SCIENCE CHINA
Mathematics

• ARTICLES •

December 2022 Vol. 65 No. 12: 2441–2458
https://doi.org/10.1007/s11425-022-2016-3

Boundary Harnack principle on nodal domains

Dedicated to Professor Jiaxing Hong on the Occasion of His 80th Birthday

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Received April 25, 2022; accepted August 5, 2022; published online September 28, 2022

Abstract We study some geometric and potential theoretic properties of nodal domains of solutions to certain uniformly elliptic equations. In particular, we establish corkscrew conditions, Carleson type estimates and boundary Harnack inequalities on a class of nodal domains.

Keywords boundary Harnack principle, nodal domain, modified Harnack chain, Carleson type estimate, frequency bound

MSC(2020) 35B05, 35B45, 31B25

Citation: Lin F H, Lin Z J. Boundary Harnack principle on nodal domains. Sci China Math, 2022, 65: 2441–2458, https://doi.org/10.1007/s11425-022-2016-3

1 Introduction

Let w be a nonzero solution of $\mathcal{L}(w) = 0$ in $B_{10}(0) \subset \mathbb{R}^n$, where $\mathcal{L} = \partial_i(a_{ij}(x)\partial_j)$ with the coefficient matrix $A(x) = (a_{ij})$ satisfying (2.2) and (2.3). Let Ω be a nodal domain of w, which is a path-connected subregion of the set $\{x \in B_{10} \mid w(x) \neq 0\}$. In order to get meaningful analytic estimates such as those presented in survey articles [2, 25], one cannot avoid dealing with the cases where Ω is a non-smooth domain. At a micro scale, Ω resembles cone like structures near each point of $\partial\Omega \cap B_{10}$ by the unique continuation property, while at larger scales, Ω could be like a highly twisted Hölder type domain with rather complicated geometrical and topological properties. In higher dimensions, even when the nodal set $Z(w) = \{x \in B_{10} \mid w(x) = 0\}$ is in a small neighborhood of a one-dimensional smooth set and hence small in the apparent geometric size, its complexity is hard to bound. For example, by Runge's theorem, one can easily construct a sequence of harmonic functions $\{w_k(x)\}$ in $\mathbb{R}^n$ $(n \geqslant 2)$ such that $w_k \to -1$ locally uniformly on Σ while $w_k \to +1$ locally uniformly on $\mathbb{R}^n \backslash \Sigma$, where Σ is a finite union of closed half-lines connecting the origin to infinity. In particular, some of the nodal domains of w_k inside B_{10} are collapsed into an arbitrarily small open neighborhood of Σ. In such cases, one cannot expect the validity of a three-spheres theorem for solutions or the validity of a uniform Carleson type estimate or the boundary Harnack principle. It is remarkable, on the other hand, that Logunov [18] proved the Nadirashvili's conjecture, which asserts that $H^{n-1}(\{x \mid w(x) = 0\} \cap B_1) \geqslant C(n) > 0$ for a harmonic

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function w with $w(0) = 0$. It means that such sequences of harmonic functions $\{w_k(x)\}$ as described above must be highly oscillating and not locally uniformly bounded.

After examining various examples one concludes that in order to carry out classical potential and elliptic partial differential equation (PDE) analysis on a nodal domain Ω similar to those in well-known cases of Lipschitz and non-tangentially accessible (NTA) domains (see [4,15]), one needs to make some additional assumptions on the solutions w and operators $\mathcal{L}$. In particular, one hopes to find a class of domains that are invariant under scaling (at least, they are so with respect to the up scalings). In recent work of Logunov and Malinnikova [19,20], it was proved that if u and v are usual harmonic functions in B_{10} with $Z(u) = Z(v)$, then the ratio $f \equiv v/u$ is analytic and satisfies the Harnack inequality and $|\nabla f|$ as well as higher-order derivatives of f validates estimates like those for typical solutions of elliptic PDEs with analytic coefficients. Similar results were proved in $\mathbb{R}^2$ in [23]. All these estimates depend on a fixed nature of the analytic variety $Z(u)$, and they are not necessarily scaling invariant. On the other hand, it is not hard to see that [20] can be generalized to the case where u and v are solutions of elliptic PDEs with real analytic coefficients.

In this paper, we consider a class of solutions w which have a fixed bound on their growth rates or a bound on their frequencies on B_{10} (see Section 2 for details). More precisely, we consider those $w \in \mathcal{S}_{N_0}(\Lambda)$ defined by (2.18), a very natural class of solutions which have been investigated in great detail for their quantitative unique continuation properties and related geometric measure estimates on the nodal and critical point sets (see Section 2). The following are main results of this paper.

1.1 Main results

Theorem 1.1. *Suppose that $\mathcal{L}u = \mathcal{L}v = 0$ in B_{10}, $N_u \leqslant N_0 < \infty$ and $0 \in Z(u) \subset Z(v)$. Then $v/u \in C^\alpha(B_1)$ for some $\alpha = \alpha(\Lambda, N_0) \in (0,1)$.*

For constants in the form of $C = C(\Lambda, N_0)$, we mean that the constants depend on N_0 and the conditions on the coefficients in (2.2) and (2.3) of the operator $\mathcal{L}$. Here, N_u is the frequency function (doubling index) of u on B_{10}, which will be reviewed in Section 2. Various equivalent notations and auxiliary lemmas are discussed in Section 2.

The above theorem is derived, as in earlier work, from the upper bound inequality

$$\sup_{B_1} |v/u| \leqslant C(\Lambda, N_0) \cdot \Big(\sup_{B_8} |v| / \sup_{B_8} |u| \Big), \tag{1.1}$$

when $Z(u) \subset Z(v)$ and $N_u \leqslant N_0$ (see Theorem 4.2). In order to get the Hölder continuity for v/u, one also needs an iterative argument involving improvements of upper and lower bounds as in [4,15]. The latter is based on the following Harnack type estimate:

$$\Big(\sup_{B_{1/8}} (v/u) - \inf_{B_1} (v/u) \Big) \leqslant C(\Lambda, N_0) \cdot \Big(\inf_{B_{1/8}} (v/u) - \inf_{B_1} (v/u) \Big). \tag{1.2}$$

To prove this Harnack type estimate, we need to show that the frequency of the function

$$v - u \cdot \inf_{B_1} (v/u)$$

is also bounded in a smaller ball like $B_{1/4}$.

The above leads us to the next more general result which says that if two solutions of two possibly different elliptic partial differential equations have the same nodal set in B_{10}, and if one of the solutions has a bounded frequency or a fixed growth rate, then the other has to have a bounded frequency and growth rate as well. We remark that it is in this latter statement that we require both operators $\mathcal{L}$ and $\mathcal{L}_1$ to have Lipschitz continuous coefficients. In fact, it can be shown that the conclusion is not valid if operators are uniformly elliptic with only bounded measurable coefficients.

Theorem 1.2. *Suppose that $\mathcal{L}(u) = \mathcal{L}_1(v) = 0$ in B_{10} and $0 \in Z(u) = Z(v)$. Also assume that $N_u \leqslant N_0 < \infty$. Then there is a positive constant $D = D(\Lambda, N_0) < \infty$ such that $N_v(0,1) \leqslant D$.*

Here, $N_v(0,1)$ is the frequency function for v and the ball $B_1(0)$. We emphasize again that $\mathcal{L}$ and $\mathcal{L}_1$ could be two different elliptic operators satisfying (2.2) and (2.3). This provides a local compactness property for a large class of solutions to such elliptic equations (see [8]).

As a direct corollary of Theorems 1.1 and 1.2, we have the following theorem.

Theorem 1.3. *Suppose that $\Delta(u) = \Delta(v) = 0$ in $\mathbb{R}^n$, u is a harmonic polynomial and $Z(u) = Z(v)$. Then there is a constant $c \in \mathbb{R}\backslash\{0\}$ such that $v = c \cdot u$.*

When u is a homogeneous harmonic polynomial, this theorem was proved (see [20, Theorem 1.2]). In Corollary 4.7, we prove, in fact, a bit stronger statement. The condition that u is a polynomial is important for [20]. For example, let $u_{a,b}(x, y, z) = \sin(z)\mathrm{e}^{ax+by}$ and $a^2+b^2 = 1$. Then harmonic functions $u_{a,b}$ share the same nodal set, but with exponential growth. The work [19] described many interesting examples of harmonic functions sharing the same nodal set either locally or globally.

In connection with harmonic/PDE analysis on non-smooth domains (see, e.g., [4, 15]), we also established Carleson type estimates like (1.1) on a single nodal domain Ω (defined by a solution u_0). It should be noted that, in general, one cannot expect continuity (or even boundedness) up to the boundary $\partial\Omega$ for the ratio v/u if v and u are solutions defined only on this single Ω (see Theorem 6.1 and Example 1.5). Another main result we establish is the following statement.

Theorem 1.4. *Let Ω be a nodal domain of a solution $u_0 \in \mathcal{S}_{N_0}(\Lambda)$ with $\mathcal{L}_0(u_0) = 0$ and $0 \in \partial\Omega$. Then there is a set consisting of a bounded number of points $\{x_1, \ldots, x_{T_0}\}$ with $T_0 = T_0(\Lambda, N_0)$ in $\Omega \cap B_2$, such that*

$$C^{-1} \cdot |\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_1) \leqslant \sum_{i}^{T_0} \omega_i \llcorner (\partial\Omega \cap B_1) \leqslant C \cdot |\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_1) \tag{1.3}$$

for some positive constant $C = C(\Lambda, N_0)$, where $\{\omega_i(\cdot)\}$'s are $\mathcal{L}_0$-harmonic measures on $\partial(\Omega \cap B_5)$ with poles at $x_i \in \Omega \cap B_2$ for $i = 1, 2, \ldots, T_0$. In particular, $\sum_i^{T_0} \omega_i \llcorner (\partial\Omega \cap B_1)$, $H^{n-1} \llcorner (\partial\Omega \cap B_1)$ and $|\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_1)$ are mutually absolutely continuous.

Note that it is necessary in general to choose more than one of such points x_i and the corresponding harmonic measures in order to have the two-sided estimates as shown in the above theorem. If one selects only one of such points (and its associated harmonic measure), then only the right half inequality of (1.3) is true in general. We also point out that the locations of these points $\{x_i\}$, while flexible, may depend on a particular nodal domain (and hence the defining function u_0). What is important is that one can always choose such points; moreover, the number of such points $\{x_i\}$ is uniformly bounded (by T_0) for any nodal domain of u_0 for all $u_0 \in \mathcal{S}_{N_0}(\Lambda)$.

