Recurrent Natural Policy Gradient for POMDPs

Semih Cayci
Department of Mathematics
RWTH Aachen University
Aachen, Germany 52062
cayci@mathc.rwth-aachen.de

Atilla Eryilmaz
Department of Electrical and Computer Engineering
The Ohio State University
Columbus, Ohio 43210
eryilmaz.2@osu.edu

Abstract

In this paper, we study a natural policy gradient method based on recurrent neural networks (RNNs) for partially-observable Markov decision processes (POMDPs), whereby RNNs are used for policy parameterization and policy evaluation to address curse of dimensionality in reinforcement learning for POMDPs. We present finite-time and finite-width analyses for both the critic (recurrent temporal difference learning), and correspondingly-operated recurrent natural policy gradient method in the near-initialization regime. Our analysis demonstrates the efficiency of RNNs for problems with short-term memory with explicit bounds on the required network widths and sample complexity, and points out the challenges in the case of long-term dependencies.

1 Introduction

Reinforcement learning for partially-observable Markov decision processes (POMDPs) has been a particularly challenging problem due to the absence of an optimal stationary policy, which leads to a curse of dimensionality as the space of non-stationary policies grows exponentially over time [26, 32]. There has been a growing interest in finite-memory policies to address the curse of dimensionality in reinforcement learning for POMDPs [47, 46, 24, 5]. Among these, recurrent neural networks (RNNs) have been shown to achieve impressive *empirical* success in solving POMDPs [43, 44, 30]. However, theoretical understanding of RNN-based RL methods for POMDPs is still in a nascent stage.

In this paper, we aim to remedy this by studying a model-free policy optimization method based on a recurrent natural actor-critic (Rec-NAC) framework (Section 5), which

- utilizes an RNN-based policy parameterization for history representation in non-stationary policies,
- incorporates an RNN-based temporal difference learning algorithm as the critic (Section 6), and
- performs policy updates by using RNN-based natural policy gradient as the actor (Section 7),

for large POMDPs. We establish non-asymptotic (finite-time, finite-width) analyses of Rec-TD (in Theorem 6.2) and Rec-NPG (Theorem 7.2 and Propositions 7.5-7.7), and prove their near-optimality in the large-network limit for problems that require short-term memory. We identify pathological cases that cause exponentially growing iteration complexity and network size (Remarks 6.3-7.3). Our analysis reveals an interesting connection between (i) the memory (i.e., long-term dependencies) in the POMDP, (ii) continuity and smoothness of the parameters of the RNN, and (iii) global near-optimality of the Rec-NPG in terms of the required network size and iterations.

1.1 Previous work

Natural policy gradient method, proposed in [23], has been extensively investigated for MDPs [1, 8, 25, 18, 6], and analyses of NPG with feedforward neural networks (FNNs) have been established

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in [42, 28, 7]. As these works consider MDPs, the policies are stationary. In our case, the analysis of RNNs and POMDPs constitute a very significant challenge.

In [46, 38, 24, 5], finite-memory policies based on sliding-window approximations of the history were investigated. Alternatively, value- and policy-based model-free approaches based on RNNs have been widely considered in the literature to solve POMDPs [27, 43, 44, 30]. However, these works are predominantly experimental, thus there is no theoretical analysis of RNN-based RL methods for POMDPs to the best of our knowledge. In this work, we also present theoretical guarantees for RNN-based NPG for POMDPs. For structural results on the hardness of RL for POMDPs, refer to [29, 38].

1.2 Notation

For a vector $\Theta = (\Theta_1^\top, \ldots, \Theta_m^\top)^\top \in \mathbb{R}^{m\cdot(d+1)},\ m, d \in \mathbb{Z}_+$ with $\Theta_i = (V_i, U_i^\top)^\top \in \mathbb{R}^{d+1}$ for $V_i \in \mathbb{R}, U_i \in \mathbb{R}^d$ and $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2_{\geq 0}$, we define $\mathcal{B}^{(m)}_{2,\infty}(\Theta, \rho) := \bigotimes_{i=1}^m \left(\mathcal{B}^{(1)}_1\left(V_i, \frac{\rho_1}{\sqrt{m}}\right), \mathcal{B}^{(d)}_2\left(U_i, \frac{\rho_2}{\sqrt{m}}\right)\right)$, where $\bigotimes$ is the Cartesian product, and $\mathcal{B}^{(d)}_p(x, \rho_0) := \{z \in \mathbb{R}^d : \|z - x\|_p \leq \rho_0\}$ for any $p \geq 1, x \in \mathbb{R}^d, \rho_0 \geq 0$. $\mathbb{M}_m$ denotes the set of all $m \times m$ diagonal matrices. $[m] := \{1, 2, \ldots, m\}$ for any $m \in \mathbb{Z}_+$. $\Delta(\mathbb{Y})$ is the space of probability distributions on a set $\mathbb{Y}$. $\mathsf{Rad}(\alpha) = \mathsf{Unif}\{-\alpha, \alpha\}$ for $\alpha \in \mathbb{R}_{\geq 0}$.

2 Preliminaries on Partially-Observable Markov Decision Processes

In this paper, we consider a discrete-time infinite-horizon partially-observable Markov decision process (POMDP) with the (nonlinear) dynamics

$$\mathbb{P}(S_{t+1} \in B | \sigma(S_k, A_k, k \leq t)) =: \mathcal{P}((S_t, A_t), B), \text{ and } \mathbb{P}(C | \sigma(S_t)) =: \phi(S_t, C),$$

for any $B \in \mathscr{B}(\mathbb{S})$ and $C \in \mathscr{B}(\mathbb{Y})$, where S_t is an $\mathbb{S}$-valued *state*, Y_t is a $\mathbb{Y}$-valued *observation*, and A_t is an $\mathbb{A}$-valued *control* process with the stochastic kernels $\mathcal{P} : \mathbb{S} \times \mathbb{A} \times \mathscr{B}(\mathbb{S}) \to [0, 1]$ and $\phi : \mathbb{S} \times \mathscr{B}(\mathbb{Y}) \to [0, 1]$. We consider finite but arbitrarily large $\mathbb{A} \subset \mathbb{R}^{d_1}, \mathbb{Y} \subset \mathbb{R}^{d_2}$ with $\mathbb{Y} \times \mathbb{A} \subset \mathcal{B}^{(d_1+d_2)}_2(0, 1)$ and $\mathbb{S}$. In this setting, the state process $(S_t)_{t\in\mathbb{N}}$ is not observable by the controller. Let

$$Z_t = \begin{cases} Y_0, & \text{if } t = 0, \\ (Z_{t-1}, A_{t-1}, Y_t), & \text{if } t > 0, \end{cases} \tag{1}$$

be the history process, which is available to the controller at time $t \in \mathbb{N}$, and $\bar{Z}_t := (Z_t, A_t) = (Y_0, A_0, \ldots, Y_t, A_t)$, be the history-action process.

Definition 2.1 (Admissible policy)**.** An admissible control policy $\pi = (\pi_t)_{t\in\mathbb{N}}$ is a sequence of measurable mappings $\pi_t : (\mathbb{Y} \times \mathbb{A})^t \times \mathbb{Y} \to \Delta(\mathbb{A})$, and the control at time t is chosen under π_t randomly as $\mathbb{P}(A_t = a | Z_t = z_t) = \pi_t(a|z_t)$, for any $z_t \in (\mathbb{Y} \times \mathbb{A})^t \times \mathbb{Y}$. We denote the class of all admissible policies by Π_{NM}.

If an action a is taken at state a, then a reward $r(s, a)$ is obtained. For simplicity, we assume that the reward is deterministic, and $\max\limits_{s,a} |r(s, a)| \leq r_\infty < \infty$.

Definition 2.2 (Value function, $\mathcal{Q}$-function, advantage function)**.** Let π be an admissible policy, and $\mu \in \Delta(\mathbb{Y})$ be an initial observation distribution. Then, the value function under π with discount factor $\gamma \in (0, 1]$ is defined as

$$\mathcal{V}^\pi_t(z_t) := \mathbb{E}^\pi\Big[\sum_{k=t}^\infty \gamma^{k-t} r(S_k, A_k)\Big| Z_t = z_t\Big], \tag{2}$$

for any $z_t \in (\mathbb{Y} \times \mathbb{A})^t \times \mathbb{Y}$. Similarly, the state-action value function (also known as $\mathcal{Q}$-function) and the advantage function under π are defined for any $\bar{z}_t \in (\mathbb{Y} \times \mathbb{A})^{t+1}$ as

$$\mathcal{Q}^\pi_t(\bar{z}_t) := \mathbb{E}^\pi\Big[\sum_{k=t}^\infty \gamma^{k-t} r(S_k, A_k)\Big| \bar{Z}_t = \bar{z}_t\Big], \text{ and } \mathcal{A}^\pi_t(z_t, a) := \mathcal{Q}^\pi_t(z_t, a) - \mathcal{V}^\pi_t(z_t). \tag{3}$$

Given an initial observation distribution $\mu \in \Delta(\mathbb{Y})$, the optimization problem is

$$\underset{\pi \in \Pi_{\mathsf{NM}}}{\text{maximize}} \int_{\mathbb{Y}} \mathcal{V}_0^{\pi}(z_0)\mu(dz_0) =: \mathcal{V}^{\pi}(\mu), \text{ where } \pi^{\star} \in \underset{\pi \in \Pi_{\mathsf{NM}}}{\arg\max}\, \mathcal{V}^{\pi}(\mu). \tag{4}$$

Remark 2.3 (Curse of history in RL for POMDPs). Note that the problem in equation 4 is significantly more challenging than its subcase of (fully-observable) MDPs since there may not exist an optimal policy which is (i) stationary, and (ii) deterministic [26, 38]. As such, the policy search is over *non-Markovian* randomized policies of type $\pi = (\pi_0, \pi_1, \ldots)$ where $\pi_t : (\mathbb{Y} \times \mathbb{A})^t \times \mathbb{Y} \to \Delta(\mathbb{A})$ depends on the history of observations $Z_t = (Y_0, A_0, Y_1, \ldots, A_{t-1}, Y_t)$ for $t \in \mathbb{N}$. In this case, direct extensions of the existing reinforcement learning methods for MDPs become intractable, even for finite $\mathbb{Y}, \mathbb{A}$: the instantaneous memory complexity of a probabilistic admissible policy $\pi \in \Pi_{\mathsf{NM}}$ at epoch $t \in \mathbb{N}$ is $\mathscr{O}(|\mathbb{Y} \times \mathbb{A}|^{t+1})$, growing exponentially over t.

Recurrent neural networks (RNNs), which involve a parametric recurrent structure to efficiently represent the process history by using finite memory, are universal approximators for sequence-to-sequence mappings [36, 15]. As such, we consider using them in an actor-critic framework for approximation in (i) value space (for the critic), and (ii) policy space (for the actor). In the following section, we formally introduce the RNN architecture that we study in this paper.

3 Elman-Type Recurrent Neural Networks

We consider an Elman-type recurrent neural network (RNN) of width $m \in \mathbb{N}$ with $\mathbf{W} \in \mathbb{R}^{m \times m}$ and $\mathbf{U} \in \mathbb{R}^{m \times d}$, where $d = d_1 + d_2$, and the rows of $\mathbf{U}$ are denoted as $U_i^{\top}$ for $i = 1, 2, \ldots, m$. Given a smooth activation function $\varrho : \mathscr{C}^2(\mathbb{R}, \mathbb{R})$ with $\|\varrho\|_\infty \leq \varrho_0, \|\varrho'\|_\infty \leq \varrho_1, \|\varrho''\|_\infty \leq \varrho_2$, we denote $\vec{\varrho} : \mathbb{R}^m \to \mathbb{R}^m : \boldsymbol{z} \mapsto \begin{pmatrix} \varrho(z_1)) \\ \vdots \\ \varrho(z_m) \end{pmatrix}$. Let $X_t = \begin{pmatrix} Y_t \\ A_t \end{pmatrix}$, which is an $\mathbb{R}^d$-valued random variable with $d = d_1 + d_2$. The central structure in an RNN is the sequence of hidden states $H_t \in \mathbb{R}^m$, which evolves according to

$$H_t(\bar{Z}_t; \mathbf{W}, \mathbf{U}) = \vec{\varrho}\Big(\mathbf{W} H_{t-1}(\bar{Z}_{t-1}; \mathbf{W}, \mathbf{U}) + \mathbf{U} X_t\Big), \tag{5}$$

with $H_0(\bar{Z}_0; \mathbf{W}, \mathbf{U}) = \vec{\varrho}(\mathbf{U} X_0)$ and $\bar{Z}_t = (X_0, \ldots, X_t)$ denoting the history. We denote the i^{th} element of H_t as $H_t^{(i)}$ for $i \in [m]$. We consider a linear readout layer with weights $c \in \mathbb{R}^m$, which leads to the output

$$F_t(\bar{Z}_t; \mathbf{W}, \mathbf{U}, c) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} c_i H_t^{(i)}(\bar{Z}_t; \mathbf{W}, \mathbf{U}). \tag{6}$$

The characteristic property of RNNs is ***weight-sharing***: throughout all time-steps $t \in \mathbb{N}$, the same weights are utilized, which enables the hidden state $(H_t)_{t>0}$ to summarize the entire history $\bar{Z}_t$ compactly with a fixed memory.

We consider diagonal $\mathbf{W}$ and general $\mathbf{U}$ in the paper, which simplifies the analysis, yet preserves the essential properties of RNNs. This diagonal structure for $\mathbf{W}$ is common in the study of deep linear networks for the aforementioned reason [16, 45], while our work also encompasses nonlinear activation functions and weight-sharing.

Following the neural tangent kernel literature, we omit the straightforward task of training the linear output layer $c \in \mathbb{R}^m$ for simplicity, and study the training dynamics of $(\mathbf{W}, \mathbf{U})$, which is the main challenge [12, 34, 3, 42]. Consequently, we denote the learnable parameters of a hidden unit $i \in [m]$ compactly as $\Theta_i = \begin{pmatrix} W_{ii} \\ U_i \end{pmatrix}$, and denote the learnable parameters of an RNN by $\Theta = \left[W_{11}, U_1^{\top}, W_{22}, U_2^{\top}, \ldots, W_{mm}, U_m^{\top}\right]^{\top} \in \mathbb{R}^{m(d+1)}$. Given learnable parameters $(\mathbf{W}, \mathbf{U})$, we denote the sequence of recurrent neural network outputs as $\boldsymbol{F}(\cdot; \mathbf{W}, \mathbf{U}) = (F_t(\cdot; \mathbf{W}, \mathbf{U}))_{t \in \mathbb{N}}$, and use Θ and $(\mathbf{W}, \mathbf{U})$ interchangeably throughout the paper.

