that our inner loop involves the supervised training of a machine learning model, while for OED it typically involves solving a PDE-constrained inverse problem.

Other related work includes ridge leverage score and randomly pivoted Cholesky sampling for kernel regression (Rudi et al., 2018; Chen et al., 2025b) as well as Bayesian optimization for sequential

918 point-by-point dataset updates (Shahriari et al., 2015). These techniques provide principled but
 919 typically label-independent sampling rules that do not depend on the target map to be learned. Instead,
 920 the present paper proposes adaptive, target-dependent, and supervised distribution design methods.
 921

922 B ADDITIONAL MATHEMATICAL BACKGROUND

924 This appendix provides more mathematical background for the present work.
 925

926 We begin with two comments about Gaussian distributions on Hilbert spaces. First, we remark that
 927 the 2-Wasserstein metric between Gaussian measures has a closed-form expression.
 928

929 **Example B.1** (2-Wasserstein metric between Gaussians). Let $\mu = \mathcal{N}(m_\mu, \mathcal{C}_\mu)$ and $\nu = \mathcal{N}(m_\nu, \mathcal{C}_\nu)$
 930 be Gaussian measures on a real separable Hilbert space $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$. Then Gelbrich (1990) gives
 931

$$930 \quad W_2^2(\mu, \nu) = \|m_\mu - m_\nu\|^2 + \text{tr}\left(\mathcal{C}_\mu + \mathcal{C}_\nu - 2(\mathcal{C}_\mu^{1/2}\mathcal{C}_\nu\mathcal{C}_\mu^{1/2})^{1/2}\right). \quad (\text{B.1})$$

932 The square root in (B.1) is the unique self-adjoint positive semidefinite (PSD) square root of a self-
 933 adjoint PSD linear operator. The trace of an operator $A: H \rightarrow H$ is given by $\text{tr}(A) = \sum_{j \in \mathbb{N}} \langle e_j, Ae_j \rangle$
 934 for any orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ of H . If A is PSD, then we can also write $\text{tr}(A) = \|A^{1/2}\|_{\text{HS}}^2$,
 935 where $B \mapsto \|B\|_{\text{HS}}^2 := \sum_{j \in \mathbb{N}} \|Be_j\|^2$ is the squared Hilbert–Schmidt norm.
 936

937 We frequently invoke (B.1) in this paper. When H is infinite-dimensional, the *Karhunen–Loève*
 938 expansion (KLE) is often useful in calculations (Stuart, 2010, Thm. 6.19, p. 533).

939 **Lemma B.2** (KLE of Gaussian measure). Let $m \in (H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ and $\mathcal{C}: H \rightarrow H$ be self-adjoint,
 940 PSD, and trace-class. Let $\{\lambda_j\}_{j \in \mathbb{N}} \subseteq \mathbb{R}_{\geq 0}$ be a non-increasing rearrangement of the eigenvalues
 941 of \mathcal{C} and $\{\phi_j\}_{j \in \mathbb{N}} \subseteq H$ be the corresponding orthonormal set of eigenvectors. If $\{\xi_j\}_{j \in \mathbb{N}}$ is an
 942 independent and identically distributed sequence with $\xi_1 \sim \mathcal{N}(0, 1)$, then
 943

$$944 \quad \text{Law}\left(m + \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} \xi_j \phi_j\right) = \mathcal{N}(m, \mathcal{C}). \quad (\text{B.2})$$

946 Generalizing Lem. B.2, there exist Karhunen–Loève expansions of non-Gaussian measures. In
 947 this case, the $\{\xi_j\}_{j \in \mathbb{N}}$ are non-Gaussian, have mean zero and variance one, and are only pairwise
 948 uncorrelated instead of independent and identically distributed (i.i.d.); still, the KLE “diagonalizes”
 949 the probability measure.
 950

951 Next, we turn our attention to general probability measures on abstract spaces. This allows us to
 952 define probability measures *over* the space of probability measures.
 953

954 **Definition B.3** (spaces of probability measures). Let (X, \mathcal{B}) be a measurable space. The space of
 955 probability measures on X is defined by
 956

$$\mathcal{P}(X) := \{\mu: \mathcal{B} \rightarrow [0, 1] \mid \mu \text{ is a probability measure}\}.$$

957 It is equipped with the σ -algebra $\mathcal{B}(\mathcal{P}(X))$ generated by the evaluation maps $e_A: \mathcal{P}(X) \rightarrow [0, 1]$
 958 given by $e_A(\mu) := \mu(A)$ for all $A \in \mathcal{B}$. Similarly, the space of probability measures on $\mathcal{P}(X)$ is
 959

$$\mathcal{P}(\mathcal{P}(X)) := \{\mathbb{Q}: \mathcal{B}(\mathcal{P}(X)) \rightarrow [0, 1] \mid \mathbb{Q} \text{ is a probability measure}\}.$$

960 Now suppose that $(\mathcal{P}(X), d)$ is a metric space. For $p \in [1, \infty)$, we say that $\mathbb{Q} \in \mathcal{P}_p(\mathcal{P}(X))$ if
 961 $\mathbb{Q} \in \mathcal{P}(\mathcal{P}(X))$ and $\mathbb{E}_{\mu \sim \mathbb{Q}}[d^p(\mu, \mu_0)] < \infty$ for some (and hence for all) $\mu_0 \in \mathcal{P}(X)$.
 962

Def. B.3 can be related to the notion of random measure from Def. 2.2 as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a
 963 probability space. Recall that a random probability measure on X is a measurable map
 964

$$965 \quad \Xi: (\Omega, \mathcal{F}) \rightarrow (\mathcal{P}(X), \mathcal{B}(\mathcal{P}(X))).$$

966 Then the *law* (or *distribution*) of Ξ is the pushforward measure $\text{Law}(\Xi) \in \mathcal{P}(\mathcal{P}(X))$ defined by
 967

$$968 \quad \text{Law}(\Xi)(E) := \mathbb{P}(\{\omega \in \Omega \mid \Xi(\omega) \in E\}) \quad \text{for all } E \in \mathcal{B}(\mathcal{P}(X)).$$

969 In this paper, we typically do not distinguish between a random measure and its law.¹
 970

We now provide three examples of random measures.
 971

¹Both concepts are closely related to *Markov kernels* (Kallenberg, 2017).

Example B.4 (empirical measure). Given i.i.d. random variables X_1, \dots, X_N taking values in measurable space (M, \mathcal{B}) with $X_1 \sim \mu$, the *empirical measure* is the random probability measure $\mu^{(N)} := \frac{1}{N} \sum_{n=1}^N \delta_{X_n}$, where $\delta_x \in \mathcal{P}(M)$ is the Dirac measure at $x \in M$. For each measurable set $A \in \mathcal{B}$, the number $\mu^{(N)}(A) \in [0, 1]$ counts the fraction of samples that fall in A . The measure $\mu^{(N)}$ is random because the samples $\{X_n\}_{n=1}^N$ are random.

Example B.5 (Dirichlet process). Let α be a finite nonnegative measure on (M, \mathcal{B}) . A *Dirichlet process* $\mu \sim \text{DP}(\alpha)$ is a random probability measure such that for every k and every measurable partition $\{A_1, \dots, A_k\}$ of M , it holds that

$$(\mu(A_1), \dots, \mu(A_k)) \sim \text{Dirichlet}(\alpha(A_1), \dots, \alpha(A_k)).$$

Example B.6 (random Gaussian measures). In the infinite-dimensional state space setting, a canonical example is that of a random Gaussian measure defined through the KLE (B.2) of its mean and covariance. Let $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ be infinite-dimensional, $\{\phi_j\}$ be an orthonormal basis of H , and $\{\lambda_j\} \subset \mathbb{R}_{\geq 0}$ and $\{\sigma_{ij}\} \subset \mathbb{R}_{\geq 0}$ be summable. Define the random measure $\mathcal{N}(m, \mathcal{C})$ by

$$m \stackrel{\text{Law}}{=} \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} \xi_j \phi_j \quad \text{and} \quad \mathcal{C} := \mathcal{R}^* \mathcal{R}, \quad \text{where} \quad \mathcal{R} \stackrel{\text{Law}}{=} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sqrt{\sigma_{ij}} \zeta_{ij} \phi_i \otimes \phi_j. \quad (\text{B.3})$$

The $\{\xi_j\}$ and $\{\zeta_{ij}\}$ are zero mean, unit variance, pairwise-uncorrelated real random variables. The outer product in (B.3) is defined by $(a \otimes b)c := \langle b, c \rangle a$. Since \mathcal{R} is a KLE in the space of Hilbert-Schmidt operators on H , it follows that $\mathcal{C} = \mathcal{R}^* \mathcal{R}$ is a valid self-adjoint PSD trace-class covariance operator with probability one. Since $\{\phi_j\}$ is fixed, the degrees of freedom in $\mathbb{Q} := \text{Law}(\mathcal{N}(m, \mathcal{C}))$ are the choice of eigenvalues $\{\lambda_j\}$ and $\{\sigma_{ij}\}$ and the distribution of the coefficients $\{\xi_j\}$ and $\{\zeta_{ij}\}$. Typically the coefficients are chosen i.i.d. from some common distribution such as the standard normal distribution $\mathcal{N}(0, 1)$ or the uniform distribution on the interval $[-\sqrt{3}, \sqrt{3}]$. The preceding construction of $\mathcal{N}(m, \mathcal{C})$ immediately extends to mixtures of Gaussian measures.

To conclude SM B, we connect back to the barycenters described in Rmk. 4.8.

Definition B.7 (Wasserstein barycenter). Fix $p \in [1, \infty)$. Given a probability measure $\mathbb{Q} \in \mathcal{P}_p(\mathcal{P}_p(\mathcal{U}))$ and a set $\mathcal{P} \subseteq \mathcal{P}_p(\mathcal{U})$, a (p, \mathcal{P}) -Wasserstein barycenter of \mathbb{Q} is any solution of

$$\inf_{\mu \in \mathcal{P}} \mathbb{E}_{\nu \sim \mathbb{Q}} W_p^p(\mu, \nu). \quad (\text{B.4})$$

If $\mathcal{P} = \mathcal{P}_p(\mathcal{U})$, then we simply write p -Wasserstein barycenter for brevity.

When \mathbb{Q} is supported on Gaussian measures, the 2-Wasserstein barycenter of \mathbb{Q} can be characterized.

Example B.8 (Gaussian barycenter). Let $\mathbb{Q} := \text{Law}(\mathcal{N}(m, \mathcal{C})) \in \mathcal{P}_2(\mathcal{P}_2(\mathcal{U}))$, where $m \in \mathcal{U}$ and $\mathcal{C}: \mathcal{U} \rightarrow \mathcal{U}$ are the random mean vector and random covariance operator of a random Gaussian measure. Then the solution $\bar{\mu}$ to $\inf_{\mu \in \mathcal{P}_2(\mathcal{U})} \mathbb{E}[W_2^2(\mu, \mathcal{N}(m, \mathcal{C}))]$ is the Gaussian $\bar{\mu} = \mathcal{N}(\mathbb{E}[m], \bar{\mathcal{C}})$ (Álvarez-Esteban et al., 2016, Thm. 2.4), where $\bar{\mathcal{C}}$ is a non-zero PSD solution to the nonlinear equation

$$\bar{\mathcal{C}} = \mathbb{E} \left[\left(\bar{\mathcal{C}}^{1/2} \mathcal{C} \bar{\mathcal{C}}^{1/2} \right)^{1/2} \right]. \quad (\text{B.5})$$

One can replace the expectation with an empirical sum and employ fixed-point iterations to approximately solve (B.5) for $\bar{\mathcal{C}}$.

C DETAILS FOR SECTION 3: LIPSCHITZ THEORY FOR OUT-OF-DISTRIBUTION ERROR BOUNDS

This appendix provides the proof of Prop. 3.1 and expands on Cor. 3.2 with Ex. C.1. We begin with the proof.

Proof of Prop. 3.1. We begin by proving the first assertion, which is similar to the result and proof of Benitez et al. (2024, Lem. 7.2, p. 23). Let $\phi(u) := \|\mathcal{G}_1(u) - \mathcal{G}_2(u)\|$. For any u and u' , the triangle inequality yields

$$\begin{aligned} |\phi(u) - \phi(u')| &\leq \|\mathcal{G}_1(u) - \mathcal{G}_2(u) - (\mathcal{G}_1(u') - \mathcal{G}_2(u'))\| \\ &\leq \|\mathcal{G}_2(u) - \mathcal{G}_2(u')\| + \|\mathcal{G}_1(u) - \mathcal{G}_1(u')\| \\ &\leq (\text{Lip}(\mathcal{G}_1) + \text{Lip}(\mathcal{G}_2)) \|u - u'\|. \end{aligned}$$

1026 Thus, $\text{Lip}(\phi) \leq \text{Lip}(\mathcal{G}_1) + \text{Lip}(\mathcal{G}_2) =: C$. By Kantorovich–Rubenstein duality, $\mathbb{E}_{u' \sim \nu'} \phi(u') - \mathbb{E}_{u \sim \nu} \phi(u) \leq C \mathbb{W}_1(\nu, \nu')$ as asserted.

1027 For the W_2 result, note that $\|\mathcal{G}_i(u)\| \leq \text{Lip}(\mathcal{G}_i)\|u\| + \|\mathcal{G}_i(0)\|$. This and the previous estimate gives

$$\begin{aligned} \phi^2(u') - \phi^2(u) &= (\phi(u') + \phi(u))(\phi(u') - \phi(u)) \\ &\leq (\|\mathcal{G}_1(u)\| + \|\mathcal{G}_1(u')\| + \|\mathcal{G}_2(u)\| + \|\mathcal{G}_2(u')\|)C\|u - u'\| \\ &\leq (C\|u\| + C\|u'\| + 2\|\mathcal{G}_1(0)\| + 2\|\mathcal{G}_2(0)\|)C\|u - u'\|. \end{aligned}$$

1034 Let π be the W_2 -optimal coupling of μ and μ' . By the Cauchy–Schwarz inequality, the expectation
1035 $\mathbb{E}_{u' \sim \mu'} \phi^2(u') - \mathbb{E}_{u \sim \mu} \phi^2(u) = \mathbb{E}_{(u, u') \sim \pi} [\phi^2(u') - \phi^2(u)]$ is bounded above by
1036

$$\begin{aligned} C\sqrt{\mathbb{E}_{(u, u') \sim \pi} (C\|u\| + C\|u'\| + 2\|\mathcal{G}_1(0)\| + 2\|\mathcal{G}_2(0)\|)^2} \sqrt{\mathbb{E}_{(u, u') \sim \pi} \|u - u'\|^2} \\ \leq C\sqrt{4C^2(\mathbf{m}_2(\mu) + \mathbf{m}_2(\mu')) + 16\|\mathcal{G}_1(0)\|^2 + 16\|\mathcal{G}_2(0)\|^2} W_2(\mu, \mu'). \end{aligned}$$

1041 The last line is due to $(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$. The second assertion is proved. \square
1042

1043 Next, we apply Cor. 3.2 in the context of Gaussian mixtures.

1044 **Example C.1** (Gaussian mixtures). Consider the *random* Gaussian measure $\nu' = \mathcal{N}(m, \mathcal{C}) \sim \mathbb{Q} \in \mathcal{P}_2(\mathcal{P}_2(\mathcal{U}))$, where $(m, \mathcal{C}) \sim \Pi$ and Π is the mixing distribution over the mean (with marginal Π_1) and covariance (with marginal Π_2). Define the Gaussian mixture $\bar{\nu} := \mathbb{E}_{(m, \mathcal{C}) \sim \Pi} \mathcal{N}(m, \mathcal{C}) \in \mathcal{P}_2(\mathcal{U})$ via duality. Let $\bar{\mu} := \mathbb{E}_{(m', \mathcal{C}') \sim \Pi'} \mathcal{N}(m', \mathcal{C}')$ be another Gaussian mixture with mixing distribution Π' (with marginals denoted by Π'_1 and Π'_2). By linearity of expectation and Fubini’s theorem, $\mathcal{E}_{\mathbb{Q}}(\mathcal{G}) = \mathcal{E}_{\delta_{\bar{\nu}}}(\mathcal{G})$; that is, average-case accuracy with respect to \mathbb{Q} is equivalent to accuracy with respect to the single test distribution $\bar{\nu}$; also see (D.2). This fact and Cor. 3.2 yield $\mathcal{E}_{\mathbb{Q}}(\mathcal{G}) \leq \mathbb{E}_{u \sim \bar{\mu}} \|\mathcal{G}^*(u) - \mathcal{G}(u)\|^2 + c(\mathcal{G}^*, \mathcal{G}, \bar{\mu}, \bar{\nu}) W_2(\bar{\mu}, \bar{\nu})$. The factor $c(\mathcal{G}^*, \mathcal{G}, \bar{\mu}, \bar{\nu})$ can be further simplified using the closed-form expression for second moments of Gaussian measures. For the W_2 factor, Lem. C.2 and coupling arguments imply that

$$\mathcal{E}_{\mathbb{Q}}(\mathcal{G}) \leq \mathbb{E}_{u \sim \bar{\mu}} \|\mathcal{G}^*(u) - \mathcal{G}(u)\|_{\mathcal{Y}}^2 + c(\mathcal{G}^*, \mathcal{G}, \bar{\mu}, \bar{\nu}) \sqrt{W_2^2(\Pi_1, \Pi'_1) + W_2^2(\Pi_2, \Pi'_2)}. \quad (\text{C.1})$$

1055 The two W_2 distances between the marginals may be further bounded with (B.1) in **SM B** under
1056 additional structural assumptions on the mixing distributions; see Rmk. C.3. We often invoke this
1057 example in numerical studies with $\mathbb{Q} = \sum_{k=1}^K w_k \delta_{\nu'_k}$ (where $K \in \mathbb{N}$, i.e., a finite mixture), weights
1058 $\sum_{k=1}^K w_k = 1$, Gaussian test measures $\{\nu'_k\}$, and degenerate training mixture $\bar{\mu} := \mathcal{N}(m_0, \mathcal{C}_0)$.
1059

1060 The following lemma is used in Ex. C.1; it is an independently interesting result.

1061 **Lemma C.2** (Lipschitz stability of mixtures in Wasserstein metric). *Let $p \in [1, \infty)$. If \mathbb{Q} and \mathbb{Q}'
1062 belong to $\mathcal{P}_p(\mathcal{P}_p(\mathcal{U}))$, then $W_p(\mathbb{E}_{\mu \sim \mathbb{Q}} \mu, \mathbb{E}_{\nu \sim \mathbb{Q}'} \nu) \leq W_p(\mathbb{Q}, \mathbb{Q}')$.*

1063 *Proof.* Let $\Pi \in \Gamma(\mathbb{Q}, \mathbb{Q}') \subseteq \mathcal{P}(\mathcal{P}_p(\mathcal{U}) \times \mathcal{P}_p(\mathcal{U}))$ be a W_p -optimal coupling with respect to the
1064 ground space $(\mathcal{P}_p(\mathcal{U}), W_p)$. By the joint convexity of the functional W_p^p and Jensen’s inequality,
1065

$$W_p^p(\mathbb{E}_{\mu \sim \mathbb{Q}} \mu, \mathbb{E}_{\nu \sim \mathbb{Q}'} \nu) = W_p^p(\mathbb{E}_{(\mu, \nu) \sim \Pi} \mu, \mathbb{E}_{(\mu, \nu) \sim \Pi} \nu) \leq \mathbb{E}_{(\mu, \nu) \sim \Pi} W_p^p(\mu, \nu) = W_p^p(\mathbb{Q}, \mathbb{Q}').$$

1066 Taking p -th roots completes the proof. \square
1067

1068 In Lem. C.2, the Wasserstein distance $W_p(\mathbb{Q}, \mathbb{Q}')$ uses $(\mathcal{P}_p(\mathcal{U}), W_p)$ as the ground metric space in
1069 Def. 2.1. Thus, it is a Wasserstein metric over the Wasserstein space.