There are two basic ingredients in proving these results. One is the validity of the corkscrew condition and the existence of modified Harnack chains for this class of nodal domains. Such geometric structural properties make these nodal domains very similar to the NTA domains (see [15]). The other is the Carleson type estimates for solutions as in [4,15,20]. We show by the doubling properties of the functions (solutions) that nodal domains possess these desired geometric properties. We then establish the boundary Harnack principle and Carleson type estimates for nonnegative solutions of uniformly elliptic operators with bounded measurable coefficients on such domains. The latter may be a useful fact for applications to some elliptic free boundary problems.

To end the descriptions of main results, let us show an example due to Leon Simon (see [13]).

Example 1.5. Let $f(z)$ be a smooth function on $\mathbb{R}$ with $|f''| < 1/2$. The function $u(x, y, z) = xy+f(z)$ satisfies the elliptic equation $\partial^2_{xx}u + \partial^2_{yy}u + \partial^2_{zz}u - (f''(z))\partial^2_{xy}u = 0$. Then the singular set of $Z(u)$, i.e., $\{x \in \mathbb{R}^3 \mid u(x) = |\nabla u(x)| = 0\}$, is $\{(0, 0, z) \in \mathbb{R}^3 \mid f(z) = f'(z) = 0\}$.

One can choose a smooth (even analytic) and sufficiently small f such that around the singular set of $Z(u)$, $Z(u)$ behaves like many double cones and u only has two nodal domains. For example, one can consider f with many local fluctuations like $(z \sin(1/z))^2$. In each hyperplane with the z-coordinate fixed, $Z(u)$ is a hyperbola if $f(z) \neq 0$ and is a joint of two crossed lines if $f(z) = 0$. The topology of the nodal domains of u and its critical set can be unbounded (in the smooth case) while the frequency of the solution u is close to 2. Carleson type estimates as well as the boundary Harnack principle are still valid among other conclusions proved here.

1.2 The structure of this paper

In Section 2, we go over some tools and basic facts that will be used in the paper, in particular, the notions of the frequency function, the doubling index and the three-spheres theorems. In Section 3, we show the corkscrew property and a modified Harnack chain property for this class of nodal domains. Our arguments generalize those in [20]. In Section 4, we first show that the ratio v/u is locally bounded near the nodal sets, and then give the proof for Theorem 1.1. We also discuss the entire solutions and prove Theorem 1.3. In Section 5, we establish the Carleson type estimates and prove Theorem 1.2. In Section 6, we discuss the boundary Harnack principle on a single given nodal domain and then we prove Theorem 1.4.

Remark 1.6. Although we only consider the elliptic operators in the divergence form in this paper, one could easily extend all the results in this paper to the elliptic operators in the non-divergence form with Lipschitz continuous leading coefficients. It would also be interesting to obtain a parabolic counterpart.

2 Preliminaries and tools

Let w be a $W^{1,2}$-solution of an elliptic equation in the divergence form in $B_{10} \subset \mathbb{R}^n$ (the Euclidean ball with the radius equal to 10 and the center at 0), i.e.,

$$\mathcal{L}(w) \equiv \operatorname{div}(A(x)\nabla w(x)) = 0, \tag{2.1}$$

where the symmetric matrix-valued function, $A(x) = (a_{ij})_{n\times n}$, satisfies

$$\lambda \cdot \mathrm{I} \leqslant A \leqslant \lambda^{-1} \cdot \mathrm{I} \tag{2.2}$$

with Lipschitz entries

$$\|a_{ij}\|_{\mathrm{Lip}} \leqslant \Lambda_1 \tag{2.3}$$

for some positive constants λ and Λ_1. In this paper, we write $\mathcal{L}$, $\mathcal{L}_1$, etc. for elliptic operators which satisfy the conditions (2.1)–(2.3). We use L, L_1, etc. to denote uniformly elliptic operators that only satisfy (2.1) and (2.2). For simplicity, we use the notation $C(\Lambda)$ to denote positive constants which depend only on λ, Λ_1 and n and call them universal constants. We use $C(\lambda)$ for constants depending only on (2.2) and the dimension n. Most of the constants appearing in this paper depend only on the dimension n, the ellipticity constant λ and the doubling constant N_0 for solutions of such uniformly elliptic operators L. By the standard interior estimates, if $\mathcal{L}(w) = 0$ in B_{10}, then w is in $C^{1,\alpha}(B_9)$ for any $\alpha \in (0,1)$. For general uniformly elliptic operators L, one infers that w is in C^α for some positive α by the De Giorgi's theorem (see [12]). We define $Z(w) \equiv \{x \in B_{10} \mid w(x) = 0\}$ as the zero set of w in B_{10}. For any point $x \in B_9$, we also define $\delta_w(x) \equiv \operatorname{dist}(x, Z(w))$ and use $\delta(x)$ if there is no ambiguity.

2.1 The frequency function and the doubling index

Let us first recall the frequency function, which goes back to the work of Agmon [1] and Almgren [3], and was further developed in [8] (see also [16]). This is a useful ingredient in estimating the size of nodal sets and the size of critical sets. We refer to [22] for more recent developments with much improved sharp results and other applications of the frequency functions. For the convenience we recall and collect a few basic facts about the frequency function and its important consequences.

The frequency function for a solution w of $\mathcal{L}(w) = 0$ is defined as

$$N_w(B_r(x_0)) = N_w(x_0, r) = \frac{r \cdot \int_{B_r(x_0)} \langle A(x)\nabla w, \nabla w\rangle dx}{\int_{\partial B_r(x_0)} \mu(x)|w|^2 d\sigma(x)}, \tag{2.4}$$

where $\sigma(x)$ is the standard surface measure on $\partial B_r(x_0)$. For simplicity, we omit the differentials in integrals if there is no ambiguity. We also set

$$H_w(x_0, r) = r^{1-n} \int_{\partial B_r(x_0)} \mu(x)|w|^2 d\sigma(x), \tag{2.5}$$

where $\mu(x) = \langle A(x)x, x\rangle/|x|^2$, $0 < \lambda < \mu(x) < n \cdot \lambda^{-1}$ and $x_0 \in B_9$. Here, $B_r(x_0)$ is the Euclidean ball with the radius equal to r and the center at x_0. If $x_0 = 0$, we use $B_r = B_r(0)$. If there is no ambiguity, we often use $N(r)$ and $H(r)$ (or $N_w(r)$ and $H_w(r)$) for simplicity. By [8], one has

$$\frac{H'}{H} = \frac{2N(r)}{r} + O(1), \tag{2.6}$$

and $O(1)$ is bounded by a universal constant $C_1 = C_1(\Lambda)$. We then have the following monotonicity theorem from [8].

Theorem 2.1. *There is a positive constant $C_2 = C_2(\Lambda)$ such that $\exp(C_2 r) \cdot N(r)$ is a nondecreasing function of r.*

A main consequence of Theorem 2.1 is the doubling estimate. By using (2.6), one has

$$\begin{aligned} \log\left(\frac{H(2R)}{H(R)}\right) &\leqslant \int_R^{2R} 2\exp(-C_2 r)\frac{\exp(C_2 r)N(r)}{r} dr + C_1 R \\ &\leqslant C(\Lambda) \cdot N(2) + C_1(\Lambda) \end{aligned} \tag{2.7}$$

for any $x_0 \in B_8$ and $R < 1$. For $|\nabla w|$, we have a similar doubling estimate, which was also derived in [8].

Theorem 2.2. *Assume that $w(0) = 0$ and $N_w(B_5) \leqslant N_0 < \infty$. Then for any $x \in B_2$ and $R \in (0, 1)$, we have*

$$\int_{B_{2R}(x)} |w|^2 dx \leqslant 2^{K_1 N_0} \int_{B_R(x)} |w|^2 dx, \tag{2.8}$$

$$\int_{B_{2R}(x)} |\nabla w|^2 dx \leqslant 2^{K_2 N_0} \int_{B_R(x)} |\nabla w|^2 dx \tag{2.9}$$

for some universal constants K_1 and K_2.

One can then easily derive the following versions of three-spheres theorems.

Theorem 2.3. *There exist universal constants K_3 and K_4 and universal constants $\alpha_1, \alpha_2 \in (0, 1)$ such that for any $x \in B_1$,*

$$\sup_{B_1(0)} |w| \leqslant K_3 \sup_{B_{1/8}(x)} |w|^{\alpha_1} \cdot \sup_{B_2(0)} |w|^{1-\alpha_1}, \tag{2.10}$$

$$\sup_{B_1(0)} |\nabla w| \leqslant K_4 \sup_{B_{1/8}(x)} |\nabla w|^{\alpha_2} \cdot \sup_{B_2(0)} |\nabla w|^{1-\alpha_2}. \tag{2.11}$$

Note that (2.10) is a consequence of (2.8) and one can get (2.11) by (2.9) (or by (2.8), the Caccioppoli estimate and the Poincaré inequality) in a similar way. Recently, in [21], Logunov and Malinnikova have improved substantially (2.10) and (2.11) by establishing a sharp Remez type estimate for solutions.

Before we proceed further, we want to point out the following equivalence of norms.

Lemma 2.4. *There are universal constants c_1 and c_2 such that for any $0 < r < 4$,*

$$\sup_{B_r(x_0)} |w|^2 \leqslant c_1 \fint_{B_{3r/2}(x_0)} |w|^2 dx \leqslant c_2 \cdot H(x_0, 2r) \tag{2.12}$$

for any $x_0 \in B_6$.

The first inequality follows from the De Giorgi's theorem [12], while the second inequality also follows from (2.6) and $N(r) \geqslant 0$, and it is a general fact for subsolutions.

In the following, we use the $\sup|\cdot|$ norm for most of the estimates. First as in [22], one defines the doubling index

$$N_D(w, B_r(x)) = \log\left(\frac{\sup_{B_r(x)}|w|}{\sup_{B_{r/2}(x)}|w|}\right) \tag{2.13}$$

for any $B_r(x) \subset B_9$. Or more generally, one defines

$$\widetilde{N}_D(w, B_r(x)) = \sup_{B_s(y)\subset B_r(x)} \log\left(\frac{\sup_{B_s(y)}|w|}{\sup_{B_{s/2}(y)}|w|}\right). \tag{2.14}$$

The doubling index $N_D(r)$ and the frequency function $N(r)$ are equivalent because of Lemma 2.4, and we have the following inequalities:

$$K_1^{-1}N(r/2) - K_2 \leqslant N_D(r) \leqslant K_1 N(2r) + K_2 \tag{2.15}$$

and

$$K_1^{-1}\widetilde{N}_D(r/2) - K_2 \leqslant N_D(r) \leqslant \widetilde{N}_D(r) \tag{2.16}$$

for some universal constants K_1 and K_2 and all $r \in (0,8)$ (see, for details, [8,16,22]). Henceforth, without ambiguity, for either L or $\mathcal{L}$, when we say $N_w \leqslant N_0$, we mean that $\widetilde{N}_D(w, B_8(0))$ is bounded by N_0. For $\mathcal{L}$, we always, doubling the size of balls if necessary, use the equivalence of $N(r)$, $N_D(r)$ and $\widetilde{N}_D(r)$.