4 Infinite-Width Limit of Diagonal Recurrent Neural Networks

In this paper, we consider a class of systems that can be efficiently approximated and learned by the class of large recurrent neural networks in the near-initialization regime following [4]. To that end, we provide the following characterization of the infinite-width limit of RNNs in order to give our results in later sections. Let $w_0 \sim \mathsf{Rad}(\alpha)$ and $u_0 \sim \mathcal{N}(0, I_d)$ be independent random variables, and $\theta := \begin{pmatrix} w_0 \\ u_0 \end{pmatrix}$. Given a history-action realization $\bar{\boldsymbol{z}} = (x_0, x_1, \dots) \in (\mathbb{Y} \times \mathbb{A})^{\mathbb{Z}_+}$, define

$$h_t(\bar{z}_t; \theta_0) := \varrho(w_0 h_{t-1}(\bar{z}_{t-1}; \theta_0) + \langle u_0, x_t \rangle),\ t > 0,$$

with $h_{-1} := 0$ (thus $h_0(\bar{z}_0; \theta_0) = \varrho(\langle u_0, x_0 \rangle)$), and $\mathcal{I}_t(\bar{z}_t; \theta_0) := \varrho'(w_0 h_{t-1}(\bar{z}_{t-1}; \theta_0) + \langle u_0, x_t \rangle)$. Then, the neural tangent random feature (NTRF) mapping[1] at time t is defined as $\psi_t(\bar{z}_t; \theta_0) := \sum_{k=0}^{t} w_0^k \begin{pmatrix} h_{t-k-1}(\bar{z}_{t-k-1}; \theta_0) \\ x_{t-k} \end{pmatrix} \bar{\mathcal{I}}_{t,k}(\bar{z}_t; \theta_0)$, with $\bar{\mathcal{I}}_{t,k}(\bar{z}_t; \theta_0) := \prod_{j=0}^{k} \mathcal{I}_{t-j}(\bar{z}_{t-j}; \theta_0)$. We also define the NTRF matrix as follows:

$$\Psi_T(\bar{\boldsymbol{z}}; \theta_0) := \begin{pmatrix} \psi_0^\top(\bar{z}_0; \theta_0) \\ \psi_1^\top(\bar{z}_1; \theta_0) \\ \vdots \\ \psi_{T-1}^\top(\bar{z}_{T-1}; \theta_0) \end{pmatrix}, \text{ with } \Psi(\bar{z}; \theta_0) := \Psi_\infty(\bar{z}; \theta_0). \tag{7}$$

Definition 4.1 (Transportation mapping)**.** Let $\mathscr{H}$ be the set of mappings $\boldsymbol{v} : \mathbb{R}^{1+d} \to \mathbb{R}^{1+d} : \theta_0 \mapsto \begin{pmatrix} v_w(w_0) \\ v_u(u_0) \end{pmatrix}$ with $\mathbb{E}[|v_w(w_0)|^2] = \frac{1}{2}\Big(|v_w(\alpha)|^2 + |v_w(-\alpha)|^2\Big) < \infty$, and $\mathbb{E}[\|v_u(u_0)\|_2^2] = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \|v_u(u)\|_2^2 e^{-\frac{1}{2}\|u\|_2^2} du < \infty$. We call $\boldsymbol{v} \in \mathscr{H}$ a transportation mapping, following [20, 21].

Definition 4.2 (Infinite-width limit)**.** We define the infinite-width limit of Elman-type RNNs as

$$\mathscr{F} := \left\{ (\mathbb{Y} \times \mathbb{A})^{\mathbb{Z}_+} \ni \bar{\boldsymbol{z}} \mapsto \mathbb{E}\left[\Psi(\bar{\boldsymbol{z}}; \theta_0) \boldsymbol{v}(\theta_0)\right] : \boldsymbol{v} \in \mathscr{H} \right\}.$$

$\mathscr{F}$ consists of $f_t^\star(\bar{z}_t; \boldsymbol{v}) = \mathbb{E}[\langle \boldsymbol{v}(\theta_0), \psi_t(\bar{z}_t; \theta_0) \rangle]$ for any $\bar{\boldsymbol{z}} \in (\mathbb{Y} \times \mathbb{A})^{\mathbb{Z}_+}$. The same transportation mapping $\boldsymbol{v}$ is used to define the mapping $f_t^\star$ at each time t, which is a characteristic feature of weight-sharing in recurrent neural networks. Also, the input $\bar{\boldsymbol{z}}$ grows over time in a concatenated way, implying $\boldsymbol{f}^\star \in \mathscr{F}$ is a representational assumption on the dynamical structure of the problem.

For any fixed time $t \in \mathbb{N}$, the completion of $\{\bar{z}_t \mapsto f_t^\star(\bar{z}_t; \boldsymbol{v}) : \boldsymbol{v} \in \mathscr{H}\}$ is exactly the reproducing kernel Hilbert space (RKHS) $\mathscr{G}_{\kappa_t}$ associated with the "recurrent" neural tangent kernel (NTK) κ_t [35, 21]. For any $t \in \mathbb{N}$, the inner product of two functions in $\mathscr{G}_{\kappa_t}$ associated with the transportation mappings $\boldsymbol{v}, \boldsymbol{v}'$ is $\langle f_t^\star(\cdot; \boldsymbol{v}), f_t^\star(\cdot; \boldsymbol{v}') \rangle_{\mathscr{H}_{\kappa_t}} = \mathbb{E}\Big[\langle \boldsymbol{v}(\theta_0), \boldsymbol{v}'(\theta_0) \rangle \Big]$. Thus, RKHS norm of $f \in \mathscr{G}_{\kappa_t}$ is $\|f\|_{\mathscr{G}_{\kappa_t}} = \sqrt{\mathbb{E}\|\boldsymbol{v}(\theta_0)\|_2^2} = \sqrt{\mathbb{E}\|v_u(u_0)\|_2^2 + \mathbb{E}|v_w(w_0)|^2}$.

Remark 4.3 (Reduction to FNNs). Consider $T = 1$:

$$\mathscr{F}_1 := \left\{ \bar{z}_0 \mapsto \mathbb{E}\left[\psi_0^\top(\bar{z}_0; \theta_0) \boldsymbol{v}(\theta_0)\right] : \boldsymbol{v} \in \mathscr{H} \right\}.$$

In this case, we exactly recover the NTK (and the associated RKHS) for single-layer FFNs [19, 42, 28]. Furthermore, since the kernel κ_0 is universal, the associated RKHS $\mathscr{G}_{\kappa_0}$ is dense in the space of continuous functions on a compact set [21].

5 Rec-NAC Algorithm for POMDPs

In this section, we present a high-level description of our Recurrent Natural Actor-Critic (Rec-NAC) Algorithm with two inner loops, critic (called Rec-TD) and actor (called Rec-NPG), for policy optimization with RNNs. The details of the inner loops of the algorithm will be given in the

[1]The feature uses a complicated weighted-sum of all past inputs $x_k, k \leq t$, leading to a discounted memory to tackle non-stationarity. x_{t-k} is scaled with $w_0^k \sim \mathsf{Rad}(\alpha)$, thus it yields a fading memory approximation of the history if $\alpha < 1$.

succeeding sections. We use an admissible policy $\pi = (\pi_t)_{t\in\mathbb{N}}$ that is parameterized by a recurrent neural network $(F_t^{\mathsf{a}}(\cdot;\Phi))_{t\in\mathbb{N}}$ of the form given in equation 6 with $m \in \mathbb{Z}_+$. To that end, let

$$\pi_t^{\Phi}(a|z_t) := \frac{\exp\left(F_t^{\mathsf{a}}((z_t,a);\Phi)\right)}{\sum_{a'\in\mathbb{A}}\exp\left(F_t^{\mathsf{a}}((z_t,a');\Phi)\right)},\ t\in\mathbb{Z}_+, \tag{8}$$

for any $z_t \in (\mathbb{Y}\times\mathbb{A})^t \times \mathbb{Y}$ and $a \in \mathbb{A}$ with the parameter $\Phi \in \mathbb{R}^{m(d+1)}$.

Rec-NAC operates as follows:

- **Initialization.** F^{a} is randomly initialized with parameter $\Phi(0) \sim \zeta_{\mathsf{init}}$ (see Def. A.1).
- **Natural policy gradient.** For $n = 0, 1, \ldots, N-1$,

  - **Critic.** Estimate $\hat{\mathcal{Q}}_t^{(n)}(\cdot) := F_t^{\mathsf{c}}(\cdot;\bar{\Theta}^{(n)})\ t<T$ of $\mathcal{Q}_t^{\pi^{\Phi(n)}}(\cdot),\ t<T$ via Rec-TD learning in Sec. 6. F^{c} is initialized independently for each n as Definition A.1.

  - **Actor.** By projected stochastic gradient descent (SGD), obtain a solution ω_n for the compatible function approximation problem

$$\min_{\omega}\mathbb{E}\sum_{t=0}^{T-1}\gamma^t|\nabla\ln\pi_t^n(A_t|Z_t)\omega - \hat{\mathcal{A}}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)|^2 \text{ such that } \omega\in\mathcal{B}_{2,\infty}(0,\rho),$$

where for any $t\in\mathbb{N}$, $\hat{\mathcal{A}}_t^{(n)}(z_t,a) := \hat{\mathcal{Q}}_t^{(n)}(z_t,a) - \underset{A'\sim\pi_t^{\Phi}(\cdot|z_t)}{\mathbb{E}}\hat{\mathcal{Q}}_t^{(n)}(z_t,A')$.

For information regarding the algorithmic tools, i.e., random initialization and max-norm regularization for RNNs, we refer to Section A.

6 Critic: Recurrent Temporal Difference Learning (Rec-TD)

In this section, we study a value prediction algorithm for policy evaluation in POMDPs, which will serve as the critic.

Policy evaluation problem. Consider the policy evaluation problem for POMDPs under a given non-Markovian policy $\pi \in \Pi_{\mathsf{NM}}$. Given an initial observation distribution $\mu\in\Delta(\mathbb{Y})$, policy evaluation aims to solve

$$\min_{\Theta}\mathcal{R}_T^{\pi}(\Theta) := \mathbb{E}_{\mu}^{\pi}\sum_{t=0}^{T-1}\gamma^t\left(F_t(\bar{Z}_t;\Theta)-\mathcal{Q}_t^{\pi}(\bar{Z}_t)\right)^2 \text{ such that } \Theta\in\Omega_{\rho,m} := \mathcal{B}_{2,\infty}^{(m)}(0,\rho), \tag{9}$$

where $T\in\mathbb{N}$ is the truncation level, and $\{F_t : t\in\mathbb{N}\}$ is an Elman-type recurrent neural network given in equation 6 – we drop the superscript a for simplicity throughout the discussion. The expectation in $\mathcal{R}_T^{\pi}(\Theta)$ is with respect to the joint probability law $P_T^{\pi,\mu}$ of the stochastic process $\{(S_t,A_t,Y_t) : t\in[0,T]\}$ where $Z_0\sim\mu$.

6.1 Recurrent TD Learning Algorithm

Given a sample trajectory $\bar{z}_T \in (\mathbb{Y}\times\mathbb{A})^{T+1}$, let

$$\delta_t(\bar{z}_{t+1};\Theta) := r_t + \gamma F_{t+1}(\bar{z}_{t+1};\Theta) - F_t(\bar{z}_t;\Theta), \tag{10}$$

be the temporal difference, and let

$$\check{\nabla}\mathcal{R}_T(\bar{z}_T;\Theta) = \sum_{t=0}^{T}\gamma^t\delta_t(\bar{z}_{t+1};\Theta)\nabla_{\Theta}F_t(\bar{z}_t;\Theta), \tag{11}$$

be the stochastic semi-gradient. Note that, despite the exponential growth in the dimension of $\bar{z}_t \in (\mathbb{Y}\times\mathbb{A})^{t+1}$ over $t\in\mathbb{N}$, the memory complexity for computing $\check{\nabla}\mathcal{R}_T(\bar{z}_T;\Theta)$ is only $\mathscr{O}(m^2+md)$.

Assumption 6.1 (Sampling oracle)**.** Given an initial state distribution μ, we assume that the system can be independently started from $S_0\sim\mu$, i.e., independent trajectories $\{(S_t,Y_t,A_t) : t\in[T]\}\sim P_T^{\pi,\mu}$ can be obtained.

Under Assumption 6.1, for $k \in \mathbb{N}$, let $\{(S_t^k, Y_t^k, A_t^k) : t \in [T]\} \sim P_T^{\pi,\mu}$ be an independent trajectory (for each $k \in \mathbb{N}$, i.e., a trajectory with an independent initial sample $S_0^k \sim \mu$), and $\{Z_t^k : t \in [T]\}$ and $\{\bar{Z}_t^k : t \in [T]\}$ be the resulting (truncated) history and history-action processes. Starting from a random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$, let

$$\check{\Theta}(k+1) = \Theta(k) + \eta \cdot \check{\nabla}\mathcal{R}_T(\bar{Z}_T^k; \Theta(k)), \tag{12}$$

for $k \in \mathbb{N}$. For **Rec-TD**, one uses $\Theta(k+1) = \check{\Theta}(k+1)$. For **Rec-TD with max-norm regularization**, one uses $\Theta(k+1) = \mathbf{Proj}_{\Omega_{\rho,m}}[\check{\Theta}(k+1)]$, for parameter $\rho = (\rho_w, \rho_u) \in \mathbb{R}^2_{>0}$.

6.2 Theoretical Analysis of Rec-TD: Finite-Time Bounds and Global Near-Optimality

In the following, we prove that Rec-TD with max-norm regularization achieves global optimality in expectation. To characterize the impact of long-term dependencies on the performance of Rec-TD, let $p_t(x) = \sum_{k=0}^{t-1} |x|^k,$ and $q_t(x) = \sum_{k=0}^{t-1}(k+1)|x|^k,\ x \in \mathbb{R}, t \in \mathbb{N}$.

Theorem 6.2 (Finite-time bounds for Rec-TD)**.** *Assume that $\{\mathcal{Q}_t^\pi : t \in \mathbb{N}\} \in \mathscr{F}$ with a transportation mapping $\boldsymbol{v} = (v_{\mathsf{w}}, v_{\mathsf{u}}) \in \mathscr{H}$ such that $\sup_{u\in\mathbb{R}^d} \|v_{\mathsf{u}}(u)\|_2 \le \nu_{\mathsf{u}}$ and $\sup_{w\in\mathbb{R}} |v_{\mathsf{w}}(w)| \le \nu_{\mathsf{w}}$. Then, for any projection radius $\rho \succeq \nu = (\nu_{\mathsf{w}}, \nu_{\mathsf{u}})$ and step-size $\eta > 0$, Rec-TD with max-norm regularization achieves the following error bound:*

$$\mathbb{E}\Big[\frac{1}{K}\sum_{k=0}^{K-1}\mathcal{R}_T^\pi(\Theta(k))\Big] \le \frac{1}{\sqrt{K}}\left(\frac{\|\nu\|_2^2}{(1-\gamma)} + \frac{C_T^{(1)}}{(1-\gamma)^3}\right) + \frac{C_T^{(2)}}{(1-\gamma)^2\sqrt{m}} + \underbrace{\frac{\gamma^T}{(1-\gamma)K}\sum_{k=0}^{K-1}\omega_{T,k}^2}_{(\heartsuit)}. \tag{13}$$

for any $K \in \mathbb{N}$, where $C_T^{(1)}, C_T^{(2)} = \mathsf{poly}\left(p_T\left(\varrho_1(\alpha + \frac{\rho_w}{\sqrt{m}})\right), \|\rho\|_2, \|\nu\|_2\right)$, are instance-dependent constants that do not depend on K, and $\omega_{t,k} := \sqrt{\mathbb{E}[(F_t(\bar{Z}_t; \Theta(k)) - \mathcal{Q}_t^\pi(\bar{Z}_t^k))^2]}$ for $t, k \in \mathbb{N}$. For the average-iterate Rec-TD with $\bar{\Theta}_K := \frac{1}{K}\sum_{k=0}^{K-1}\Theta(k)$, we have

$$\mathbb{E}\left[\mathcal{R}_T^\pi\left(\bar{\Theta}_K\right)\right] \le \frac{10}{(1-\gamma)\sqrt{K}}\left(\|\nu\|_2^2 + \frac{C_T^{(1)}}{(1-\gamma)^2}\right) + \frac{10C_T^{(2)}}{(1-\gamma)^2\sqrt{m}} + \frac{10\gamma^T}{(1-\gamma)K}\sum_{k=0}^{K-1}\omega_{T,k}^2.$$

The proof of Theorem 6.2 can be found in Section B.

From Proposition 7.1, we observe that the exact natural policy gradient update would require a large T. As noted in [13], the spectral radius of $\{\mathbf{W}(k) : k \in \mathbb{N}\}$ determines the degree of long-term dependencies in the problem as it scales H_t. Consistent with this observation, our bounds have a strong dependency on $\alpha_m := \alpha + \frac{\rho_{\mathsf{w}}}{\sqrt{m}} \ge \lambda_{\mathsf{max}}(\mathbf{W}(k)) = \|\mathbf{W}(k)\|_{\infty,\infty}$ for any $k \in \mathbb{N}$.

Remark 6.3 (When is Rec-TD efficient? Impact of long-term dependencies). Note that both constants $C_T^{(1)}, C_T^{(2)}$ polynomially depend on $p_T\left(\varrho_1\alpha_m\right)$. Let $\varepsilon > 0$ be any given target error.