1070 Last, we remark on structural assumptions that allow us to further bound (C.1) in Ex. C.1.

1071 **Remark C.3** (KLE mixing distributions). Instate the setting and notation of Ex. C.1. Furthermore, let
1072 $m \sim \Pi_1$ and $m' \sim \Pi'_1$ have centered (possibly non-Gaussian) KLEs
1073

$$m \stackrel{\text{Law}}{=} \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} \xi_j \phi_j \quad \text{and} \quad m' \stackrel{\text{Law}}{=} \sum_{j \in \mathbb{N}} \sqrt{\lambda'_j} \xi'_j \phi_j, \quad \text{where } \xi_j \stackrel{\text{i.i.d.}}{\sim} \eta \quad \text{and} \quad \xi'_j \stackrel{\text{i.i.d.}}{\sim} \eta'.$$

In the preceding display, we further use the notation from (B.3) in Ex. B.6; thus, $\{\xi_j\}$ and $\{\xi'_j\}$ have zero mean and unit variance, with laws in $\mathcal{P}_2(\mathbb{R})$. Let $\rho \in \Gamma(\eta, \eta')$ be the W_2 -optimal coupling. Similar to Garbuno-Inigo et al. (2023, Lem. 4.1, pp. 13–14), let $\pi \in \Gamma(\Pi_1, \Pi'_1)$ be a coupling with the property that $\pi = \text{Law}(h, h')$, where $h = \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} z_j \phi_j$ and $h' = \sum_{j \in \mathbb{N}} \sqrt{\lambda'_j} z'_j \phi_j$ in law and $(z_j, z'_j) \stackrel{\text{i.i.d.}}{\sim} \rho$. Then by (B.1) and the triangle inequality,

$$\begin{aligned} W_2^2(\Pi_1, \Pi'_1) &\leq \mathbb{E}_{(m, m') \sim \pi} \|m - m'\|^2 \\ &\leq 2 \mathbb{E}_\rho \sum_{j \in \mathbb{N}} \lambda_j |\xi_j - \xi'_j|^2 + 2 \mathbb{E}_{\eta'} \sum_{j \in \mathbb{N}} |\xi'_j|^2 \left| \sqrt{\lambda_j} - \sqrt{\lambda'_j} \right|^2 \\ &= 2W_2^2(\eta, \eta') \sum_{j \in \mathbb{N}} \lambda_j + 2 \sum_{j \in \mathbb{N}} \left| \sqrt{\lambda_j} - \sqrt{\lambda'_j} \right|^2. \end{aligned}$$

This estimate shows that Π_1 and Π'_1 are close in W_2 if η and η' are close in W_2 and $\{\sqrt{\lambda_j}\}$ and $\{\sqrt{\lambda'_j}\}$ are close in $\ell^2(\mathbb{N}; \mathbb{R})$. We conclude this remark by upper bounding the term $W_2^2(\Pi_2, \Pi'_2)$ appearing (C.1). Following (B.3) from Ex. B.6, suppose that $\mathcal{C} = \mathcal{R}^* \mathcal{R}$ and $\mathcal{C}' = (\mathcal{R}')^* \mathcal{R}'$, where

$$\mathcal{R} \stackrel{\text{Law}}{=} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sqrt{\sigma_{ij}} \zeta_{ij} \phi_i \otimes \phi_j, \quad \mathcal{R}' \stackrel{\text{Law}}{=} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sqrt{\sigma'_{ij}} \zeta'_{ij} \phi_i \otimes \phi_j, \quad \zeta_{ij} \stackrel{\text{i.i.d.}}{\sim} \varrho, \quad \text{and} \quad \zeta'_{ij} \stackrel{\text{i.i.d.}}{\sim} \varrho'$$

are KLE expansions in the Hilbert space $\text{HS}(\mathcal{U})$ of Hilbert–Schmidt operators from \mathcal{U} into itself. Using (B.1) and Cockayne & Duncan (2021, Lem. SM1.2, p. SM2), it holds that

$$\begin{aligned} W_2^2(\Pi_2, \Pi'_2) &= \inf_{\gamma \in \Gamma(\Pi_2, \Pi'_2)} \mathbb{E}_{(\mathcal{C}, \mathcal{C}') \sim \gamma} W_2^2(\mathcal{N}(0, \mathcal{C}), \mathcal{N}(0, \mathcal{C}')) \\ &\leq \inf_{\gamma \in \Gamma(\Pi_2, \Pi'_2)} \mathbb{E}_\gamma \|\mathcal{R} - \mathcal{R}'\|_{\text{HS}}^2. \end{aligned}$$

Then a nearly identical coupling argument to the one used for $W_2(\Pi_1, \Pi'_1)$ delivers the bound

$$W_2^2(\Pi_2, \Pi'_2) \leq 2W_2^2(\varrho, \varrho') \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sigma_{ij} + 2 \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \left| \sqrt{\sigma_{ij}} - \sqrt{\sigma'_{ij}} \right|^2.$$

D DETAILS FOR SECTION 4: PRACTICAL ALGORITHMS FOR TRAINING DISTRIBUTION DESIGN

This appendix expands on the content of Sec. 4 and provides the deferred proofs of the theoretical results from that section. It begins in **SM D.1** by identifying a degeneracy of the bilevel problem formulation when infinite data are available. **SM D.2** concerns the bilevel minimization algorithm from Sec. 4.1 and **SM D.3** pertains to the alternating minimization algorithm from Sec. 4.2.

D.1 NECESSITY OF FINITE DATA

Although Secs. 3 and 4 are formulated at the infinite data level, i.e., with full access to \mathbb{Q} and the candidate training distribution ν , the proposed formulation is only advantageous when ν is accessible from a finite number of samples. Of course, this is the case in practice (and moreover \mathbb{Q} will often only be accessible from samples as well); recall Rem. 4.1. Our *optimize-then-discretize* approach of Secs. 3 and 4 simplifies the exposition and notation, while allowing us to derive continuum algorithms that still work at the finite data level. Regardless, it is instructive to see how the necessity of finite data emerges from a theoretical viewpoint.

To this end, recall the functional $\mathcal{E}_{\mathbb{Q}}$ from (3.3). Define the infinite mixture probability measure

$$\nu_{\mathbb{Q}} := \mathbb{E}_{\nu' \sim \mathbb{Q}} [\nu'] \tag{D.1}$$

via duality with bounded continuous test functions. By linearity, one can check that

$$\mathcal{E}_{\mathbb{Q}}(\mathcal{G}) = \mathbb{E}_{u' \sim \nu_{\mathbb{Q}}} \|\mathcal{G}^*(u') - \mathcal{G}(u')\|_{\mathcal{Y}}^2. \tag{D.2}$$

1134 This display shows that the average OOD accuracy with respect to \mathbb{Q} is equal to the standard test
 1135 error with respect to the test mixture distribution $\nu_{\mathbb{Q}}$.
 1136

1137 Now define the optimal value $V_{\mathbb{Q}}^* := \inf_{\mathcal{G} \in \mathcal{H}} \mathcal{E}_{\mathbb{Q}}(\mathcal{G})$. By the definition of the trained model in (4.1),
 1138 it holds that $V_{\mathbb{Q}}^* = \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu_{\mathbb{Q}})})$. The inequality $V_{\mathbb{Q}}^* \leq \inf_{\nu} \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu)})$ holds because $V_{\mathbb{Q}}^*$ is the optimal
 1139 value over all elements from \mathcal{H} . Moreover, $\inf_{\nu} \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu)}) \leq \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu_{\mathbb{Q}})}) = V_{\mathbb{Q}}^*$ due to optimality over
 1140 training distributions. We deduce that

$$\inf_{\nu} \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu)}) = \inf_{\mathcal{G} \in \mathcal{H}} \mathcal{E}_{\mathbb{Q}}(\mathcal{G}). \quad (\text{D.3})$$

1141 That is, the optimal value of the objective over distributions versus the objective over models coincide.
 1142 Write $\widehat{\nu}$ for a solution of $\inf_{\nu} \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu)})$ from (4.1). The preceding display then says
 1143

$$\mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\widehat{\nu})}) = \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}^{(\nu_{\mathbb{Q}})}). \quad (\text{D.4})$$

1144 This does not imply that $\widehat{\nu}$ equals $\nu_{\mathbb{Q}}$. Indeed, these infinite-dimensional optimization problems over
 1145 possibly noncompact sets could admit multiple minima. What (D.4) does suggest is that the mixture
 1146 $\nu_{\mathbb{Q}}$ (which only depends on \mathbb{Q}) is an equally good solution for making the OOD accuracy small
 1147 compared to the adaptive measure $\widehat{\nu}$ (which depends on \mathcal{G}^* , \mathcal{H} , and \mathbb{Q}), assuming full knowledge of
 1148 \mathbb{Q} and infinite training data. In this case, there is no reason to construct $\widehat{\nu}$ if $\nu_{\mathbb{Q}}$ is known.

1149 However, for each N , writing $\nu^N = \frac{1}{N} \sum_{n=1}^N \delta_{u_n}$ for the empirical measure corresponding to i.i.d.
 1150 samples $u_n \sim \nu$, we can instead consider
 1151

$$\widehat{\nu}_N \in \arg \min \left\{ \mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}_N^{(\nu)}) \mid \nu \in \mathcal{P} \subseteq \mathcal{P}_2(\mathcal{U}) \quad \text{and} \quad \widehat{\mathcal{G}}_N^{(\nu)} \in \arg \min_{\mathcal{G} \in \mathcal{H}} \mathbb{E}_{u \sim \nu^N} \|\mathcal{G}^*(u) - \mathcal{G}(u)\|_{\mathcal{Y}}^2 \right\} \quad (\text{D.5})$$

1152 for some search space \mathcal{P} . The preceding argument leading to the degenerate behavior shown in
 1153 (D.4) no longer holds for the empirical bilevel problem (D.5) due to the finite data constraint. It is
 1154 now possible that models trained on samples from $\widehat{\nu}_N$ can outperform models trained on samples
 1155 from the mixture $\nu_{\mathbb{Q}}$. Indeed, we observe the superiority of $\widehat{\nu}_N$ over $\nu_{\mathbb{Q}}$ even for simple Gaussian
 1156 parametrizations in Secs. 5 and F.1. More generally, the *test distribution* $\nu_{\mathbb{Q}}$ is not the optimal
 1157 distribution from which to generate training data for regression problems. This fact is known to the
 1158 numerical analysis community (Adcock, 2024; Cohen & Migliorati, 2017), building off of variance
 1159 reduction and importance sampling ideas (Agapiou et al., 2017) for Monte Carlo and quasi-Monte
 1160 Carlo methods (Caflisch, 1998; Dick et al., 2013). It remains true for (linear) operator learning, where
 1161 de Hoop et al. (2023) suggests that one should pick a training data distribution over input functions
 1162 whose samples are rougher than function samples from the test distribution. This implies that the
 1163 training distribution support should (strictly) contain the support of the test distribution. However,
 1164 there are fundamental limits on how much improvement one can expect to obtain by adjusting the
 1165 training data distribution (Grohs et al., 2025; Kovachki et al., 2024a).
 1166

1167 Another practical takeaway from the preceding discussions is that if data from \mathbb{Q} are very limited and
 1168 fixed once and for all, then a finite sample approximation will not capture the mixture $\nu_{\mathbb{Q}}$ well. This
 1169 could lead to a poor performing model when trained on the aggregate empirical mixture samples. On
 1170 the other hand, the proposed approach delivers a distribution $\widehat{\nu}_N$ that ideally can be sampled from at
 1171 will (leading to many more training samples, e.g., a generative model) and uses information about \mathcal{G}^*
 1172 and \mathcal{H} to compensate for limited information about \mathbb{Q} .
 1173

1174 D.2 DERIVATIONS FOR SUBSECTION 4.1: EXACT BILEVEL FORMULATION

1175 **Proofs.** Recall the use of the adjoint method in real (i.e., $\mathcal{Y} = \mathbb{R}$) RKHS \mathcal{H} with kernel κ from
 1176 Sec. 4.1. For a fixed $\nu \in \mathcal{P}_2(\mathcal{U})$, the data misfit $\Psi: \mathcal{H} \rightarrow \mathbb{R}$ for this ν is
 1177

$$\mathcal{G} \mapsto \Psi(\mathcal{G}) := \frac{1}{2} \|\iota_{\nu} \mathcal{G} - \mathcal{G}^*\|_{L_{\nu}^2(\mathcal{U})}^2 = \mathbb{E}_{u \sim \nu} |\mathcal{G}(u) - \mathcal{G}^*(u)|^2. \quad (\text{D.6})$$

1178 This is the inner objective in (4.1). In the absence of RKHS norm penalization, typically one must
 1179 assume the existence of minimizers of Ψ (Rudi et al., 2015, Assump. 1, p. 3). The next lemma
 1180 characterizes them.

1188 **Lemma D.1** (inner minimization solution). *Fix a set $P \subseteq \mathcal{P}_2(\mathcal{U})$ of probability measures. Suppose*
 1189 *that for every $\nu \in P$, there exists $\widehat{\mathcal{G}}^{(\nu)} \in \mathcal{H}$ such that $\Psi(\widehat{\mathcal{G}}^{(\nu)}) = \inf_{\mathcal{G} \in \mathcal{H}} \Psi(\mathcal{G})$, where Ψ is as in*
 1190 *(D.6). Then $\widehat{\mathcal{G}}^{(\nu)}$ satisfies the compact operator equation $\mathcal{K}_\nu \widehat{\mathcal{G}}^{(\nu)} = \iota_\nu^* \mathcal{G}^*$ in \mathcal{H} .*

1192 *Proof.* The result follows from convex optimization in separable Hilbert space. Since we assume
 1193 minimizers of Ψ exist, they are critical points. The Frechét derivative $D\Psi: \mathcal{H} \rightarrow \mathcal{H}^*$ is
 1194

$$1195 \mathcal{G} \mapsto D\Psi(\mathcal{G}) = \langle \iota_\nu^*(\iota_\nu \mathcal{G} - \mathcal{G}^*), \cdot \rangle_\kappa.$$

1196 At a critical point \mathcal{G} , we have $D\Psi(\mathcal{G}) = 0$ in \mathcal{H}^* . This means

$$1197 \langle \iota_\nu^*(\iota_\nu \mathcal{G} - \mathcal{G}^*), h \rangle_\kappa = 0 \quad \text{for all } h \in \mathcal{H}.$$

1198 By choosing $h = \iota_\nu^*(\iota_\nu \mathcal{G} - \mathcal{G}^*)$, it follows that $\iota_\nu^* \iota_\nu \mathcal{G} = \mathcal{K}_\nu \mathcal{G} = \iota_\nu^* \mathcal{G}^*$ in \mathcal{H} as asserted. \square

1200 Next, recall the Lagrangian \mathcal{L} from (4.2). We prove Lem. 4.2 on the adjoint state equation.
 1201

1202 *Proof of Lem. 4.2.* The action of the Frechét derivative with respect to the second coordinate is
 1203

$$1204 D_2 \mathcal{L}(\nu, \mathcal{G}, \lambda) = \mathbb{E}_{\nu' \sim \mathbb{Q}} \langle \iota_{\nu'}^*(\iota_{\nu'} \mathcal{G} - \mathcal{G}^*), \cdot \rangle_\kappa + \langle \mathcal{K}_\nu \lambda, \cdot \rangle_\kappa \in \mathcal{H}^*.$$

1205 Setting the preceding display equal to $0 \in \mathcal{H}^*$, we deduce that

$$1206 \mathcal{K}_\nu \lambda = \mathbb{E}_{\nu' \sim \mathbb{Q}} [\iota_{\nu'}^*(\mathcal{G}^* - \iota_{\nu'} \mathcal{G})] \quad \text{in } \mathcal{H}.$$

1207 Simplifying the right-hand side using the definition of ι_ν^* completes the proof. \square

1209 We are now in a position to prove Prop. 4.3, which computes the first variation of $\mathcal{J}: \nu \mapsto$
 1210 $\mathcal{L}(\nu, \widehat{\mathcal{G}}^{(\nu)}, \lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)}))$ in the sense of Santambrogio (2015, Def. 7.12, p. 262).

1212 *Proof of Prop. 4.3.* The first variation (Santambrogio, 2015, Def. 7.12, p. 262) of \mathcal{L} with respect to
 1213 the variable ν is given by the function

$$1215 D_1 \mathcal{L}(\nu, \mathcal{G}, \lambda) = (\mathcal{G} - \mathcal{G}^*) \lambda$$

1216 that maps from \mathcal{U} to \mathbb{R} , which is independent of ν given \mathcal{G} and λ . Now let $\widehat{\mathcal{G}}^{(\nu)}$ be as in Lem. D.1 and
 1217 $\lambda^{(\nu)}$ be as in Lem. 4.2. Due to the smoothness of \mathcal{L} (4.2) with respect to both \mathcal{G} and ν —in particular,
 1218 \mathcal{L} is convex in \mathcal{G} and linear in ν —the envelope theorem (Afriat, 1971) ensures that

$$1220 (D\mathcal{J})(\nu) = (D_1 \mathcal{L})|_{(\nu, \mathcal{G}, \lambda) = (\nu, \widehat{\mathcal{G}}^{(\nu)}, \lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)}))} = (\widehat{\mathcal{G}}^{(\nu)} - \mathcal{G}^*) \lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)})$$

1221 as asserted. \square

1223 **Remark D.2** (Wasserstein gradient). As discussed in Rmk. 4.4, it is clear that

$$1224 \nabla[(D\mathcal{J})(\nu)] = (\nabla \widehat{\mathcal{G}}^{(\nu)} - \nabla \mathcal{G}^*) \lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)}) + (\widehat{\mathcal{G}}^{(\nu)} - \mathcal{G}^*) \nabla(\lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)}))$$

1226 is the Wasserstein gradient of \mathcal{J} (Santambrogio, 2015, Chp. 8.2). Its computation requires the usual
 1227 \mathcal{U} -gradient of the trained model $\widehat{\mathcal{G}}^{(\nu)}$, the true map \mathcal{G}^* , and the optimal adjoint state $\lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)})$.

1228 We now turn to the parametric training distribution setting. Let $\nu = \nu_\vartheta$, where $\vartheta \in \mathcal{V}$ and
 1229 $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}}, \|\cdot\|_{\mathcal{V}})$ is a parameter Hilbert space as introduced in Sec. 4.1. The main result of Sec. 4.1
 1230 is Thm. 4.5. Before proving that theorem, we require the following lemma.

1231 **Lemma D.3** (differentiation under the integral). *Let the density $p_\vartheta := d\nu_\vartheta/d\mu_0$ be such that*
 1232 *$(u, \vartheta) \mapsto p_\vartheta(u)$ is sufficiently regular. Then for all bounded continuous functions $\phi \in C_b(\mathcal{U})$ and all*
 1233 *$\vartheta \in \mathcal{V}$, it holds that*

$$1234 \nabla_\vartheta \int_{\mathcal{U}} \phi(u) \nu_\vartheta(du) = \int_{\mathcal{U}} \phi(u) \left(\nabla_\vartheta \log p_\vartheta(u) \right) \nu_\vartheta(du). \quad (\text{D.7})$$

1237 *Proof.* By the assumed smoothness and absolute continuity with respect to μ_0 ,

$$1239 \nabla_\vartheta \int_{\mathcal{U}} \phi(u) \nu_\vartheta(du) = \int_{\mathcal{U}} \phi(u) \nabla_\vartheta p_\vartheta(u) \mu_0(du).$$

1241 Shrinking the domain of integration to the support of ν_ϑ , multiplying and dividing the rightmost
 1242 equality by $p_\vartheta(u)$, and using the logarithmic derivative formula delivers the assertion. \square

1242 With the preceding lemma in hand, we are now able to prove Thm. 4.5.
 1243

1244 *Proof of Thm. 4.5.* We use the chain rule for Frechét derivatives to compute $D\mathbf{J}: \mathcal{V} \rightarrow \mathcal{V}^*$ and hence
 1245 the gradient $\nabla \mathbf{J}: \mathcal{V} \rightarrow \mathcal{V}$. Define $Z: \mathcal{V} \rightarrow \mathcal{P}_2(\mathcal{U})$ by $Z(\vartheta) = \nu_\vartheta$. Then Lem. D.3 and duality allow
 1246 us to identify DZ with the (possibly signed) \mathcal{V} -valued measure

$$1247 \quad (DZ(\vartheta))(du) = (\nabla_\vartheta \log p_\vartheta(u))Z(\vartheta)(du).$$

1248 We next apply the chain rule for Frechét derivatives to the composition $\mathbf{J} = \mathcal{J} \circ Z$. This yields
 1249

$$1250 \quad D\mathbf{J}(\vartheta) = (D\mathcal{J})(Z(\vartheta)) \circ (DZ)(\vartheta) \in \mathcal{V}^*$$

1252 for any $\vartheta \in \mathcal{V}$. To interpret this expression, let $\vartheta' \in \mathcal{V}$ be arbitrary. Then formally,
 1253

$$1254 \quad (D\mathbf{J}(\vartheta))(\vartheta') = \int_{\mathcal{U}} (\widehat{\mathcal{G}}^{(Z(\vartheta))}(u) - \mathcal{G}^*(u))(\lambda^{(Z(\vartheta))}(\widehat{\mathcal{G}}^{(Z(\vartheta))}))(u) \langle (DZ(\vartheta))(du), \vartheta' \rangle_{\mathcal{V}} \\ 1255 \\ 1256 \quad = \left\langle \int_{\mathcal{U}} (\widehat{\mathcal{G}}^{(\nu_\vartheta)}(u) - \mathcal{G}^*(u))(\lambda^{(\nu_\vartheta)}(\widehat{\mathcal{G}}^{(\nu_\vartheta)}))(u) (\nabla_\vartheta \log p_\vartheta(u))\nu_\vartheta(du), \vartheta' \right\rangle_{\mathcal{V}}. \\ 1257$$

1258 The formula (4.4) for $\nabla \mathbf{J}$ follows. \square
 1259

1260 **Discretized implementation.** A practical, discretized implementation of the parametric bilevel
 1261 gradient descent scheme Alg. 1 further requires regularization of compact operator equations and
 1262 finite data approximations. When the inner objective of (4.1) is approximated over a finite data
 1263 set, we have the following regularized solution for the model update obtained from kernel ridge
 1264 regression. The lemma uses the compact vector and matrix notation $U = (u_1, \dots, u_N)^\top \in \mathcal{U}^N$,
 1265 $\mathcal{G}^*(U)_n = \mathcal{G}^*(u_n)$ for every n , and $\kappa(U, U)_{ij} = \kappa(u_i, u_j)$ for every i and j .