Finally, let us give another application of these statements above. It is a growth estimate of $|w(x)|$ in terms of $\delta(x)$ for x near the nodal set of the solution w, which will be an important ingredient in our paper.

Theorem 2.5. *Suppose that $L(w) = 0$ in B_{10} for L only satisfying* (2.2) *with $Z(w) \cap B_4 \neq \emptyset$ and $N_w \leqslant N_0 < \infty$. Then there exist positive constants $A_1(\lambda)$, $A_2(\lambda, N_0)$ and $\alpha(\lambda) \in (0,1)$ such that*

$$A_1 \cdot \sup_{B_8}|w| \cdot \operatorname{dist}^{\alpha}(x, Z(w)) \geqslant |w(x)| \geqslant A_2 \cdot \sup_{B_8}|w| \cdot \operatorname{dist}(x, Z(w))^{N_0} \tag{2.17}$$

for any $x \in B_2$.

Proof. We can assume that $\sup_{B_8}|w| = 1$. The inequality on the left-hand side follows directly from the De Giorgi's theorem (see [12]). For the one on the right-hand side, let $r = \operatorname{dist}(x, Z(w))$. Then the usual Harnack inequality implies that $\sup_{B_{r/2}(x)}|w| \leqslant h(\lambda)\cdot|w(x)|$ for some $h(\lambda) > 1$. By the definition of $N_w \leqslant N_0$, we know that $2^{-kN_0}\cdot\sup_{B_{2^{k-1}\cdot r}(x)}|w| \leqslant \sup_{B_{r/2}(x)}|w|$ for all $k \in \mathbb{Z}_+$, which yields the conclusion. $\square$

Remark 2.6. One can easily find scaled versions of the above growth estimate on balls of size r. For operators with analytic coefficients (hence solutions are also analytic in the interior), the above growth estimate can be derived from the Łojasiewicz inequality as in [20]. However, all the constants involved depend on the real analytic nature of the variety $Z(w)$. It is thus not so convenient to obtain uniform estimates when the nodal sets $Z(w)$ or operators involved are perturbed. If the coefficients are Lipschitz continuous, the gradients of the solutions w satisfy the same growth estimates (see [8]).

2.2 A compact class of solutions

Our second tool builds on the compactness of a class of solutions to any elliptic equations satisfying (2.2) and (2.3), which are defined as follows:

$$\mathcal{S}_{N_0}(\Lambda) \equiv \left\{ w \in W^{1,2} \,\middle|\, \mathcal{L}(w) = 0 \text{ in } B_{10},\ \mathcal{L} \text{ satisfies (2.2) and (2.3)},\ N_w \leqslant N_0,\ \sup_{B_8}|w| = 1 \right\}. \tag{2.18}$$

This is a compact family in the local $C^{1,\alpha}$-metric. A direct consequence is the compactness of their zero sets, i.e.,

$$\mathcal{F}_{N_0}(\Lambda) = \{Z(w) \cap \overline{B_8} \mid w \in \mathcal{S}_{N_0}(\Lambda)\} \tag{2.19}$$

is compact under the Hausdorff distance.

The class $\mathcal{S}_{N_0}(\Lambda)$ is usually used to give upper bounds for the size of nodal sets or the size of critical sets. Let us summarize these estimates into the following statements:

$$H^{n-1}(Z(w) \cap B_4) \leqslant P_1(\Lambda, N_0) \tag{2.20}$$

and

$$H^{n-2}(S(w) \cap B_4) \leqslant P_2(\Lambda, N_0) \tag{2.21}$$

for any $w \in \mathcal{S}_{N_0}(\Lambda)$. Here, $S(w) \equiv \{x \in B_9 \mid w(x) = |\nabla w|(x) = 0\}$ and the two positive constants P_1 and P_2 depend only on Λ and N_0.

There are several important contributions for these two estimates (see, for example, [5, 7, 10, 14, 16]). The best estimates up to date are $P_1 = M_1(\Lambda) \cdot N_0^\alpha$ for some $\alpha = \alpha(n) > 1$ and $P_2 = \exp(M_2(\Lambda) \cdot N_0^2)$, which are in [21], [17] and [24] separately. It is worth pointing out that Cheeger et al. [5] and Naber and Valtorta [24] also established estimates on the Minkowski content, i.e., the volume of a small neighborhood of $Z(w)$ and $S(w)$. Moreover, the set $S(w)$ can be replaced by $C(w)$, the set of all the points $x \in B_9$ such that $|\nabla w(x)| = 0$ (see, for example, [5, 11]).

3 The modified Harnack chain and the corkscrew condition

In this section, we show some geometric properties of nodal domains. Surprisingly, some of them are similar to properties of NTA domains [15], which have been influential in potential analysis on non-smooth domains and which have applications to many problems including the regularity of free boundaries. For a domain to be NTA, it needs to satisfy two assumptions called the corkscrew condition and the Harnack chain condition. It is not hard to find examples of nodal domains that are not NTA. In some sense, typical nodal domains are like Lipschitz cones at sufficiently small scales and at larger scales they are more like twisted Hölder domains with complicated topology. For the class of uniformly elliptic operators with bounded measurable coefficients, so long as the solutions that are considered satisfy this additional doubling property (2.8) , the associated nodal domains will satisfy a corkscrew condition and a modified Harnack chain condition. In the (uniformly) analytic case, it was proved in [20]. Our proof of the following statement is a generalization of that in [20]. It builds on the natural scaling invariant property for this class of nodal domains.

Theorem 3.1. *Suppose that $L(w) = 0$ in B_{10} with $0 \in Z(w)$ and $N_w \leqslant N_0 < \infty$. Then for any nodal domain Ω of w with $\Omega \cap B_1 \neq \emptyset$ and any $x \in \Omega \cap B_1$, there is a chain of points $\{x_i\}_{i=0}^m \subset \Omega$ with $x_0 = x$ and satisfying the following properties: for $i = 0, 1, \ldots, m-1$,*

(1) (*modified Harnack chain*)

(i) $|w(x_{i+1})| \geqslant C_3(\lambda, N_0)|w(x_i)|$ *for some* $C_3 > 1$;

(ii) $|x_{i+1} - x_i| \leqslant (1 - \theta_0(\lambda, N_0)) \cdot \delta(x_i)$ *for some* $\theta_0 \in (0, 1)$, $x_i \in B_2$ *and* $\delta(x_i) \leqslant 1/4$;

(iii) $x_m \in B_3 \backslash B_2$ *or* $x_m \in B_2$ *but* $\delta(x_m) > 1/4$;

(iv) $m \leqslant -\xi_1(\lambda, N_0) \log(\delta(x_0)) + \xi_2(\lambda, N_0)$ *for some* $\xi_1, \xi_2 > 0$;

(2) (*corkscrew condition*) $\delta(x_m) > c_4(\lambda, N_0)$ *for some* $c_4 \in (0, 1/4)$, *and hence* $B_4 \cap \Omega$ *contains a ball of radius* $c_4/2$.

If one considers all the nodal domains of w that intersect with B_1, the second statement in the above theorem exactly implies the two-sided corkscrew condition as in the definition of NTA domains.

The first statement in the above theorem leads to modified Harnack chains. One does have that the values of $w(x_i)$ grow geometrically. But it only implies that x_i's stay away from $Z(w)$ (in a same nodal domain) by a power of its distance to the boundary of the nodal domain. This latter geometric picture is consistent with Theorem 2.5. In this connection, we find that there is an interesting connection with the hyperbolic metric defined on the nodal domains, which is the Euclidean metric multiplied by the conformal factor w^{-2}. But we shall not explore it in this paper.

Lemma 3.2. *Suppose that $L(w) = 0$ in B_{10} with $0 \in Z(w)$ and $N_w \leqslant N_0 < \infty$. Then there are constants $C_3 = C_3(\lambda, N_0) > 1$ and $\theta_0 = \theta_0(\lambda, N_0) \in (0, 1)$ such that for any $x \in B_2$ with $w(x) \neq 0$ and $\delta(x) \leqslant 1/4$, there is an $\tilde{x} \in B_3$ with $|x - \tilde{x}| \leqslant (1 - \theta_0) \cdot \delta(x)$ and $|w(\tilde{x})| > C_3|w(x)|$.*

Proof. Suppose that $w(x) > 0$ and let $\delta = \delta(x)$. Set $\epsilon \equiv (\sup_{B_{(1-\theta)\delta(x)}} w)/w(x) - 1 > 0$ with a positive and small θ to be chosen later. Since $L(w(\cdot) - w(x)) = 0$, by the usual Harnack inequality,

$$\sup_{B_{(1-2\theta)\delta(x)}} |w(\cdot) - w(x)| \leqslant C(\lambda, \theta) \cdot \sup_{B_{(1-\theta)\delta(x)}} (w(\cdot) - w(x)) = C(\lambda, \theta) \cdot \epsilon w(x). \tag{3.1}$$

By the definition of $N_w \leqslant N_0$ and the usual Harnack inequality, we see

$$\sup_{B_{2\delta}(x)} |w| \leqslant 4^{N_0} \sup_{B_{\delta/2}(x)} |w| \leqslant 4^{N_0} \cdot C(\lambda) \cdot w(x). \tag{3.2}$$

On the other hand, since there is an $x_* \in Z(w)$ such that $|x - x_*| = \delta$, the De Giorgi's theorem yields

$$\sup_{B_{4\theta\delta}(x_*)} |w| \leqslant C(\lambda) \cdot \theta^\alpha \sup_{B_{2\delta}(x)} |w| \leqslant C(\lambda) 4^{N_0} \cdot \theta^\alpha \cdot w(x) \tag{3.3}$$

for some $\alpha = \alpha(\lambda)$ and for every $\theta \in (0, 1/16)$. Now we choose a $\theta = \theta(\lambda, N_0) \in (0, 1)$ such that $C(\lambda)4^{N_0} \cdot \theta^\alpha < 1/2$ in (3.3).