- **Short-term memory.** If $\alpha_m < \frac{1}{\varrho_1}$, then it is easy to see that $p_T(\varrho_1\alpha_m) \le \frac{1}{1-\varrho_1\alpha_m}$. Thus, the extra term $(\heartsuit)$ in equation 13 vanishes at a geometric rate as $T \to \infty$, yet m (network-width) and K (iteration-complexity) are $\tilde{\mathscr{O}}(1/\varepsilon^2)$. Rec-TD is efficient in that case.
- **Long-term memory.** If $\alpha_m > \frac{1}{\varrho_1}$, as $T \to \infty$, both m and K grow at a rate $\mathscr{O}\left((\varrho_1\alpha_m)^T/\varepsilon^2\right)$ while the extra term $(\heartsuit)$ in equation 13 vanishes at a geometric rate. As such, the required network size and iterations grow at a geometric rate with T in systems with long-term memory, constituting the pathological case for Rec-TD.

Finally, note that the additional term $(\heartsuit)$ in Theorem 6.2 is unique to Rec-TD learning, and stems from the use of bootstrapping in reinforcement learning.

The performance of Rec-TD is studied numerically in Random-POMDP instances in Section C.

7 Actor: Recurrent Natural Policy Gradient (Rec-NPG) for POMDPs

The goal is to solve the following problem:

$$\underset{\Theta\in\mathbb{R}^{m(d+1)}}{\text{maximize}}\ \mathcal{V}^{\pi^{\Phi}}(\mu) \text{ such that } \Phi\in\Omega_{\rho,m}, \tag{PO}$$

for a given initial distribution $\mu\in\Delta(\mathbb{Y})$ and $\rho\in\mathbb{R}^2_{>0}$. $\pi^\star$ denotes an optimal policy.

7.1 Recurrent Natural Policy Gradient for POMDPs

In this section, we describe the recurrent natural policy gradient (Rec-NPG) algorithm for non-Markovian reinforcement learning. As proved in Prop. D.2, a policy gradient for POMDPs is

$$\nabla_\Phi \mathcal{V}^{\pi^\Phi}(\mu) := \mathbb{E}_\mu^{\pi^\Phi}\sum_{t=0}^{\infty}\gamma^t \mathcal{Q}_t^{\pi^\Phi}(Z_t,A_t)\nabla_\Phi \ln \pi_t^\Phi(A_t|Z_t).$$

Fisher information matrix under a policy π^Φ is defined as

$$G_\mu(\Phi) := \mathbb{E}_\mu^{\pi^\Phi}\sum_{t=0}^{\infty}\gamma^t\nabla \ln\pi_t^\Phi(A_t|Z_t)\nabla^\top \ln\pi_t^\Phi(A_t|Z_t),$$

for an initial observation distribution $\mu\in\Delta(\mathbb{Y})$. Rec-NPG updates the policy parameters by

$$\Phi(n+1) = \Phi(n) + \eta\cdot G_\mu^+(\Phi(n))\nabla_\Phi\mathcal{V}^{\pi^{\Phi(n)}}(\mu), \tag{14}$$

for an initial parameter $\Phi(0)$ and step-size $\eta>0$, where G^+ denotes the Moore-Penrose inverse of a matrix G. This update rule is in the same spirit as the NPG introduced in [23], however, due to the non-Markovian nature of POMDPs, it has significant complications that we will address.

In order to avoid computationally-expensive policy updates in equation 14, we utilize the following extension of the compatible function approximation in [23] to the case of non-Markovian policies.

Proposition 7.1 (Compatible function approximation for non-Markovian policies)**.** *For any $\Phi\in\mathbb{R}^{m(d+1)}$ and initial observation distribution μ, let*

$$\mathcal{L}_\mu(w;\Phi) = \mathbb{E}_\mu^{\pi^\Phi}\sum_{t=0}^{\infty}\gamma^t\big(\nabla^\top \ln\pi_t^\Phi(A_t|Z_t)\omega - \mathcal{A}_t^{\pi^\Phi}(\bar{Z}_t)\big)^2, \tag{15}$$

for $\omega\in\mathbb{R}^{m(d+1)}$. Then, we have

$$G_\mu^+(\Phi)\nabla_\Phi\mathcal{V}^{\pi^\Phi}(\mu) \in \underset{\omega\in\mathbb{R}^{m(d+1)}}{\arg\min}\ \mathcal{L}_\mu(\omega;\Phi). \tag{16}$$

Path-based compatible function approximation with truncation. For general (non-Markovian) problems as in equation 15, we use a path-based method under truncation for a given $T\in\mathbb{N}$ with $\ell_T(\omega;\Phi,\mathcal{Q}) := \sum_{t=0}^{T-1}\gamma^t(\nabla\ln\pi_t^\Phi(A_t|Z_t)\omega - \mathcal{A}_t(Z_t,A_t))^2$, where $\mathcal{A}_t(z_t,a_t) = \mathcal{Q}_t(z_t,a_t) - \sum_{a\in\mathbb{A}}\pi_t^\Phi(a|z_t)\mathcal{Q}_t(z_t,a)$. Given a policy with parameter $\Phi(n)$ and the corresponding output of the critic (Rec-TD with the average-iterate $\bar{\Theta}^{(n)} := \frac{1}{K_{\mathsf{td}}}\sum_{k<K_{\mathsf{td}}}\Theta^{(n)}(k)$): $\hat{\mathcal{Q}}^{(n)}(\cdot) := F_t(\cdot;\bar{\Theta}^{(n)})$, the actor aims to solve the following problem:

$$\min_\omega \mathbb{E}\Big[\ell_T\left(\omega;\Phi(n),\hat{\mathcal{Q}}^{(n)}\right)\Big|\bar{\Theta}^{(n)},\Phi(n),\ldots,\Phi(0)\Big] \text{ such that } \omega\in\mathcal{B}_{2,\infty}^{(m)}(0,\rho).$$

To that end, we utilize stochastic gradient descent (SGD) to solve the above problem. Let $\bar{Z}_T^{n,k}\sim P_T^{\pi^{\Phi(n)},\mu}$ be an independent random sequence for $k=0,1,\ldots$, and let

$$\tilde{\omega}_n(k+1) = \hat{\omega}_n(k) - \eta_{\mathsf{sgd}}\nabla_\omega\ell_T\big(\hat{\omega}_n(k);\Phi(n),\hat{\mathcal{Q}}^{(n)}\big), \text{ and } \hat{\omega}_n(k+1) = \mathbf{Proj}_{\mathcal{B}_{2,\infty}^{(m)}(0,\rho)}[\tilde{\omega}_n(k+1)],$$

with $\hat{\omega}_n(0)=0$. Then, a biased stochastic approximation of the natural policy gradient is obtained as $\omega_n := \frac{1}{K_{\mathsf{sgd}}}\sum_{k<K_{\mathsf{sgd}}}\hat{\omega}_n(k)$, and the policy update is performed as

$$\Phi(n+1) = \Phi(n) + \eta_{\mathsf{npg}}\cdot\omega_n.$$

In the following, we present a non-asymptotic analysis of the above approach.

7.2 Theoretical Analysis of Rec-NAC for POMDPs

We establish an error bound on the best-iterate for the Rec-NPG. The significance of the following result is two-fold: (i) it will explicitly connect the optimality gap to the compatible function approximation error, and (ii) it will explicitly show the impact of truncation on the performance of path-based policy optimization for the non-stationary case.

Theorem 7.2. *Assume that $P_T^{\pi^\star,\mu} \ll P_T^{\pi^{\Phi(n)},\mu}$, $n < N$, and let $\kappa := \max_{0\leq n<N} \left\| \frac{P_T^{\pi^\star,\mu}}{P_T^{\pi^{\Phi(n)},\mu}} \right\|_\infty$. We have the following result under Rec-NPG after $N \in \mathbb{Z}_+$ steps with step-size $\eta_{\mathsf{npg}} = 1/\sqrt{N}$ with projection radius $\rho \in \mathbb{R}^2_{>0}$:*

$$\min_{0\leq n<N} \mathbb{E}_0[\mathcal{V}^{\pi^\star}(\mu) - \mathcal{V}^{\pi^{\Phi(n)}}(\mu)] \leq \frac{\ln|\mathbb{A}|}{(1-\gamma)\sqrt{N}} + \sqrt{p_T(\gamma)}\mathbb{E}_0\left[\frac{1}{N}\sum_{n=0}^{N-1}\left(\kappa\varepsilon^T_{\mathsf{cfa}}(\Phi(n),\omega_n)\right)^{\frac{1}{2}}\right]$$
$$+ \frac{2\gamma^T r_\infty}{(1-\gamma)^2} + \sum_{t<T}\gamma^t \frac{\mathbf{poly}(\|\rho\|_2, L_t, \beta_t, \Lambda_t, \chi_t)}{m^{1/4}} + \frac{\|\rho\|_2^2}{2\sqrt{N}}\sum_{t<T}\gamma^t L_t^2,$$

where $\varepsilon^T_{\mathsf{cfa}}(\Phi,\omega) := \mathbb{E}_\mu^{\pi^{\Phi(n)}} \sum_{t<T}\gamma^t|\nabla^\top \ln \pi_t^\Phi(A_t|Z_t)\omega - \mathcal{A}_t^{\pi^\Phi}(Z_t,A_t)|^2$, and the sequence $(L_t,\beta_t,\Lambda_t,\chi_t)_t$ is defined in Lemma B.1.

Remark 7.3. We have the following remarks.

- The effectiveness of Rec-NPG is proportional to the approximation power of the RNN used for policy parameterization, as reflected in $\varepsilon^T_{\mathsf{cfa}}$ in Theorem 7.2. We further characterize this error term in Prop. 7.5-7.7.
- The terms $L_t, \beta_t, \Lambda_t, \chi_t$ grow at a rate $p_t(\varrho_1\alpha_m)$. Thus, if $\alpha_m > \varrho_1^{-1}$, then m and N should grow at a rate $(\alpha_m\varrho_1)^T$, implying the curse of dimensionality (more generally, it is known as the exploding gradient problem [13]). On the other hand, if $\alpha_m < \varrho_1^{-1}$, then $L_t, \beta_t, \Lambda_t, \chi_t$ are all $\mathscr{O}(1)$ for all t, implying efficient learning of POMDPs. This establishes a very interesting connection between the memory in the system, the continuity and smoothness of the RNN with respect to its parameters, and the optimality gap under Rec-NPG.
- The term $\frac{2\gamma^T r_\infty}{(1-\gamma)^2}$ is due to truncating the trajectory at T, and vanishes with large T.

Remark 7.4. The quantity κ in the above theorem is the so-called concentrability coefficient in policy gradient methods [1, 2, 42], and determines the complexity of exploration. Note that it is defined in terms of path probabilities $P_T^{\pi,\mu}$ in the non-stationary setting.

In the following, we decompose the compatible function approximation error $\varepsilon^T_{\mathsf{cfa}}$ into the approximation error for the RNN and the statistical errors. To that end, let

$$\varepsilon_{\mathsf{app},n} = \inf\left\{\mathbb{E}\sum_{t=0}^{T-1}\gamma^t\left|\nabla^\top F_t(\bar{Z}_t;\Phi(0))\omega - \mathcal{Q}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)\right|^2 : \omega \in \mathcal{B}^{(m)}_{2,\infty}(0,\rho)\right\},$$

be the approximation error where the expectation is with respect to $P_T^{\pi^{\Phi(n)},\mu}$,

$$\varepsilon_{\mathsf{td},n} = \mathbb{E}[\mathcal{R}_T^{\pi^{\Phi(n)}}(\bar{\Theta}^{(n)})|\Phi(k), k\leq n],$$

be the error in the critic (see equation 9), and finally let

$$\varepsilon_{\mathsf{sgd},n} = \mathbb{E}[\ell_T(\omega_n;\Phi(n),\hat{\mathcal{Q}}^{(n)})|\bar{\Theta}^{(n)},\Phi(k),k\leq n] - \inf_w \mathbb{E}[\ell_T(\omega;\Phi(n),\hat{\mathcal{Q}}^{(n)})|\bar{\Theta}^{(n)},\Phi(k),k\leq n],$$

be the error in the policy update via compatible function approximation.

Proposition 7.5 (Error decomposition for $\varepsilon^T_{\mathsf{cfa}}$). *We have*

$$\mathbb{E}\left[\mathbb{E}_\mu^{\pi^{\Phi(n)}}\left[\ell_T(\omega_n;\Phi(n),\mathcal{Q}^{(n)})\right]\middle|\Phi(k),k\leq n\right] \leq \frac{8\|\rho\|_2^2}{m}\sum_{t=0}^{T-1}\gamma^t\beta_t^2 + 8\varepsilon_{\mathsf{app},n} + 6\varepsilon_{\mathsf{td},n} + 2\varepsilon_{\mathsf{sgd},n}.$$

for any $n \in \mathbb{Z}_+$.

From Theorem 6.2, we have $\varepsilon_{\mathsf{td},n} \leq \mathbf{poly}(p_T(\varrho_1\alpha_m))\mathscr{O}\left(\frac{1}{\sqrt{K_{\mathsf{td}}}} + \frac{1}{\sqrt{m_{\mathsf{critic}}}} + \gamma^T\right)$ with $\eta_{\mathsf{td}} = \mathscr{O}(1/\sqrt{K_{\mathsf{td}}})$, and by Theorem 14.8 in [37], we have $\varepsilon_{\mathsf{sgd},n} \leq \mathbf{poly}(p_T(\varrho_1\alpha_m), \|\rho\|_2)\mathscr{O}(1/\sqrt{K_{\mathsf{sgd}}})$ with $\eta_{\mathsf{td}} = \mathscr{O}(1/\sqrt{K_{\mathsf{td}}})$. As such, the statistical errors in the critic and the policy update (i.e., $\varepsilon_{\mathsf{td},n}, \varepsilon_{\mathsf{sgd},n}$) can be made arbitrarily small by using larger $K_{\mathsf{td}}, K_{\mathsf{sgd}}$ and larger m_{critic}. The remaining quantity to characterize is the approximation error, which is of critical importance for a small optimality gap as shown in Theorem 7.2 and Proposition 7.5. In the following, we will provide a finer characterization of $\varepsilon_{\mathsf{app},n}$ and identify a class of POMDPs that can be efficiently solved by Rec-NPG.

Assumption 7.6. For an index set J and $\nu \in \mathbb{R}^2_{>0}$, we consider a class $\mathscr{H}_{J,\nu}$ of transportation mappings $\left\{ \boldsymbol{v}^{(j)} \in \mathscr{H} : j \in J, \begin{pmatrix} \sup_{w\in\mathbb{R}, j\in J} |v_{\mathsf{w}}^{(j)}(w)| \\ \sup_{u\in\mathbb{R}^d, j\in J} \|v_{\mathsf{u}}^{(j)}(u)\|_2 \end{pmatrix} \leq \begin{pmatrix} \nu_{\mathsf{w}} \\ \nu_{\mathsf{u}} \end{pmatrix} \right\}$, and the infinite-width limit $\mathscr{F}_{J,\nu} := \{\bar{z} \mapsto \mathbb{E}[\Psi(\bar{z};\theta_0)\boldsymbol{v}(\theta_0)] : \boldsymbol{v} \in \mathbf{Conv}(\mathscr{H}_{J,\nu})\}$, where $\Psi(\cdot;\theta_0)$ is the NTRF matrix.

We assume that there exists an index set J and $\nu \in \mathbb{R}^2_{>0}$ such that $\mathcal{Q}^{\pi^{\Phi(n)}} \in \mathscr{F}_{J,\nu}$ for all $n \in \mathbb{N}$.