1266 **Lemma D.4** (model update). *Fix a regularization strength $\sigma > 0$. Let $\{u_n\}_{n=1}^N \sim \nu^{\otimes N}$. Then*

$$1268 \quad \widehat{\mathcal{G}}_N^{(\nu, \sigma)} := \sum_{n=1}^N \kappa(\cdot, u_n) \beta_n \approx \widehat{\mathcal{G}}^{(\nu)}, \quad \text{where } \beta = (\kappa(U, U) + N\sigma^2 I_N)^{-1} \mathcal{G}^*(U) \in \mathbb{R}^N. \quad (\text{D.8})$$

1271 *Proof.* To numerically update \mathcal{G} by solving $\mathcal{K}_\nu \mathcal{G} = \iota_\nu^* \mathcal{G}^*$ in \mathcal{H} for fixed ν , we first regularize the
 1272 inverse with a small nugget parameter $\sigma > 0$. Then we define
 1273

$$1274 \quad \widehat{\mathcal{G}}^{(\nu, \sigma)} := (\mathcal{K}_\nu + \sigma^2 \text{Id}_{\mathcal{H}})^{-1} \iota_\nu^* \mathcal{G}^* = \iota_\nu^* (\iota_\nu \iota_\nu^* + \sigma^2 \text{Id}_{L_\nu^2})^{-1} \mathcal{G}^* \approx \widehat{\mathcal{G}}^{(\nu)}.$$

1275 The second equality is due to the usual Woodbury push-through identity for compact linear operators.
 1276 Let $\nu^{(N)} := \frac{1}{N} \sum_{n=1}^N \delta_{u_n}$ be the empirical measure with $u_n \stackrel{\text{i.i.d.}}{\sim} \nu$. We identify $L_{\nu^{(N)}}^2$ with \mathbb{R}^N
 1277 equipped with the usual (i.e., unscaled) Euclidean norm. The action of $\mathcal{K}_{\nu^{(N)}}$ on a function $h \in \mathcal{H}$ is
 1278

$$1279 \quad \mathcal{K}_{\nu^{(N)}} h = \frac{1}{N} \sum_{n=1}^N \kappa(\cdot, u_n) h(u_n) =: \frac{1}{N} \kappa(\cdot, U) h(U) \quad (\text{D.9})$$

1282 and similar for $\iota_{\nu^{(N)}}^*$ acting on \mathbb{R}^N . Define $\widehat{\mathcal{G}}_N^{(\nu, \sigma)} := \widehat{\mathcal{G}}^{(\nu^{(N)}, \sigma)}$. By the preceding displays,
 1283

$$1284 \quad \widehat{\mathcal{G}}_N^{(\nu, \sigma)} = \sum_{n=1}^N \kappa(\cdot, u_n) \beta_n, \quad \text{where } \beta = (\kappa(U, U) + N\sigma^2 I_N)^{-1} \mathcal{G}^*(U),$$

1287 which recovers standard kernel ridge regression under square loss and Gaussian likelihood. This
 1288 implies the asserted approximation $\widehat{\mathcal{G}}_N^{(\nu, \sigma)} \approx \widehat{\mathcal{G}}^{(\nu)}$. \square
 1289

1290 In Lem. D.4, we expect that $\widehat{\mathcal{G}}_N^{(\nu, \sigma)} \rightarrow \widehat{\mathcal{G}}^{(\nu)}$ as $N \rightarrow \infty$ and $\sigma \rightarrow 0$. This lemma takes care of the
 1291 finite data approximation of the model. Next, we tackle the adjoints.
 1292

1293 When discretizing the integral in the gradient (4.4) with Monte Carlo sampling over the training data,
 1294 we recognize that only the values of $\lambda = \lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)})$ at the training points are needed and not $\lambda \in \mathcal{H}$
 1295 itself. That is, we never need to query λ away from the current training data samples. The next lemma
 1296 exploits this fact to numerically implement the adjoint state update.

1296 **Lemma D.5** (adjoint state update). *For $J \in \mathbb{N}$ and a sequence $\{M_j\} \subseteq \mathbb{N}$, define $\mathbb{Q}^{(J)} :=$
 1297 $\frac{1}{J} \sum_{j=1}^J \delta_{\mu_j^{(M_j)}}$, where $\mu_j^{(M_j)} := \frac{1}{M_j} \sum_{m=1}^{M_j} \delta_{v_m^j}$, $V^j := \{v_m^j\}_{m=1}^{M_j} \sim \mu_j^{\otimes M_j}$, and $\{\mu_j\}_{j=1}^J \sim \mathbb{Q}^{\otimes J}$.
 1298 Then with $\widehat{\mathcal{G}}_N^{(\nu, \sigma)}$ and U as in (D.8), the adjoint values $(\lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)}))(U) \in \mathbb{R}^N$ are approximated by
 1299*

1300
$$\widehat{\lambda}_{N,J}^{(\nu, \sigma)}(U) := \left(\kappa(U, U) + N\sigma^2 I_N \right)^{-1} \frac{N}{J} \sum_{j=1}^J \frac{1}{M_j} \kappa(U, V^j) \left(\mathcal{G}^*(V^j) - \widehat{\mathcal{G}}_N^{(\nu, \sigma)}(V^j) \right). \quad (\text{D.10})$$

1304 *Proof.* Let $\mu \sim \mathbb{Q}$, where $\mathbb{Q} \in \mathcal{P}_2(\mathcal{P}_2(\mathcal{U}))$ is fixed. For some M , let $\mu^{(M)} = \frac{1}{M} \sum_{m=1}^M \delta_{v_m}$ be
 1305 the empirical measure with $V = \{v_m\}_{m=1}^M \sim \mu^{\otimes M}$. Consider the case $J = 1$ so that $\mathbb{Q}^{(1)} = \delta_{\mu^{(M)}}$;
 1306 the general case for $J > 1$ as asserted in the lemma follows by linearity. With $\nu = \nu^{(N)}$ being
 1307 the empirical data measure corresponding to $U \in \mathcal{U}^N$, $\mathcal{G} = \widehat{\mathcal{G}}_N^{(\nu, \sigma)}$ being the trained model from
 1308 Lem. D.4, and $\mathbb{Q}^{(1)}$ in place of \mathbb{Q} , the adjoint equation (4.3) from Lem. 4.2 is
 1309

1311
$$\mathcal{K}_{\nu^{(N)}} \lambda = \frac{1}{M} \sum_{m=1}^M \kappa(\cdot, v_m) \left(\mathcal{G}^*(v_m) - \widehat{\mathcal{G}}_N^{(\nu, \sigma)}(v_m) \right).$$

1314 By evaluating the preceding display at the N data points $U = \{u_n\}_{n=1}^N$ and using (D.9), we obtain

1315
$$\frac{1}{N} \kappa(U, U) \lambda(U) = \frac{1}{M} \kappa(U, V) \left(\mathcal{G}^*(V) - \widehat{\mathcal{G}}_N^{(\nu, \sigma)}(V) \right).$$

1317 This is a linear equation for the unknown $\lambda(U) \in \mathbb{R}^N$. Finally, we regularize the equation by replacing
 1318 $\kappa(U, U)$ with $\kappa(U, U) + \sigma^2 I_N$. We denote the solution of the resulting system by $\widehat{\lambda}_{N,1}^{(\nu, \sigma)}(U)$, which
 1319 is our desired approximation (D.10) of $(\lambda^{(\nu)}(\widehat{\mathcal{G}}^{(\nu)}))(U) \in \mathbb{R}^N$. \square
 1320

1322 In Lem. D.5, the vector $\widehat{\lambda}_{N,J}^{(\nu, \sigma)}(U) \in \mathbb{R}^N$ depends on ν (through the empirical measure $\nu^{(N)} :=$
 1323 $\frac{1}{N} \sum_{n=1}^N \delta_{u_n}$), the regularization parameter σ , the current model $\widehat{\mathcal{G}}_N^{(\nu, \sigma)}$, and the random empirical
 1324 measure $\mathbb{Q}^{(J)}$ (which itself depends on the sequence $\{M_j\}_{j=1}^J$). The formula (D.10) requires the
 1325 inversion of the same regularized kernel matrix as in the model update (D.8). Thus, it is advantageous
 1326 to pre-compute a factorization of this matrix so that it may be reused in the adjoint update step.
 1327

1328 Finally, we perform the numerically implementable training data distribution update by inserting
 1329 the approximations $\widehat{\mathcal{G}}_N^{(\nu, \sigma)}$ and $\widehat{\lambda}_{N,J}^{(\nu, \sigma)}(U)$ from Lems. D.4–D.5 and empirical measure $\nu_\vartheta^{(N)}$ into the
 1330 exact gradient formula (4.4) for $\nabla \mathcal{J}$. This yields the approximate gradient

1331
$$\widehat{\nabla \mathcal{J}}(\vartheta) := \int_{\mathcal{U}} \left(\widehat{\mathcal{G}}_N^{(\nu_\vartheta, \sigma)}(u) - \mathcal{G}^*(u) \right) \widehat{\lambda}_{N,J}^{(\nu_\vartheta, \sigma)}(u) \left(\nabla_\vartheta \log p_\vartheta(u) \right) \nu_\vartheta^{(N)}(du), \quad (\text{D.11})$$

1334 where $\nu_\vartheta^{(N)}$ is the empirical measure of the same training data points used to obtain both the
 1335 trained model and adjoint state vector, and we identify the vector $\widehat{\lambda}_{N,J}^{(\nu, \sigma)}(U) \in \mathbb{R}^N$ with a function
 1336 $\widehat{\lambda}_{N,J}^{(\nu, \sigma)} \in L^2_{\nu_\vartheta^{(N)}}$. This leads to Alg. 3. In this practical algorithm, we allow for a varying number N_k
 1337 of training samples from $\nu_{\vartheta(k)}$ and a varying regularization strength σ_k at each iteration k in Line 4.
 1338 The integrals in (D.11) and Line 4 of Alg. 3 are with respect to the empirical measure; hence, they
 1339 equal the equally-weighted average over the finite data samples that make up the empirical measure.
 1340

1341 One limitation of the practical bilevel gradient descent scheme Alg. 3 as written is that it is negatively
 1342 impacted by overfitting in the model update step, as discussed in the following remark.
 1343

1344 **Remark D.6** (overfitting and vanishing gradients). A drawback of approximating the integral in the
 1345 exact gradient (4.4) with a Monte Carlo average over the i.i.d. training samples in (D.11) and Line 4
 1346 of Alg. 3 is that the resulting gradient approximation incorporates the training error residual. If the
 1347 trained model overfits to the data, then this residual becomes extremely small, which slows down
 1348 the convergence of gradient descent significantly. In the extreme case, the gradient vanishes if the
 1349 model perfectly fits the training data. Thus, the proposed method as currently formulated would
 likely fail for highly overparametrized models. To address this problem, one solution is to hold out a
 “validation” subset of the training data pairs for the adjoint state and Monte Carlo integral calculations;

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Algorithm 3 Gradient Descent on a Parametrized Bilevel Objective: Discretized Scheme

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1: **Initialize:** Parameter $\vartheta^{(0)}$, step sizes $\{t_k\}$, nuggets $\{\sigma_k\}$, sample sizes $\{N_k\}$, measure $\mathbb{Q}^{(J)}$ 2: **for** $k = 0, 1, 2, \dots$ **do**3: **Sample and label:** Construct the empirical measure $\nu_{\vartheta^{(k)}}^{(N_k)}$ and label vector $\mathcal{G}^*(U) \in \mathbb{R}^{N_k}$ 4: **Gradient step:** Update the training distribution's parameters via

$$\vartheta^{(k+1)} = \vartheta^{(k)} - t_k \int_{\mathcal{U}} \left(\widehat{\mathcal{G}}_{N_k}^{(\nu_{\vartheta^{(k)}}, \sigma_k)}(u) - \mathcal{G}^*(u) \right) \widehat{\lambda}_{N_k, J}^{(\nu_{\vartheta^{(k)}}, \sigma_k)}(u) (\nabla_{\vartheta} \log p_{\vartheta^{(k)}}(u)) \nu_{\vartheta^{(k)}}^{(N_k)}(du)$$

5: **if** stopping criterion is met **then**6: Return $\vartheta^{(k+1)}$ **then break**

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the remaining data should be used to train the model. Another approach is to normalize the size of the approximate gradient by its current magnitude. This adaptive normalization aims to avoid small gradients. In the numerical experiments of Sec. 5, we use kernel ridge regression with strictly positive regularization; this prevents interpolation of the data and helps avoid the vanishing gradient issue.

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D.3 DERIVATIONS FOR SUBSECTION 4.2: UPPER-BOUND MINIMIZATION

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Recall the alternating minimization algorithm Alg. 2 from Sec. 4.2. We provide details about its numerical implementation. For gradient-based implementations, the first variation of the upper bound objective (4.8) is required. This is the content of Prop. 4.9, which we now prove.

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Proof of Prop. 4.9. Since the first term $\int_{\mathcal{U}} f(u) \nu(du)$ in the objective F in (4.8) is linear in the argument ν , the first variation of that term is simply f itself (Santambrogio, 2015, Def. 7.12, p. 262). The chain rule of Euclidean calculus and a similar first variation calculation delivers the second term in the asserted derivative (4.9). The third and final term arising from the chain rule requires the first variation of $\nu \mapsto \mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu, \nu')$. By Santambrogio (2015, Prop. 7.17, p. 264, applied with cost $c(x, y) = \|x - y\|_{\mathcal{U}}^2/2$), this first variation equals the Kantorovich potential $2\phi_{\nu, \nu'}$, where the asserted characterization of $\phi_{\nu, \nu'}$ is due to Brenier's theorem; see Chewi (2024, Thm. 1.3.8, p. 27) or (Villani, 2021). Finally, we exchange expectation over \mathbb{Q} and differentiation using dominated convergence due to the assumed regularity of the set \mathbb{P} . \square

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The Wasserstein gradient of F is useful for interacting particle discretizations. We have the following corollary of Prop. 4.9.

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Corollary D.7 (Wasserstein gradient). *Instate the hypotheses of Prop. 4.9. The Wasserstein gradient ∇DF at $\nu \in \mathbb{P} \subseteq \mathcal{P}_2(\mathcal{U})$ is given by the function mapping any $u \in \mathcal{U}$ to the vector*

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$$((\nabla DF)(\nu))(u) = \nabla f(u) + \sqrt{\frac{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu, \nu')}{1 + m_2(\nu)}} u + \sqrt{\frac{1 + m_2(\nu)}{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu, \nu')}} (u - \mathbb{E}_{\nu' \sim \mathbb{Q}} T_{\nu \rightarrow \nu'}(u)).$$

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Alternatively, one can derive standard gradients when working with a parametrized family $\{\nu_{\vartheta}\}_{\vartheta \in \mathcal{V}}$ of candidate training distributions. The next lemma instantiates this idea in the specific setting of a finite-dimensional Gaussian family.

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Lemma D.8 (chain rule: Gaussians). *Instate the hypotheses of Prop. 4.9. Suppose that $\mathcal{U} \subseteq \mathbb{R}^d$. Let $\nu_{\vartheta} := \mathcal{N}(m_{\vartheta}, C_{\vartheta})$ and $F(\vartheta) := F(\nu_{\vartheta})$, where F is given in (4.8). Write $p_{\vartheta} : \mathcal{U} \rightarrow \mathbb{R}_{\geq 0}$ for the density of ν_{ϑ} with respect to any σ -finite dominating measure. Then for any parameter $\vartheta \in \mathcal{V}$, it*

1404 holds that

$$\begin{aligned}
 \nabla F(\vartheta) &= \int_{\mathcal{U}} f(u) (\nabla_{\vartheta} \log p_{\vartheta}(u)) \nu_{\vartheta}(du) \\
 &\quad + \frac{1}{2} \sqrt{\frac{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu_{\vartheta}, \nu')}{1 + \|m_{\vartheta}\|_{\mathcal{U}}^2 + \text{tr}(\mathcal{C}_{\vartheta})}} \int_{\mathcal{U}} \frac{\|u\|_{\mathcal{U}}^2}{2} (\nabla_{\vartheta} \log p_{\vartheta}(u)) \nu_{\vartheta}(du) \\
 &\quad + \sqrt{\frac{1 + \|m_{\vartheta}\|_{\mathcal{U}}^2 + \text{tr}(\mathcal{C}_{\vartheta})}{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu_{\vartheta}, \nu')}} \int_{\mathcal{U}} \frac{1}{2} \langle u, (I_d - \mathbb{E}_{\mathbb{Q}}[A'_{\vartheta}])u \rangle_{\mathcal{U}} (\nabla_{\vartheta} \log p_{\vartheta}(u)) \nu_{\vartheta}(du) \\
 &\quad + \sqrt{\frac{1 + \|m_{\vartheta}\|_{\mathcal{U}}^2 + \text{tr}(\mathcal{C}_{\vartheta})}{\mathbb{E}_{\nu' \sim \mathbb{Q}} W_2^2(\nu_{\vartheta}, \nu')}} \int_{\mathcal{U}} \langle u, \mathbb{E}_{\mathbb{Q}}[A'_{\vartheta}]m_{\vartheta} - \mathbb{E}_{\mathbb{Q}}[m'] \rangle_{\mathcal{U}} (\nabla_{\vartheta} \log p_{\vartheta}(u)) \nu_{\vartheta}(du),
 \end{aligned}$$

1417 where $\nu' = \mathcal{N}(m', \mathcal{C}') \sim \mathbb{Q}$ is a random Gaussian and $A'_{\vartheta} := \mathcal{C}_{\vartheta}^{-1/2} (\mathcal{C}_{\vartheta}^{1/2} \mathcal{C}' \mathcal{C}_{\vartheta}^{1/2})^{1/2} \mathcal{C}_{\vartheta}^{-1/2}$.

1419 *Proof.* The W_2 -optimal transport map from $\nu = \mathcal{N}(m_1, \mathcal{C}_1)$ to $\nu' = \mathcal{N}(m_2, \mathcal{C}_2)$ is

$$1421 \quad u \mapsto T_{\nu \rightarrow \nu'}(u) := m_2 + A(u - m_1), \quad \text{where } A := \mathcal{C}_1^{-1/2} (\mathcal{C}_1^{1/2} \mathcal{C}_2 \mathcal{C}_1^{1/2})^{1/2} \mathcal{C}_1^{-1/2}.$$

1423 Furthermore, $\phi_{\nu, \nu'} = \|\cdot\|_{\mathcal{U}}^2/2 - \varphi_{\nu, \nu'}$, where $\nabla \varphi_{\nu, \nu'} = T_{\nu \rightarrow \nu'}$ by Brenier's theorem (Chewi, 2024, Thm. 1.3.8, p. 27). Upon integration, we find that

$$1425 \quad u \mapsto \phi_{\nu, \nu'}(u) = \frac{\|u\|^2}{2} - \frac{1}{2} \langle u, Au \rangle - \langle u, m_2 - Am_1 \rangle.$$

1428 This formula is unique up to a constant. Now let $\nu := \nu_{\vartheta} = \mathcal{N}(m_{\vartheta}, \mathcal{C}_{\vartheta})$ and $\nu' = \mathcal{N}(m', \mathcal{C}') \sim \mathbb{Q}$ be as in the hypotheses of the lemma. By the chain rule for Frechét derivatives,

$$1430 \quad \nabla F(\vartheta) = \int_{\mathcal{U}} (DF(\nu_{\vartheta}))(u) (\nabla_{\vartheta} \log p_{\vartheta}(u)) \nu_{\vartheta}(du)$$

1433 for each ϑ . Applying (4.9) from Prop. 4.9, closed-form formulas for the second moments of Gaussian distributions, and the Fubini–Tonelli theorem to exchange integrals completes the proof. \square

1435 Lem. D.8 requires the gradient $\nabla_{\vartheta} \log p_{\vartheta}$ of the log density with respect to the parameter ϑ . In general, this gradient has no closed form and must be computed numerically, e.g., with automatic differentiation. However, when only the mean of the Gaussian family is allowed to vary, the following remark discusses an analytical expression for the gradient.