Then for any $y \in B_{4\theta\delta}(x_*) \cap B_{(1-2\theta)\delta}(x)$, by (3.1) and (3.3), we know

$$(1 - C(\lambda, \theta)\epsilon) \cdot w(x) \leqslant w(y) \leqslant \frac{1}{2} \cdot w(x), \tag{3.4}$$

which yields $\epsilon \geqslant c > 0$ for some positive $c = c(\lambda, \theta) = c(\lambda, N_0)$. □

With Lemma 3.2, we can proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. For (i) and (ii), one simply applies Lemma 3.2 iteratively. This iteration that satisfies both (i) and (ii) has to end after finitely many steps. We let m as the smallest positive integer such that the corresponding x_m satisfies Theorem 3.1(iii).

For (iv), the upper bound of m, by (i) and Theorem 2.5, we find

$$(C_3)^m \cdot A_2 \delta(x)^{N_0} \cdot \sup_{B_8} |w| \leqslant (C_3)^m |w(x)| \leqslant |w(x_m)| \leqslant \sup_{B_8} |w|, \tag{3.5}$$

which is equivalent to

$$m \leqslant (-N_0 \log(\delta(x)) - \log(A_2))/\log(C_3). \tag{3.6}$$

Since $C_3 > 1$, we get the desired ξ_1 and ξ_2.

For the corkscrew condition, we first assume that $\delta(x_m) \leqslant 1/4$ and $\sup_{B_8} |w| = 1$. From Theorem 2.5, there are $A_1(\lambda), A_2(\lambda, N_0) > 0$ such that

$$A_1 \cdot \delta(y)^\alpha \geqslant |w(y)| \geqslant A_2 \cdot \delta(y)^{N_0} \tag{3.7}$$

for any $y \in B_2$. Hence, it suffices to show that $|w(x_m)| \geqslant C(\lambda, N_0) > 0$. Because

$$|x_0 - x_m| \leqslant \sum_{i=0}^{m-1} |x_i - x_{i+1}| \tag{3.8}$$

and

$$|x_i - x_{i+1}| \leqslant \delta(x_i) \leqslant A_2^{-1/N_0} |w(x_i)|^{1/N_0} \leqslant A_2^{-1/N_0} |w(x_m)|^{1/N_0} \cdot C_3^{(i-m)/N_0}, \tag{3.9}$$

also, $|x_0 - x_m| \geqslant 2 - 1 = 1$ and $\sum_{i=0}^{m-1} C_3^{(i-m)/N_0}$ is bounded by $1/(C_3^{1/N_0} - 1)$, we get a desired lower bound for $|w(x_m)|$. □

A direct corollary of the corkscrew condition is the local boundedness of the number of nodal domains.

Corollary 3.3. *Suppose that $L(w) = 0$ in B_{10} with $0 \in Z(w)$ and $N_w \leqslant N_0 < \infty$. Then the number of nodal domains in B_4 which have nonempty intersections with B_1 is bounded by a positive integer $T_0 = T_0(\lambda, N_0)$.*

4 Boundary Harnack, Hölder continuity and entire solutions

We use the corkscrew property of nodal domains to provide versions of the boundary Harnack principle. We first observe the following lemma.

Lemma 4.1. *Assume that $\mathcal{L}u = \mathcal{L}v = 0$ in B_{10} with $\sup_{B_8}|u| = \sup_{B_8}|v| = 1$ and $N_D(u, B_8) \leqslant N_0 < \infty$. If $1 < 2^{N_0+1} < m < \infty$, then $N_D(mu - v, B_8) \leqslant N_0 + 2 < \infty$.*

Proof. It holds that

$$\frac{\sup_{B_8}|mu-v|}{\sup_{B_4}|mu-v|} \leqslant \frac{m+1}{m\cdot \sup_{B_4}|u| - 1} \leqslant \frac{m+1}{m\cdot 2^{-N_0}-1} \leqslant 2\frac{m+1}{m\cdot 2^{-N_0}} \leqslant 4\cdot 2^{N_0}. \tag{4.1}$$

This completes the proof. □

Theorem 4.2. *Suppose that $\mathcal{L}u = \mathcal{L}v = 0$ in B_{10} and $0 \in Z(u) \subset Z(v)$ with $\sup_{B_8}|u| = \sup_{B_8}|v| = 1$. If $N_u \leqslant N_0 < \infty$, then there is a positive constant $C = C(\Lambda, N_0) < \infty$ such that $|v/u| \leqslant C$ in $B_1 \backslash Z(u)$.*

Proof. First, we show that there is a large $C = C(\Lambda, N_0)$ such that $Cu - v$ has the same nodal domains as u in B_1. Set $\delta(x) \equiv \delta_{Z(u)}(x)$.

Let $E = \{x \in B_3 \mid (Cu - v)(x)\cdot u(x) > 0\}$. We can first assume that $C > 2^{N_0+1}$ as in the previous lemma and then by (2.16) get $N_{Cu-v} \leqslant \mathcal{N}_0 \equiv K_1(\Lambda)(N_0 + 2) + K_2(\Lambda)$ for some $K_1, K_2 > 0$. Let $E_1 = \{x \in B_3 \mid \delta(x) > c_4/8\}$, where $c_4 = c_4(\mathcal{N}_0, \Lambda)$ is the same constant appearing in the corkscrew condition of Theorem 3.1 for $Cu - v$. By Theorem 2.5, we have $|u(x)| > A_2(c_4/8)^{N_0} \equiv c$ for any $x \in E_1$. Let us fix $C = 2\max\{c^{-1}, 2^{N_0+1}\}$. For this C, we have $E_1 \subset E$ because for any $x \in E_1$, we have $|Cu(x)| > 2$, and then $(Cu - v)(x)$ and $u(x)$ must have the same sign.

For this fixed C, assume that there is an $x \in B_1$ such that $u(x) > 0$ but $(Cu - v)(x) \leqslant 0$. Note that if $(Cu - v)(x) = 0$, by the strong maximum principle and unique continuation, we can always choose another point y arbitrarily close to x with $(Cu - v)(y) < 0$ but $u(y) > 0$. So we assume that $(Cu - v)(x) < 0$. Therefore, this x is in a negative nodal domain Ω of $Cu - v$ in B_3, which means that $Cu - v < 0$ in Ω and Ω is in the complement of E. On the other hand, since x is in a positive nodal domain of u in B_3, which we denote by Ω_1, Ω is contained in Ω_1 and is certainly not connected with other nodal domains of u. So $\Omega \subset \Omega_1 \backslash E$. By the corkscrew property as in Theorem 3.1 for $Cu - v$ (note that the doubling index is bounded by $N_0 + 2$ independent of large C), there is a point $x_m \in \Omega \cap B_3$ such that $\delta(x_m) \geqslant \operatorname{dist}(x_m, Z(Cu - v)) > c_4$, which is clearly impossible by our construction of E_1 and the fact that $E_1 \subset E$. Hence we have proved that $B_1 \backslash Z(u) \subset E$, which means that $Cu - v$ has the same nodal domains as u in B_1.

Similarly, for the same C, we can show that $Cu + v$ has the same nodal domains as u in B_1. Hence, $|v/u| \leqslant C$ in $B_1 \backslash Z(u)$. □

Corollary 4.3. *Suppose that $\mathcal{L}u = \mathcal{L}v = 0$ in B_{10}, $0 \in Z(u) = Z(v)$ and $\sup_{B_8}|u| = \sup_{B_8}|v| = 1$ with $N_u \leqslant N_0$ and $N_v \leqslant N_0$ for some positive $N_0 < \infty$. Then there is a positive constant $C = C(\Lambda, N_0) < \infty$ such that $C^{-1} \leqslant |v/u| \leqslant C$ in $B_1 \backslash Z(u)$.*

Proof. Switch the positions of u and v in Theorem 4.2. □

Remark 4.4. We should note that both Theorem 4.2 and Corollary 4.3 remain true when $\mathcal{L}$ is replaced by L (see Theorem 6.1 in Section 6). On the other hand, by Theorem 5.1 that we will prove in the next section, we can drop the assumption that $N_v \leqslant N_0$ because Theorem 5.1 implies that $N_v \leqslant D(\Lambda, N_0) < \infty$ on B_1. Consequently, one can prove the boundedness of $|v/u|$ on $B_{1/10} \backslash Z(u)$ as in Theorem 4.2.

Next, we show that v/u satisfies a strong maximum principle, which was noted in [19, Remark 2.8] for the case $\mathcal{L} = \Delta$.

Theorem 4.5. *Suppose that $\mathcal{L}u = \mathcal{L}v = 0$ in B_{10} and $Z(u) \subset Z(v)$. Then $\sup_{B_8} v/u$ cannot be achieved at $x \in B_8$ if v/u is not a constant.*

Proof. Denote $\sup_{B_8} v/u$ by M. We consider $Mu - v$. We can assume that $Mu - v \not\equiv 0$. For $x \in B_8$, if $u(x) > 0$, then $M \geqslant v(x)/u(x)$ and then $Mu(x) - v(x) \geqslant 0$. By the usual strong maximum principle, we know $Mu(x) - v(x) > 0$. Similarly, if $u(x) < 0$, we have $Mu(x) - v(x) < 0$. Hence, M is not achieved at $x \in B_8 \backslash Z(u)$. These also tell us that $Z(u) \cap B_8 = Z(Mu - v) \cap B_8$.

Now, for any $x_0 \in Z(u) \cap B_8$, consider $B_{10r}(x_0)$ for some r small enough. By Theorem 4.2,

$$\begin{aligned}\inf_{B_r(x_0)\backslash Z(u)} \left|M - \frac{v}{u}\right| &= \inf_{B_r(x_0)\backslash Z(u)} \left|\frac{Mu - v}{u}\right| \\ &= \frac{1}{\sup_{B_r(x_0)\backslash Z(u)} |u/(Mu - v)|} \\ &\geqslant C \cdot \frac{\sup_{B_{8r}(x_0)} |Mu - v|}{\sup_{B_{8r}(x_0)} |u|} > 0, \end{aligned} \tag{4.2}$$

where C is a positive constant depending on Λ and $\widetilde{N}_D(Mu - v, B_{10r}(x_0)) < \infty$. Hence, we conclude that in $B_r(x_0)$, $M > v/u$. Then $M > (v/u)(x)$ for any x strictly inside B_8. □

To end this section, we are going to work on the continuity of v/u. If we only need the continuity of v/u at some point $x_0 \in Z(u)$, we can consider u and $v - (v/u)(x_0) \cdot u$ in Theorem 4.2 and use the Taylor expansions of u and v at x_0 (see [9] for the Taylor expansion and [19] for more in the case of harmonic functions). But in this way, the continuity scale depends on the point x_0. In the $\mathbb{R}^2$ case, this way also gives differentiability of v/u since the formal gradient of v/u at $x_0 \in S(u) = \{x \mid u(x) = |\nabla u| = 0\}$ is 0.