This representational assumption states that the $\mathcal{Q}$-functions under all policies $\pi^{\Phi(n)}$ throughout the Rec-NPG iterations n can be represented by convex combinations of a *fixed* set of mappings in the function class $\mathscr{F}$ indexed by J. Richness of J as measured by a Rademacher complexity will play an important role in bounding $\varepsilon_{\mathsf{app},n}$. To that end, for $\bar{z}_t = (z_t, a_t) \in (\mathbb{Y} \times \mathbb{A})^{t+1}$, let

$$G_t^{\bar{z}_t} = \{\phi \mapsto \nabla_\phi^\top H_t^{(1)}(\bar{z}_t;\phi)\boldsymbol{v}(\phi) : \boldsymbol{v} \in \mathscr{H}_{J,\nu}\}, \text{ and } \mathfrak{Rad}_m(G_t^{\bar{z}_t}) = \mathop{\mathbb{E}}_{\substack{\epsilon\sim\mathsf{Rad}^m(1)\\ \Phi(0)\sim\zeta_{\mathsf{init}}}} \sup_{g\in G_t^{\bar{z}_t}} \sum_{i=1}^m \frac{\epsilon_i}{m} g(\Phi_i(0)).$$

Note that $\boldsymbol{v} \in \mathscr{H}_{J,\nu}$ above can be replaced more with $\boldsymbol{v} \in \mathbf{Conv}(\mathscr{H}_{J,\nu})$ without any loss. In that case, since the mapping $\boldsymbol{v}^{(j)} \mapsto f_t^\star(\bar{z}_t;\boldsymbol{v}^{(j)}) \in G_t^{\bar{z}_t}$ is linear, $G_t^{\bar{z}_t}$ is replaced with $\mathbf{Conv}(G_t^{\bar{z}_t})$ without changing the Rademacher complexity [31].

The following proposition provides a finer characterization of the function approximation error.

Proposition 7.7. *Under Assumption 7.6, if $\rho \succeq \nu$, then*

$$\epsilon_{\mathsf{app},n} \leq \frac{1}{1-\gamma}\left(2 \max_{0\leq t<T} \max_{\bar{z}_t\in(\mathbb{Y}\times\mathbb{A})^{t+1}} \mathfrak{Rad}_m(G_t^{\bar{z}_t}) + L_T\|\rho\|_2\sqrt{\frac{\ln\left(2T|\mathbb{Y}\times\mathbb{A}|^T/\delta\right)}{m}}\right)^2,$$

for all n simultaneously with probability at least $1-\delta$ over the random initialization for any $\delta \in (0,1)$.

Remark 7.8. Two interesting cases that lead to a vanishing approximation error (as $m \to \infty$), thus global near-optimality, are as follows.

- **Finite J.** If $|J| < \infty$, then Proposition 7.7 reduces to [7] (with $T = 1$ for FNNs) with the complexity term $\mathscr{O}\left(\sqrt{\frac{\ln(|J|/\delta)}{m}}\right)$ by the finite-class lemma [31]. In this case, the $\mathcal{Q}$-functions throughout $n = 0, 1, \ldots$ lie in the convex hull of $|J|$ basis functions in $\mathscr{F}$ generated by $\{\boldsymbol{v}^{(j)} \in \mathscr{H} : j \in J\}$.
- **Linear transportation mappings.** For a fixed map $\varpi : \mathbb{R}^{d+1} \to \mathbb{R}^{(d+1)\times(d+1)}$, let $\boldsymbol{v}^{(b)}(\theta) = \langle\varpi(\theta), b\rangle,\ b \in J$ where $J \subset \mathbb{R}^{d+1}$ is compact. The approximation error vanishes at a rate $\mathscr{O}(\frac{1}{\sqrt{m}})$.

The proof of Prop. 7.7 and a discussion on the necessity of uniform bounds for the actor network in policy parameterization within the policy optimization framework can be found in Appendix 5.

8 Conclusion

In this work, we have studied RNN-based policy evaluation and policy optimization methods with finite-time analyses. An important limitation of Rec-NPG is that it does not provide an effective solution in POMDPs that require long-term memory as we point out in Remarks 6.3-7.3. As an extension of this work, theoretical analyses of more complicated LSTM- [17] and GRU-based [10] natural policy gradient algorithms can be considered as a future work. Alternatively, the study of hard- and soft-attention mechanisms to address the limitations of the RNNs [33] in policy optimization is also a very interesting future direction.

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A Algorithmic Tools for Recurrent Neural Networks

A.1 Random Initialization for Recurrent Neural Networks

One key concept is random initialization, which is widely used in practice [13] and yields the basis of the kernel analysis [19, 9]. In this work, we assume that m is even, and use the following symmetric initialization [9].

Definition A.1 (Symmetric random initialization)**.** Let $c_i \sim \mathsf{Rad}(1), V_i \sim \mathsf{Rad}(\alpha), U_i(0) \sim \mathcal{N}(0, I_d)$ independently for all $i \in \{1, 2, \dots, m/2\}$ and independently from each other, and $c_i = -c_{i-m/2}, V_i = V_{i-m/2}$ and $U_i(0) = U_{i-m/2}(0)$ for $i \in \{m/2+1, \dots, m\}$. Then, $(\mathbf{W}(0), \mathbf{U}(0), c)$ is called a symmetric random initialization where $\mathbf{W}(0) = \mathsf{diag}_m(V)$ and $U_i^\top(0)$ is the i^{th}-row of $\mathbf{U}(0)$.

The symmetrization ensures that $F_t(\bar{z}_t; \mathbf{W}(0), \mathbf{U}(0), c) = 0$ for any $t \geq 0$ and input $\bar{z}_t$.

A.2 Max-Norm Regularization for Recurrent Neural Networks

Max-norm regularization, proposed by [39], has been shown to be very effective across a broad spectrum of deep learning problems [40, 14]. In this work, we incorporate max-norm regularization (around the random initialization) into the recurrent natural policy gradient for sharp convergence guarantees. To that end, given a random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$ as in Definition A.1 and a vector $\rho = (\rho_{\mathsf{w}}, \rho_{\mathsf{u}})^\top \in \mathbb{R}^2_{>0}$ of projection radii, we define the compactly-supported set of weights $\Omega_{\rho,m} \subset \mathbb{R}^{m(d+1)}$ as

$$\Omega_{\rho,m} = \mathcal{B}^{(m)}_{2,\infty}(\Theta(0), \rho). \tag{17}$$

Given any symmetric random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$ and $\rho \in \mathbb{R}^2_{>0}$, the set $\Omega_{\rho,m}$ is a compact and convex subset of $\mathbb{R}^{m(d+1)}$, and for any $\Theta \in \Omega_{\rho,m}$, we have

$$\max_{1\leq i\leq m} |W_{ii} - W_{ii}(0)| \leq \frac{\rho_{\mathsf{w}}}{\sqrt{m}},$$
$$\max_{1\leq i\leq m} \|U_i - U_i(0)\| \leq \frac{\rho_{\mathsf{u}}}{\sqrt{m}}.$$

Let

$$\mathbf{Proj}_{\Omega_{\rho,m}}[\Theta] = \left[\mathop{\arg\min}_{w\in\mathcal{B}_2\left(W_{ii}(0), \frac{\rho_{\mathsf{w}}}{\sqrt{m}}\right)} |W_{ii} - w_i|, \mathop{\arg\min}_{u_i\in\mathcal{B}_2\left(U_i(0), \frac{\rho_{\mathsf{u}}}{\sqrt{m}}\right)} \|\mathbf{U}_i - u_i\|_2\right]_{i\in[m]} \tag{18}$$

As such, the projection operator $\mathbf{Proj}_{\Omega_{\rho,m}}[\cdot]$ onto $\Omega_{\rho,m}$ is called the max-norm projection (or regularization).

Note that we have $\|\mathbf{W} - \mathbf{W}(0)\|_2 \leq \rho_{\mathsf{w}}$, $\|\mathbf{U} - \mathbf{U}(0)\|_2 \leq \rho_{\mathsf{u}}$ and $\|\Theta - \Theta(0)\|_2 \leq \|\rho\|_2$ in the ℓ_2 geometry for any $\Theta \in \Omega_{\rho,m}$. Therefore, although the max-norm parameter class $\Omega_{\rho,m} \subset \{\Theta \in \mathbb{R}^{m(d+1)} : \|\Theta - \Theta(0)\|_2 \leq \|\rho\|_2\}$, the ℓ_2-projected [3, 42, 28] and max-norm projected [7] optimization algorithms recover exactly the same function class (i.e., RKHS associated with the neural tangent kernel studied in [21, 41], see Section 4).

B Proofs for Section 6

An important quantity in the analysis of recurrent neural networks is the following:

$$\Gamma_t^{(i)}(\bar{z}_t;\Theta) := W_{ii} H_t^{(i)}(\bar{z}_t;\Theta),$$

for any hidden unit $i \in [m]$ and $\Theta \in \mathbb{R}^{m(d+1)}$. The following Lipschitzness and smoothness results for $\Theta_i \mapsto H_t^{(i)}(\bar{z}_t;\Theta)$ and $\Theta_i \mapsto \Gamma_t^{(i)}(\bar{z}_t;\Theta)$.

Lemma B.1 (Local continuity of hidden states; Lemma 1-2 in [4])**.** *Given $\rho \in \mathbb{R}^2_{>0}$ and $\alpha \geq 0$, let $\alpha_m = \alpha + \frac{\rho_w}{\sqrt{m}}$. Then, for any $\bar{\boldsymbol{z}} \in (\mathbb{Y} \times \mathbb{A})^{\bar{Z}_+}$ with $\sup_{t\in\mathbb{N}} \left\| \begin{pmatrix} y_t \\ a_t \end{pmatrix} \right\|_2 \leq 1$, $t \in \mathbb{N}$ and $i \in [m]$,*

- *$\Theta_i \mapsto H_t^{(i)}(\bar{z}_t;\Theta)$ is L_t-Lipschitz continuous with $L_t = (\varrho_0^2+1)\varrho_0^2 \cdot p_t^2(\alpha_m\varrho_1)$,*
- *$\Theta_i \mapsto H_t^{(i)}(\bar{z}_t;\Theta)$ is β_t-smooth with $\beta_t = \mathscr{O}\left(d \cdot p(\alpha_m\varrho_1) \cdot q(\alpha_m\varrho_1)\right)$,*
- *$\Theta_i \mapsto \Gamma_t^{(i)}(\bar{z}_t;\Theta)$ is Λ_t-Lipschitz with $\Lambda_t = \sqrt{2}(\varrho_0 + 1 + \alpha_m L_t)$,*
- *$\Theta_i \mapsto \Gamma_t^{(i)}(\bar{z}_t;\Theta)$ is χ_t-smooth with $\chi_t = \sqrt{2}(L_t + \alpha_m\beta_t)$,*

in $\Omega_{\rho,m}$. Consequently, for any $\Theta \in \Omega_{\rho,m}$,

$$\sup_{\bar{\boldsymbol{z}}\in\bar{\mathbb{H}}_\infty} \max_{0\leq t\leq T} |F_t(\bar{z}_t;\Theta)| \leq L_T \cdot \|\rho\|_2,\ T \in \mathbb{N}, \tag{19}$$

$$\sup_{\bar{\boldsymbol{z}}\in\bar{\mathbb{H}}_\infty} |F_t^{\mathsf{Lin}}(\bar{z}_t;\Theta) - F_t(\bar{z}_t;\Theta)| \leq \frac{2}{\sqrt{m}}(\varrho_2\Lambda_t^2 + \varrho_1\chi_t)\|\Theta - \Theta(0)\|_2^2,\ t \in \mathbb{N}, \tag{20}$$

$$\sup_{\bar{\boldsymbol{z}}\in\bar{\mathbb{H}}_\infty} \left\langle \nabla F_t(\bar{z}_t;\Theta) - \nabla F_t(\bar{z}_t;\Theta(0)), \Theta - \bar{\Theta} \right\rangle \leq \frac{2\beta_t^2\|\rho\|_2^2}{\sqrt{m}}, \tag{21}$$

with probability 1 over the symmetric random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$.

Lemma B.2 (Approximation error between RNN-NTRF and RNN-NTK)**.** *Let $f^\star \in \mathscr{F}$ with the transportation mapping $\boldsymbol{v} \in \mathscr{H}$, and let*

$$\bar{\Theta}_i = \Theta_i(0) + \frac{1}{\sqrt{m}} c_i \boldsymbol{v}(\Theta_i(0)), i \in [m]. \tag{22}$$

for any symmetric random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$ in Def. A.1. Let

$$F_t^{\mathsf{Lin}}(\cdot;\Theta) = \nabla_\Theta F_t(\cdot;\Theta(0)) \cdot (\Theta - \Theta(0)).$$

If $P_T^{\pi,\mu}$ induces a compactly-supported marginal distribution for $X_t, t \in \mathbb{N}$ such that $\|X_t\|_2 \leq 1$ a.s. and $\{\bar{Z}_t : t \in \mathbb{N}\}$ is independent from the random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$, then we have

$$\mathbb{E}\left[\mathbb{E}_\mu^\pi\left[\left(f_t^\star(\bar{Z}_t) - F_t^{\mathsf{Lin}}(\bar{Z}_t;\bar{\Theta})\right)^2\right]\right] \leq \frac{2\|\nu\|_2^2(1+\varrho_0^2)p_t^2(\alpha\varrho_1)}{m}, \tag{23}$$

where the outer expectation is with respect to the random initialization $(\mathbf{W}(0), \mathbf{U}(0), c)$.

Proof. For any hidden unit $i \in [m]$, let

$$\zeta_i = \left\langle \boldsymbol{v}(\Theta_i(0)), \sum_{k=0}^{t} {W_{ii}}^k(0) \begin{pmatrix} H_{t-k-1}^{(i)}(\bar{Z}_{t-k-1}, \Theta_i(0)) \\ X_{t-k} \end{pmatrix} \prod_{j=0}^{k} \mathcal{I}_{t-j}(\bar{Z}_{t-j};\Theta_i(0)) \right\rangle.$$

Then, it is straightforward to see that

$$F_t^{\mathsf{Lin}}(\bar{Z}_t;\bar{\Theta}) = \frac{1}{m}\sum_{i=1}^{m}\zeta_i, \tag{24}$$

and $\mathbb{E}[\zeta_i|\bar{Z}_t] = \mathbb{E}[f_t^\star(\bar{Z}_t)|\bar{Z}_t]$ almost surely. Note that $\{\zeta_i : i \in [m/2]\}$ is independent and identically distributed and $\zeta_i = \zeta_{i+m/2}$ for any $i \in [m/2]$. Also, with probability 1 we have

$$\begin{aligned}
|\zeta_i| &\overset{(\spadesuit)}{\leq} \|\boldsymbol{v}(\Theta_i(0))\|_2 \cdot \left\| \sum_{k=0}^{t} W_{ii}{}^k(0) \begin{pmatrix} H_{t-k-1}^{(i)}(\bar{Z}_{t-k-1},\Theta_i(0)) \\ X_{t-k} \end{pmatrix} \prod_{j=0}^{k} \mathcal{I}_{t-j}(\bar{Z}_{t-j};\Theta_i(0)) \right\|_2, \\
&\overset{(\clubsuit)}{\leq} \|\boldsymbol{v}(\Theta_i(0))\|_2 \sum_{k=0}^{t-1} \alpha^k \varrho_1^{k+1}\sqrt{1+\varrho_0^2}, \\
&\overset{(\diamondsuit)}{\leq} \|\nu\|_2 \cdot \varrho_1 \cdot \sqrt{1+\varrho_0^2} \cdot p_t(\alpha\varrho_1),
\end{aligned}$$

where ($\spadesuit$) follows from Cauchy-Schwarz inequality, ($\clubsuit$) follows from the uniform bound $\sup_{z\in\mathbb{R}} |\varrho(z)| \leq \varrho_1$ and almost-sure bounds $\|X_k\|_2 \leq 1$ and $|W_{ii}(0)| \leq \alpha$, and ($\clubsuit$) follows from $\boldsymbol{v} \in \mathscr{H}_\nu$. From these bounds,

$$\mathrm{Var}(\zeta_i) \leq \mathbb{E}[\mathbb{E}_\mu^\pi[|\zeta_i|^2]] \leq \|\nu\|_2^2\varrho_1^2(1+\varrho_0)^2 p_t^2(\alpha\varrho_1),\ i \in [m]. \tag{25}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}_\mu^\pi\left[\left(f_t^\star(\bar{Z}_t) - F_t^{\mathsf{Lin}}(\bar{Z}_t;\bar{\Theta})\right)^2\right]\right] &= \mathbb{E}_\mu^\pi\left[\mathbb{E}\left[\left|\frac{1}{m}\sum_{i=1}^{m}\left(\zeta_i - \mathbb{E}[\zeta_i|\bar{Z}_t]\right)\right|^2\right]\right], \\
&= \mathbb{E}_\mu^\pi\left[\mathbb{E}\left[\left|\frac{2}{m}\sum_{i=1}^{m/2}\left(\zeta_i - \mathbb{E}[\zeta_i|\bar{Z}_t]\right)\right|^2\right]\right], \\
&= \frac{4}{m^2}\mathbb{E}_\mu^\pi \sum_{i=1}^{m/2}\sum_{j=1}^{m/2} \mathbb{E}\left[\left(\zeta_i - \mathbb{E}[\zeta_i|\bar{Z}_t]\right)\left(\zeta_j - \mathbb{E}[\zeta_j|\bar{Z}_t]\right)\right], \\
&= \frac{4}{m^2}\mathbb{E}_\mu^\pi \sum_{i=1}^{m/2} \mathrm{Var}(\zeta_i) \leq \frac{2}{m}\|\nu\|_2^2\varrho_1^2(1+\varrho_0)^2 p_t^2(\alpha\varrho_1),
\end{aligned}$$

where the first identity is from Fubini's theorem, the second identity is from the symmetricity of the random initialization, the fourth identity is due to the independent initialization for $i \leq m/2$, and the inequality is from the bound in equation 25.