1440 **Remark D.9** (Gaussian mean parametrization). Let $\mathcal{U} \subseteq \mathbb{R}^d$. If $\vartheta = m$ so that $\nu_{\vartheta} = \nu_m = \mathcal{N}(m, \mathcal{C}_0)$ for fixed \mathcal{C}_0 , then $\nabla_m \log p_m(u) = \mathcal{C}_0^{-1}(u - m)$, where p_m is the Lebesgue density of ν_m .

1443 E OPERATOR LEARNING ARCHITECTURES

1445 In this appendix, we provide detailed descriptions of the various deep operator learning architectures used in the paper. These are the DeepONet (SM E.1), NIO (SM E.2), and the newly proposed AMINO (SM E.3).

1449 E.1 DEEPONET: DEEP OPERATOR NETWORK

1451 Deep operator networks (DeepONet) (Lu et al., 2021) provide a general framework for learning 1452 nonlinear operators $\mathcal{G}^*: u \mapsto \mathcal{G}^*(u)$ from data. A DeepONet \mathcal{G}_{θ} achieves this via two jointly trained 1453 subnetworks:

- 1454 • **Branch network B:** takes as input the values of the input function u at a fixed set of sensor 1455 locations $\{x^{(j)}\}_{j=1}^J$ and outputs a feature vector $b \in \mathbb{R}^p$ for some fixed latent dimension p .
- 1456 • **Trunk network T:** takes as input a query point z in the output domain and outputs feature 1457 vector $t(z) \in \mathbb{R}^p$.

1458 The operator prediction at function u and query point z is then formed by the inner product of these
 1459 features, yielding
 1460

$$1461 \mathcal{G}_\theta(u)(z) = \langle \mathbf{B}(u(x^{(1)}), \dots, u(x^{(J)})), \mathbf{T}(z) \rangle_{\mathbb{R}^p} = \sum_{k=1}^p \mathbf{B}_k(u(x^{(1)}), \dots, u(x^{(J)})) \mathbf{T}_k(z). \quad (\text{E.1})$$

1462

1463 In practice, a constant bias vector is added to the preceding display. During ERM training as in
 1464 (2.1), the weights θ of both neural networks \mathbf{B} and \mathbf{T} are optimized together to minimize the average
 1465 discrepancy between $\mathcal{G}_\theta(u)(z)$ and ground truth values $\mathcal{G}^*(u)(z)$ over a dataset of input-output
 1466 function pairs. This branch-trunk architecture enables DeepONet to approximate a broad class of
 1467 operators, including the Neumann-to-Dirichlet (NtD) map in EIT and the conductivity-to-solution
 1468 map in Darcy flow considered in this work; see Sec. 5 and **SM F**.
 1469

1470 **E.2 NIO: NEURAL INVERSE OPERATOR**
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1472 Traditional approaches for constructing direct solvers for inverse problems, such as the D-bar method
 1473 for EIT (Siltanen et al., 2000), have laid the foundation for direct solver approximations utilizing
 1474 modern machine learning techniques. Among these, the Neural Inverse Operator (NIO) (Molinaro
 1475 et al., 2023) framework stands out as a significant advancement. In the context of operator learning,
 1476 many inverse problems can be formulated as a mapping from an operator Λ_a , dependent on a physical
 1477 parameter a , to the parameter a itself, i.e., $\Lambda_a \mapsto a$. In practical scenarios, the operator Λ_a is not
 1478 directly accessible; instead, we observe its action on a set of input-output function pairs $\{(f_n, g_n)\}_n$,
 1479 where $g_n = \Lambda_a(f_n)$. This observation leads to a two-step problem: first, infer the operator Λ_a from
 1480 the data pairs, and second, deduce the parameter a from the inferred operator. This process can be
 1481 further generalized by considering the operator as a pushforward map $(\Lambda_a)_\#$, which transforms an
 1482 input distribution μ of functions to an output distribution ν_a of functions, i.e., $(\Lambda_a)_\# \mu = \nu_a$; see
 1483 Sec. 2 for the definition of the pushforward operation. Consequently, a measure-centric machine
 1484 learning approach to solving the inverse problem can be reformulated as learning the mapping
 1485 $(\mu, \nu_a) \mapsto a$ (Nelsen & Yang, 2025, Sec. 2.4.1).
 1486

1486 Conditioned on a fixed input distribution μ , NIO is designed to learn mappings from output measures
 1487 ν_a to functions a , i.e., $\nu_a \mapsto a$, effectively relaxing the operator-to-function problem to a distribution-
 1488 to-function problem. Since direct manipulation of measures is computationally infeasible, NIO
 1489 circumvents this by leveraging finite data approximations. Specifically, it utilizes a composition of
 1490 DeepONets and Fourier Neural Operators (FNO) (Li et al., 2021) to approximate the inverse mapping
 1491 from an empirical measure/approximation $\widehat{\nu}_a$ to the underlying parameter function a .
 1492

1493 **E.3 AMINO: A MEASURE-THEORETIC INVERSE NEURAL OPERATOR**

1494 Despite the innovative architecture of NIO, it exhibits certain limitations, particularly concerning its
 1495 handling of input distributions. NIO primarily focuses on the output distribution ν_a during evaluation,
 1496 neglecting variability and influence of the input distribution μ . This oversight can lead to models that
 1497 perform well on in-distribution (ID) data but lack robustness when faced with OOD scenarios. To
 1498 address this shortcoming, we introduce *A Measure-theoretic Inverse Neural Operator* (AMINO), a
 1499 framework that explicitly accounts for the input distribution in the inverse problem. By formulating
 1500 the inverse task in a fully measure-theoretic setting and taking into account the input distribution,
 1501 AMINO aims to improve OOD accuracy. Like NIO, AMINO combines a DeepONet and an FNO.
 1502 The key distinction is that AMINO incorporates both input and output functions as concatenated pairs
 1503 $\{(f_n, g_n)\}_n \stackrel{\text{i.i.d.}}{\sim} (\text{Id}, \Lambda_a)_\# \mu$ which are passed to the branch network, as illustrated in Fig. 7. See
 1504 (Nelsen & Yang, 2025, Sec. 4.2.3) for a more detailed description of NIO and AMINO.
 1505

1506 **F DETAILS FOR SECTION 5: NUMERICAL RESULTS**
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1508 This appendix describes detailed experimental setups for all numerical tests from Sec. 1 and Sec. 5
 1509 in the main text. Regarding function approximation, **SM F.1** covers the bilevel gradient descent
 1510 experiments. The remaining appendices concern operator learning. **SM F.2** details the AMINO
 1511 experiments. **SM F.3** and **SM F.4** describe the results of the alternating minimization algorithm when
 applied to learning linear NtD maps and the nonlinear solution operator of Darcy flow, respectively.

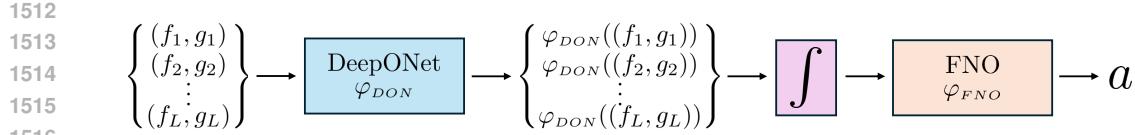


Figure 7: Schematic of the AMINO architecture, a variant of NIO, which combines a DeepONet, averaging, and an FNO module. The model takes as input the paired function samples $\{(f_e, g_e)\}_{e=1}^L$ drawn from the joint distribution $(\text{Id}, \Lambda_a)_\# \mu$ and outputs the target parameter a .

Last, **SM F.5** presents results from applying nonparametric particle-based AMA to learn a parameter-to-solution map for the radiative transport equation.

F.1 BILEVEL MINIMIZATION FOR FUNCTION APPROXIMATION

We now describe the setting of the numerical results reported in Sec. 5 for Alg. 1 and its numerical version Alg. 3. The goal of the experiments is to find the optimal Gaussian training distribution $\nu_\vartheta = \mathcal{N}(m, \mathcal{C}) \in \mathcal{P}_2(\mathbb{R}^d)$ with respect to the bilevel OOD objective (4.1), specialized to the task of function approximation. We optimize over the mean and covariance $\vartheta = (m, \mathcal{C})$. For all $x \in \mathbb{R}^d$, the target test functions $\mathcal{G}^* = g_i: \mathbb{R}^d \rightarrow \mathbb{R}$ for $i \in \{1, 2, 3, 4\}$ that we consider are given by

$$g_1(x) = \prod_{j=1}^d \frac{|4x_j - 2| + (j-2)/2}{1 + (j-2)/2}, \quad (\text{F.1a})$$

$$g_2(x) = 10 \sin(\pi x_1 x_2) + 20(x_3 - 1/2)^2 + 10x_4 + 5x_5, \quad (\text{F.1b})$$

$$g_3(x) = \sqrt{(100x_1)^2 + \left(x_3(520\pi x_2 + 40\pi) - \frac{1}{(520\pi x_2 + 40\pi)(10x_4 + 1)}\right)^2}, \quad \text{and} \quad (\text{F.1c})$$

$$g_4(x) = \sum_{l=1}^{1000} c_l \kappa(x, x_l). \quad (\text{F.1d})$$

The first function comes from (Dunbar et al., 2025) and the middle two from (Potts & Schmischke, 2021; Saha et al., 2023). They are common benchmarks in the approximation theory literature; g_1 is called the Sobol G function, g_2 the Friedmann-1 function, and g_3 the Friedmann-2 function. Although these functions are typically defined on compact domains (e.g., $[0, 1]^d$ or $[-\pi, \pi]^d$), we instead consider approximating them on average with respect to Gaussian distributions (or mixtures thereof) on the whole of \mathbb{R}^d . The final function g_4 is a linear combination of kernel sections for some kernel function $\kappa: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$. We sample (and then fix) the coefficients $c_l \stackrel{\text{i.i.d.}}{\sim} \text{Unif}([-1, 1])$ and the centers $x_l \stackrel{\text{i.i.d.}}{\sim} \text{Unif}([-4, 4])^{\otimes d}$. We allow the dimension d to be arbitrary for all test functions; in particular, the functions g_2 and g_3 are constant in certain directions for large enough d .

Model architecture and optimization algorithm. In all bilevel minimization experiments, we use the squared exponential kernel function $\kappa: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_{>0}$ defined by

$$(x, x') \mapsto \kappa(x, x') := \exp\left(-\frac{\|x - x'\|_2^2}{\ell^2}\right), \quad (\text{F.2})$$

where $\ell > 0$ is a scalar lengthscale hyperparameter. While not tuned for accuracy, we do adjust ℓ on the order of $O(1)$ to $O(10)$ depending on the target function g_i to avoid blow up of the bilevel gradient descent iterations. In particular, we set $\ell = 1$ for g_1 , $\ell = 3$ for g_2 , $\ell = 2/1.1$ for g_3 , and $\ell = 5$ for g_4 . We also experimented with the less regular Laplace kernel $(x, x') \mapsto \exp(-\|x - x'\|_1/\ell)$; although giving similar accuracy as κ , the Laplace kernel led to slower bilevel optimization. Thus, we opted to report all results using the squared exponential kernel κ . This same kernel is used to define g_4 in (F.1), which belongs to the RKHS of κ as a result. Given a candidate set of training samples, the resulting kernel method is trained on these samples via kernel ridge regression as in Lem. D.4. We solve all linear systems with lower triangular direct solves using Cholesky factors of

1566 the regularized kernel matrices. No iterative optimization algorithms are required to train the kernel
 1567 regressors. When fitting the kernel models to data outside of the bilevel gradient descent loop (see
 1568 the center and right columns of Fig. 8, for instance), we set the ridge strength to be $\sigma^2 = 10^{-3}/N$,
 1569 where N is the current training sample size.
 1570

1571 **Training and optimization procedure.** Moving on to describe the specific implementation details
 1572 of Alg. 3, we initialize $\vartheta^{(0)} = (m_0, I_d)$. We take $m_0 = \frac{1}{2}(1, \dots, 1)^\top \in \mathbb{R}^d$ in all setups except
 1573 for the one concerning test function g_1 , where we set $m_0 = (0, \dots, 0)^\top$. We decrease the gradient
 1574 descent step sizes $\{t_k\}$ in Alg. 3 according to a cosine annealing scheduler (Loshchilov & Hutter,
 1575 2017) with initial learning rate of 10^{-2} and final learning rate of 0. The regularization parameters
 1576 $\{\sigma_k\}$ are also selected according to the cosine annealing schedule with initial value 10^{-3} and final
 1577 value 10^{-7} . Although the sequence $\{N_k\}$ of training sample sizes could also be scheduled, we opt to
 1578 fix $N_k = 250$ for all k in Alg. 3 for simplicity. No major differences were observed when choosing
 1579 $\{N_k\}$ according to some cosine annealing schedule. The identification of more principled ways to
 1580 select the sequences $\{\sigma_k\}$, $\{t_k\}$, and $\{N_k\}$ to optimally balance cost and accuracy is an important
 1581 direction for future work.
 1582

1583 For the probability measure $\mathbb{Q} \in \mathcal{P}_2(\mathcal{P}_2(\mathbb{R}^d))$ over test distributions, we choose it to be the
 1584 empirical measure

$$1585 \mathbb{Q} := \frac{1}{K} \sum_{k=1}^K \delta_{\nu'_k}. \quad (F.3)$$

1586 The test measures $\nu'_k \in \mathcal{P}_2(\mathbb{R}^d)$ in (F.3) are deterministic and fixed. In applications, these could be
 1587 the datasets for which we require good OOD accuracy. However, to synthetically generate the $\{\nu'_k\}$
 1588 in practice, we set
 1589

$$1590 \nu'_k := \mathcal{N}(m'_k, \mathcal{C}'_k), \quad \text{where } m'_k \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, I_d) \quad \text{and} \quad \mathcal{C}'_k \stackrel{\text{i.i.d.}}{\sim} \text{Wishart}_d(I_d, d+1) \quad (F.4)$$

1591 for each $k = 1, \dots, K$. Once the K realizations of these distributions are generated, we view them
 1592 as fixed once and for all. We assume that K and \mathbb{Q} are fixed, but we only have access to the empirical
 1593 measure $\hat{\mathbb{Q}}$ defined by (F.3) but with each ν'_k replaced by the empirical measure $\frac{1}{M} \sum_{m=1}^M \delta_{v_m^{(k)}}$
 1594 generated by samples $\{v_m^{(k)}\}_{m=1}^M \sim \nu'_k^{\otimes M}$. In the experiments, we set $M = 5000$ and either $K = 10$
 1595 or $K = 1$ (a single test distribution). To create a fair test set, we use a fixed 500 sample validation
 1596 subset to update the adjoint state as in Lem. D.5 and to monitor the validation OOD accuracy during
 1597 gradient descent. The remaining 4500 samples are used to evaluate the final OOD test accuracy of
 1598 our trained models; see (F.5).
 1599

1600 Based on the available samples $\{v_m^{(k)}\}_{1 \leq m \leq M, 1 \leq k \leq K}$ associated to the meta test distribution \mathbb{Q} , we
 1601 construct six adaptive and nonadaptive baseline distributions for comparison:
 1602

1. (Normal) The Gaussian measure $\mathcal{N}(m_0, I_d)$ which is independent of \mathbb{Q} ;
2. (Barycenter) The empirical W_2 barycenter of \mathbb{Q} (which is Gaussian), as described in Ex. B.8;
3. (Mixture) The non-Gaussian empirical mixture of \mathbb{Q} given by $\frac{1}{K} \sum_{k=1}^K \mathcal{N}(\hat{m}_k, \hat{\mathcal{C}}_k)$, where
 \hat{m}_k and $\hat{\mathcal{C}}_k$ are the empirical mean and covariance of the points $\{v_m^{(k)}\}_{m=1}^M$, respectively.
Note that we enforce the mixture components to be Gaussian instead of empirical measures;
4. (Uniform) The d -dimensional uniform distribution $\text{Unif}([0, 1]^d) := \text{Unif}([0, 1])^{\otimes d}$, which is
independent of \mathbb{Q} .
5. (nCoreSet) Nonadaptive coresset described in SM F.6 with features $\kappa(\cdot, v_m) \in \mathcal{H}_\kappa$ for v_m
belonging to the fixed pool of $500K$ validation points. New indices m are selected according
to the maxmin sequential update (F.40) with features as above and distance between features
induced by the RKHS norm of κ . We use point-by-point updates and an initial random
coreset of size one.
6. (aCoreSet) Adaptive greedy coresset described in SM F.6 with features $\{c_n \kappa(u_n, v_m)\}_{n=1}^N \in \mathbb{R}^N$ for
 v_m belonging to the fixed pool of $500K$ validation points. The numbers $\{c_n\}$ are
the kernel method coefficients trained on points $\{u_n\}$, which makes this an adaptive method.
New indices m are selected according to the maxmin sequential update (F.40) with features

1620 as above and distance between features induced by the Euclidean norm on \mathbb{R}^N . We select
 1621 new points from the pool in batches of size ten and used an initial random coresset of size six.
 1622

1623 The pool-based active learning baselines are based on the work of Musekamp et al. (2025) and results
 1624 for these methods are shown in Sec. 5 and in Fig. 10.

1625 It remains to describe the optimization of the Gaussian family $\nu_\vartheta = \mathcal{N}(m, \mathcal{C})$ with parameter
 1626 $\vartheta = (m, \mathcal{C})$. The closed form update for the mean follows from Rmk. D.9. For the covariance matrix,
 1627 we optimize over the lower triangular Cholesky factor L of $\mathcal{C} = LL^\top \in \mathbb{R}^{d \times d}$ instead of \mathcal{C} itself.
 1628 This removes redundant parameters due to the symmetry of \mathcal{C} . We enforce strict positive definiteness
 1629 of \mathcal{C} by replacing all nonpositive diagonal entries of L with a threshold value of 10^{-7} . Other than this
 1630 constraint, each entry of L is optimized over the whole of \mathbb{R} . Recalling the log density formula (4.5)
 1631 from Ex. 4.6, we use automatic differentiation to compute $\nabla_L \log(p_{(m, LL^\top)})$; this factor appears in
 1632 the definition of the approximate gradient (D.11) and in Alg. 3.

1633
 1634 **Model evaluation.** We evaluate the performance of all kernel regressors $\widehat{\mathcal{G}}_N^{(\nu, \sigma)}$ regularized with
 1635 strength σ^2 and trained with N i.i.d. samples from ν with respect to the root relative average OOD
 1636 squared error, which is defined as

$$1637 \text{Err} := \sqrt{\frac{\mathbb{E}_{\widehat{\nu} \sim \widehat{\mathbb{Q}}} \mathbb{E}_{u' \sim \widehat{\nu}} |\mathcal{G}^*(u') - \widehat{\mathcal{G}}_N^{(\nu, \sigma)}(u')|^2}{\mathbb{E}_{\widehat{\nu} \sim \widehat{\mathbb{Q}}} \mathbb{E}_{u' \sim \widehat{\nu}} |\mathcal{G}^*(u')|^2}} \approx \sqrt{\frac{\mathcal{E}_{\mathbb{Q}}(\widehat{\mathcal{G}}_N^{(\nu, \sigma)})}{\mathcal{E}_{\mathbb{Q}}(0)}}. \quad (F.5)$$

1641 In the preceding display, we observe that Err employs a Monte Carlo discretization of the integrals
 1642 appearing in $\mathcal{E}_{\mathbb{Q}}$ by averaging over the empirical meta measure $\widehat{\mathbb{Q}}$, which is composed of K test
 1643 measures. Each individual test measure is itself an empirical measure with 4500 atoms at points that
 1644 are unseen during the adjoint state calculation. See the preceding paragraph for details about our
 1645 choice of \mathbb{Q} . We report values of Err in Figs. 2, 8, 10, 12, 14 and in Tab. 1.

1646 Supplementary numerical results are given in Figs. 8–15. Fig. 8 reports the same information that
 1647 Fig. 2 does, except now for the remaining functions $\{g_i\}$. The conclusions from these numerical
 1648 results are mostly the same. The left column shows the training history over 50 iterations; the
 1649 “seen” curve computes the OOD error with the \mathbb{Q} samples used in the bilevel algorithm, while the
 1650 “unseen” curve computes the OOD error on a held-out set of \mathbb{Q} samples. The third row shows a
 1651 slight “generalization gap” for $\mathcal{G}^* = g_2$ ($d = 8$). The optimized Gaussian outperforms alternative
 1652 distributions except for the g_3 ($d = 5$) and g_4 cases, where the empirical \mathbb{Q} mixture eventually
 1653 overtakes it. Fig. 10 shows that this behavior is largely mitigated if 1000 iterations of gradient descent
 1654 are used to find the optimal Gaussian instead of 50. We remark that in Fig. 2, the leftmost subplot
 1655 showing the training history is designed to highlight the long-time behavior (i.e., 1000 iterations) of
 1656 the bilevel algorithm. In practice, we only run the algorithm for $O(10)$ to $O(100)$ gradient descent
 1657 iterations; this is reflected in the leftmost column of Figs. 8 and 12.