Here, we are going to show the Hölder continuity of v/u, and the proof is quite standard if we also apply the conclusion of Theorem 5.1 which will be proven in the next section.

Theorem 4.6. *Suppose that $\mathcal{L}u = \mathcal{L}v = 0$ in B_{10}, $N_u \leqslant N_0 < \infty$ and $0 \in Z(u) \subset Z(v)$. Then $v/u \in C^\alpha(B_{1/10})$ for some $\alpha = \alpha(\Lambda, N_0) \in (0,1)$.*

Proof. We are going to show the oscillation decay estimate at 0. If

$$\sup_{B_{1/100}} \frac{v}{u} \leqslant \frac{1}{2}\left(\sup_{B_1} \frac{v}{u} + \inf_{B_1} \frac{v}{u}\right), \tag{4.3}$$

then

$$\sup_{B_{1/100}} \frac{v}{u} - \inf_{B_{1/100}} \frac{v}{u} \leqslant \frac{1}{2}\left(\sup_{B_1} \frac{v}{u} - \inf_{B_1} \frac{v}{u}\right). \tag{4.4}$$

If

$$\sup_{B_{1/100}} \frac{v}{u} \geqslant \frac{1}{2}\left(\sup_{B_1} \frac{v}{u} + \inf_{B_1} \frac{v}{u}\right), \tag{4.5}$$

we consider $v^*(x) = (v - (\inf_{B_1}(v/u)) \cdot u)(x/10)$ and $u^*(x) = u(x/10)$. Note that u^* and v^* have the same zero set in B_{10} by the proof of Theorem 4.5, $v^*u^* \geqslant 0$ and $N_{u^*} \leqslant N_u \leqslant N_0$. By Theorem 5.1, $N_{v^*} \leqslant D = D(\Lambda, N_0)$ in B_1. Then by Corollary 4.3, with a larger constant $C = C(\Lambda, D) = C(\Lambda, N_0)$ in it, we can show

$$\inf_{B_{1/100}} \frac{v}{u} - \inf_{B_1} \frac{v}{u} = \inf_{B_{1/10}} \frac{v^*}{u^*} \geqslant C^{-2} \sup_{B_{1/10}} \frac{v^*}{u^*} \geqslant \frac{1}{2C^2}\left(\sup_{B_1} \frac{v}{u} - \inf_{B_1} \frac{v}{u}\right), \tag{4.6}$$

and then

$$\sup_{B_{1/100}} \frac{v}{u} - \inf_{B_{1/100}} \frac{v}{u} \leqslant \left(1 - \frac{1}{2C^2}\right)\left(\sup_{B_1} \frac{v}{u} - \inf_{B_1} \frac{v}{u}\right). \tag{4.7}$$

This completes the proof. □

A direct corollary of Theorem 4.6 and (4.7) is the following Liouville theorem for the case $\mathcal{L} = \Delta$. In this case, all the constants $C(\Lambda, N_0)$ will be replaced by $C(n, N_0)$ so that we can do both blow-ups and blow-downs. All the theorems in this section are valid with constants of the form $C(n, N_0)$.

Corollary 4.7. *Suppose that $\Delta(u) = \Delta(v) = 0$ in $\mathbb{R}^n$, $N_u(0,r) < N_0 < \infty$ for all $r > 0$ and $Z(u) \subset Z(v)$. Then there is a $\beta = \beta(n, N_0) \in (0,1)$ such that if*

$$\liminf_{r\to\infty} r^{-\beta} \cdot \sup_{B_r} \frac{v}{u} < \infty, \tag{4.8}$$

we have $v = c \cdot u$ for some $c \in \mathbb{R}$. In particular, if $Z(u) = Z(v)$, the condition (4.8) *will be satisfied, and then there is a constant $c \in \mathbb{R}\backslash\{0\}$ such that $v = c \cdot u$.*

Proof. If $Z(u) \neq Z(v)$, we may assume that $(v/u)(0) = 0$. Then if $\sup_{B_r} |v/u| = -\inf_{B_r} v/u$, one can define $M = \sup_{B_{100r}} v/u$ and consider $M - v/u = (Mu - v)/u$ on B_{100r}. Since $Mu - v$ and u have the same zero set in B_{100r}, by Theorem 5.1 and Corollary 4.3, there is a constant $C = C(n, N_0) > 1$ such that

$$M + \sup_{B_r} \left|\frac{v}{u}\right| = \sup_{B_r} \frac{Mu - v}{u} \leqslant C \cdot \inf_{B_r} \frac{Mu - v}{u} = C \cdot M - \sup_{B_r} \frac{v}{u} \leqslant C \cdot M. \tag{4.9}$$

Hence, there is always a constant $M_1 = M_1(n, N_0) > 1$ such that

$$\sup_{B_r} \left|\frac{v}{u}\right| \leqslant M_1 \cdot \sup_{B_{100r}} \frac{v}{u}. \tag{4.10}$$

By the proof of Theorem 4.6 and (4.7), there is a constant $\theta = \theta(n, N_0) \in (0, 1)$ such that

$$\sup_{B_r} \frac{v}{u} - \inf_{B_r} \frac{v}{u} \leqslant \theta^k \cdot \left(\sup_{B_{100^k r}} \frac{v}{u} - \inf_{B_{100^k r}} \frac{v}{u} \right) \leqslant \theta^k \cdot (2M_1) \cdot \sup_{100^{k+1} r} \frac{v}{u} \tag{4.11}$$

for all $r > 0$ and $k \in \mathbb{Z}_+$. By choosing $\beta = \beta(n, N_0) \in (0, 1)$ such that $\theta \cdot 100^\beta < 1$, we see that the statement follows if we let $k \to \infty$ and then $r \to \infty$.

If $Z(u) = Z(v)$, by Corollary 4.3 and Theorem 5.1, we have

$$\sup_{B_r} \left|\frac{v}{u}\right| \leqslant C \cdot \inf_{B_r} \left|\frac{v}{u}\right| \leqslant C \cdot \left|\frac{v}{u}\right|(0) \tag{4.12}$$

for some $C = C(n, N_0) > 0$ and all $r > 0$. Denote the right-hand side of (4.12) by M_2. By the first inequality of (4.11), we know

$$\sup_{B_r} \frac{v}{u} - \inf_{B_r} \frac{v}{u} \leqslant \theta^k \cdot \left(\sup_{B_{100^k r}} \frac{v}{u} - \inf_{B_{100^k r}} \frac{v}{u} \right) \leqslant \theta^k \cdot M_2. \tag{4.13}$$

The statement follows if we let $k \to \infty$ and then $r \to \infty$. We note that the above proof involves only controls of growth of both u and v at infinity. If one uses the fact that the operator is the standard Laplacian, then the hypothesis on u implies that u is a harmonic polynomial. If the ratio v/u grows like a power of r, then v is also a harmonic polynomial. The conclusions can also be derived directly by working with polynomials and blow-downs. □

5 Uniform bounds on frequency functions for solutions with the same zero set

In this section, all the elliptic operators $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_0$ satisfy the conditions (2.2) and (2.3). Our main result is the following theorem.

Theorem 5.1. *Suppose that $\mathcal{L}(u) = \mathcal{L}_1(v) = 0$ in B_{10} and $0 \in Z(v) = Z(u) = Z$. Also assume that $N_u \leqslant N_0 < \infty$. Then there is a positive constant $D = D(\Lambda, N_0) < \infty$ such that*

$$\log \left(\frac{\sup_{B_1} |v|}{\sup_{B_{1/2}} |v|} \right) \leqslant D. \tag{5.1}$$

In order to prove this theorem, we first need a Carleson type estimate, which is always a key ingredient for the boundary Harnack principle (see, for example, [4,15,20]). The proof for this Lemma 5.2 is inspired by [4].

Lemma 5.2. *Suppose that $L(u) = 0$ in B_{10}, $0 \in Z(u) = Z$ and $N_u \leqslant N_0 < \infty$. Assume that Ω is a nodal domain of u in B_3 which satisfies $\Omega \cap B_{1/2} \neq \emptyset$. Then if $L_1(v) = 0$ in Ω, $v > 0$ in Ω and $v = 0$ on $Z \cap \partial\Omega$, there exist constants $M = M(\lambda, N_0) > 0$ and $c = c(\lambda, N_0) > 0$ such that the following estimate holds*:

$$\sup_{B_{1/2}\cap\Omega} v \leqslant M \sup_{y\in B_2\cap\Omega,\delta(y)\geqslant c} v(y). \tag{5.2}$$

In particular, if $L_1(v) = 0$ in B_{10} and $Z(v) = Z$, then

$$\sup_{B_{1/2}} |v| \leqslant M \sup_{y\in B_2,\delta(y)\geqslant c} |v|(y). \tag{5.3}$$

Proof. Take $c = c_4/2$ for c_4 in the corkscrew condition of Theorem 3.1. Assume that

$$\sup\{|v(y)| \mid y \in B_2 \cap \Omega, \delta(y) \geqslant c\} = 1.$$

Then we prove that $\sup\{|v(y)| \mid y \in B_{1/4} \cap \Omega\} \leqslant M$ for some $M = M(\lambda, N_0)$.

First, we claim that for any $x \in B_1 \cap \Omega$, there are $\alpha_1(\lambda, N_0) > 0$ and $\alpha_2(\lambda, N_0) > 0$ such that

$$v(x) \leqslant \alpha_2 \cdot \delta(x)^{-\alpha_1}. \tag{5.4}$$

This claim follows from Theorem 3.1(iv) by a backward iteration along the Harnack chain. Indeed, since the length of the modified Harnack chain associated with x is bounded by $-\xi_1 \log(\delta(x)) + \xi_2$ and $\delta(x_m) \geqslant c_4$, if we apply the usual Harnack inequality along this modified Harnack chain, we get

$$v(x) \leqslant h^m \cdot v(x_m) \leqslant h^{-\xi_1 \log(\delta(x))+\xi_2} \cdot 1 \tag{5.5}$$

for some $h = h(\lambda, \theta_0) = h(\lambda, N_0) > 1$, which is the constant in the Harnack inequality for this class of elliptic operators.