□

Proposition B.3 (Non-stationary Bellman equation)**.** *For $\pi \in \Pi_{\mathsf{NM}}$, we have*

$$\mathcal{Q}_t^\pi(\bar{z}_t) = \mathbb{E}^\pi\left[r(S_t,A_t) + \gamma\mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1})\middle|\bar{Z}_t = \bar{z}_t\right] = \mathbb{E}^\pi\left[r(S_t,A_t) + \gamma\mathcal{V}_{t+1}^\pi(Z_{t+1})\middle|\bar{Z}_t = \bar{z}_t\right],$$

for any $t \in \mathbb{Z}_+$.

Proof of Theorem 6.2. Since $\{\mathcal{Q}_t^\pi : t \in \mathbb{N}\} \in \mathscr{F}$, let the point of attraction $\bar{\Theta}$ be defined as in equation 22, and the potential function be defined as

$$\Psi(\Theta) = \|\Theta - \bar{\Theta}\|_2^2. \tag{26}$$

Then, from the non-expansivity of the projection operator onto the convex set $\Omega_{\rho,m}$, we have the following inequality:

$$\begin{aligned}
\Psi(\Theta(k+1)) \leq \Psi(\Theta(k)) + 2\eta\sum_{t=0}^{T-1}\gamma^t\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\left\langle\nabla F_t(\bar{Z}_t^k;\Theta(k)),\Theta(k)-\bar{\Theta}\right\rangle \\
+ 2\eta^2\|\check{\mathcal{R}}_T(\bar{Z}_T^k;\Theta(k))\|_2^2. \quad (27)
\end{aligned}$$

Let $\check{\mathbb{E}}_t^k[\cdot] := \mathbb{E}[\cdot|\Theta(k), \dots, \Theta(0), \bar{Z}_t^k]$. Then, we obtain

$$\mathbb{E}[\Psi(\Theta(k+1) - \Psi(\Theta(k))] \leq 2\eta\mathbb{E}\Big[\sum_{t=0}^{T-1} \gamma^t \underbrace{\check{\mathbb{E}}_t^k[\delta_t(\bar{Z}_{t+1}^k; \Theta(k))] \left\langle \nabla F_t(\bar{Z}_t^k; \Theta(k)), \Theta(k) - \bar{\Theta} \right\rangle}_{(\spadesuit)_t}\Big] + \eta^2 \mathbb{E} \underbrace{\|\check{\nabla}\mathcal{R}_T(\bar{Z}_T^k; \Theta(k))\|_2^2}_{(\clubsuit)}. \quad (28)$$

Bounding $\mathbb{E}(\spadesuit)_t$. By using the Bellman equation in the non-Markovian setting (cf. Proposition B.3), notice that

$$\begin{aligned}\check{\mathbb{E}}_t^k \delta_t(\bar{Z}_{t+1}^k; \Theta(k)) &= \check{\mathbb{E}}_t^k[r_t^k + \gamma F_{t+1}(\bar{Z}_{t+1}^k; \Theta(k)] - F_t(\bar{Z}_t^k; \Theta(k)),\\ &= \gamma \check{\mathbb{E}}_t^k \left[F_{t+1}(\bar{Z}_{t+1}^k; \Theta(k)) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)\right] + \mathcal{Q}_t^\pi(\bar{Z}_t) - F_t(\bar{Z}_t^k; \Theta(k)).\end{aligned}$$

Secondly, we perform a change-of-feature as follows:

$$\left\langle \nabla F_t(\bar{Z}_t^k; \Theta(k)), \Theta(k) - \bar{\Theta} \right\rangle = \left\langle \nabla F_t(\bar{Z}_t^k; \Theta(0)), \Theta(k) - \bar{\Theta} \right\rangle + \mathsf{err}_{t,k}^{(1)}, \quad (29)$$

where

$$\mathsf{err}_{t,k}^{(1)} := \left\langle \nabla F_t(\bar{Z}_t^k; \Theta(k)) - \nabla F_t(\bar{Z}_t^k; \Theta(0)), \Theta(k) - \bar{\Theta} \right\rangle, \text{ and } |\mathsf{err}_{t,k}^{(1)}| \leq \frac{2\beta_t^2 \|\rho\|_2^2}{\sqrt{m}} \leq \frac{2\beta_T^2 \|\rho\|_2^2}{\sqrt{m}},$$

by Lemma B.1. Furthermore,

$$\begin{aligned}\left\langle \nabla F_t(\bar{Z}_t^k; \Theta(0)), \Theta(k) - \bar{\Theta} \right\rangle &= F_t^{\mathsf{Lin}}(\bar{Z}_t^k; \Theta(k)) - F_t^{\mathsf{Lin}}(\bar{Z}_t^k; \bar{\Theta}), \quad (30)\\ &= F_t(\bar{Z}_t^k; \Theta(k)) - \mathcal{Q}_t^\pi(\bar{Z}_t^k) + \mathsf{err}_{t,k}^{(2)} + \mathsf{err}_{t,k}^{(3)} \quad (31)\end{aligned}$$

where

$$\begin{aligned}\mathsf{err}_{t,k}^{(2)} &:= F_t^{\mathsf{Lin}}(\bar{Z}_t^k; \Theta(k)) - F_t(\bar{Z}_t^k; \Theta(k)),\\ \mathsf{err}_{t,k}^{(3)} &:= -F_t^{\mathsf{Lin}}(\bar{Z}_t^k; \bar{\Theta}) + \mathcal{Q}_t^\pi(\bar{Z}_t^k).\end{aligned}$$

Thus,

$$\begin{aligned}(\spadesuit)_t = &-(\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k; \Theta(k)))^2 + +\check{\mathbb{E}}_t^k \delta_t(\bar{Z}_{t+1}^k; \Theta(k)) \sum_{j=1}^{3} \mathsf{err}_{t,k}^{(j)}\\ &+ \gamma \check{\mathbb{E}}_t^k \left[F_{t+1}(\bar{Z}_{t+1}^k; \Theta(k)) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)\right] \cdot (\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k; \Theta(k))).\end{aligned}$$

By equation 19, we have

$$\sup_{\bar{\boldsymbol{z}} \in \bar{\mathbb{H}}_\infty} |\delta_t(\bar{z}_{t+1}; \Theta(k))| \leq r_\infty + 2L_T \|\rho\|_2 =: \delta_{\mathsf{max}}$$

Now, let $\omega_{t,k} := \left(\mathbb{E}[(\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k; \Theta(k)))^2]\right)^{1/2}$, where the expectation is over the joint distribution of $\Theta(k)$ and $\bar{Z}_T^k$. Then,

$$\mathbb{E}[(\spadesuit)_t] \leq -\omega_{t,k}^2 + \gamma \omega_{t+1,k} \omega_{t,k} + \delta_{\mathsf{max}} \sum_{j=1}^{3} \mathbb{E}|\mathsf{err}_{t,k}^{(j)}|.$$

From equation 20, we have

$$\mathbb{E}|\mathsf{err}_{t,k}^{(2)}| \leq \frac{2}{\sqrt{m}} (\varrho_2 \Lambda_T^2 + \varrho_1 \chi_T) \|\rho\|_2^2.$$

From the approximation bound in Lemma B.2, we get

$$\mathbb{E}|\mathsf{err}_{t,k}^{(3)}| \leq \sqrt{\mathbb{E}|\mathsf{err}_{t,k}^{(3)}|^2} \leq \frac{2\|\nu\|_2 \sqrt{1 + \varrho_0^2} \cdot p_T(\alpha \varrho_1)}{\sqrt{m}}.$$

Also, note that $\omega_{t+1,k}\omega_{t,k} \leq \frac{1}{2}(\omega_{t,k}^2 + \omega_{t+1,k}^2)$. Putting these together, we obtain the following bound for every $t \in \{0, 1, \ldots, T-1\}$:

$$\mathbb{E}[(\spadesuit)_t] \leq -\omega_{t,k}^2 + \frac{\gamma}{2}(\omega_{t+1,k}^2 + \omega_{t,k}^2) + \delta_{\mathsf{max}} \cdot \frac{C_T}{\sqrt{m}},$$

where

$$C_T := 2\beta_T^2\|\rho\|_2^2 + 2(\varrho_2\Lambda_T^2 + \varrho_1\chi_T)\|\rho\|_2^2 + 2\|\nu\|_2\sqrt{1+\varrho_0^2} \cdot p_T(\alpha\varrho_1).$$

Hence, we obtain the following upper bound:

$$\begin{aligned}\sum_{t=0}^{T-1}\gamma^t\mathbb{E}[(\spadesuit)_t] &\leq -(1-\gamma/2)\sum_{t<T}\gamma^t\omega_{t,k}^2 + \frac{\delta_{\mathsf{max}} \cdot C_T}{(1-\gamma)\sqrt{m}} + \underbrace{\frac{1}{2}\sum_{t<T}\gamma^{t+1}\omega_{t+1,k}^2}_{\leq \frac{1}{2}(\sum_{t<T}\gamma^t\omega_{t,k}^2 + \gamma^T\omega_{T,k}^2)} \\ &\leq -\frac{1-\gamma}{2}\sum_{t<T}\gamma^t\omega_{t,k}^2 + \frac{1}{2}\gamma^T\omega_{T,k}^2 + \frac{C_T \cdot \delta_{\mathsf{max}}}{(1-\gamma)\sqrt{m}}. \end{aligned} \tag{32}$$

Bounding $\mathbb{E}[(\clubsuit)]$. Using the triangle inequality, we obtain:

$$\|\sum_{t<T}\gamma^t\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\nabla F_t(\bar{Z}_t;\Theta(k))\|_2 \leq \sum_{t<T}\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\Theta(k))| \cdot \|\nabla F_t(\bar{Z}_t;\Theta(k))\|_2.$$

Since $\Theta(k) \in \Omega_{\rho,m}$ for every $k \in \mathbb{N}$ as a consequence of the max-norm regularization, we have

$$|\delta_t(\bar{Z}_{t+1}^k;\Theta(k))| \leq \delta_{\mathsf{max}} = r_\infty + 2L_T\|\rho\|_2,$$

$$\|\nabla F_t(\bar{Z}_t^k;\Theta(k))\|_2^2 = \frac{1}{m}\sum_{i=1}^m\|\nabla_{\Theta_i}H_t^{(i)}(\bar{Z}_t^k;\Theta(k))\|_2^2 \leq L_t^2 \leq L_T^2,$$

for every $t < T$ with probability 1 since $\Theta_i \mapsto H_t^{(i)}(\bar{z}_t;\Theta_i)$ is L_t-Lipschitz continuous by Lemma B.1. Hence, we obtain:

$$\|\check{\nabla}\mathcal{R}_T(\bar{Z}_T^k;\Theta(k))\|_2 \leq \frac{\delta_{\mathsf{max}}L_T}{1-\gamma}. \tag{33}$$

Final step. Now, taking expectation over $(\bar{Z}_t^k, \Theta(k))$ in equation 28, and substituting equation 32 and equation 33, we obtain:

$$\mathbb{E}[\Psi(\Theta(k+1)) - \Psi(\Theta(k))] \leq -\eta(1-\gamma)\sum_{t=0}^{T-1}\gamma^t\omega_{t,k}^2 + \eta\gamma^T\omega_{T,k}^2 + \eta\frac{\delta_{\mathsf{max}} \cdot C_T}{(1-\gamma)\sqrt{m}} + \eta^2\frac{\delta_{\mathsf{max}}^2 L_T^2}{(1-\gamma)^2},$$

for every $k \in \mathbb{N}$. Note that $\Psi(\Theta(0)) \leq \|\nu\|_2^2$. Thus, telescoping sum over $k = 0, 1, \ldots, K-1$ yields

$$\frac{1}{K}\sum_{k=0}^{K-1}\mathcal{R}_T(\Theta(k)) \leq \frac{\|\nu\|_2^2}{\eta(1-\gamma)K} + \frac{\eta\delta_{\mathsf{max}}^2 L_T^2}{(1-\gamma)^3} + \frac{\delta_{\mathsf{max}} \cdot C_T}{(1-\gamma)^2\sqrt{m}} + \frac{\gamma^T}{(1-\gamma)K}\sum_{k=0}^{K-1}\omega_{T,k}^2. \tag{34}$$

The final inequality in the proof stems from the linearization result Lemma B.2, and directly follows from

$$\mathcal{R}_T\left(\frac{1}{K}\sum_{k<K}\Theta(k)\right) \leq \frac{4}{K}\sum_{k<K}\mathcal{R}_T(\Theta(k)) + \frac{6}{\sqrt{m}}\left(\varrho_2\Lambda_T^2 + \varrho_1\chi_T\right)\|\rho\|_2^2,$$

which directly follows from [4], Corollary 1. □

In the following, we study the error under mean-path Rec-TD learning algorithm.

Theorem B.4 (Finite-time bounds for mean-path Rec-TD). *For $K \in \mathbb{N}$, with the step-size choice $\eta = \frac{(1-\gamma)^2}{64L_T^2}$, mean-path Rec-TD learning achieves the following error bound:*

$$\begin{aligned}\mathbb{E}\left[\frac{1}{K}\sum_{k<K}\mathcal{R}_T^{\boldsymbol{\pi}}(\Theta(k))\right] \leq \frac{2\|\nu\|_2^2}{(1-\gamma)\eta K} + \frac{\gamma^T\omega_{T,k}}{1-\gamma} + \frac{C_T\delta_{\mathsf{max}}}{(1-\gamma)^2\sqrt{m}} \\ + \eta\Big(\frac{(C_T')^2}{m} + 16\gamma^{2T}L_T^4(\|\rho\|_2^2 + \|\nu\|_2^2)\Big),\end{aligned}$$

where C_T' and L_T are terms that do not depend on K.

Theorem B.4 indicates that if a noiseless semi-gradient is used in Rec-TD, then the rate can be improved from $\mathscr{O}\left(\frac{1}{\sqrt{K}}\right)$ to $\mathscr{O}\left(\frac{1}{K}\right)$, indicating the potential limits of using variance-reduction schemes.