1658 While the optimized Gaussian is shown to consistently outperform nonadaptive distributions when
 1659 \mathbb{Q} in (F.3) is composed of $K = 10$ atoms (Figs. 8 and 10) and N is large enough, this is less true
 1660 for a single test distribution ($K = 1$), as seen in Figs. 12 and 14 when N is sufficiently large. In
 1661 this case, the barycenter and mixture distributions are the same test distribution and lead to trained
 1662 models with lower OOD error. When N is small and many iterations are used to find the optimal
 1663 Gaussian, the optimal Gaussian can achieve smaller OOD error than models trained on the test
 1664 distribution. Understanding for what \mathcal{G}^* , \mathbb{Q} , N , and gradient descent iteration count makes this
 1665 desirable improvement possible is an interesting and important question for future work.

1666 To understand what properties the optimal Gaussian absorbs from \mathcal{G}^* and \mathbb{Q} , Figs. 9, 11, 13, and 15
 1667 visualize the learned covariance matrices, barycenter covariances, and mixture covariances. For
 1668 instance, the learned covariance puts high weight on the x_3 coordinate in the g_2 case (Fig. 11, rows 2–
 1669 3, diagonal entry of matrix) even though the barycenter and mixture do not. This makes sense because
 1670 g_2 is quadratic in x_3 and lower order in all other variables (recall (F.1)); thus, the learned covariance
 1671 picks out the highest order term in the ground truth map. Similarly, for g_3 the learned covariance
 1672 assigns high importance to the x_2 and x_3 coordinates which are also fast growing. In all cases (except
 1673 for g_4), the learned covariance differs substantially from the mixture or test distribution’s covariance.
 These covariance visualization figures highlight an additional benefit of our \mathcal{G}^* -dependent estimators:
 they lead to interpretable training distributions tailored to the structure of the problem.

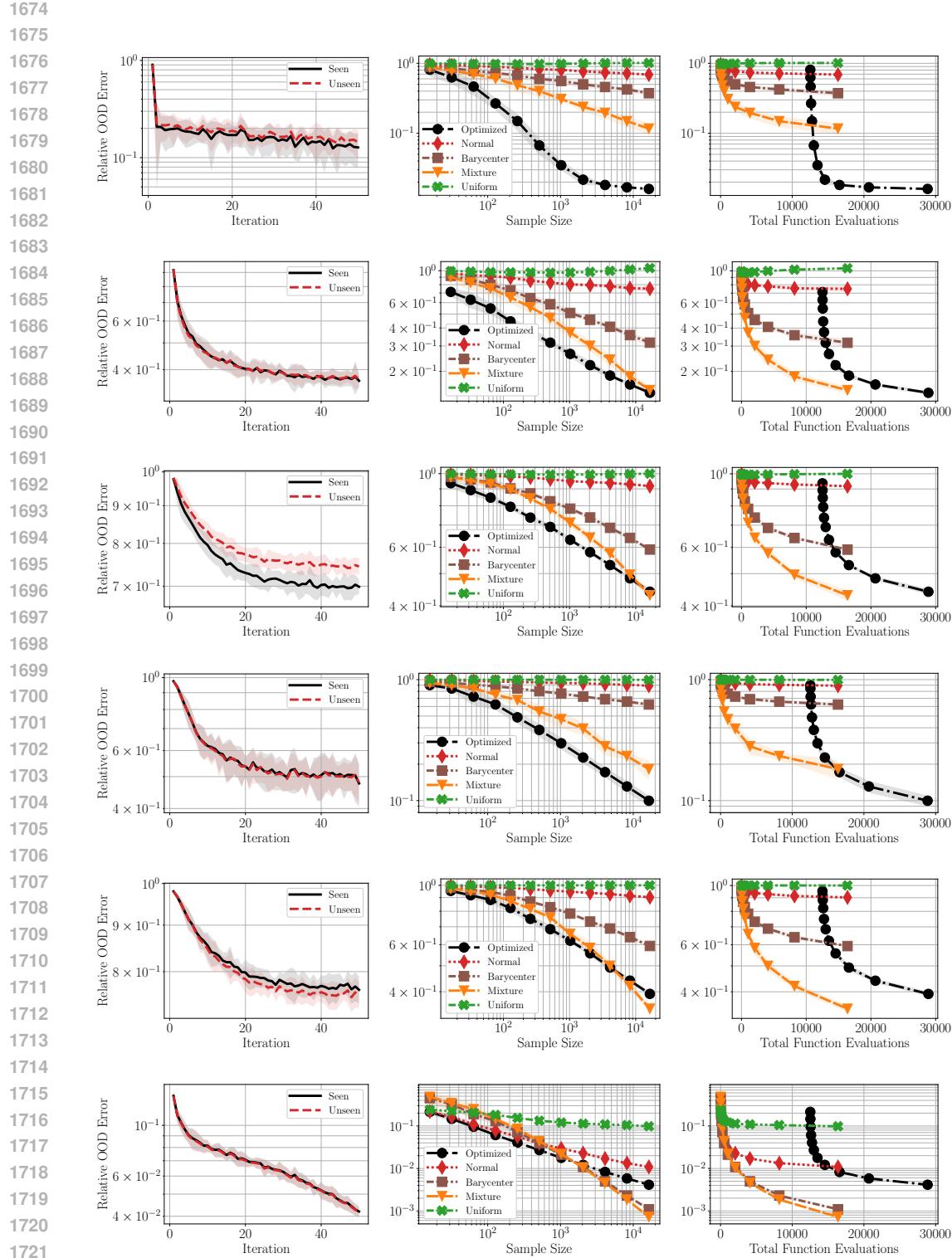
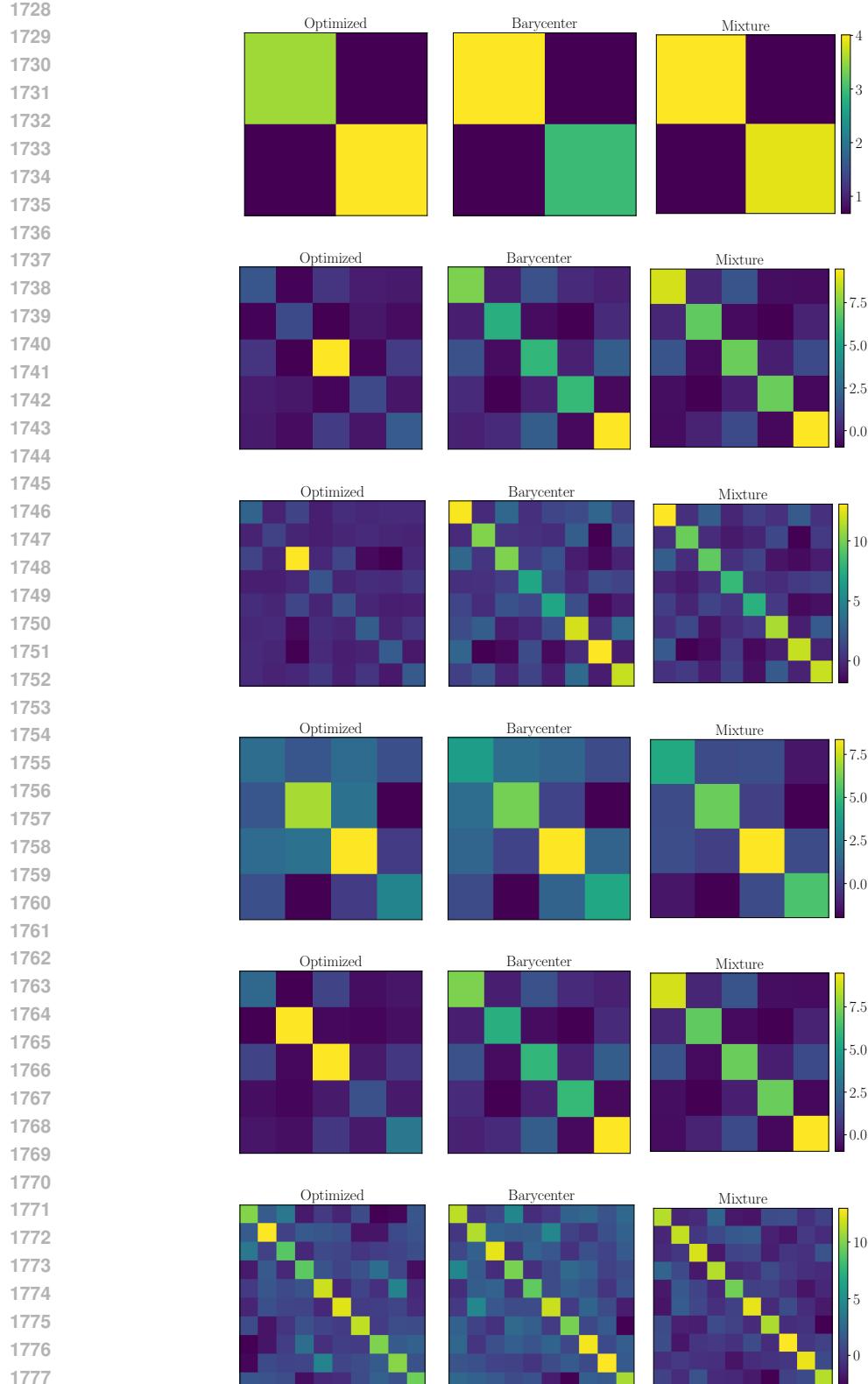


Figure 8: Bilevel Alg. 3 applied to the various ground truths $\{g_i\}$ with $K = 10$ test atoms and 50 gradient descent iterations. (Left column) Evolution of Err from (F.5) vs. iterations. (Center column) Err of model trained on N samples from optimal ν_ϑ vs. nonadaptive distributions. (Right column) Same as center, except incorporating the additional function evaluation cost incurred from Alg. 1. From top to bottom, each row represents target functions $\{g_i\}$ corresponding to those listed in each column of Tab. 1 (from left to right). Shading represents two standard deviations away from the mean Err over 10 independent runs.



1778 Figure 9: Bilevel Alg. 3 applied with $K = 10$ test atoms and 50 gradient descent iterations. Each column
 1779 visualizes the covariance matrix of the optimized (left), empirical $\mathbb{Q} W_2$ barycenter (center), and empirical \mathbb{Q}
 1780 mixture distributions (right). From top to bottom, each row represents target functions $\{g_i\}$ corresponding to
 1781 those listed in each column of Tab. 1 (from left to right).

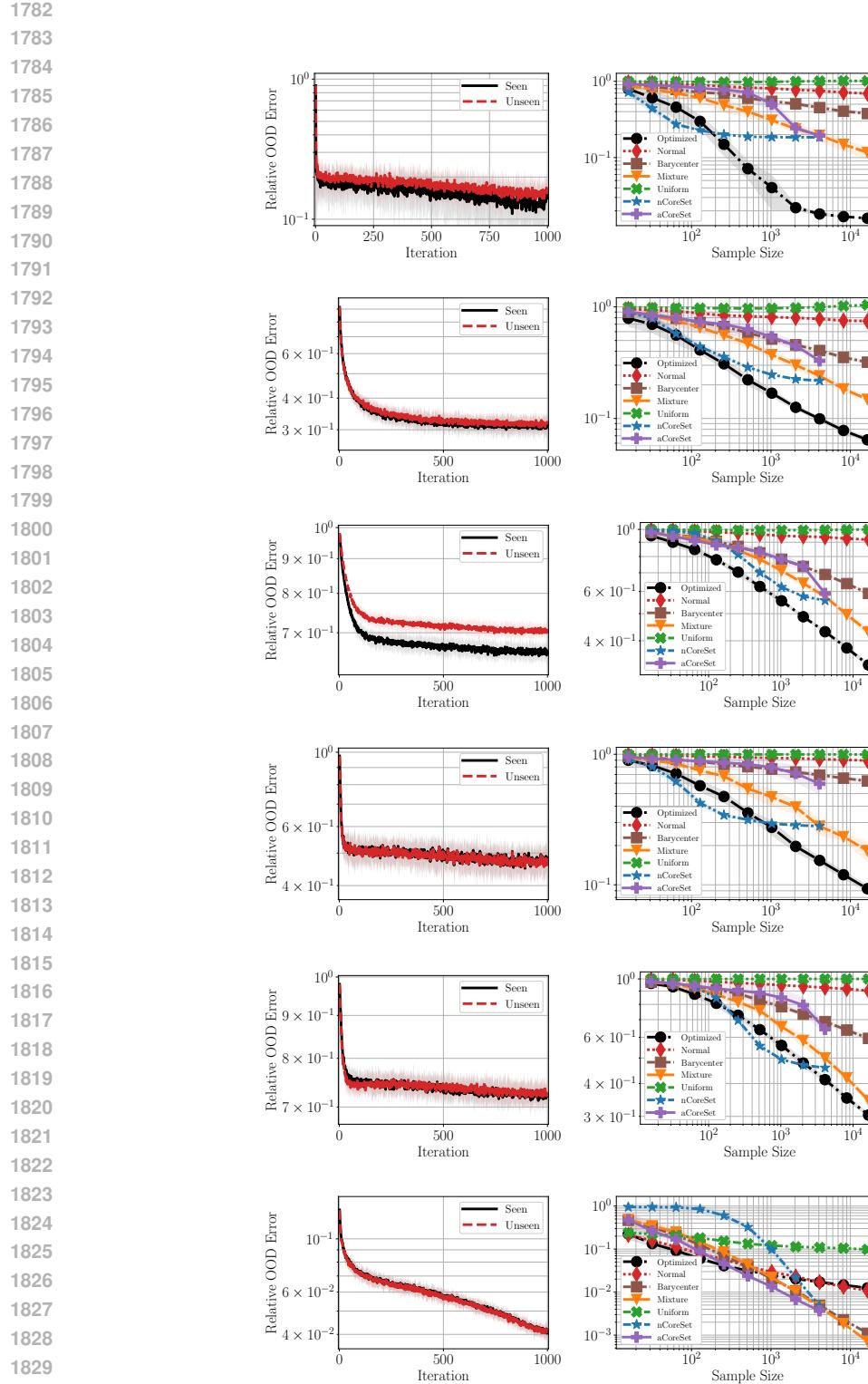


Figure 10: Bilevel Alg. 3 applied to the various ground truths $\{g_i\}$ with $K = 10$ test atoms and 1000 gradient descent iterations. (Left column) Evolution of Err from (F.5) vs. iterations. (Right column) Err of model trained on N samples from optimal ν_θ vs. baseline distributions. From top to bottom, each row represents target functions $\{g_i\}$ corresponding to those listed in each column of Tab. 1 (from left to right). Shading represents two standard deviations away from the mean Err over 10 independent runs.

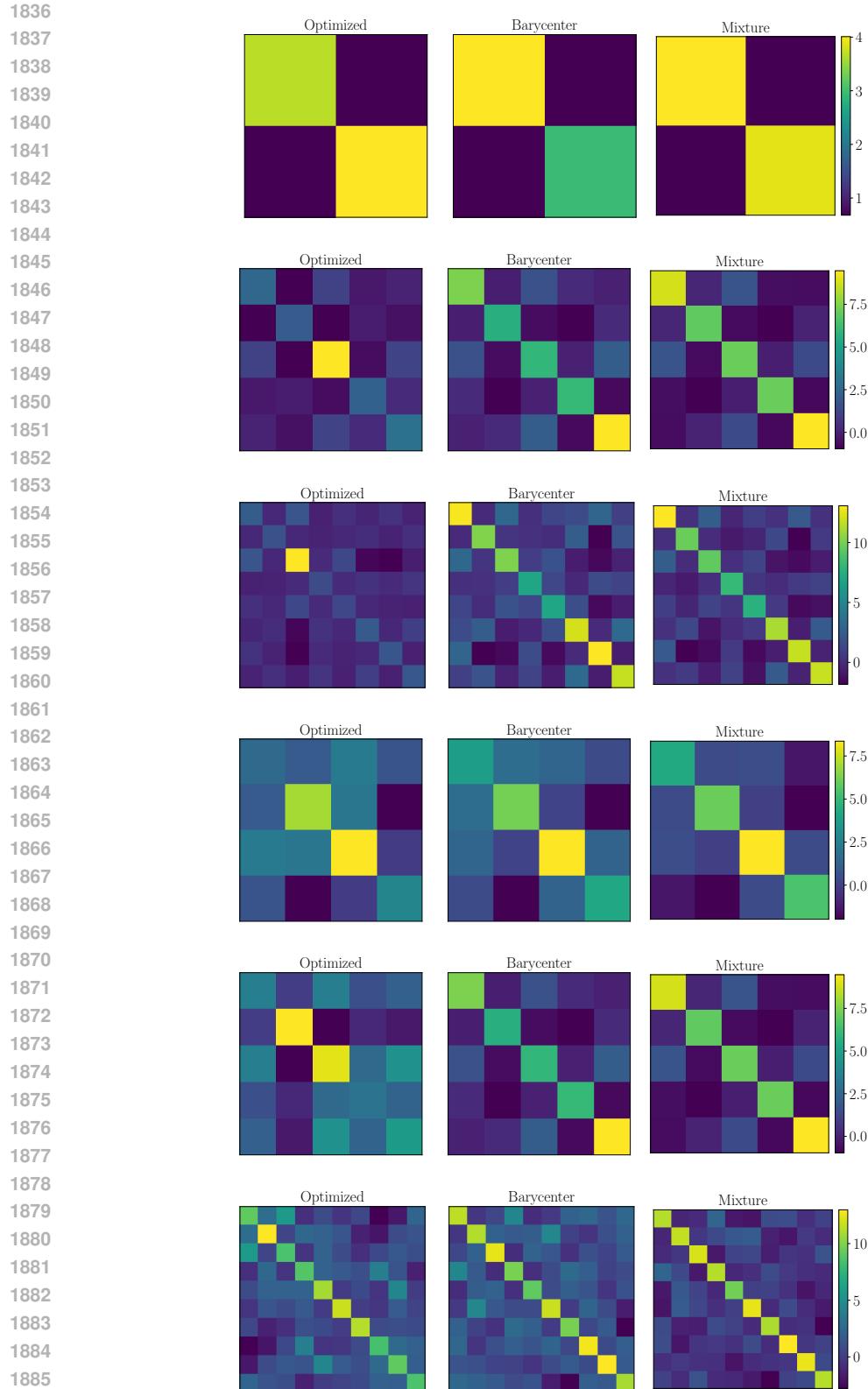


Figure 11: Bilevel Alg. 3 applied with $K = 10$ test atoms and 1000 gradient descent iterations. Each column visualizes the covariance matrix of the optimized (left), empirical \mathbb{Q} W_2 barycenter (center), and empirical \mathbb{Q} mixture distributions (right). From top to bottom, each row represents target functions $\{g_i\}$ corresponding to those listed in each column of Tab. 1 (from left to right).

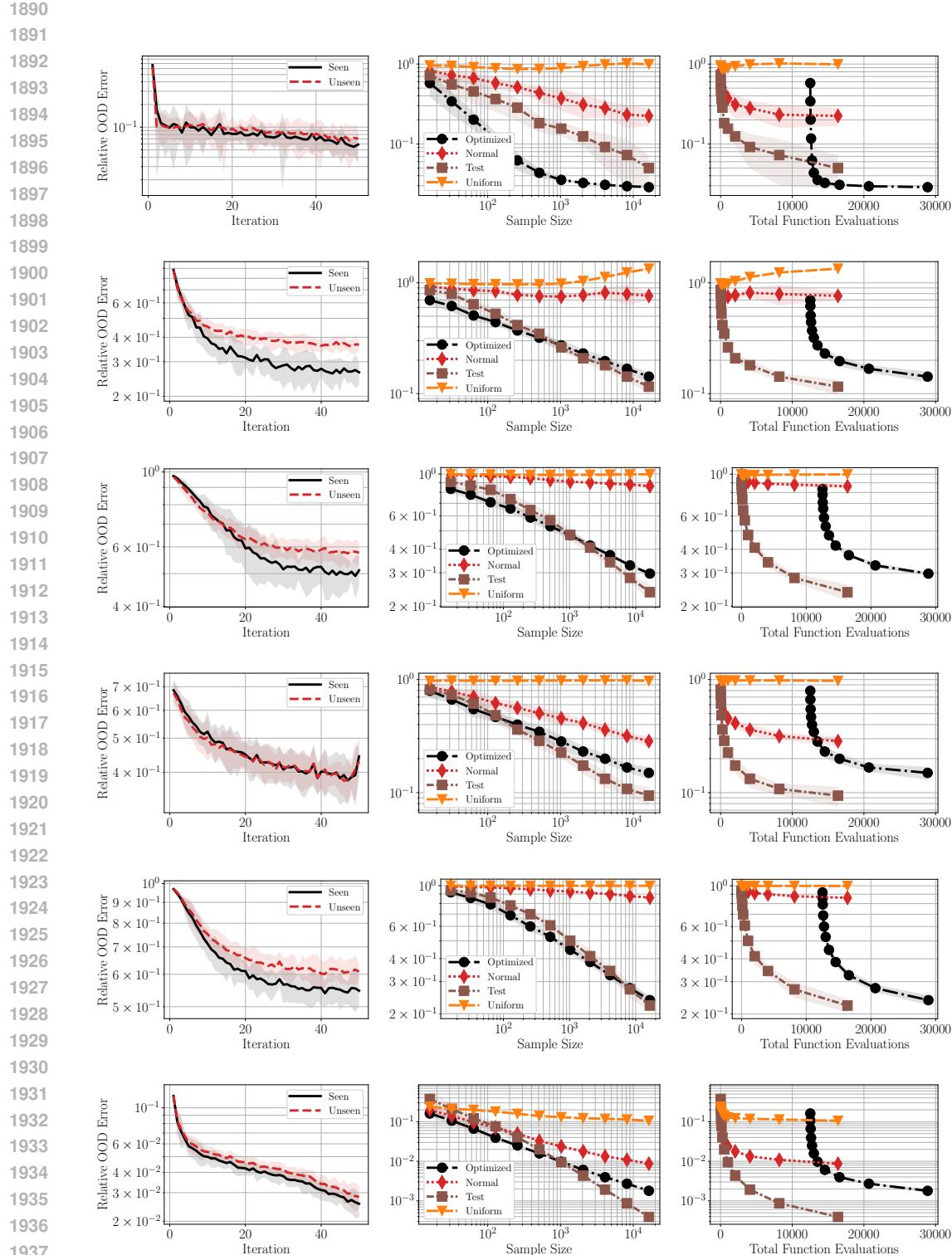


Figure 12: Bilevel Alg. 3 applied to the various ground truths $\{g_i\}$ with $K = 1$ test atoms and 50 gradient descent iterations. (Left column) Evolution of Err from (F.5) vs. iterations. (Center column) Err of model trained on N samples from optimal ν_ϑ vs. nonadaptive distributions. (Right column) Same as center, except incorporating the additional function evaluation cost incurred from Alg. 1. From top to bottom, each row represents target functions $\{g_i\}$ corresponding to those listed in each column of Tab. 1 (from left to right). Shading represents two standard deviations away from the mean Err over 10 independent runs.