Next, we need the following standard elliptic estimate for subsolutions: if $L(w) \geqslant 0$ in B_2, $w \geqslant 0$ in B_2 and $|\{x \in B_2 \mid w(x) = 0\}| \geqslant \epsilon > 0$, then $\sup_{B_1} w \leqslant \theta \cdot \sup_{B_2} w$ for some $\theta = \theta(\lambda, \epsilon) \in (0, 1)$.

We now follow the same type arguments as in [4]. Assume that for some $y_0 \in B_{1/2} \cap \Omega$ and $|v(y_0)| = M_0 > 1$, then one has $\delta(y_0) < c$. Consider the ball $B_{3\delta(y_0)}(y_0)$, on which v may be regarded as a nonnegative subsolution if we extend v to be 0 out of Ω. By the corkscrew condition of Theorem 3.1, $B_{3\delta(y_0)}(y_0)\backslash\Omega$ contains a ball of radius $\delta(y_0) \cdot r$ with some small $r = r(\lambda, N_0) > 0$. Hence, by the above estimate for nonnegative subsolutions, there is a $y_1 \in B_{3\delta(y_0)}(y_0) \cap \Omega$ such that $v(y_1) \geqslant \theta^{-1} v(y_0) = \theta^{-1} M_0$ for a $\theta = \theta(\lambda, r) = \theta(\lambda, N_0) \in (0, 1)$. Consequently, $\delta(y_1) < c$ so long as y_1 is also in B_2.

We can continue this process to find $y_2, y_3, \ldots$ so long as they all stay inside B_2. Let us estimate $|y_0 - y_i|$ for $i \geqslant 0$. Note that our construction gives $|y_i - y_{i+1}| \leqslant 3\delta(y_i)$. By (5.4), if $y_i \in B_1 \cap \Omega$, then

$$\delta(y_i) \leqslant \beta_1 \cdot v(y_i)^{-\beta_2} \leqslant \beta_1 \cdot \theta^{\beta_2 i} \cdot v(y_0)^{-\beta_2} = \beta_1 \cdot \theta^{\beta_2 i} \cdot M_0^{-\beta_2} \tag{5.6}$$

for some $\beta_1 = \beta_1(\lambda, N_0) > 0$ and $\beta_2 = \beta_2(\lambda, N_0) > 0$. Since $\theta < 1$, the last terms on the right-hand side form a convergent geometric series, and we can sum all of them up for $i = 1, 2, \ldots$

If $M_0^{\beta_2} \geqslant 30\beta_1/(1 - \theta^{\beta_2})$, then $|y_0 - y_i| \leqslant 1/10$ for all $i \geqslant 0$, and then all y_i's stay in $B_1 \cap \Omega$. This is a contradiction since $v(y_i) \geqslant \theta^{-i} M_0 \to \infty$ as $i \to \infty$. □

We can now proceed with the proof of Theorem 5.1. The strategy is quite simple. The first step is to use Lemma 5.2 to push the point where the solution v takes approximate maximum values away from the nodal set. Next, we apply the Harnack inequality along paths fully contained in a nodal domain of v (or equivalently, a nodal domain of u), which connects points in a larger ball far away from the zero set to points where v reaches approximate maximums inside a smaller ball. The difficulty is to avoid neck-like tiny regions in the process of connecting these points so that it can be done in a quantitatively controlled manner.

Proof of Theorem 5.1. We need to consider the following family:

$$\mathcal{S}_{N_0}(\Lambda) \equiv \Big\{ w \;\Big|\; \mathcal{L}(w) = 0 \text{ in } B_{10},\, \mathcal{L} \text{ satisfies (2.2) and (2.3)},\, N_w \leqslant N_0,\, \sup_{B_8} |w| = 1 \Big\}, \tag{5.7}$$

which is a compact family in the local $C^{1,\alpha}$-metric.

We can then prove the statement by contradiction. If the theorem failed, assume that $\{u_n\} \subset \mathcal{S}_{N_0}(\Lambda)$ with $\sup_{B_8} |u_n| = 1$ and $0 \in Z(u_n) = Z_n$. v_n satisfies that $\mathcal{L}_n(v_n) = 0$ in B_{10} and $Z(v_n) = Z_n$ with

$$\log\left(\frac{\sup_{B_1} |v_n|}{\sup_{B_{1/2}} |v_n|}\right) \to \infty. \tag{5.8}$$

By compactness of the class $\mathcal{S}_{N_0}(\Lambda)$, we can assume that $u_n \to u_0 \in \mathcal{S}_{N_0}(\Lambda)$. Note that $0 \in Z(u_0)$ since the convergence is in the local $C^{1,\alpha}$-metric.

Let $Z_0 = Z(u_0)$. We make a partition of $B_2\backslash Z_0$ in terms of nodal domains. Let us assume that $B_2\backslash Z_0 = \bigsqcup_{i=1}^{T}(\Omega_i \cap B_2)$, where Ω_i $(i = 1, \ldots, T)$ are disjoint nodal domains of u_0 in B_3 such that $\Omega_i \cap B_2 \neq \emptyset$. Note that $T \leqslant T_0 = T_0(\lambda, N_0)$ by Corollary 3.3. If we divide $[3/2, 2)$ into $[3/2 + (j-1)/(4T_0), 3/2 + j/(4T_0))$, $j = 1, \ldots, 4T_0$, there exists a $j = j(Z_0)$ such that for each Ω_i, if $\Omega_i \cap B_{3/2+j/(4T_0)} \neq \emptyset$, then $\Omega_i \cap B_{3/2+(j-1)/4T_0} \neq \emptyset$. We denote the subset of subindices of these Ω_i by I_0. Hence, we can set $\eta = 3/2 + (j-1)/(4T_0) + 1/(8T_0) \in (3/2, 2)$ and $\epsilon = 1/(100T_0) \ll 1$. We focus on the ball B_η. The point here is that those Ω_i with $i \in I_0$ are path-connected, form a partition of B_η and also have a nonempty intersection with $B_{\eta-10\epsilon}$.

By using Lemma 5.2 to push maximum points away from zero sets locally, one can show that for some positive $M = M(\lambda, N_0)$ and $c(\lambda, N_0)$,

$$\sup_{B_{\eta-2\epsilon}} |v_n| \leqslant M \sup_{y\in B_{\eta-\epsilon}, \delta_n(y)\geqslant c} |v_n|(y), \tag{5.9}$$

where $\delta_n(y) = \mathrm{dist}(y, Z_n)$. Assume that the maximal value of the right-hand side of (5.9) is achieved by y_n. Note that when n is large enough, $\{y \in B_{\eta-\epsilon} \mid \delta_n(y) \geqslant c\}$ is contained in

$$\{y \in B_{\eta-\epsilon} \mid \delta_0(y) = \mathrm{dist}(y, Z_0) \geqslant c/2\}.$$

Hence, we can assume that $\delta_0(y_n) \geqslant c/2$.

Because for each $i \in I_0$, $\Omega_i \cap B_{\eta-10\epsilon} \neq \emptyset$, by the corkscrew condition of Theorem 3.1, there is a small ball of radius $r = r(\lambda, N_0) > 0$ with center x_i inside $\Omega_i \cap B_{\eta-4\epsilon}$. Let $d = \min\{c/4, r/2\}$. There is a constant $\mu = \mu(d, Z_0) > 0$ such that for any two points x and y in $\Omega_i(d) \equiv \Omega_i \cap \{y \in B_{3-d} \mid \delta_0(y) \geqslant d\}$, x and y are connected by a path γ, which is fully contained in $\Omega_i(\mu) \equiv \Omega_i \cap \{y \in B_{3-\mu} \mid \delta_0(y) \geqslant \mu\}$. The existence of such a μ and the Harnack inequality lead to the desired conclusion. In fact, if we use dyadic cubes of side length $\mu/10$ to cover B_{10}, those cubes which intersect with $\Omega_i(\mu)$ are fully contained in $\Omega_i(\mu/2)$, and the number of cubes is bounded by $Q \equiv C(n)\mu^{-n}$ (here n is the dimension and not to be confused with the subindices).

Hence, when the subindex n of u_n is large enough, each $\Omega_i(\mu/2)$ is fully contained in a single nodal domain of u_n by Theorem 2.5. Since $\delta_0(y_n) \geqslant c/2$, y_n is contained in an $\Omega_i(d)$ for an $i \in I$. y_n and x_i are then connected by a path $\gamma_{n,i}$ fully contained in $\Omega_i(\mu)$, which is covered by Q cubes with side length $\mu/10$. We can then apply the Harnack inequality Q times along $\gamma_{n,i}$, and get

$$|v_n|(y_n) \leqslant h^Q |v_n|(x_i) \leqslant h^Q \sup_{B_{\eta-4\epsilon}} |v_n| \tag{5.10}$$

for some $h = h(\lambda) > 1$.

Combining (5.9) and (5.10), we see

$$\sup_{\eta-2\epsilon} |v_n| \leqslant M \cdot h^Q \sup_{\eta-4\epsilon} |v_n|, \tag{5.11}$$

which contradicts (5.8) by Theorem 2.1. $\square$

6 Analysis on a single nodal domain

In this section, we fix a single domain and discuss properties of solutions on this domain. More precisely, let $L_0(u_0)=0$ in B_{10}, $0\in Z(u_0)=Z_0$ and $N_{u_0}\leqslant N_0<\infty$. We consider a nodal domain Ω of u_0 in B_5 with $0\in\partial\Omega$. We use the notation $\delta(x)\equiv\operatorname{dist}(x,Z_0)=\operatorname{dist}(x,\partial\Omega)$ for $x\in\Omega$.

6.1 The boundary Harnack inequality on a given nodal domain

Theorem 6.1. *Suppose that $L(u)=L(v)=0$ in Ω, $u>0$ on Ω and $u=v=0$ on $\partial\Omega\cap B_3$ continuously. Then there are positive constants $M=M(\lambda,N_0)$ and $r=r(\lambda,N_0)$ such that*

$$\left|\frac{v}{u}\right|\leqslant M\cdot\frac{\sup_{B_1\cap\Omega}|v|}{\inf_{y\in B_2\cap\Omega,\delta(y)\geqslant r}u(y)}\tag{6.1}$$

on $B_{1/4}\cap\Omega$. In particular, if $v>0$ on Ω and

$$0<C_1\leqslant v,u\leqslant C_2\tag{6.2}$$

on $\{x\in B_2\cap\Omega\mid\delta(x)\geqslant r\}$, then

$$\frac{C_1}{C_2}\cdot M^{-2}\leqslant\frac{v}{u}\leqslant\frac{C_2}{C_1}\cdot M^2\tag{6.3}$$

on $B_{1/4}\cap\Omega$.