Proof of Theorem B.4. At any iteration $k \in \mathbb{N}$, let

$$\bar{\nabla}\mathcal{R}_T(\Theta(k)) := \mathbb{E}_\mu^\pi\Big[\check{\nabla}\mathcal{R}(\bar{Z}_t^k;\Theta(k))\Big], \tag{35}$$

be the ***mean-path semi-gradient***. First, note that

$$\|\bar{\nabla}\mathcal{R}_T(\Theta(k))\|_2^2 \leq 2\|\bar{\nabla}\mathcal{R}_T(\Theta(k)) - \bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2 + 2\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2. \tag{36}$$

Bounding $\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2$. For any $k \in \mathbb{N}, t \leq T$, we have

$$\begin{aligned}\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})|\bar{Z}_t^k,\Theta(0),c\big] = \gamma\mathbb{E}[F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)|\bar{Z}_t^k,\Theta(0),c]& \\ + \mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\bar{\Theta}).&\end{aligned}$$

Since $\|\nabla F_t(\bar{z}_t;\bar{\Theta})\|_2 \leq L_t$, the following inequality holds:

$$\begin{aligned}\big\|\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})\nabla F_t(\bar{Z}_t^k;\bar{\Theta})\big]\big\|_2 &\leq \mathbb{E}\,\big\|\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})|\bar{Z}_t^k,\Theta(0),c\big]\nabla F_t(\bar{Z}_t^k;\bar{\Theta})\big\|_2\,, \\ &\leq L_T\mathbb{E}\,\big|\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})|\bar{Z}_t^k,\Theta(0),c\big]\big|\,, \\ &\leq L_T\left(\gamma\mathbb{E}\,\big|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)\big| + \mathbb{E}\,\big|\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\bar{\Theta})\big|\right), \end{aligned} \tag{37}$$

where we used Jensen's inequality, the law of iterated expectations, and triangle inequality. From the above inequality, we obtain

$$\begin{aligned}\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2 &\overset{①}{\leq} \sum_{t=0}^{T-1}\gamma^t\big\|\mathbb{E}\big[\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})\nabla F_t(\bar{Z}_t^k;\bar{\Theta})\big]\big\|_2, \\ &\overset{②}{\leq} L_T\gamma\sum_{t<T}\gamma^t\mathbb{E}|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)| + L_T\sum_{t<T}\gamma^t\mathbb{E}|\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\bar{\Theta})|, \\ &\overset{③}{\leq} \frac{L_T}{\sqrt{1-\gamma}}\Big(\gamma\mathbb{E}\sqrt{\sum_{t<T}\gamma^t|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)|^2} + \mathbb{E}\sqrt{\sum_{t<T}\gamma^t|F_t(\bar{Z}_t^k;\bar{\Theta}) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)|^2}\Big), \\ &\overset{④}{\leq} \frac{L_T}{\sqrt{1-\gamma}}\Big(\gamma\sqrt{\mathbb{E}\sum_{t<T}\gamma^t|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - \mathcal{Q}_{t+1}^\pi(\bar{Z}_{t+1}^k)|^2} + \sqrt{\mathbb{E}\sum_{t<T}\gamma^t|F_t(\bar{Z}_t^k;\bar{\Theta}) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)|^2}\Big), \\ &\overset{⑤}{\leq} \frac{\sqrt{2}(1+\gamma)L_T}{\sqrt{1-\gamma}}\frac{\|\nu\|_2\sqrt{1+\varrho_0^2}\cdot p_T(\varrho_1\alpha)}{\sqrt{m}}.\end{aligned}$$

where ① follows from triangle inequality, ② follows from equation 37, ③ follows from Cauchy-Schwarz inequality and the monotonicity of the geometric series $T \mapsto \sum_{t<T}\gamma^t$, ④ follows from Jensen's inequality, and finally ⑤ follows from Lemma B.2. Hence, we obtain

$$\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2 \leq \frac{8L_T^2\|\nu\|_2^2(1+\varrho_0^2)p_T^2(\varrho_1\alpha)}{(1-\gamma)m}. \tag{38}$$

Bounding $\|\bar{\nabla}\mathcal{R}_T(\Theta(k)) - \bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2$. First, note that

$$\begin{aligned}&\|\bar{\nabla}\mathcal{R}_T(\Theta(k)) - \bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2 \\ &\qquad = \|\mathbb{E}\Big[\sum_{t<T}\gamma^t\left(\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\nabla F_t(\bar{Z}_t^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})\nabla F_t(\bar{Z}_t^k;\bar{\Theta})\right)\Big]\|_2\end{aligned}$$

We make the following decomposition for each $t < T$:

$$\begin{aligned}&\delta_t(\bar{Z}_{t+1}^k;\Theta(k))\nabla F_t(\bar{Z}_t^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})\nabla F_t(\bar{Z}_t^k;\bar{\Theta}) \\ &\qquad = \delta_t(\bar{Z}_{t+1}^k;\Theta(k))\left(\nabla F_t(\bar{Z}_t^k;\Theta(k)) - \nabla F_t(\bar{Z}_t^k;\bar{\Theta})\right) \\ &\qquad\qquad + \nabla F_t(\bar{Z}_t^k;\Theta(k))\left(\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta}) - \delta_t(\bar{Z}_{t+1}^k;\Theta(k))\right)\end{aligned} \tag{39}$$

By Lemma B.1, we have $|\delta_t(\bar{Z}_{t+1}^k;\Theta)| \le \delta_{\max}$ and $\|\nabla F_t(\bar{Z}_t^k;\Theta)\|_1 \le L_t \le L_T$ almost surely for any $\Theta \in \Omega_{\rho,m}$, which holds for $\Theta(k)$ (due to the max-norm projection) and $\bar{\Theta}$. As such, by triangle inequality,

$$
\begin{aligned}
&\|\bar{\nabla}\mathcal{R}_T(\Theta(k)) - \bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2 \\
&\qquad \le \sum_{t<T}\gamma^t \delta_{\max}\frac{\beta_t^2\mathbb{E}\|\Theta(k)-\bar{\Theta}\|_2^2}{m} + \sum_{t<T}\gamma^t L_t \mathbb{E}|\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta}) - \delta_t(\bar{Z}_{t+1}^k;\Theta(k))|, \\
&\qquad \le \underbrace{\frac{\delta_{\max}\beta_T^2(\|\rho\|_2^2+\|\nu\|_2^2)}{m(1-\gamma)}}_{=:\frac{C_T^{(4)}}{m}} + L_T\mathbb{E}\left[\sum_{t=0}^{T-1}\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\bar{\Theta}) - \delta_t(\bar{Z}_{t+1}^k;\Theta(k))|\right] \quad (40)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{t<T}\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})| \\
&\qquad = \sum_{t<T}\gamma^t\Big(|F_{t+1}(\bar{Z}_{t+1}^k;\bar{\Theta}) - F_{t+1}(\bar{Z}_{t+1}^k;\Theta(k))| + |F_t(\bar{Z}_t^k;\bar{\Theta}) - F_t(\bar{Z}_t^k;\Theta(k))|\Big), \\
&\qquad \le 2\sum_{t<T}\gamma^t\Big|F_t(\bar{Z}_t^k;\bar{\Theta}) - F_t(\bar{Z}_t^k;\Theta(k))\Big| + \gamma^T L_T\|\Theta(k)-\bar{\Theta}\|_2, \quad (41)
\end{aligned}
$$

where the second line follows from the Lipschitz continuity of $\Theta \mapsto F_t(\cdot;\Theta)$. Then, adding and subtracting $\mathcal{Q}_t^\pi$ to each term, we obtain

$$
\begin{aligned}
&\sum_{t<T}\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})| \\
&\le 2\sum_{t<T}\gamma^t\left(|F_t(\bar{Z}_t^k;\bar{\Theta}) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)| + |\mathcal{Q}_t^\pi(\bar{Z}_t^k) - F_t(\bar{Z}_t^k;\Theta(k))|\right) + \gamma^T L_T\|\Theta(k)-\bar{\Theta}\|_2. \quad (42)
\end{aligned}
$$

Taking expectation, we obtain

$$
\begin{aligned}
\mathbb{E}\sum_{t<T}\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})| &\le \frac{2}{\sqrt{1-\gamma}}\sqrt{\mathbb{E}\left[\sum_{t<T}\gamma^t|F_t(\bar{Z}_t^k;\Theta(k)) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)|^2\right]} \\
&+ \frac{2}{\sqrt{1-\gamma}}\sqrt{\mathbb{E}\left[\sum_{t<T}\gamma^t|F_t(\bar{Z}_t^k;\bar{\Theta}) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)|^2\right]} + \gamma^T L_T\|\Theta(k)-\bar{\Theta}\|_2.
\end{aligned}
$$

By Lemma B.2 and equation 20, we have

$$
\mathbb{E}|F_t(\bar{Z}_t^k;\bar{\Theta}) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)|^2 \le \frac{4}{m}\|\nu\|_2^2\varrho_1^2(1+\varrho_0)^2p_t^2(\alpha\varrho_1) + \frac{4}{m}(\varrho_2\Lambda_T^2 + \varrho_1\chi_T)^2\|\rho\|_2^4,
$$

for any $t < T$. Thus,

$$
\begin{aligned}
\mathbb{E}\sum_{t<T}\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})| &\le \frac{2}{\sqrt{1-\gamma}}\sqrt{\mathbb{E}\left[\sum_{t<T}\gamma^t|F_t(\bar{Z}_t^k;\Theta(k)) - \mathcal{Q}_t^\pi(\bar{Z}_t^k)|^2\right]} \\
+ \frac{1}{\sqrt{m}}\underbrace{\frac{4}{\sqrt{(1-\gamma)^3}}\left(\|\nu\|_2\varrho_1(1+\varrho_0)p_T(\alpha\varrho_1) + (\varrho_2\Lambda_T^2+\varrho_1\chi_T)\|\rho\|_2^2\right)}_{=:C_T^{(3)}} &+ \gamma^T L_T\underbrace{\|\Theta(k)-\bar{\Theta}\|_2}_{\le\|\rho\|_2+\|\nu\|_2}.
\end{aligned}
$$

This results in the following bound:

$$
\mathbb{E}\sum_{t<T}\left[\gamma^t|\delta_t(\bar{Z}_{t+1}^k;\Theta(k)) - \delta_t(\bar{Z}_{t+1}^k;\bar{\Theta})|\right] \le \frac{2}{\sqrt{1-\gamma}}\sqrt{\mathcal{R}_T(\Theta(k))} + \frac{C_T^{(3)}}{\sqrt{m}} + \gamma^T L_T(\|\rho\|_2 + \|\nu\|_2). \quad (43)
$$

Substituting the local smoothness result in equation 43 into equation 40, we obtain

$$\|\bar{\nabla}\mathcal{R}_T(\Theta(k))-\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2 \leq L_T\left(\frac{2}{\sqrt{1-\gamma}}\sqrt{\mathcal{R}_T(\Theta(k))}+\frac{C_T^{(3)}}{\sqrt{m}}+\gamma^T L_T(\|\rho\|_2+\|\nu\|_2)\right)+\frac{C_T^{(4)}}{m}.$$

Thus, we obtain

$$\|\bar{\nabla}\mathcal{R}_T(\Theta(k))-\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2 \leq \frac{16L_T^2}{1-\gamma}\mathcal{R}_T(\Theta(k))+\frac{4(C_T^{(3)})^2L_T^2+4(C_T^{(4)})^2}{m}+8\gamma^{2T}L_T^4(\|\rho\|_2^2+\|\nu\|_2^2). \tag{44}$$

Using equation 38 and equation 44 together, we obtain

$$\begin{aligned}\|\bar{\nabla}\mathcal{R}_T(\Theta(k))\|_2^2 &\leq 2\|\bar{\nabla}\mathcal{R}_T(\Theta(k))-\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2+2\|\bar{\nabla}\mathcal{R}_T(\bar{\Theta})\|_2^2,\\ &\leq \frac{32L_T^2\mathcal{R}_T(\Theta(k))}{1-\gamma}+\frac{(C_T')^2}{m}+16\gamma^{2T}L_T^4(\|\rho\|_2^2+\|\nu\|_2^2).\end{aligned} \tag{45}$$

In the final step, we use equation 28, equation 32 and equation 45 together:

$$\begin{aligned}\mathbb{E}\left[\Psi(\Theta(k+1))-\Psi(\Theta(k))\right] \leq -\eta(1-\gamma)\mathbb{E}\mathcal{R}_T(\Theta(k))+\eta\gamma^T\omega_{T,k}+\eta\frac{C_T\delta_{\mathsf{max}}}{(1-\gamma)\sqrt{m}}\\ +\eta^2\left(\frac{32L_T^2\mathbb{E}\mathcal{R}_T(\Theta(k))}{1-\gamma}+\frac{(C_T')^2}{m}+16\gamma^{2T}L_T^4(\|\rho\|_2^2+\|\nu\|_2^2)\right),\end{aligned} \tag{46}$$

where the expectation is over the random initialization. Choosing $\eta=\frac{(1-\gamma)^2}{64L_T^2}$, we obtain

$$\begin{aligned}\mathbb{E}[\Psi(\Theta(k+1))-\Psi(\Theta(k))] \leq -\frac{\eta(1-\gamma)}{2}\mathbb{E}\mathcal{R}_T(\Theta(k))+\eta\gamma^T\omega_{T,k}+\eta\frac{C_T\delta_{\mathsf{max}}}{(1-\gamma)\sqrt{m}}\\ +\eta^2\left(\frac{(C_T')^2}{m}+16\gamma^{2T}L_T^4(\|\rho\|_2^2+\|\nu\|_2^2)\right).\end{aligned} \tag{47}$$

Telescoping sum over $k=0,1,\ldots,K-1$, and re-arranging terms, we obtain:

$$\begin{aligned}\mathbb{E}\left[\frac{1}{K}\sum_{k<K}\mathcal{R}_T(\Theta(k))\right] \leq \frac{2\|\nu\|_2^2}{(1-\gamma)\eta K}+\frac{\gamma^T\omega_{T,k}}{1-\gamma}+\frac{C_T\delta_{\mathsf{max}}}{(1-\gamma)^2\sqrt{m}}\\ +\eta\left(\frac{(C_T')^2}{m}+16\gamma^{2T}L_T^4(\|\rho\|_2^2+\|\nu\|_2^2)\right).\end{aligned} \tag{48}$$

□

C Numerical Experiments for Rec-TD

In the following, we will demonstrate the numerical performance of Rec-TD for a given non-Markovian policy π^{greedy}.

POMDP setting. We consider a randomly-generated finite POMDP instance with $|\mathbb{S}|=|\mathbb{Y}|=8$, $|\mathbb{A}|=4$, $r(s,a)\sim\mathsf{Unif}[0,1]$ for all $(s,a)\in\mathbb{S}\times\mathbb{A}$. For a fixed ambient dimension $d=8$, we use a random feature mapping $(y,a)\mapsto\varphi(y,a)\sim\mathcal{N}(0,I_d)$, $\forall(y,a)\sim\mathbb{Y}\times\mathbb{A}$.

Greedy policy. Let

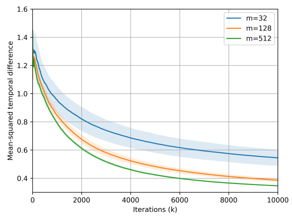
$$j^\star(t)\in\arg\max_{0\leq j<t} r_j,$$

be the instance before t at which the maximum reward was obtained, and let

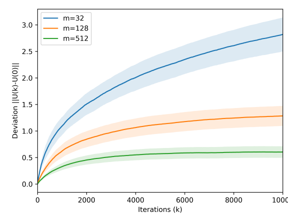
$$\pi_t^{\mathsf{greedy}}(a|Z_t)=\begin{cases}\frac{1}{|\mathbb{A}|}, & \text{w.p. } \min\{\frac{2+t}{10},p_{\mathsf{exp}}\},\\ \mathbb{1}_{a=A_{j^\star(t)}}, & \text{w.p. } 1-\min\{\frac{2+t}{10},p_{\mathsf{exp}}\},\end{cases} \tag{49}$$

be the greedy policy with a user-specified exploration probability $p_{\mathsf{exp}}\in(0,1)$. The long-term dependencies in this greedy policy is obviously controlled by p_{exp}: a small exploration probability will make the policy (thus, the corresponding $\mathcal{Q}$-functions) more history-dependent. Since the exact computation of $(\mathcal{Q}_t^\pi)_{t\in\mathbb{N}}$ is highly intractable for POMDPs, we use (empirical) mean-square temporal difference (MSTD) [2] as a surrogate loss.