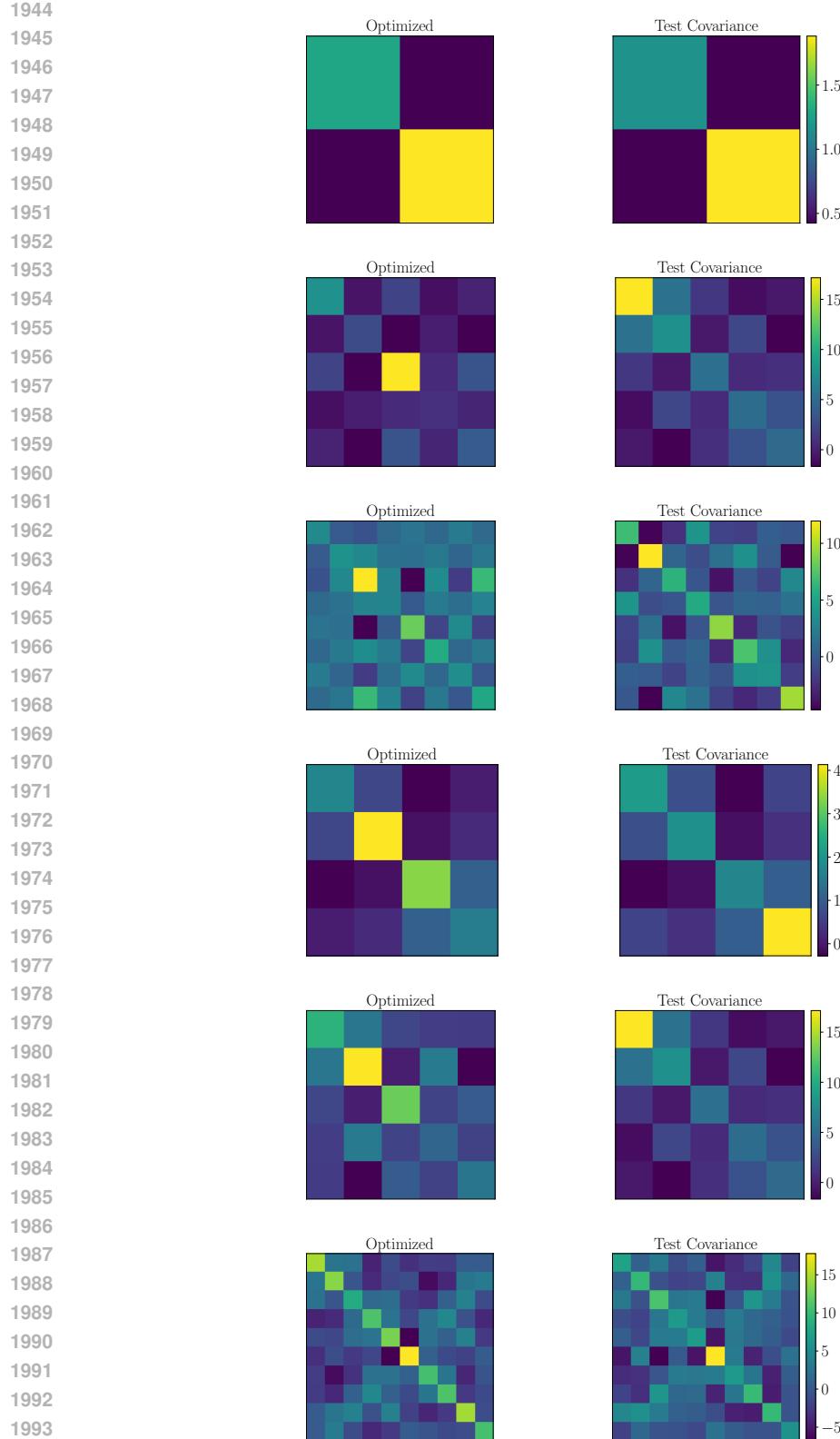


Figure 13: Bilevel Alg. 3 applied with $K = 1$ test atoms and 50 gradient descent iterations. Each column visualizes the covariance matrix of the optimized (left) and empirical test distributions (right), respectively. From top to bottom, each row represents target functions $\{g_i\}$ corresponding to those listed in each column of Tab. 1 (from left to right).

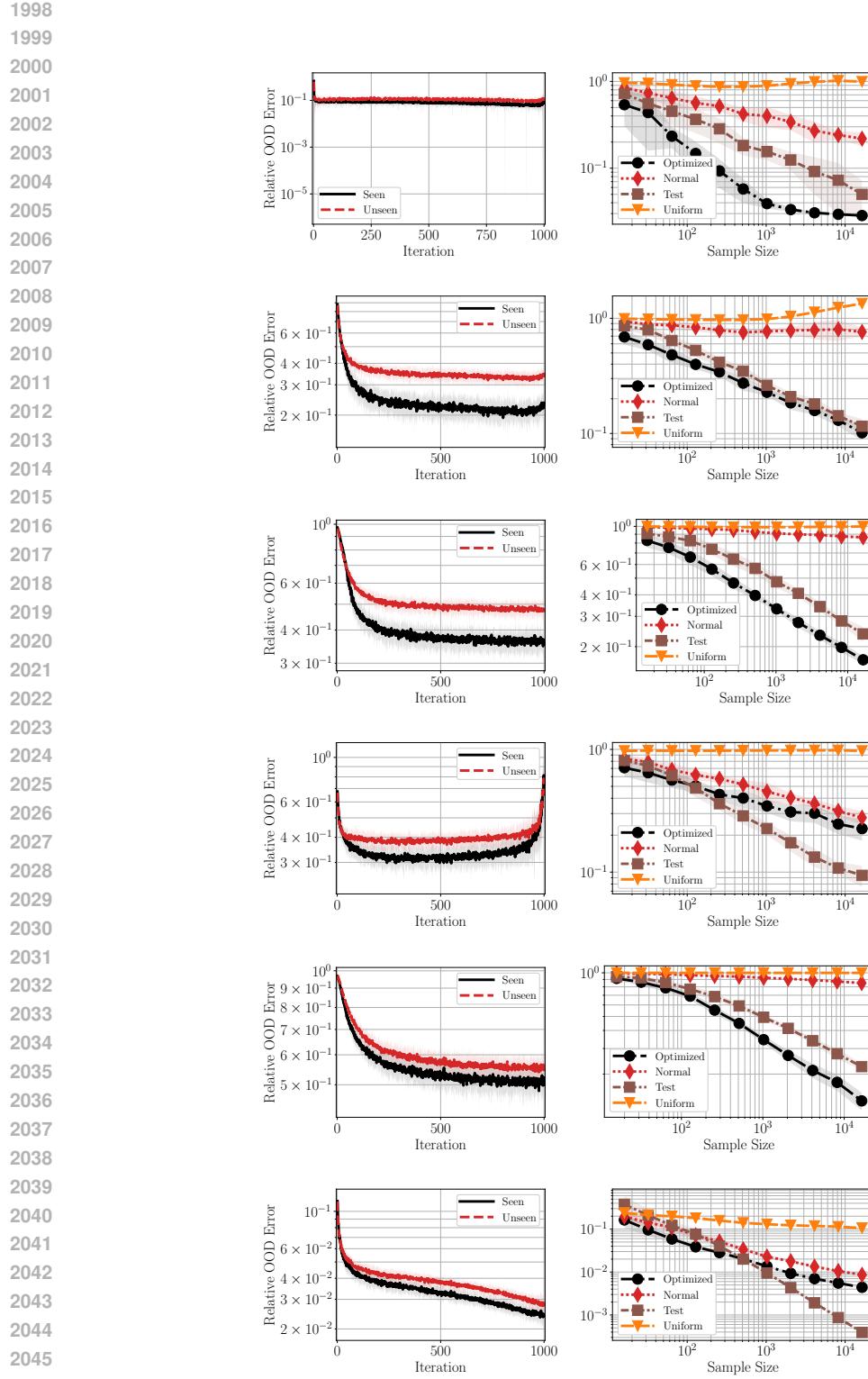


Figure 14: Bilevel Alg. 3 applied to the various ground truths $\{g_i\}$ with $K = 1$ test atoms and 1000 gradient descent iterations. (Left column) Evolution of Err from (F.5) vs. iterations. (Right column) Err of model trained on N samples from optimal ν_θ vs. nonadaptive distributions. From top to bottom, each row represents target functions $\{g_i\}$ corresponding to those listed in each column of Tab. 1 (from left to right). Shading represents two standard deviations away from the mean Err over 10 independent runs.

2051



Figure 15: Bilevel Alg. 3 applied with $K = 1$ test atoms and 1000 gradient descent iterations. Each column visualizes the covariance matrix of the optimized (left) and empirical test distributions (right), respectively. From top to bottom, each row represents target functions $\{g_i\}$ corresponding to those listed in each column of Tab. 1 (from left to right).

Computational resources. All bilevel minimization experiments are performed in Python using the PyTorch framework in `float32` single precision on a machine equipped with an NVIDIA GeForce RTX 4090 GPU (24 GB VRAM) running CUDA version 12.4, Intel i7-10700K CPU (8 cores/16 threads) running at 4.9 GHz, and 64 GB of DDR4-3200 CL16 RAM. The experiments required less than one total CPU and GPU hour to complete. An additional five CPU and GPU hours were expended to tune hyperparameters and set up the experiments. We ensure reproducibility of our numerical results by appropriately setting PyTorch backends to be deterministic and using fixed seeds for the relevant random number generators.

F.2 AMINO FOR LEARNING ELECTRICAL IMPEDANCE TOMOGRAPHY

In all machine learning tasks, the choice of training distribution plays a critical role. In practice, testing distributions can span a wide range, making training on the entire space infeasible. Instead, we must identify an optimal training distribution that enables good generalization across diverse testing scenarios.

To illustrate this, we consider the EIT problem with the AMINO solver; see **SM E.3**. We start with the following forward problem setup taken from Molinaro et al. (2023) for EIT on the domain $\Omega := (0, 1)^2 \subset \mathbb{R}^2$. The model is the two-dimensional elliptic PDE boundary value problem

$$\begin{cases} \nabla \cdot (a \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \quad (\text{F.6})$$

For one conductivity realization a defined by

$$x \mapsto a(x) = \exp\left(\sum_{k=1}^m c_k \sin(k\pi x_1) \sin(k\pi x_2)\right), \quad (\text{F.7})$$

where $m = \lfloor \bar{m} \rfloor$, $\bar{m} \sim \text{Unif}([1, 5])$, and $\{c_k\} \sim \text{Unif}([-1, 1]^m)$, we take $L \in \mathbb{N}$ different Dirichlet boundary conditions $\{f_\ell\}_{\ell=1}^L$ for f in (F.6). These are given by

$$x \mapsto f_\ell(x) = \cos(\omega(x_1 \cos(\theta_\ell) + x_2 \sin(\theta_\ell))) \quad (\text{F.8})$$

with $\omega \sim \text{Unif}([\pi, 3\pi])$ and $\theta_\ell := 2\pi\ell/L$ for $\ell = 1, \dots, L$. We denote by Λ_a the DtN operator that maps the Dirichlet data of the PDE, f_ℓ , to the Neumann boundary data, $a \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega}$, where u is the PDE solution and \mathbf{n} is the outward normal unit vector. Here, the Dirichlet boundary conditions are functions drawn from a probability distribution μ . AMINO learns the map from the joint Dirichlet–Neumann distribution $(\text{Id}, \Lambda_a)_\# \mu$ to the conductivity a .

Suppose the general distribution of interest for the Dirichlet boundary condition is generated according to the law of f_l in (F.8) with $\omega \sim \text{Unif}([\frac{\pi}{2}, 7\pi])$. A reasonable distribution to train upon is this very same distribution. However, suppose we were constrained to focus and train on distributions corresponding to $\text{Law}(\omega)$ supported only on a subset of the domain $[\frac{\pi}{2}, 7\pi]$. In that case, the choice of the distribution will greatly affect the OOD error when testing with $\omega \sim \text{Unif}([\frac{\pi}{2}, 7\pi])$. To be more concrete, we will focus on four training distributions for ω , which directly determines the input Dirichlet boundary condition as in **SM E.3**: $\text{Unif}([\frac{\pi}{2}, \pi])$, $\text{Unif}([\frac{7\pi}{2}, 4\pi])$, $\text{Unif}([5\pi, \frac{11\pi}{2}])$, and $\text{Unif}([\frac{13\pi}{2}, 7\pi])$. After obtaining the well-trained neural operators (in the sense that the ID generalization error is small enough) corresponding to each training distribution, we study the OOD error of the trained neural operators for Dirichlet data obtained by drawing $\omega \sim \text{Unif}([\frac{\pi}{2}, 7\pi])$ in (F.8).

Model architecture and optimization algorithm. We train four distinct AMINO models; see **SM E.3** to recall AMINO and **SM E.1** for its DeepONet component. Each model comprises three jointly trained components:

- **Branch encoder:** Eight sequential convolutional blocks with LeakyReLU activations:
 - `ConvBlock(1 → 64, kernel = (1, 7), stride = (1, 2), pad = (0, 3))`
 - `ConvBlock(64 → 128, kernel = (1, 3), stride = (1, 2), pad = (0, 1))`
 - `ConvBlock(128 → 128, kernel = (1, 3), pad = (0, 1))`
 - `ConvBlock(128 → 256, kernel = (1, 3), stride = (1, 2), pad = (0, 1))`

- 2160 – ConvBlock($256 \rightarrow 256$, kernel = $(1, 3)$, pad = $(0, 1)$)
- 2161 – ConvBlock($256 \rightarrow 512$, kernel = $(1, 3)$, stride = $(1, 2)$, pad = $(0, 1)$)
- 2162 – ConvBlock($512 \rightarrow 512$, kernel = $(1, 3)$, pad = $(0, 1)$)
- 2163 – ConvBlock($512 \rightarrow 512$, kernel = $(4, 5)$, pad = 0).

2164 The feature map is then flattened and passed through a linear layer to produce the p -
2165 dimensional branch output.

- 2166 • **Trunk network:** An 8-layer multilayer perceptron (MLP) with 512 neurons per hidden
2167 layer, LeakyReLU activations, zero dropout, and a final output dimension of $p = 100$.
- 2168 • **Fourier Neural Operator (FNO):** A single spectral convolution layer with 128 channels
2169 and 32 Fourier modes.

2171 All parameters are learned using the AdamW optimizer with a learning rate of 10^{-3} .

2172 **Training and optimization procedure.** The four neural operator models “Model 1”, “Model 2”,
2173 “Model 3”, and “Model 4” are trained using four different input parameter distributions for $\text{Law}(\omega)$,
2174 respectively: $\text{Unif}([\frac{\pi}{2}, \pi])$, $\text{Unif}([\frac{7\pi}{2}, 4\pi])$, $\text{Unif}([5\pi, \frac{11\pi}{2}])$, and $\text{Unif}([\frac{13\pi}{2}, 7\pi])$; see (F.8) to recall
2175 how ω is reflected in the actual input distributions on function space. For each distribution, 4096
2176 training samples are generated. Each sample consists of $L = 100$ Dirichlet–Neumann boundary
2177 condition pairs with the underlying conductivity discretized on a 70×70 grid for the domain
2178 $\Omega = (0, 1)^2$. After 500 training epochs, the trained models are evaluated on 1024 test samples of the
2179 distribution for the Dirichlet boundary condition (F.8) determined by $\omega \sim \text{Unif}([\frac{\pi}{2}, 7\pi])$.
2180

2181 **Model evaluation.** For both ID and OOD settings, we report the average relative L^1 error given as

$$2184 \text{Expected Relative } L^1 \text{ Error} := \mathbb{E}_{a \sim \text{Law}(a)} \left[\frac{\|a - \mathcal{G}_\theta((\text{Id}, \Lambda_a) \# \mu)\|_{L^1(\Omega)}}{\|a\|_{L^1(\Omega)}} \right], \quad (F.9)$$

2185 where $\text{Law}(a)$ is the distribution of a from (F.7) and \mathcal{G}_θ is the AMINO model. Using the same
2186 setup as in E.3, we approximate (F.9) with a Monte Carlo average over $N' = 1024$ test conductivity
2187 samples $a_n \stackrel{\text{i.i.d.}}{\sim} \text{Law}(a)$ and each with $L = 100$ pairs of corresponding Dirichlet and Neumann
2188 boundary data. That is, the expected relative L^1 error is approximately given by the average
2189

$$2190 \frac{1}{N'} \sum_{n=1}^{N'} \frac{\left\| a_n - \mathcal{G}_\theta \left(\frac{1}{L} \sum_{\ell=1}^L \delta_{(f_\ell^{(n)}, g_\ell^{(a_n)})} \right) \right\|_{L^1}}{\|a_n\|_{L^1}}, \text{ where } \{(f_\ell^{(n)}, g_\ell^{(a_n)})\}_{\ell=1}^L \sim ((\text{Id}, \Lambda_{a_n}) \# \mu)^{\otimes L}. \quad (F.10)$$

2191 The $L^1(\Omega)$ norms in the preceding display are approximated with Riemann sum quadrature over the
2192 equally-spaced 70×70 grid.

2193 Tab. 2 illustrates how OOD performance varies with the choice of training distribution. Model 1
2194 achieves strong ID performance but performs poorly in the OOD setting. Model 2 exhibits consistent
2195 yet mediocre performance across both ID and OOD evaluations. Model 4 performs well in the
2196 ID setting but shows a noticeable drop in OOD performance. Model 3 offers the best balance,
2197 achieving low error in both ID and OOD scenarios. Notably, despite being trained on a Dirichlet data
2198 distribution with a small support regarding the parameter ω , Model 3 generalizes significantly better
2199 to the larger test domain $[\frac{\pi}{2}, 7\pi]$ for the parameter ω .

2200 These results are further illustrated in Fig. 1, which presents two representative conductivity samples.
2201 The top row displays the predicted conductivities from each of the four models when evaluated on
2202 Dirichlet–Neumann data pairs sampled from the same distribution used during training. In this ID
2203 setting, all four models accurately recover the true conductivity. In contrast, the bottom row shows
2204 predictions when the Dirichlet data is drawn from the OOD setting, i.e., $\omega \sim \text{Unif}([\frac{\pi}{2}, 7\pi])$, and
2205 the joint Dirichlet–Neumann data distribution also changes accordingly. Under this distributional
2206 shift, Models 1, 2, and 4 exhibit noticeable degradation in performance, while Model 3 maintains a
2207 reasonable level of accuracy. We use this test to demonstrate the importance of carefully selecting the
2208 training data distribution to achieve robust OOD performance.