We prove this theorem with cubes in the place of balls for convenience. We consider cubes Q_s with center 0 and side length $2s$, and we define $K_s\equiv\Omega\cap Q_s$ and $A_s\equiv\{x\in K_s\mid\delta(x)\geqslant\delta\cdot s\}$, where $\delta=\delta(\lambda,N_0)\ll1$ will be chosen in the following lemma. The argument is inspired by [6] for NTA domains.

Lemma 6.2. *There exist $M_0=M_0(\lambda,N_0)>0$ and $\delta=\delta(\lambda,N_0)>0$ such that if w is a solution to $L(w)=0$ in K_1, not necessarily positive, which vanishes on $\partial\Omega\cap B_1$, and $w\geqslant M_0$ on A_1, $w\geqslant-1$ on K_1, then we have $w\geqslant M_0\cdot a$ on $A_{1/2}$ and $w\geqslant-a$ on $K_{1/2}$ for some small $a=a(\lambda,N_0)>0$.*

Proof. First, we construct the lower bound on $A_{1/2}$. Pick an $x_0\in A_{1/2}$. Then there is a modified Harnack chain $\{x_0,x_1,\dots,x_m\}$, which we get in Theorem 3.1. In the corkscrew condition of Theorem 3.1, we show that $\delta(x_m)\geqslant c_4=c_4(\lambda,N_0)$. We also find

$$m\leqslant-\xi_1\log(\delta(x_0))+\xi_2\leqslant-\xi_1\log(\delta/2)+\xi_2\tag{6.4}$$

in Theorem 3.1(iv) for $\xi_1=\xi_1(\lambda,N_0)>0$ and $\xi_2=\xi_2(\lambda,N_0)$. Hence, if we assume that $\delta<c_4$ first, by the Harnack inequality along this chain with the constant $h(\lambda,\theta_0)=h(\lambda,N_0)>1$, we have

$$w(x_0)\geqslant(M_0+1)\cdot h^{\xi_1\log(\delta/2)-\xi_2}-1.\tag{6.5}$$

We choose

$$a=(1/2)\cdot h^{\xi_1\log(\delta/2)-\xi_2}.$$

Then when $M_0\geqslant1/a$, we have

$$w(x_0)\geqslant M_0\cdot a.\tag{6.6}$$

Then we show that $w\geqslant-a$ on $K_{1/2}$ for suitable δ. Let $x_0\in K_{1-2\delta}\backslash A_1$. Consider the cube $Q(x_0,2\delta)$. By Theorem 3.1, there is a small ball with radius $\delta\cdot c$ for some $c=c(\lambda,N_0)$ in $Q(x_0,2\delta)\backslash\Omega$, where $w^-=0$. Hence, by the weak Harnack inequality we mentioned in the proof of Lemma 5.2 and $w\geqslant-1$ on K_1,

$$w^-(x_0)\leqslant(1-c_1)\sup_{Q(x_0,2\delta)}w^-\leqslant(1-c_1)\tag{6.7}$$

for some $c_1=c_1(\lambda,N_0)\in(0,1)$. Hence, $w^-\leqslant(1-c_1)$ in $K_{1-2\delta}$. By iteration, we get $w^-\leqslant(1-c_1)^t$ in $K_{1-2t\delta}$, and then

$$w\geqslant-(1-c_1)^{\frac{1}{8\delta}}\quad\text{on }K_{1/2}.\tag{6.8}$$

Since $\delta \cdot \log(\delta) \to 0$ as $\delta \to 0$, we can choose a small $\delta = \delta(\lambda, N_0)$ such that

$$(1-c_1)^{\frac{1}{8\delta}} \leqslant a = (1/2) \cdot h^{\xi_1 \log(\delta/2) - \xi_2}. \tag{6.9}$$

This completes the proof. □

By the above Lemma 6.2, and by iterating the same arguments on $K_{2^{-t}}$ and $A_{2^{-t}}$, we can conclude that $w > 0$ in $\{x \in K_1 \mid \delta(x) \geqslant 2\delta|x|\}$. Because one can vary the centers of K_s and A_s, hence w is positive in $K_{1/4}$.

Proof of Theorem 6.1. Set $w \equiv Cu - v$. We choose suitable C so that w satisfies the assumptions of Lemma 6.2.

For (6.1), the statement can be proved by choosing $M \geqslant M_0 + 1$, $r \leqslant \delta$ with M_0 and δ in Lemma 6.2 and choosing $C = M \cdot \sup_{K_1} |v| \cdot (\inf_{y \in K_2, \delta(y) \geqslant r} u(y))^{-1}$.

For (6.3), by Lemma 5.2,

$$\sup_{K_1} v \leqslant M_1 \cdot \sup_{y \in K_{3/2}, \operatorname{dist}(y, Z_0) \geqslant c} v \tag{6.10}$$

for some positive $M_1 = M_1(\lambda, N_0)$ and $c = c(\lambda, N_0)$. Hence, if we choose $r = \min\{c, \delta\}$ and $M = \max\{M_1, M_0 + 1\}$ with M_0 and δ in Lemma 6.2, we find

$$\sup_{K_1} v \leqslant M \cdot C_2. \tag{6.11}$$

Then we choose $C = C_1^{-1} C_2 M^2$ and the conclusions of the theorem follow. □

Remark 6.3. The strong maximum principle holds for v/u by a similar proof as in Theorem 4.5. An interesting part is that $\sup_{\Omega \cap B_1} v/u$ may not be achieved on $\partial\Omega \cap B_1$.

Corollary 6.4. *Suppose that $L_0(v) = 0$ in Ω, $v > 0$ on Ω and $v = 0$ on $\partial\Omega \cap B_3$ continuously. Then there are positive constants $C = C(\lambda, N_0)$ and $r = r(\lambda, N_0)$ such that*

$$C^{-1} \cdot \delta(x)^{N_0} \cdot \inf_{y \in B_2 \cap \Omega, \delta(y) \geqslant r} v(y) \leqslant v(x) \leqslant C \cdot \delta^{\alpha}(x) \cdot \sup_{B_1 \cap \Omega} v \tag{6.12}$$

in $B_{1/4} \cap \Omega$.

Proof. These follow from considering v and u_0 in Theorems 6.1 and 2.5. □

Remark 6.5. As we can see in (6.3) of Theorem 6.1, the upper bound depends on the ratio C_2/C_1, which also depends on the set $\{x \in K_2 \mid \delta(x) \geqslant r\}$. Since Ω is connected, one can apply the usual Harnack inequality on this set. So C_2/C_1 is actually a quantity depending on the shape of the single nodal domain Ω. Is it possible that C_2/C_1 could be controlled by some constants only depending on N_0? The answer is no and we have the following counterexample. Consider

$$\Omega_\epsilon \equiv \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 > -\epsilon, |x| < 1\},$$

which is the part of one nodal domain of $u_\epsilon(x, y) = x^2 - y^2 + \epsilon$ in B_1. It has a thin and short neck region around the origin. Let v_ϵ be the solution of the following Dirichlet problem:

$$\Delta v_\epsilon = 0 \quad \text{on } \Omega_\epsilon \tag{6.13}$$

and

$$v_\epsilon = 1 \quad \text{on } \{x = 1\} \cap \partial\Omega_\epsilon, \quad v_\epsilon = 0 \quad \text{on } \{-1 \leqslant x < 1\} \cap \partial\Omega_\epsilon. \tag{6.14}$$

We notice that $v_\epsilon > 0$ in Ω_ϵ, $v_\epsilon > C(n) > 0$ when $x > 1/2$ and $|y| < 1/2$, and v_ϵ is very close to 0 when $x < 0$. As $\epsilon \to 0$, v_ϵ tends to 0 on $\{x < 0\}$, which means that $u_\epsilon/v_\epsilon \to \infty$. One can also consider $v_\epsilon(-x, y)$ for a similar purpose on the part of the nodal domain with $x > 0$. If one replaces y by $(y_1, \dots, y_{n-1})$, one finds examples in the dimension n. On the other hand, if one replaces x by (x_1, x_2) and y by (y_1, y_2), then there is no problem when ϵ goes to zero. In the latter case, Ω_ϵ is quantitatively connected (independent of small ϵ) (see Definition 6.7).

Nodal domains are path-connected by their definitions. Examples in Remark 6.5 show that they can easily degenerate and decompose into several smaller nodal domains even for a sequence of nodal domains of solutions in $\mathcal{S}_{N_0}(\Lambda)$. Consequently, many analytic estimates on a nodal domain of a solution $u_0 \in \mathcal{S}_{N_0}(\Lambda)$ are not uniform (depending only on Λ and N_0). On the other hand, even a single nodal domain is degenerate and decomposed into several smaller nodal domains, the number of such small nodal domains is again locally uniformly bounded by a constant $T_0(\lambda, N_0)$ (see Corollary 3.3).

Inspired by our proofs of Theorem 5.1 and Lemma 5.2, if we use dyadic cubes with side length $r/10$ to cover B_{10} with $r = r(\lambda, N_0)$ chosen in Theorem 6.1, those cubes which have nonempty intersections with $\{y \in B_2 \mid \delta(y) \geqslant r\}$ form several big chunks E_i $(i = 1, \ldots, T)$ with each E_i path-connected and $T \leqslant C(n)r^{-n}$, but different E_i and E_j are disjoint. Then we give another interesting upper bound in the following corollary.

Corollary 6.6. *Let u_0 be the solution at the beginning of Section 6 and Ω be a nodal domain of u_0 in B_5 with $0 \in \partial\Omega$. Also suppose that $L_0(v) = 0$ in Ω, $v > 0$ on Ω and $v = 0$ on $\partial\Omega \cap B_3$ continuously. Then there is a positive constant $C = C(\lambda, N_0)$ such that*

$$\frac{v(x)}{u_0(x)} \leqslant C \cdot \max\left\{\frac{v(x_1)}{u_0(x_1)}, \ldots, \frac{v(x_T)}{u_0(x_T)}\right\} \tag{6.15}$$

in $B_{1/4} \cap \Omega$, where x_i is an arbitrary point inside E_i for each $i = 1, \ldots, T$.