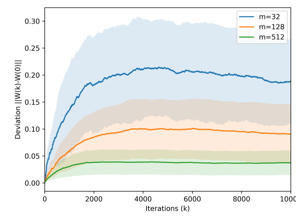
Example 1 (Short-term memory). We first consider the performance of Rec-TD with learning rate $\eta = 0.05$, discount factor $\gamma = 0.7$ and RNNs with various choices of network width m. For $p_{\mathsf{exp}} = 0.8$, the performance of Rec-TD is demonstrated in Figure 1. Consistent with the



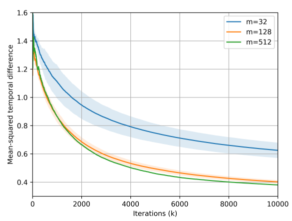
(a) Mean-squared TD, $T = 8$



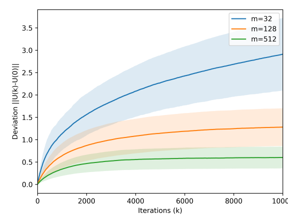
(b) $\frac{1}{m}\|\mathbf{U}(k)-\mathbf{U}(0)\|_{2,1}, T = 8$.



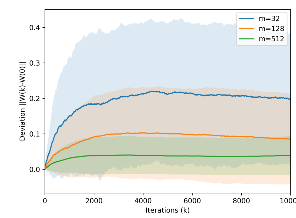
(c) $\frac{1}{m}\|\mathbf{W}(k) - \mathbf{W}(0)\|_{2,1}, T = 8$.



(d) Mean-squared TD, $T = 32$



(e) $\frac{1}{m}\|\mathbf{U}(k) - \mathbf{U}(0)\|_{2,1}, T = 32$.



(f) $\frac{1}{m}\|\mathbf{W}(k) - \mathbf{W}(0)\|_{2,1}, T = 32$.

Figure 1: Mean-square TD and parameter movement under Rec-TD for the case $p_{\mathsf{min}} = 0.8$ and $\gamma = 0.7$. The mean curve and confidence intervals (90%) in Figures 1a and 1d stem from 5 trials. The 90% confidence intervals in Figures 1b-1c and 1e-1f correspond to deviations (i.e., $\|U_i(k) - U_i(0)\|_2$ and $|W_{ii}(k) - W_{ii}(0)|$) across different units $i \in [m]$ in a single trial.

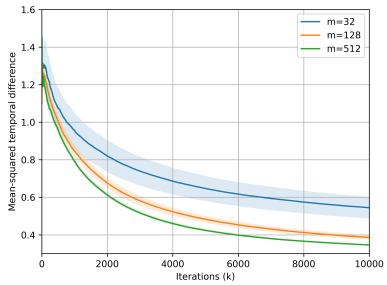
theoretical results in Theorem 6.2, Rec-TD (1) achieves smaller error with larger network width m, (2) requires smaller deviation from the random initialization $\Theta(0)$, which is known as the *lazy training* phenomenon. Since $\|\mathbf{W}(k)\|_{2,\infty} \leq 1$ due to large enough p_{exp} that avoids long-term dependencies, the problem exhibits a weak memory behavior. This is observed in Figures 1d-1f without a visible increase in the MSTD performance despite a significant 3-fold increase in T, consistent with the theoretical findings in Theorem 6.2.

Example 2 (Long-term memory). In the second example, we consider the same POMDP with a discount factor $\gamma = 0.9$. The exploration probability is reduced to $p_{\mathsf{exp}} = 0.3$, which leads to longer dependency on the history. This impact can be observed in Figure 2b-2d, which implies a larger spectral radius compared to Example 1 (in comparison with Figures 1c-1f). As a consequence of the long-term dependencies, increasing T from 8 to 32 leads to a dramatic increase in the MSTD unlike the weak-memory system in Example 1. The impact of a larger network size (i.e., m) is very significant in this example: choosing $m = 512$ leads to a dramatic improvement in the performance.

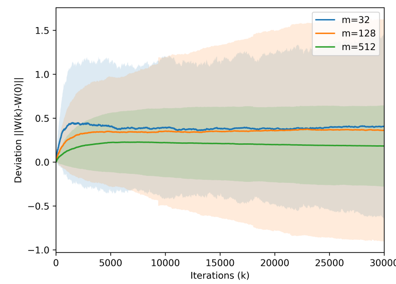
D Policy Gradients under Partial Observability

In this section, we will provide basic results for policy gradients under POMDPs, which is critical to develop the natural policy gradient method for POMDPs.

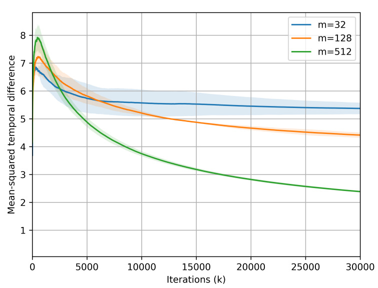
[2] the empirical mean of independently sampled $\left\{\frac{1}{k}\sum_{s<k}\hat{\mathcal{R}}_T^{\mathsf{TD}}(\Theta(s)) : k \in \mathbb{N}\right\}$ where $\hat{\mathcal{R}}_T^{\mathsf{TD}}(\Theta(k)) = \sum_{t=0}^{T-1}\gamma^t\delta_t^2(\bar{Z}_t^k;\Theta(k))$.



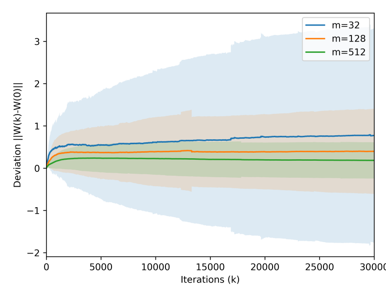
(a) Mean-squared TD, $T = 8$



(b) $\frac{1}{m}\|\mathbf{W}(k) - \mathbf{W}(0)\|_{2,1}$, $T = 8$.



(c) Mean-squared TD, $T = 32$



(d) $\frac{1}{m}\|\mathbf{W}(k) - \mathbf{W}(0)\|_{2,1}$, $T = 32$.

Figure 2: Mean-square TD and parameter deviation under Rec-TD for the case $p_{\mathsf{min}} = 0.3$ and $\gamma = 0.9$. The mean curve and confidence intervals (90%) in Figures 2a and 2c stem from 5 trials. The 90% confidence intervals in Figures 2b and 2d correspond to deviations (i.e., $|W_{ii}(k) - W_{ii}(0)|$) across different units $i \in [m]$ in a single trial.

Proposition D.1. *Let $\pi' \in \Pi_{\mathsf{NM}}$ be an admissible policy, and let $\bar{Z}_T \sim P_T^{\pi',\mu}$. Then, for any $t < T$, conditional distribution of S_t given $\bar{Z}_t$ is independent of π'. Furthermore, for any $\pi \in \Pi_{\mathsf{NM}}$, the conditional distribution of $r(S_t, A_t) + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1})$ given $\bar{Z}_t$ is independent of π'.*

Proof of Prop. D.1. Let the belief at time $t \in \mathbb{N}$ be defined as

$$b_t(s) := \mathbb{P}(S_t = s | \bar{Z}_t). \tag{50}$$

For any non-stationary admissible policy π, the belief function is policy-independent. To see this, note that

$$\begin{aligned}
\mathbb{P}(S_t = s_t, \bar{Z}_t = \bar{z}_t) = & \sum_{(s_0,\ldots,s_{t-1}) \in \mathbb{S}^t} \mathbb{P}(S_0 = s_0 | Y_0 = y) \pi_0(a_0 | z_0) \\
& \times \prod_{k=0}^{t-1} \mathcal{P}(s_{k+1} | s_k, a_k) \phi(y_{k+1} | s_{k+1}) \pi_{k+1}(a_{k+1} | z_{k+1}), \\
= & \left(\prod_{k=0}^{t} \pi_k(a_k | z_k) \right) \sum_{(s_0,\ldots,s_{t-1}) \in \mathbb{S}^t} \mathbb{P}(S_0 = s_0 | Y_0 = y) \\
& \times \prod_{k=0}^{t-1} \mathcal{P}(s_{k+1} | s_k, a_k) \phi(y_{k+1} | s_{k+1}),
\end{aligned}$$

since $\prod_{k=0}^{t} \pi_k(a_k|z_k)$ does not depend on the summands $(s_0, \ldots, s_{t-1})$ – note that we use the notation $\mathcal{P}(s_{k+1}|s_k, a_k) := \mathcal{P}(s_k, a_k, \{S_{k+1} = s_{k+1}\})$ and $\phi(y_k|s_k) := \phi(s_k, \{Y_k = y_k\})$. Thus,

$$b_t(s_t) = \frac{\sum_{(s_0,\ldots,s_{t-1}) \in \mathbb{S}^t} \mathbb{P}(S_0 = s_0 | Y_0 = y) \prod_{k=0}^{t-1} \mathcal{P}(s_{k+1} | s_k, a_k) \phi(y_{k+1} | s_{k+1})}{\sum_{(s_0',\ldots,s_{t-1}',s_t') \in \mathbb{S}^{t+1}} \mathbb{P}(S_0 = s_0' | Y_0 = y) \prod_{k=0}^{t-1} \mathcal{P}(s_{k+1}' | s_k', a_k) \phi(y_{k+1} | s_{k+1}')},$$

independent of π. As such, we have

$$\begin{aligned}\mathbb{E}^{\pi'}[r_t+\gamma\mathcal{V}^\pi(Z_{t+1})|\bar{Z}_t] &= \sum_{s\in\mathbb{S}} b_t(s)\mathbb{E}^{\pi'}[r_t+\gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})|\bar{Z}_t=\bar{z}_t, S_t=s],\\ &= \sum_{s_t,s_{t+1}\in\mathbb{S}}\sum_{y\in\mathbb{Y}} b_t(s_t)\left(r(s_t,A_t)+\gamma\mathcal{P}(s_{t+1}|s_t,A_t)\phi(y|s_{t+1})\mathcal{V}^\pi_{t+1}(Z_t,y_{t+1})\right),\\ &= \mathbb{E}[r_t+\gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})|\bar{Z}_t=\bar{z}_t],\end{aligned}$$

in other words, the conditional distribution of $r(S_t,A_t)+\gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})$ given $\{\bar{Z}_t=\bar{z}_t\}$ is independent of π'. We also know from Prop. B.3 that

$$\mathbb{E}^{\pi'}[r_t+\gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})|\bar{Z}_t=\bar{z}_t] = \mathbb{E}[r_t+\gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})|\bar{Z}_t=\bar{z}_t] = \mathcal{Q}^\pi_t(\bar{z}_t).$$

□

The next result generalizes the policy gradient theorem to POMDPs. We note that there is an extension of REINFORCE-type policy gradient for POMDPs in [44]. The following result is a different and improved version as it ① provides a variance-reduced unbiased estimate of the policy gradient for POMDPs, and more importantly ② yields the compatible function approximation (Prop. 7.1) that yields natural policy gradient (NPG) for POMDPs.

Proposition D.2 (Policy gradient – POMDPs)**.** *For any* $\Phi\in\mathbb{R}^{m(d+1)}$*, we have*

$$\nabla_\Phi\mathcal{V}^{\pi^\Phi}(\mu) = \mathbb{E}^{\pi^\Phi}_\mu\left[\sum_{t=0}^\infty\gamma^t\cdot\mathcal{Q}^{\pi^\Phi}_t(Z_t,A_t)\cdot\nabla_\Phi\ln\pi^\Phi_t(A_t|Z_t)\right], \tag{51}$$

for any $\mu\in\Delta(\mathbb{Y})$.

Proof of Prop. D.2. For any $t\in\mathbb{N}$, we have

$$\mathcal{V}^{\pi^\Phi}_t(z_t) = \sum_{a_t}\pi^\Phi_t(a_t|z_t)\mathcal{Q}^{\pi^\Phi}_t(z_t,a_t), \tag{52}$$

by Prop. B.3. Thus, we obtain

$$\begin{aligned}\nabla\mathcal{V}^{\pi^\Phi}_t(z_t) &= \sum_{a_t}\pi^\Phi_t(a_t|z_t)\nabla\ln\pi^\Phi_t(a_t|z_t)\mathcal{Q}^{\pi^\Phi}_t(z_t,a_t)+\sum_{a_t}\pi^\Phi_t(a_t|z_t)\nabla\mathcal{Q}^{\pi^\Phi}_t(z_t,a_t),\\ &= \mathbb{E}^{\pi^\Phi}[\nabla\ln\pi^\Phi_t(A_t|Z_t)\mathcal{Q}^{\pi^\Phi}_t(Z_t,A_t)+\nabla\mathcal{Q}^{\pi^\Phi}_t(Z_t,A_t)|Z_t=z_t]. \end{aligned}\tag{53}$$

Now, note that

$$\begin{aligned}\mathcal{Q}^{\pi^\Phi}_t(z_t,a_t) &= \mathbb{E}[r(S_t,A_t)+\gamma\mathcal{V}^{\pi^\Phi}_{t+1}(Z_{t+1})|\bar{Z}_t=(z_t,a_t)],\\ &= \sum_{s_t}b_t(s_t)\left(r(s_t,a_t)+\gamma\sum_{s_{t+1}}\mathcal{P}(s_{t+1}|s_t,a_t)\sum_{y_{t+1}}\phi(y_{t+1}|s_{t+1})\mathcal{V}^{\pi^\Phi}_{t+1}(z_{t+1})\right),\end{aligned}$$

where $z_{t+1}=(z_t,a_t,y_{t+1})$. As a consequence of Prop. D.1, we have $\nabla_\Phi\sum_{s_t}b_t(s_t)r(s_t,a_t)=0$, and also

$$\begin{aligned}\nabla_\Phi\mathcal{Q}^{\pi^\Phi}_t(z_t,a_t) &= \gamma\sum_{s_t}b_t(s_t)\sum_{s_{t+1}}\mathcal{P}(s_{t+1}|s_t,a_t)\sum_{y_{t+1}}\phi(y_{t+1}|s_{t+1})\nabla_\Phi\mathcal{V}^{\pi^\Phi}_{t+1}(z_{t+1}),\\ &= \gamma\mathbb{E}[\nabla\ln\pi^\Phi_{t+1}(A_{t+1}|Z_{t+1})\mathcal{Q}^{\pi^\Phi}_{t+1}(Z_{t+1},A_{t+1})+\nabla_\Phi\mathcal{Q}^{\pi^\Phi}_{t+1}(Z_{t+1},A_{t+1})|\bar{Z}_t=(z_t,a_t)],\\ &= \gamma\mathbb{E}^{\pi^\Phi}\Big[\sum_{k=t+1}^\infty\gamma^{k-t-1}\nabla_\Phi\ln\pi^\Phi_k(A_k|Z_k)\mathcal{Q}^{\pi^\Phi}_k(Z_k,A_k)\Big|\bar{Z}_t=(z_t,a_t)\Big].\end{aligned}$$

Using the above recursive formula for $\nabla_\Phi\mathcal{Q}^{\pi^\Phi}_t$ along with the law of iterated expectations in equation 53, we obtain

$$\nabla_\Phi\mathcal{V}^{\pi^\Phi}_t(z_t) = \mathbb{E}^{\pi^\Phi}\Big[\sum_{k=t}^\infty\gamma^{k-t}\nabla_\Phi\ln\pi^\Phi_k(A_k|Z_k)\mathcal{Q}^{\pi^\Phi}_k(Z_k,A_k)\Big|Z_t=z_t\Big]. \tag{54}$$

Since we have $\mathcal{V}^\pi := \mathcal{V}_0^\pi$, and also $\nabla_\Phi \mathcal{V}^{\pi^\Phi}(\mu) = \nabla_\Phi \sum_{z_0} \mu(z_0)\mathcal{V}^{\pi^\Phi}(z_0) = \sum_{z_0} \mu(z_0)\nabla_\Phi \mathcal{V}^{\pi^\Phi}(z_0)$ by the linearity of gradient, we conclude the proof.