Table 2: Average relative L^1 error of AMINO models under ID and OOD settings for the EIT inverse problem solver. Model 1 is trained on $\omega_1 \sim \text{Unif}([\frac{\pi}{2}, \pi])$, Model 2 on $\omega_2 \sim \text{Unif}([\frac{7\pi}{2}, 4\pi])$, Model 3 on $\omega_3 \sim \text{Unif}([5\pi, \frac{11\pi}{2}])$, and Model 4 on $\omega_4 \sim \text{Unif}([\frac{13\pi}{2}, 7\pi])$. In the OOD setting, all models are evaluated on Dirichlet data from $\omega \sim \text{Unif}([\frac{\pi}{2}, 7\pi])$. One standard deviation is reported. See Fig. 1 for a visual comparison on two representative samples.

	Model 1	Model 2	Model 3	Model 4
In-Distribution (ID)	0.034 ± 0.019	0.081 ± 0.028	0.021 ± 0.010	0.017 ± 0.009
Out-of-Distribution (OOD)	0.796 ± 0.251	0.078 ± 0.047	0.066 ± 0.038	0.132 ± 0.047

Computational resources. The AMINO experiment was conducted on a machine equipped with an AMD Ryzen 9 5950X CPU (16 cores / 32 threads), 128 GB of RAM, and an NVIDIA RTX 3080Ti GPU (10 GB VRAM) running with CUDA version 12.1. Twelve CPU and GPU hours total were needed for the training and testing of all four models.

F.3 NTD LINEAR OPERATOR LEARNING

In this example from Sec. 5, we begin by defining a conductivity field $a: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$x \mapsto a(x) = \exp \left(\sum_{j=1}^{M+1} \sum_{k=1}^{M+1} \frac{2\sigma}{((j\pi)^2 + (k\pi)^2 + \tau^2)^{\alpha/2}} z_{jk} \sin(j\pi x_1) \sin(k\pi x_2) \right) \quad (\text{F.11})$$

with fixed parameters $\alpha = 2$, $\tau = 3$, $\sigma = 3$ and random coefficients $z_{jk} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Once these random coefficients are drawn, they are fixed throughout the example. Thus, in this experiment, our realization of the conductivity a remains fixed; see the left panel of Fig. 3 for a visualization of this a . Our goal is to develop neural operator approximations of the NtD linear operator $\mathcal{G}^* := \Lambda_a^{-1}$ that maps the Neumann boundary condition to the Dirichlet boundary condition according to the PDE (F.6) (with Neumann data g given instead of Dirichlet data f). Using the alternating minimization algorithm (AMA, Alg. 2) from Sec. 4.2, we want to optimize the training distribution over the Neumann boundary conditions to achieve small OOD error (3.3).

To evaluate the OOD error, we need to assign a random measure \mathbb{Q} . Let $\mathbb{Q} = \frac{1}{K} \sum_{k=1}^K \delta_{\nu'_k}$, where $\{\nu'_k\}_{k=1}^K$ are distributions over functions. For a fixed k , the function $g \sim \nu'_k$ is a Neumann boundary condition. It takes the form $g: \partial\Omega \rightarrow \mathbb{R}$, where

$$t \mapsto g(r(t)) = 10 \sin(\omega_k(t - \frac{1}{2})) + \sum_{j=1}^{N_{\text{basis}}} \frac{\sigma_k}{((j\pi)^2 + \tau_k^2)^{\alpha_k/2}} z_j \sqrt{2} \sin(j\pi t). \quad (\text{F.12})$$

We view $r: [0, 1] \rightarrow \partial\Omega$ as a 1D parametrization of the boundary $\partial\Omega \subset \mathbb{R}^2$ for the domain $\Omega := (0, 1)^2$. The entire domain is discretized on an $M \times M$ grid, where M is as in (F.11). Here $z_j \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$, $\alpha_k = 1.5$ for all k , and $N_{\text{basis}} = 4M - 4$. We set the grid parameter to be $M = 128$ and sample $K = 64$ test distributions in this example. The parameters $\omega_k \stackrel{\text{i.i.d.}}{\sim} \rho([-2\pi, 2\pi])$ and $\tau_k \stackrel{\text{i.i.d.}}{\sim} \rho([0, 50])$ are sampled from a custom distribution $\rho = \rho([x_{\min}, x_{\max}])$ and held fixed throughout the algorithm. The custom distribution ρ has the probability density function

$$p(x) = C \sin(x - \frac{\pi}{2}) + 1 \quad (\text{F.13})$$

for $x \in [x_{\min}, x_{\max}] \subset \mathbb{R}$, where C is a normalizing constant ensuring that p integrates to 1 over the domain $[x_{\min}, x_{\max}]$. Samples are generated by using inverse transform sampling. Any time a Neumann boundary condition g is sampled according to (F.12) and (F.13), we subtract the spatial mean of g from itself to ensure that the resulting function integrates to zero. This is necessary to make the Neumann PDE problem—as well as the NtD operator itself—well-posed.

The training distribution ν is assumed to generate functions of the form

$$t \mapsto \tilde{g}(t) = 10 \sin(\omega(t - \frac{1}{2})) + \sum_{j=1}^{N_{\text{basis}}} \frac{\sigma}{((j\pi)^2 + \tau^2)^{\alpha/2}} \tilde{z}_j \sqrt{2} \sin(j\pi t), \quad (\text{F.14})$$

2268 where $\tilde{z}_j \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$, $\alpha = 2$, $\tau = 3$, and $\sigma = \tau^{3/2}$. That is, $\nu = \text{Law}(\tilde{g})$ with \tilde{g} as in (F.14). The
 2269 frequency parameter ω appearing in the mean term of (F.14) is optimized using Alg. 2 to identify a
 2270 training distribution that minimizes average OOD error with respect to \mathbb{Q} .
 2271

2272 Note that g and \tilde{g} are finite-term truncations of the KLE of Gaussian measures with Matérn-like
 2273 covariance operators; recall Lem. B.2. In particular, for \tilde{g} , and similarly for g , the covariance operator
 2274 of the Gaussian measure has eigenpairs

$$2275 \quad \lambda_j = \frac{\sigma^2}{((j\pi)^2 + \tau^2)^\alpha} \quad \text{and} \quad t \mapsto \phi_j(t) = \sqrt{2} \sin(j\pi t) \quad \text{for } j = 1, 2, \dots, N_{\text{basis}}. \quad (\text{F.15})$$

2278 Using (F.15) and (B.1), the squared 2-Wasserstein distance in Prop. 3.1 reduces to
 2279

$$2280 \quad W_2^2(\nu, \nu'_k) = \int_0^1 \left(10 \sin\left(\omega\left(t - \frac{1}{2}\right)\right) - 10 \sin\left(\omega_k\left(t - \frac{1}{2}\right)\right) \right)^2 dt + \sum_{j=1}^{N_{\text{basis}}} \left(\sqrt{\lambda_j} - \sqrt{\lambda_{j,k}} \right)^2, \quad (\text{F.16})$$

2284 where $\lambda_{j,k} = \sigma_k^2 / ((j\pi)^2 + \tau_k^2)^{\alpha_k}$. We approximate the integral with standard quadrature rules.
 2285 Moreover, the second moment of ν , and similarly for ν'_k , is

$$2287 \quad \mathbf{m}_2(\nu) = \int_0^1 \left(10 \sin\left(\omega\left(t - \frac{1}{2}\right)\right) \right)^2 dt + \sum_{j=1}^{N_{\text{basis}}} \lambda_j. \quad (\text{F.17})$$

2290 These closed-form reductions follow directly from the finite-dimensional KLE and Ex. B.1.
 2291

2292 Furthermore, we estimate the Lipschitz constants of both the true NtD operator \mathcal{G}^* and the learned
 2293 model \mathcal{G}_θ —which appear in the multiplicative factor c from Prop. 3.1—by sampling 250 realizations
 2294 of Neumann input function pairs (g_1, g_2) from 11 candidate distributions ν . We denote the finite set
 2295 of these pairs of realizations by S . We then use the quantity

$$2296 \quad \max_{(g_1, g_2) \in S} \frac{\|\mathcal{G}(g_1) - \mathcal{G}(g_2)\|_{L^2(\partial\Omega)}}{\|g_1 - g_2\|_{L^2(\partial\Omega)}} \quad (\text{F.18})$$

2299 to estimate the Lipschitz constant of map $\mathcal{G} \in \{\mathcal{G}^*, \mathcal{G}_\theta\}$. These Lipschitz constants are then fixed
 2300 throughout the algorithm.

2301 **Model architecture and optimization algorithm.** The model architecture is a DeepONet comprising
 2302 a branch network with five fully connected layers of sizes $(4M - 4), 512, 512, 512$, and 200 , and
 2303 a trunk network with five fully connected layers of sizes $2, 128, 128, 128$, and 200 , respectively; see
 2304 SM E.1 for details. All layers use ReLU activation functions. Weights are initialized using the Glorot
 2305 normal initializer. The model is trained with the Adam optimizer with a learning rate of 10^{-3} . After
 2306 the model-training half step in the alternating minimization scheme Alg. 2, the parameter ω in (F.14)
 2307 is optimized using SciPy’s default implementation of differential evolution with a tolerance of 10^{-7} .
 2308 Due to the presence of many local minima in the optimization landscape, this global optimization
 2309 algorithm is necessary.

2311 **Training and optimization procedure.** The initial value for ω is set to 8π . For every model-
 2312 training step, 500 pairs of Neumann and Dirichlet data are generated using the current value of ω . For
 2313 practical reasons, the model is initially trained for 2000 iterations of Adam. If the AMA loss has not
 2314 improved relative to the previous iteration, training continues for an additional 2000 iterations. This
 2315 process is repeated up to 10 times, after which the algorithm proceeds to the parameter optimization
 2316 step for the training distribution ν . During distribution parameter optimization, the AMA loss is
 2317 evaluated using 500 newly generated data pairs, used specifically to compute the first term of the
 2318 objective (4.7). To ensure deterministic behavior, the 500 sets of random coefficients $\{z_j\}_{j=1}^{N_{\text{basis}}}$ are
 2319 fixed separately for training and optimization. This results in 1000 total sets of coefficients fixed for
 2320 the entire algorithm. This process is repeated across 80 independent runs using the same conductivity
 2321 but different realizations of the random coefficients $\{z_j\}$. The relative AMA loss and relative OOD
 2322 error is computed for each run; see (F.19).

2322 **Model evaluation.** The relative AMA loss is defined as the algorithm’s objective value (4.7)
 2323 normalized by its initial value. The middle panel of Fig. 3 illustrates the rapid decay of this
 2324 normalized loss over subsequent iterations. The relative OOD loss is defined as
 2325

$$2326 \text{Relative OOD Error} := \sqrt{\frac{\mathbb{E}_{\nu' \sim \mathbb{Q}} \mathbb{E}_{g \sim \nu'} \|\mathcal{G}^* g - \mathcal{G}_\theta(g)\|_{L^2}^2}{\mathbb{E}_{\nu' \sim \mathbb{Q}} \mathbb{E}_{g \sim \nu'} \|\mathcal{G}^* g\|_{L^2}^2}}, \quad (F.19)$$

2328 which measures the squared prediction error of the model \mathcal{G}_θ relative to the ground truth NtD operator
 2329 \mathcal{G}^* on average with respect to the given random measure \mathbb{Q} .
 2330

2331 The relative OOD error is estimated using Monte Carlo sampling with $K = 64$ and $N' = 500$ as
 2332

$$2333 \text{Relative OOD Error} \approx \sqrt{\frac{\frac{1}{K} \sum_{k=1}^K \frac{1}{N'} \sum_{n=1}^{N'} \|\mathcal{G}^* g_{k,n} - \mathcal{G}_\theta(g_{k,n})\|_{L^2}^2}{\frac{1}{K} \sum_{k=1}^K \frac{1}{N'} \sum_{n=1}^{N'} \|\mathcal{G}^* g_{k,n}\|_{L^2}^2}}, \quad (F.20)$$

2335 where $g_{k,n}$ is the n^{th} i.i.d. sample from ν'_k . Further numerical quadrature is used to discretize the
 2336 $L^2(\partial\Omega)$ norms in the preceding display.
 2337

2338 After 80 independent runs of Alg. 2, a 95% confidence interval for the true relative OOD error was
 2339 computed using a Student’s t -distribution as seen in the middle panel of Fig. 3.
 2340

2341 **Computational resources.** The computational resources here are the same as those reported in
 2342 SM F.2, except three total CPU and GPU hours were needed for all independent runs.
 2343

2344 F.4 DARCY FLOW FORWARD OPERATOR LEARNING

2345 In this example from Sec. 5, we are interested in learning another operator that is relevant and related
 2346 to the EIT problem. In contrast with the previous example in SM F.3 where we fix the conductivity,
 2347 the Darcy flow forward operator that we consider is nonlinear and maps the log-conductivity function
 2348 $\log(a)$ to the PDE solution u of

$$2349 \begin{cases} -\nabla \cdot (a \nabla u) = 1 & \text{in } \Omega, \\ 2350 u = 0 & \text{on } \partial\Omega. \end{cases} \quad (F.21)$$

2352 That is, $\mathcal{G}^* : \log(a) \mapsto u$. Here, $\Omega := (0, 1)^2$.
 2353

2354 Again, we first assign a random measure \mathbb{Q} to evaluate the OOD performance. We take $\mathbb{Q} =$
 2355 $\frac{1}{K} \sum_{k=1}^K \delta_{\nu'_k}$, where each test distribution ν'_k is defined by parameters sampled i.i.d. from custom
 2356 distributions (F.13): $m_k \sim \rho([0, 1])$, $\alpha_k \sim \rho([2, 3])$, and $\tau_k \sim \rho([1, 4])$. The scaling parameter is
 2357 set as $\sigma_k = \tau_k^{\alpha_k - 1}$. We set $K = 20$ and the grid parameter $M = 64$. Given these parameters, each
 2358 random log-conductivity function $\log(a')$ from test distribution ν'_k has the form
 2359

$$2360 x \mapsto \log(a'(x)) = m_k + \sum_{i=1}^{M+1} \sum_{j=1}^{M+1} \frac{2\sigma_k}{((i\pi)^2 + (j\pi)^2 + \tau_k^2)^{\alpha_k/2}} z'_{ij} \sin(i\pi x_1) \sin(j\pi x_2), \quad (F.22)$$

2362 where $z'_{ij} \sim \mathcal{N}(0, 1)$ independently. Thus, a is lognormal. Sample log-conductivities $\log(a)$ from
 2363 training distribution ν are assumed to follow the same form
 2364

$$2365 x \mapsto \log(a(x)) = m + \sum_{i=1}^{M+1} \sum_{j=1}^{M+1} \frac{2\sigma}{((i\pi)^2 + (j\pi)^2 + \tau^2)^{\alpha/2}} z_{ij} \sin(i\pi x_1) \sin(j\pi x_2), \quad (F.23)$$

2367 with fixed parameters $\alpha = 2$, $\tau = 3$, and $\sigma = 3$ and $z_{ij} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Alg. 2 is used to optimize the
 2368 mean shift parameter $m \in \mathbb{R}$ in order to reduce the OOD error. As in SM F.3, $\log(a)$ is modeled via a
 2369 finite KLE of a Gaussian measure with Matérn-like covariance operator. Thus, by working with $\log(a)$
 2370 directly, we can exploit (B.1) to write the squared 2-Wasserstein distance factor in objective (4.7) in
 2371 closed-form.
 2372

2373 **Model architecture and optimization algorithm.** The same DeepONet architecture and training
 2374 setup from SM F.3 are used here, except the branch network has layer sizes M^2 , 256, 256, 128, and
 2375 200. After the model-training half step in Alg. 2, the scalar parameter m is optimized using PyTorch’s
 Adam optimizer with a learning rate of 10^{-3} .
 2376

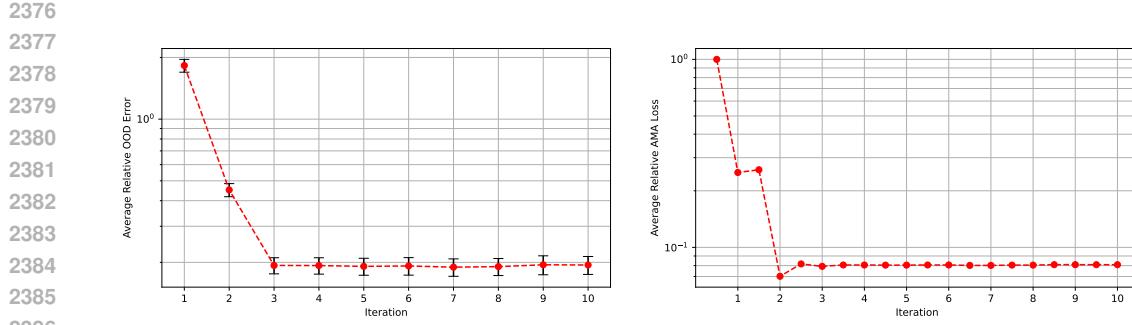


Figure 16: Average relative OOD error and average relative loss of Alg. 2 on the Darcy flow forward problem. (Left) After 13 independent runs, decay of the average relative OOD error of the model when trained on the optimal distribution identified at each iteration; a 95% confidence interval of the true relative OOD error is provided at each iteration. (Right) Decay of average AMA loss defined in (4.7) vs. iteration relative to the same loss at initialization.

Training and optimization procedure. The initial value of m is set to zero. The training and optimization procedure follows the same structure as in SM F.3, but with the following differences: 200 conductivity-solution pairs are generated for training using the current value of m . The model is initially trained for 5000 iterations. If the AMA loss has not improved relative to the previous iteration, training continues for an additional 5000 iterations. This process is repeated up to three times, after which the algorithm proceeds to the distribution parameter optimization step. During distribution parameter optimization, 500 steps are taken. Then the AMA loss is evaluated using three newly generated data pairs. If the AMA loss has not improved relative to the previous iteration, another 500 steps are taken, up to 1500 total steps. After this, the procedure returns to model training again. To ensure deterministic behavior, 203 sets of random coefficients $\{z_{ij}\}$ are fixed across training and optimization. The preceding process is repeated across 13 independent runs, each using different realizations of the coefficients. The average relative OOD error and average relative AMA loss are computed for each run.

Model evaluation. After 13 independent runs, the average relative OOD error and relative AMA loss are computed and are shown in Fig. 16. In Fig. 4, we visualize the pointwise absolute error for iterations 1, 2, 3, and 8. At these iterations, the DeepONet model is trained using the optimal training distribution produced by AMA. The model is then evaluated on the same fixed test conductivity. The progressive reduction in the spatial error across these panels of Fig. 4 corroborates the quantitative decrease in the relative OOD error, which converges to approximately 4% in our final model.

Computational resources. The computational resources here are the same as those reported in SM F.3, except three total CPU and GPU hours were needed for all independent runs.

F.5 RADIATIVE TRANSPORT OPERATOR LEARNING

In this example from Sec. 5, we consider the hyperbolic PDE for radiative transport governing the particle density $u(z, v)$ at position $z \in \Omega \subset \mathbb{R}^d$ and velocity $v \in V \subset \mathbb{R}^d$:

$$\begin{cases} v \cdot \nabla_z u(z, v) = \frac{1}{\varepsilon} a(z) Q[u](z, v), & (z, v) \in \Omega \times V, \\ u(z, v) = \phi(z, v), & (z, v) \in \Gamma_-, \end{cases} \quad (\text{F.24})$$

where a is the scattering coefficient, ε is the Knudsen number, and the collision operator $u \mapsto Q[u]$ is given by

$$Q[u](z, v) := \int_V \mathcal{K}(v, v') u(z, v') dv' - u(z, v). \quad (\text{F.25})$$

In this example, we use the constant kernel $\mathcal{K}(v, v') = \frac{1}{|V|}$. The inflow boundary is defined as

$$\Gamma_- := \{(z, v) \in \partial\Omega \times V \mid -\mathbf{n}_z \cdot v \geq 0\}, \quad (\text{F.26})$$

where \mathbf{n}_z denotes the outward normal at $z \in \partial\Omega$.

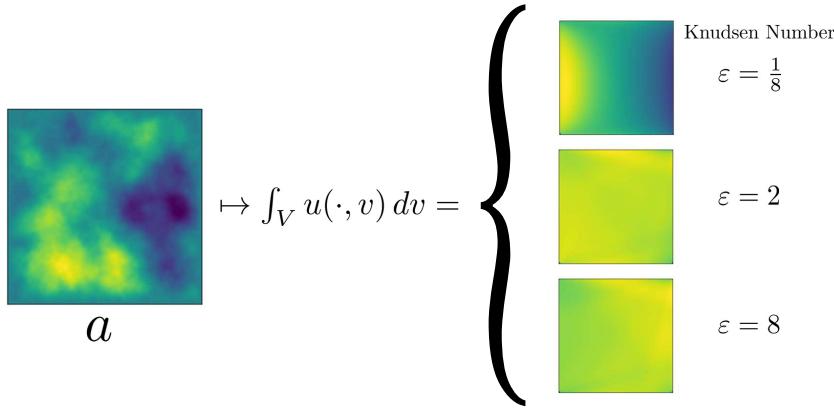


Figure 17: Mapping from scattering coefficient to spatial density in the radiative transport equation for three Knudsen numbers ε . Larger ε values yield a more hyperbolic regime.