Proof. On the right-hand side of (6.1), by Lemma 5.2, for some $M = M(\lambda, N_0)$ we have

$$\sup_{B_1 \cap \Omega} v \leqslant M \cdot \sup_{y \in B_{3/2} \cap \Omega, \delta(y) \geqslant r} v. \tag{6.16}$$

Assume that the maximal value on the right-hand side of the above inequality is achieved by a point $y_1 \in E_1$. Then by the usual Harnack inequality inside E_1 and the fact that the number of all the dyadic cubes is also bounded by $C(n)r^{-n}$, there is a constant $C = C(\lambda, N_0)$ such that

$$v(y_1) \leqslant C \cdot v(x_1). \tag{6.17}$$

By Theorem 2.5 and the Harnack inequality, there is a constant $c = c(\lambda, N_0, r) = c(\lambda, N_0)$ such that

$$\inf_{y \in B_2 \cap \Omega, \delta(y) \geqslant r} u_0(y) \geqslant c \cdot u_0(x_1). \tag{6.18}$$

By combining the above three inequalities and (6.1), we obtain (6.15). □

The above discussions inspire one to introduce the notion of the quantitative connectedness in the following definition.

Definition 6.7. We say that the nodal domain Ω is quantitatively connected, if there are positive constants $\delta_1 = \delta_1(\lambda, N_0) \leqslant \delta_2 = \delta_2(\lambda, N_0) \leqslant r/2$ such that for any $x_0 \in \overline{\Omega}$, any $s > 0$ and any pair of points x and y in $\Omega \cap B_s(x_0)$ with $\delta(x) \geqslant \delta_2 \cdot s$ and $\delta(y) \geqslant \delta_2 \cdot s$, they can be connected by a path totally contained inside $\Omega \cap B_s(x_0) \cap \{z \mid \delta(z) \geqslant s \cdot \delta_1\}$.

If Ω is quantitatively connected, then it is easy to show that in Corollary 6.6, one can give an upper bound by an arbitrary $v(x_i)/u_0(x_i)$. With the assumptions in Theorem 6.1, one can show the Hölder continuity of v/u to the boundary $\partial\Omega$ if Ω is quantitatively connected.

6.2 Some other properties and connections to other typical domains

Apart from the corkscrew property and the modified Harnack chain obtained in Section 3, the nodal domain Ω of a solution $u_0 \in \mathcal{S}_{N_0}(\Lambda)$ has several other properties that are important for classical potential analysis on non-smooth domains. Let us recall a few of such properties here.

Property 6.8. For $u_0 \in \mathcal{S}_{N_0}(\Lambda)$, $\partial\Omega \cap B_5$ is Ahlfors regular. Indeed, the following upper bound

$$H^{n-1}(B_s(x) \cap \partial\Omega) \leqslant H^{n-1}(B_s(x) \cap Z(u_0)) \leqslant C(\Lambda, N_0) \cdot s^{n-1} \tag{6.19}$$

for all $x \in \partial\Omega \cap B_5$ and $s \in (0,1)$, follows from the geometric measure estimate (2.20) (see, for example, [5, 7, 11, 14, 16, 17, 21, 24]). The lower bound follows from the corkscrew condition that

$$|\Omega \cap B_s(x)| \geqslant C(\Lambda, N_0) \cdot s^n, \tag{6.20}$$

$$|\Omega^c \cap B_s(x)| \geqslant C(\Lambda, N_0) \cdot s^n \tag{6.21}$$

for some $C(\Lambda, N_0) > 0$ and the relative isoperimetric inequality

$$H^{n-1}(\partial\Omega \cap B_s(x)) \geqslant C(n) \cdot (\min\{|\Omega \cap B_s(x)|, |\Omega^c \cap B_s(x)|\})^{\frac{n-1}{n}}. \tag{6.22}$$

It is also clear from the proofs in [14] and [11] that the following is true.

Property 6.9. $\partial\Omega \cap B_5$ is uniformly rectifiable. In fact, there is an $\epsilon_0 = \epsilon_0(\Lambda, N_0) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $Z(u_0) \cap B_5$ can be decomposed into two parts. One big part is a C^1-hypersurface with the C^1-structure depending on ϵ, and the other small part has H^{n-1} Hausdorff measures less than ϵ.

Finally, we examine some basic properties of harmonic measures with poles in Ω. For any pole $x_0 \in \{x \in \Omega \cap B_2 \mid \delta(x) \geqslant r/2\}$ with $r = r(\lambda, N_0)$ chosen in Theorem 6.1, one can easily show that

$$G(x_0, x) \leqslant C(\lambda, N_0) \cdot u_0(x) \tag{6.23}$$

for $x \in \Omega \cap B_1$ by the maximum principle on $(\Omega \cap B_5) \backslash \overline{B_{r/4}(x_0)}$. Here, $G(x_0, \cdot)$ is the Green function of L_0 on $\Omega \cap B_5$. Hence, by the definition of the harmonic measure, we know

$$\omega_{x_0} \llcorner (\partial\Omega \cap B_1) \leqslant C(\lambda, N_0) \cdot |\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_1). \tag{6.24}$$

Here, ω_{x_0} is the Harmonic measure on $\partial(\Omega \cap B_5)$ with the pole x_0.

By the estimate (2.21) and the gradient estimates for u_0, one sees that

$$|\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_2) \ll H^{n-1} \llcorner (\partial\Omega \cap B_2) \ll |\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_2). \tag{6.25}$$

On the other hand, by Corollary 6.6 and Lemma 5.2, one can prove

$$\sum_{i=1}^{T} \omega_{x_i} \llcorner (\partial\Omega \cap B_2) \geqslant C(\lambda, N_0) \cdot |\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_2). \tag{6.26}$$

Hence, we conclude the following theorem.

Theorem 6.10. *Let Ω be a nodal domain of a solution $u_0 \in \mathcal{S}_{N_0}(\Lambda)$ with $0 \in \partial\Omega$. Then there is a set of points $\{x_1, \dots, x_T\}$ chosen in Corollary 6.6 with $T \leqslant T_0(\lambda, N_0) < \infty$ in $\Omega \cap B_2$ such that*

$$C^{-1} \cdot |\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_1) \leqslant \sum_{i}^{T} \omega_i \llcorner (\partial\Omega \cap B_1) \leqslant C \cdot |\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_1) \tag{6.27}$$

for some positive constant $C = C(\lambda, N_0)$, where $\{\omega_i(\cdot)\}$ are harmonic measures on $\partial(\Omega \cap B_5)$ with poles $x_i \in \Omega \cap B_2$ for $i = 1, 2, \dots, T$. In particular, $\sum_i^T \omega_i \llcorner (\partial\Omega \cap B_1)$, $H^{n-1} \llcorner (\partial\Omega \cap B_1)$ and $|\nabla u_0| \cdot H^{n-1} \llcorner (\partial\Omega \cap B_1)$ are mutually absolutely continuous.

Acknowledgements This work was supported by National Science Foundation of USA (Grant No. DMS 1955249).

References

1 Agmon S. Unicité et convexité dans les problèmes différentiels. Montréal: Presses de l'Université se Montréal, 1966

2 Aikawa H. Potential analysis on nonsmooth domains—Martin boundary and boundary Harnack principle. In: Complex Analysis and Potential Theory. CRM Proceedings and Lecture Notes, vol. 55. Providence: Amer Math Soc, 2012, 235–253

3 Almgren F J Jr. Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. In: Minimal Submanifolds and Geodesics. Amsterdam-New York: North-Holland, 1979, 1–6

4 Caffarelli L, Fabes E, Mortola S, et al. Boundary behavior of nonnegative solutions of elliptic operators in divergence form. Indiana Univ Math J, 1981, 30: 621–640

5 Cheeger J, Naber A, Valtorta D. Critical sets of elliptic equations. Comm Pure Appl Math, 2015, 68: 173–209

6 De Silva D, Savin O. A short proof of boundary Harnack principle. J Differential Equations, 2020, 269: 2419–2429

7 Donnelly H, Fefferman C. Nodal sets of eigenfunctions on Riemannian manifolds. Invent Math, 1988, 93: 161–183

8 Garofalo N, Lin F H. Monotonicity properties of variational integrals, A_p weights and unique continuation. Indiana Univ Math J, 1986, 35: 245–268

9 Han Q. Singular sets of solutions to elliptic equations. Indiana Univ Math J, 1994, 43: 983–1002

10 Han Q, Hardt R, Lin F H. Geometric measure of singular sets of elliptic equations. Comm Pure Appl Math, 1998, 51: 1425–1443

11 Han Q, Lin F H. Rank zero and rank one sets of harmonic maps. Methods Appl Anal, 2000, 7: 417–442

12 Han Q, Lin F H. Elliptic Partial Differential Equations. Providence: Amer Math Soc, 2011

13 Hardt R, Hoffmann-Ostenhof M, Hoffmann-Ostenhof T, et al. Critical sets of solutions to elliptic equations. J Differential Geom, 1999, 51: 359–373

14 Hardt R, Simon L. Nodal sets for solutions of elliptic equations. J Differential Geom, 1989, 30: 505–522

15 Jerison D S, Kenig C E. Boundary behavior of harmonic functions in non-tangentially accessible domains. Adv Math, 1982, 46: 80–147

16 Lin F H. Nodal sets of solutions of elliptic and parabolic equations. Comm Pure Appl Math, 1991, 44: 287–308

17 Logunov A. Nodal sets of Laplace eigenfunctions: Polynomial upper estimates of the Hausdorff measure. Ann of Math (2), 2018, 187: 221–239

18 Logunov A. Nodal sets of Laplace eigenfunctions: Proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture. Ann of Math (2), 2018, 187: 241–262

19 Logunov A, Malinnikova E. On ratios of harmonic functions. Adv Math, 2015, 274: 241–262

20 Logunov A, Malinnikova E. Ratios of harmonic functions with the same zero set. Geom Funct Anal, 2016, 26: 909–925

21 Logunov A, Malinnikova E. Quantitative propagation of smallness for solutions of elliptic equations. In: Proceedings of the International Congress of Mathematicians 2018. Singapore: World Scientific, 2019, 2391–2411

22 Logunov A, Malinnikova E. Lecture notes on quantitative unique continuation for solutions of second order elliptic equations. arXiv:1903.10619, 2019

23 Mangoubi D. A gradient estimate for harmonic functions sharing the same zeros. Electron Res Announc Math Sci, 2014, 21: 62–71

24 Naber A, Valtorta D. Volume estimates on the critical sets of solutions to elliptic PDEs. Comm Pure Appl Math, 2017, 70: 1835–1897

25 Toro T. Analysis and geometry on non-smooth domains. Rev Acad Colombiana Cienc Exact Fís Natur, 2017, 41: 521–527