On three separate experiments with error results given in Fig. 5, we set the Knudsen number ε equal to 1/8, 2, and 8. Changing ε puts us in different physical regimes as seen in Fig. 17. The inflow boundary condition is given by

$$\phi(z, v) = \begin{cases} (1 + 0.5 \sin(\pi y/L)) \max(v \cdot \mathbf{n}_z, 0), & \text{if } z = 0, \\ 0, & \text{elsewhere on } \Gamma_- . \end{cases} \quad (\text{F.27})$$

We seek the optimal input training distribution ν for learning the nonlinear operator that maps the scattering coefficient function $a(\cdot)$ to the spatial domain density $\int_V u(\cdot, v) dv$.

In this example, $\Omega := (0, 1)^2$ is the unit square and $V := \mathbb{S}^1 = \{(\cos \theta, \sin \theta) \mid \theta \in [-\pi, \pi]\}$ is the unit circle. Thus, we parametrize v by θ , i.e., $v(\theta) = (\cos \theta, \sin \theta)$ for $\theta \in [-\pi, \pi]$, $\int_V u(\cdot, v) dv = \int_{-\pi}^{\pi} u(\cdot, v(\theta)) d\theta$, and $v \cdot \nabla_z = (\cos \theta) \partial_{x_1} + (\sin \theta) \partial_{x_2}$. We first generate samples from \mathbb{Q} to then evaluate the OOD performance of the model trained on ν . We set $K = 3$. Each distribution $\nu'_k \sim \mathbb{Q}$ is set to be $(\exp(\#) \gamma_k)$, where γ_k is a Gaussian process parametrized by σ_k , α , and τ . Thus, samples from ν'_k have the form

$$a'(x) = \exp\left(f_k(x) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{2\sigma_k}{((i\pi)^2 + (j\pi)^2 + \tau^2)^{\alpha/2}} z_{ij} \sin(i\pi x_1) \sin(j\pi x_2)\right), \quad (\text{F.28})$$

where $z_{ij} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. The mean f_k of each distribution γ_k is selected as follows. For a fixed centered Gaussian process γ , draw $f_k \stackrel{\text{i.i.d.}}{\sim} \gamma$ for each k . Then these f_k functions are then fixed once and for all at the beginning of the experiment. By supplying distinct nonzero means to each of the K test distributions, the problem of finding the optimal training distribution becomes harder. Indeed, the test distributions now exhibit not only covariance shifts, but also mean shifts. We set $\tau = 1$ and $\alpha = 2$ for all $K = 3$ test distributions, and set $\sigma_1 = 1$, $\sigma_2 = 2$, and $\sigma_3 = 3$. Note that the parameters of γ are set to $\tau = 1$, $\alpha = 2$, and $\sigma = 1$.

Model architecture and optimization algorithm. Following Alg. 2, we represent ν as an empirical measure of N equally weighted particles: $\nu = \frac{1}{N} \sum_{q=1}^N \delta_{a_q}$. Each particle a_q is a function of the form

$$a_q(x) = \exp\left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{qij} \cdot 2 \sin(i\pi x_1) \sin(j\pi x_2)\right), \quad (\text{F.29})$$

where the coefficients $c_q := \{c_{qij}\}_{i,j \geq 1}$ belong to ℓ^2 . Rather than optimizing the particles a_q directly, we optimize their representations in the coefficient space through the transformation

$$a_q = \exp(Lc_q), \quad (\text{F.30})$$

where $L: \ell^2 \rightarrow L^2(\Omega)$ is the linear operator mapping coefficients to the series in (F.29) and \exp is applied pointwise. The distribution ν is thus obtained as the pushforward $\nu = (\exp \circ L)_\# \mu$, where μ is the distribution over coefficient sequences. We can then write the loss as $\tilde{F}(\mu) := F((\exp \circ L)_\# \mu) = F(\nu)$, where F is defined as the upper bound in Prop. 3.1, i.e., $\tilde{F} = F \circ (\exp \circ L)_\#$. For each step k , the discretized Wasserstein gradient flow update rule with respect to $\{c_q\}_{q=1}^N$ is then

$$c_q^{(k+1)} = c_q^{(k)} - \eta^{(k)} \nabla_{\ell^2} \left[D\tilde{F}(\mu_N^{(k)}) \right] (c_q^{(k)}), \quad \text{where } \mu_N^{(k)} := \frac{1}{N} \sum_{q'=1}^N \delta_{c_{q'}^{(k)}} \quad (\text{F.31})$$

and $\eta^{(k)}$ is the step size. By the chain rule and linearity of pushforward,

$$D\tilde{F}(\mu) = DF((\exp \circ L)_\# \mu) \circ (\exp \circ L) \quad (\text{F.32})$$

for each μ , where DF denotes the derivative from Prop. 4.9. Thus, we can write the update rule (F.31) in terms of the known derivative DF .

The ℓ^2 gradient with respect to c_q is computed by a combination of PyTorch’s autograd and the adjoint state method. After updating $\{c_q\}_{q=1}^N$, we train a model with discretized versions of particles $\{a_q\}_{q=1}^N$ denoted as $\{\hat{a}_q\}_{q=1}^N$. Each particle a_q is discretized as an $m \times m$ matrix. Similar to the setups from SM F.3 and SM F.4, the model architecture is a DeepONet. However, the branch network now has 6 fully connected layers of sizes m^2 , 100, 100, 100, 64, and 32, and the trunk network has 6 fully connected layers of sizes 2, 100, 100, 100, 64 and 32.

Training and optimization procedure. During optimization, we work with N samples from each distribution $\{\nu_k\}_{k=1}^K$ rather than the distributions themselves. For computational simplicity, we set the number of samples from each ν'_k equal to the number of particles in the training distribution.

The particles $\{a_q\}_{q=1}^N$ are discretized by truncating the number of terms in its expansion, i.e.,

$$a_q(x) = \exp \left(\sum_{i=1}^{N_{\text{truncate}}} \sum_{j=1}^{N_{\text{truncate}}} c_{qij} \cdot 2 \sin(i\pi x_1) \sin(j\pi x_2) \right). \quad (\text{F.33})$$

The coefficient sequences $\{c_q\}_{q=1}^N$ are truncated to $N_{\text{truncate}} = 20$ for both indices i and j , yielding sequences of 100 coefficients each. These are initialized by sampling from $\mathcal{N}(0, 10^{-6})$.

We discretize Ω on a 51×51 grid ($m = 51$) and V with 12 different directions, i.e. $\Delta\theta = \pi/6$, so we approximate the collision term by

$$\begin{aligned} Q[u_q] &= \frac{1}{|V|} \int_V u_q(z, v') dv' - u_q(z, v) \\ &\approx \frac{1}{|V|} \sum_{i=1}^{12} u_q(\cdot, \theta_i) \Delta\theta - u_q \\ &= \frac{\pi/6}{2\pi} \sum_{i=1}^{12} u_q(\cdot, \theta_i) - u_q \\ &= \frac{1}{12} \sum_{i=1}^{12} u_q(\cdot, \theta_i) - u_q \end{aligned}$$

and the spatial-domain density by

$$\tilde{u}_q := \int_V u_q(\cdot, v) dv \approx \sum_{i=1}^{12} u_q(\cdot, \theta_i) \Delta\theta. \quad (\text{F.34})$$

Initialize the step size as $\eta = 1 \times 10^{-6}$. At each iteration of Alg. 2:

2538 Table 3: Radiative transport operator learning with particle-based Alg. 2. Reported are mean relative OOD
 2539 errors with 95% confidence intervals computed over 10 independent runs for three separate models with varying
 2540 Knudsen number ε . All models have the same DeepONet architecture and are trained with $N = 120$ samples
 2541 for 5,000 epochs. What makes each model different is their training distribution. The initial model was trained
 2542 with random particles $c_{qij} \sim \mathcal{N}(0, 10^{-6})$, the optimized model was trained with optimized particles from
 2543 particle-based AMA, and the benchmark model is a model trained with samples from the mixture of test
 2544 distributions, i.e., $\nu_{\mathbb{Q}} = \frac{1}{K} \sum_{k=1}^K \nu'_k$.

ε	Initial	Optimized	Benchmark
1/8	$8.65 \times 10^{-2} \pm 8.18 \times 10^{-3}$	$1.55 \times 10^{-2} \pm 9.72 \times 10^{-4}$	$1.62 \times 10^{-2} \pm 5.34 \times 10^{-3}$
2	$8.76 \times 10^{-2} \pm 8.63 \times 10^{-3}$	$1.29 \times 10^{-2} \pm 1.20 \times 10^{-3}$	$1.24 \times 10^{-2} \pm 2.69 \times 10^{-3}$
8	$9.09 \times 10^{-2} \pm 4.39 \times 10^{-3}$	$1.13 \times 10^{-2} \pm 1.33 \times 10^{-3}$	$1.16 \times 10^{-2} \pm 2.26 \times 10^{-3}$

1. Compute the true outputs $\{\tilde{u}_q\}_{q=1}^N$ as $m \times m$ matrices using the current $\{c_q\}_{q=1}^N$.
2. Train the DeepONet for 5,000 epochs with an 80/20 train-test split.
3. Update $\{c_q\}_{q=1}^N$ according to Eqn. (F.31).
4. If the loss $F(\nu)$ increases, halve the step size and recompute the update. The decreased step size carries over to the next iteration.

The stopping condition for the algorithm is if $\|\text{step}^{(k)}\|_2 < \text{tol}$, where $\text{step}^{(k)} = \eta^{(k)} \nabla_{\ell^2} [D\tilde{F}(\mu_N^{(k)})]$ and $\text{tol} = 1 \times 10^{-6}$. Note that in this example, convergence usually occurs in about seven iterations.

For the loss computation in Step 4 above, the second moment of a distribution ν is estimated with N samples by

$$m_2(\nu) = \mathbb{E}_{a \sim \nu} \|a\|_{L^2(\Omega)}^2 \approx \frac{1}{N} \sum_{i=1}^N \|a_i\|_{L^2(\Omega)}^2, \quad (\text{F.35})$$

where $\|a_i\|_{L^2(\Omega)} = \int_{\Omega} a_i(z)^2 dz$ is estimated with the trapezoidal rule. Note that for our empirical training distribution, F.35 is not an approximation, but instead an equality. We estimate the Wasserstein distance $W_2(\nu, \nu')$ between distributions ν and ν' using N samples from each, with the ground cost function taken to be $c(a, a') = \|a - a'\|_{L^2(\Omega)}^2$.

Model evaluation. We compare models trained on the optimized distribution at each iteration of particle-based AMA with a benchmark model trained on N samples from the mixture of test distributions $\nu_{\mathbb{Q}} = \frac{1}{K} \sum_{k=1}^K \nu'_k$ (Eqn. D.1), using the same DeepONet architecture and 5,000 training epochs for all models. Performance is measured by the relative OOD error (Eqn. F.19), approximated with the original N samples from each ν'_k . As shown in Fig. 5, the optimized model improves substantially after particle-based AMA, achieving performance comparable to the benchmark and even outperforming the benchmark in small sample-size regimes. Given that AMA minimizes an upper bound, we are not directly minimizing the OOD error which can account for why the optimized distribution does not outperform the benchmark distribution for most sample sizes. Tab. 3 reports 95% confidence intervals for relative OOD errors over 10 independent trials at $N = 120$, demonstrating that the optimized model reduces its relative OOD error by 82% from its original value on average for $\varepsilon = \frac{1}{8}$, 85% for $\varepsilon = 2$, and 88% for $\varepsilon = 8$. For this particular sample size, the runtime, shown in Fig. 18, averages 20.6 minutes for $\varepsilon = \frac{1}{8}$, 26.7 minutes for $\varepsilon = 2$, and 26.1 minutes for $\varepsilon = 8$.

Computational resources. All experiments were conducted on AWS using Galaxy’s Deep Learning AMI GPU, PyTorch 2.0.1, and Ubuntu 20.04-prod-ada4rwvf26mda with an EC2 instance type of g5.12xlarge. The experiments for the three different ε values were run in parallel, with the $\varepsilon = 2$ case taking the longest at 27 hours. Thus, a total of approximately 27 GPU-hours (and corresponding CPU usage) were required for all independent runs with three GPUs running in parallel.

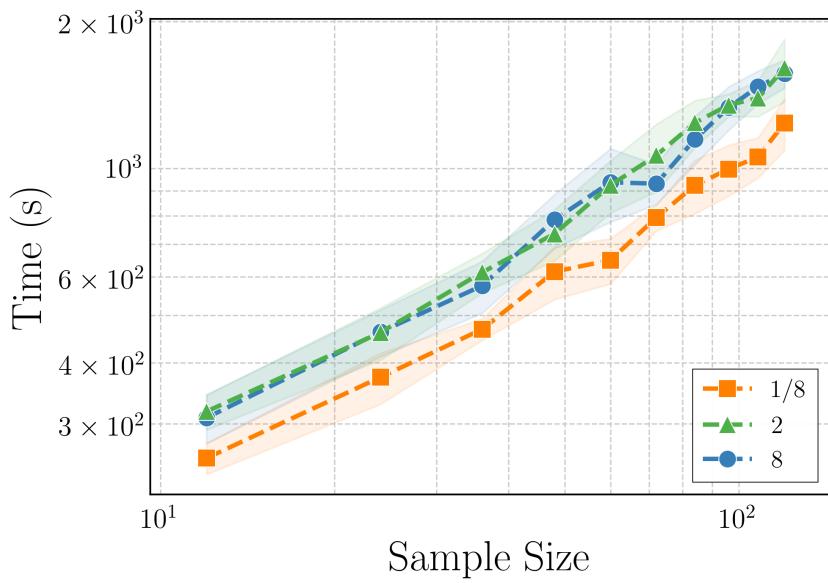


Figure 18: Runtime of particle-based AMA for the radiative transport equation with sample size N at three Knudsen numbers $\varepsilon \in \{1/8, 2, 8\}$. Error bars indicate 95% confidence intervals over 10 trials.

F.6 BURGERS' EQUATION OPERATOR LEARNING

In the final example from Sec. 5, we consider the viscous Burgers' equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2} \quad (\text{F.36})$$

with viscosity $\kappa = 0.1$ on the periodic domain $x \in [0, 2\pi]_{\text{per}}$. Our goal is to learn the operator that maps the initial condition $a := u(\cdot, 0)$ to the solution $u := u(\cdot, 1)$. This benchmark problem is taken from Nelsen & Stuart (2024); Li et al. (2021); Kovachki et al. (2023).

Following Alg. 2, we model the training data distribution ν as a Gaussian process with a Matérn covariance kernel. Samples from ν are generated according to

$$a(x) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{n/2} \frac{\sigma}{(j^2 + \tau^2)^{\alpha/2}} (z_j^{(s)} \sin(jx) + z_j^{(c)} \cos(jx)), \quad (\text{F.37})$$

where $z_j^{(s)}$ and $z_j^{(c)}$ are i.i.d. samples from $\mathcal{N}(0, 1)$ and $n = 8192$ is the discretization size. Only $n/2$ modes are retained due to the Nyquist limit.

Model architecture and optimization algorithm. We adopt the DeepONet architecture described in SM F.3. The branch network has layer sizes $n, 1024, 512, 256, 128, p$ and the trunk network has layer sizes $1, 128, 128, 128, 128, p$ where $p = 128$. Training uses Adam with initial learning rate 10^{-3} and inverse-time decay (decay steps = 2000, decay rate = 0.9). We add ℓ_2 regularization via weight decay 10^{-3} and apply early stopping on the test loss with patience 1000 and min_delta = 10^{-4} . The maximum number of training iterations is 10,000. After the model-training half-step in Alg. 2, we optimize the Matérn covariance parameters (α, τ, σ) by minimizing the upper bound in Prop. 3.1 using `scipy.optimize.minimize`, with gradients from Lem. D.8.

Training and optimization procedure. We initialize the parameters at $\alpha = 3, \tau = 1.4$, and $\sigma = 5$. All experiments use the fixed publicly available dataset `burgers_data_R10.mat` from Kovachki et al. (2023), which contains 2048 input-output sample pairs. We reserve 10% of the data (205 samples) as a held-out set for estimating the relative OOD error and treat the remaining 1843 samples as i.i.d. draws from a test distribution which we assume to be Gaussian. To optimize the training distribution ν , we generate $N = 1843$ samples from the current parameter values, train the

2646 DeepONet, and then update the parameters using this model and test samples. We repeat this process
 2647 until the improvement in the objective function falls below 3×10^{-3} . This occurs at iteration 10,
 2648 yielding optimal parameters $\alpha = 2.906$, $\tau = 1.456$, $\sigma = 5.005$.
 2649

2650 **Model evaluation.** After identifying the optimal training distribution, we compare the model
 2651 trained on N samples from this distribution with models trained using two pool-based active learning
 2652 strategies, Query-by-Committee (QbC) and CoreSet. We treat the 1843 test samples as the pool, i.e.,
 2653 $N_{\text{pool}} = 1843$. Each method begins with 10 randomly selected samples, after which we fully train
 2654 the model, select an additional $n_{\text{select}} = 40$ samples according to the method-specific criterion, and
 2655 retrain the model from scratch. For subsequent iterations we use $n_{\text{select}} = 50$ until each method
 2656 reaches a total of 1000 training samples. We run 10 independent trials of this experiment. The
 2657 selection criteria for each strategy are summarized below.
 2658

2659 **Selection criteria.** The pool contains N_{pool} candidate inputs. Let n_{select} denote the number of
 2660 points selected at each acquisition step and let $\Delta x = 2\pi/n$ be the spatial grid spacing.
 2661

- 2662 • **Query-by-Committee (QbC).** Given an ensemble of $M = 4$ models that produce
 2663 predictions $u_i^{(m)} \in \mathbb{R}^n$ for pool sample i and model m , define the ensemble mean
 2664 $\bar{u}_i = \frac{1}{M} \sum_{m=1}^M u_i^{(m)}$. The uncertainty score is the average squared L^2 deviation from
 2665 the mean:

$$\text{uncertainty}(i) := \frac{1}{M} \sum_{m=1}^M \|u_i^{(m)} - \bar{u}_i\|_{L^2}^2 \quad \text{and} \quad \|v\|_{L^2} := \sqrt{\Delta x \sum_{k=1}^n v_k^2}. \quad (\text{F.38})$$

2666 Select the n_{select} pool points with largest $\text{uncertainty}(i)$.
 2667

- 2668 • **CoreSet (greedy k -center on last-layer features).** Let $k := n_{\text{select}}$. For each pool input,
 2669 compute a last-layer feature vector $f_i \in \mathbb{R}^p$ from the DeepONet evaluated at point a_i from
 2670 the pool, apply a Gaussian sketch with $d = 32$, i.e.,
 2671

$$U \sim \mathcal{N}(0, 1)^{d \times p}, \quad s_i = \frac{f_i U^\top}{\sqrt{d}} \in \mathbb{R}^d, \quad (\text{F.39})$$

2672 and normalize $\tilde{s}_i = s_i / \|s_i\|_2$. Initialize the selected set S_0 with one random index and
 2673 iteratively add
 2674

$$i_{t+1} = \arg \max_{i \in \text{pool} \setminus S_t} \min_{j \in S_t} \|\tilde{s}_i - \tilde{s}_j\|_2 \quad (\text{F.40})$$

2675 until $|S_T| = k$. Return the indices in S_T .
 2676

2677 For each model trained on N samples, performance is measured by the relative OOD error (Eqn. F.19),
 2678 approximated with the 205 held-out samples.
 2679

2680 Fig. 6 compares the model trained on N samples from the optimal distribution with the two active-
 2681 learning-based models. The results show that the model trained on the optimal distribution performs
 2682 comparably to the active learning methods. A key advantage of the AMA approach is that it allows
 2683 sampling beyond the size of the fixed data pool, whereas pool-based active learning methods are
 2684 restricted to selecting at most the available pool samples. We remark that generally within the field
 2685 of active learning and adaptive sampling, one cannot expect to always outperform nonadaptive/i.i.d.
 2686 sampling, especially in high dimensions. The target mapping and choice of deployment regime \mathbb{Q}
 2687 will determine the gains, if any, of active sampling (Adcock & Brugliapaglia, 2022).
 2688

2689 **Computational resources.** All experiments were conducted on AWS using Galaxy’s Deep Learning
 2690 AMI GPU, PyTorch 2.0.1, and Ubuntu 20.04-prod-ada4rwvf26mda with an EC2 instance type of
 2691 g5.12xlarge. The experiments for AMA and the two different active learning methods were run in
 2692 parallel, with the QbC method taking the longest at 11 hours. Thus, a total of approximately 11
 2693 GPU-hours (and corresponding CPU usage) were required for all independent runs with four GPUs
 2694 running in parallel.
 2695