Note on the baseline. Similar to the case of fully-observable MDPs, adding a baseline $q_t^{\pi^\Phi}(z_t)$ to the $\mathcal{Q}$-function does not change the policy gradients since $\sum_a \pi_t(a|z_t)\nabla \ln \pi_t^\Phi(a|z_t) q_t^{\pi^\Phi}(z_t) = q_t^{\pi^\Phi}(z_t)\sum_a \nabla \pi_t^\Phi(a|z_t) = q_t^{\pi^\Phi}(z_t)\nabla \sum_a \pi_t^\Phi(a|z_t) = 0$. Thus, we also have

$$\nabla_\Phi \mathcal{V}^{\pi^\Phi}(\mu) = \mathbb{E}_\mu^{\pi^\Phi}\left[\sum_{t=0}^\infty \gamma^t \mathcal{A}_t^{\pi^\Phi}(Z_t, A_t)\nabla_\Phi \ln \pi_t^\Phi(A_t|Z_t)\right], \tag{55}$$

which uses $q_t^{\pi^\Phi} = \mathcal{V}_t^{\pi^\Phi}$ as the baseline, akin to the fully-observable case. □

The following result extends the compatible function approximation theorem in [23] to POMDPs.

Proof of Prop. 7.1. The proof is identical to [23]. By first-order condition for optimality, we have

$$2\mathbb{E}_\mu^{\pi^\Phi}\sum_{t=0}^\infty \gamma^t \nabla \ln \pi_t^\Phi(A_t|Z_t)\left(\nabla^\top \ln \pi_t^\Phi(A_t|Z_t)\omega^\star - \mathcal{A}_t^{\pi^\Phi}(\bar{Z}_t)\right)$$
$$= 2\left(G_\mu(\Phi)\omega^\star - \nabla_\Phi \mathcal{V}^{\pi^\Phi}(\mu)\right) = 0,$$

which concludes the proof. □

E Theoretical Analysis of Rec-NPG

First, we prove structural results for RNNs in the kernel regime, which will be key in the analysis later.

E.1 Log-Linearization of SOFTMAX Policies Parameterized by RNNs

The key idea behind the neural tangent kernel (NTK) analysis is linearization around the random initialization. To that end, let

$$F_t^{\mathsf{Lin}}(\bar{z}_t;\Theta) := \langle \nabla F_t(\bar{z}_t;\Theta(0)), \Theta - \Theta(0)\rangle, \tag{56}$$

for any $\Theta \in \mathbb{R}^{m(d+1)}$. We define the log-linearized policy as follows:

$$\tilde{\pi}_t^\Phi(a|z_t) := \frac{\exp(F_t^{\mathsf{Lin}}(z_t, a;\Phi))}{\sum_{a'\in\mathbb{A}}\exp(F_t^{\mathsf{Lin}}(z_t, a';\Phi))},\ t\in\mathbb{N}. \tag{57}$$

The first result bounds the Kullback-Leibler divergence between π_t^Φ and its log-linearized version $\tilde{\pi}_t^\Phi$. In the case of FNNs with ReLU activation functions, a similar result was presented in [7]. The following result extends this idea to (i) RNNs, and (ii) smooth activation functions.

Proposition E.1 (Log-linearization error)**.** *For any $t \in \mathbb{N}$ and $(z_t, a) \in (\mathbb{Y}\times\mathbb{A})^{t+1}$, we have*

$$\sup_{(z_t,a)\in(\mathbb{Y}\times\mathbb{A})^{t+1}}\left|\ln\frac{\tilde{\pi}_t^\Phi(a|z_t)}{\pi_t^\Phi(a|z_t)}\right| \leq \frac{6}{\sqrt{m}}\left(\Lambda_t^2\varrho_2 + \chi_t\varrho_1\right)\|\Phi - \Phi(0)\|_2^2, \tag{58}$$

for any $t \in \mathbb{N}$. Consequently, we have $\pi_t(\cdot|z_t) \ll \tilde{\pi}_t(\cdot|z_t)$ and $\tilde{\pi}_t(\cdot|z_t) \ll \pi_t(\cdot|z_t)$, and

$$\max\left\{\mathscr{D}_{\mathsf{KL}}(\pi_t^\Phi(\cdot|z_t)\|\tilde{\pi}_t^\Phi(\cdot|z_t)), \mathcal{D}_{\mathsf{KL}}(\tilde{\pi}_t^\Phi(\cdot|z_t)\|\pi_t^\Phi(\cdot|z_t))\right\} \leq \frac{6}{\sqrt{m}}\left(\Lambda_t^2\varrho_2 + \chi_t\varrho_1\right)\|\Phi - \Phi(0)\|_2^2, \tag{59}$$

for all $z_t \in (\mathbb{Y}\times\mathbb{A})^{t+1}$ and $t \in \mathbb{N}$.

Proof. Fix $(z_t, a) \in (\mathbb{Y}\times\mathbb{A})^{t+1}$. By the log-sum inequality [11], we have

$$\ln\frac{\sum_a \exp(F_t^{\mathsf{Lin}}(z_t, a;\Phi))}{\sum_a \exp(F_t(z_t, a;\Phi))} \leq \sum_{a\in\mathbb{A}}\tilde{\pi}_t^\Phi(a|z_t)\left(F_t^{\mathsf{Lin}}(z_t, a;\Phi) - F_t(z_t, a;\Phi)\right).$$

Using the same argument, we obtain

$$\left|\ln \frac{\sum_a \exp(F_t^{\mathsf{Lin}}(z_t, a; \Phi))}{\sum_a \exp(F_t(z_t, a; \Phi))}\right| \leq \sum_{a \in \mathbb{A}} \left(\tilde{\pi}_t^{\Phi}(a|z_t) + \pi_t^{\Phi}(a|z_t)\right) \cdot \left|F_t^{\mathsf{Lin}}(z_t, a; \Phi) - F_t(z_t, a; \Phi)\right|. \tag{60}$$

Thus, we have

$$\left|\ln \frac{\tilde{\pi}_t^{\Phi}(a|z_t)}{\pi_t^{\Phi}(a|z_t)}\right| \leq (1 + \tilde{\pi}_t^{\Phi}(a|z_t) + \pi_t^{\Phi}(a|z_t)) \left|F_t^{\mathsf{Lin}}(z_t, a; \Phi) - F_t(z_t, a; \Phi)\right|.$$

By using Lemma B.1, we have $\sup_{\bar{z}_t \in (\mathbb{Y}\times\mathbb{A})^{t+1}} \left|F_t^{\mathsf{Lin}}(\bar{z}_t'; \Phi) - F_t(\bar{z}_t'; \Phi)\right| \leq \frac{2}{\sqrt{m}}(\Lambda_t^2 \varrho_2 + \chi_t \varrho_1)\|\Phi - \Phi(0)\|_2^2$. By using the last two inequalities together, and noting that $1 + \tilde{\pi}_t^{\Phi}(a|z_t) + \pi_t^{\Phi}(a|z_t) \leq 3$, we conclude that

$$\left|\ln \frac{\tilde{\pi}_t^{\Phi}(a|z_t)}{\pi_t^{\Phi}(a|z_t)}\right| \leq \frac{6}{\sqrt{m}}(\Lambda_t^2 \varrho_2 + \chi_t \varrho_1)\|\Phi - \Phi(0)\|_2^2.$$

Since the righthand-side of the above inequality is independent of (z_t, a), we deduce that the result holds for all (z_t, a), thus concluding the proof. □

The following result will be important in establishing the Lyapunov drift analysis of Rec-NPG.

Proposition E.2 (Smoothness of $\ln \tilde{\pi}_t^{\Phi}(a|z_t)$)**.** *For any $t \in \mathbb{N}$, we have*

$$\sup_{(z_t, a) \in (\mathbb{Y}\times\mathbb{A})^{t+1}} \|\nabla \ln \tilde{\pi}_t^{\Phi}(a|z_t) - \nabla \ln \tilde{\pi}_t^{\Phi'}(a|z_t)\|_2 \leq L_t^2 \|\Phi - \Phi'\|_2,$$

for any $\Phi, \Phi' \in \mathbb{R}^{m(d+1)}$.

Proof. Consider a general log-linear parameterization

$$p_\theta(x) \propto \exp(\phi_x^\top \theta),\ x \in \mathbf{X}.$$

Then, if $\sup_{x \in \mathbf{X}} \|\phi_x\|_2 \leq B < \infty$, then $\theta \mapsto \ln p_\theta(x)$ has B^2-Lipschitz continuous gradients for each $x \in \mathbf{X}$ [1]. The remaining part is to prove a uniform upper bound for $\|\nabla_\Phi F_t(\bar{z}_t; \Phi(0))\|_2$. To that end, notice that

$$\nabla_{\Phi_i} F_t(\bar{z}_t; \Phi(0)) = \frac{1}{\sqrt{m}} c_i \nabla H_t^{(i)}(\bar{z}_t; \Phi(0)),\ \bar{z}_t \in (\mathbb{Y}\times\mathbb{A})^{t+1}, i \in [m].$$

From the local Lipschitz continuity result in Lemma B.1, we have $\sup_{\bar{z}_t : \max_{j \leq t} \|(y_j, a_j)\|_2 \leq 1} \|\nabla_{\Phi_i} H_t^{(i)}(\bar{z}_t; \Phi(0))\|_2 \leq L_t$ for any $i \in [m]$. Thus, for any $\bar{z}_t$, we have

$$\|\nabla_\Phi F_t(\bar{z}_t; \Phi(0))\|_2^2 = \frac{1}{m} \sum_{i=1}^m \|\nabla_{\Phi_i} H_t^{(i)}(\bar{z}_t; \Phi(0))\|_2^2 \leq L_t^2. \tag{61}$$

□

E.2 Theoretical Analysis of Rec-NPG

For any $\pi \in \Pi_{\mathsf{NM}}$, we define the potential function as

$$\mathscr{L}(\pi) := \mathbb{E}_\mu^{\pi^\star}\left[\sum_{t=0}^{T-1} \gamma^t \mathscr{D}_{\mathsf{KL}}\left(\pi_t^\star(\cdot|Z_t) \| \pi_t(\cdot|Z_t)\right)\right]. \tag{62}$$

Then, we have the following drift inequality.

Proposition E.3 (Drift inequality)**.** *For any $n \in \mathbb{N}$, the drift can be bounded as follows:*

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) \le -\eta_{\mathsf{npg}}(\mathcal{V}^{\pi^\star}(\mu) - \mathcal{V}^{\pi^{\Phi(n)}}(\mu))$$

$$\underbrace{-\eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star}\left[\sum_{t=0}^{T-1}\gamma^t\left(\nabla^\top \ln \pi_t^{\Phi(n)}(A_t|Z_t)\omega_n - \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)\right)\right]}_{①}$$

$$+\underbrace{\eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star}\sum_{t=T}^{\infty}\gamma^t\mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)}_{②}\underbrace{-\eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\left(\nabla\ln\tilde{\pi}_t^{\Phi(n)}(A_t|Z_t) - \nabla\ln\pi_t^{\Phi(n)}(A_t|Z_t)\right)^\top\omega_n}_{③}$$

$$+\frac{1}{2}\eta_{\mathsf{npg}}^2\|\rho\|_2^2\sum_{t=0}^{T-1}\gamma^t L_t^2 + \frac{12\|\rho\|_2^2}{\sqrt{m}}\sum_{t=0}^{T-1}\gamma^t(\Lambda_t^2\varrho_2 + \chi_t\varrho_1).$$

Proof. First, note that the drift can be expressed as

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) = \mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\sum_{a\in\mathbb{A}}\pi_t^\star(A_t|Z_t)\ln\frac{\pi_t^{\Phi(n)}(A_t|Z_t)}{\pi_t^{\Phi(n+1)}(A_t|Z_t)}.$$

Then, with a log-linear transformation,

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) = \mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\sum_{a\in\mathbb{A}}\pi_t^\star(A_t|Z_t)$$
$$\times\left(\ln\frac{\tilde{\pi}_t^{\Phi(n)}(A_t|Z_t)}{\tilde{\pi}_t^{\Phi(n+1)}(A_t|Z_t)} + \ln\frac{\pi_t^{\Phi(n)}(A_t|Z_t)}{\tilde{\pi}_t^{\Phi(n)}(A_t|Z_t)} + \ln\frac{\tilde{\pi}_t^{\Phi(n+1)}(A_t|Z_t)}{\pi_t^{\Phi(n+1)}(A_t|Z_t)}\right).$$

By using the log-linearization bound in Prop. E.1 twice in the above inequality, we obtain

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) \le \mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\sum_{a\in\mathbb{A}}\pi_t^\star(A_t|Z_t)\ln\frac{\tilde{\pi}_t^{\Phi(n)}(A_t|Z_t)}{\tilde{\pi}_t^{\Phi(n+1)}(A_t|Z_t)}$$
$$+\frac{12}{\sqrt{m}}\sum_{t=0}^{T-1}\gamma^t(\Lambda_t^2\varrho_2 + \chi_t\varrho_1)\|\rho\|_2^2. \quad (63)$$

By the smoothness result in Prop. E.2, we have

$$|\ln\tilde{\pi}_t^{\Phi(n+1)}(a_t|z_t) - \ln\tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla\ln\tilde{\pi}_t^{\Phi(n)}(a_t|z_t)(\Phi(n+1) - \Phi(n))| \le \frac{1}{2}L_t^4\|\Phi(n+1) - \Phi(n)\|_2^2.$$

Thus, we obtain

$$-\eta_{\mathsf{npg}}^2 L_t^4\|\rho\|_2^2 \le -\eta_{\mathsf{npg}}^2 L_t^4\|\omega_n\|_2^2 \le -\ln\frac{\tilde{\pi}_t^{\Phi(n)}(a_t|z_t)}{\tilde{\pi}_t^{\Phi(n+1)}(a_t|z_t)} - \eta_{\mathsf{npg}}\nabla^\top\ln\tilde{\pi}_t^{\Phi(n)}(a_t|z_t)\omega_n,$$

because of the max-norm gradient clipping that yields $\|\omega_n\|_2 \le \|\rho\|_2$ and $\Phi(n+1) = \Phi(n) + \eta_{\mathsf{npg}}\omega_n$ for any $n \in \mathbb{N}$. Using this in equation 63, we get

$$\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) \le -\eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star}\sum_{t=0}^{T-1}\gamma^t\nabla^\top\ln\tilde{\pi}_t^{\Phi(n)}(a_t|z_t)\omega_n$$
$$+\frac{12}{\sqrt{m}}\sum_{t=0}^{T-1}\gamma^t(\Lambda_t^2\varrho_2 + \chi_t\varrho_1)\|\rho\|_2^2 + \frac{1}{2}\eta_{\mathsf{npg}}^2 L_t^4\|\rho\|_2^2. \quad (64)$$

An important technical result that will be useful in our analysis is the *pathwise* performance difference lemma, which was originally developed in [22] for fully-observable MDPs.

Lemma E.4 (Pathwise Performance Difference Lemma)**.** *Let* $\Phi, \Phi' \in \mathbb{R}^{m(d+1)}$ *be two parameters. Then, we have*

$$\mathcal{V}^{\pi^{\Phi'}}(\mu) - \mathcal{V}^{\pi^{\Phi}}(\mu) = \mathbb{E}_\mu^{\pi^{\Phi'}} \sum_{t=0}^{\infty} \gamma^t \mathcal{A}_t^{\pi^{\Phi}}(Z_t, A_t).$$

The proof of Lemma E.4 is an extension of [1] to non-stationary policies, and can be found at the end of this subsection.

Using Lemma E.4 in equation 64, we obtain

$$\begin{aligned}\mathscr{L}(\pi^{\Phi(n+1)}) - \mathscr{L}(\pi^{\Phi(n)}) &\leq -\eta_{\mathsf{npg}}(\mathcal{V}^{\pi^\star}(\mu) - \mathcal{V}^{\pi^{\Phi(n)}}(\mu)) \\ &\quad - \eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star} \sum_{t=0}^{T-1} \gamma^t \left(\nabla^\top \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t)\omega_n - \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)\right) \\ &\quad + \eta_{\mathsf{npg}}\mathbb{E}_\mu^{\pi^\star} \sum_{t=T}^{\infty} \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t) + \frac{12}{\sqrt{m}} \sum_{t=0}^{T-1} \gamma^t(\Lambda_t^2 \varrho_2 + \chi_t \varrho_1)\|\rho\|_2^2 + \frac{1}{2}\eta_{\mathsf{npg}}^2 L_t^4 \|\rho\|_2^2. \quad (65)\end{aligned}$$

Finally, we replace the term $\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t)$ with $\nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)$ by including the corresponding error term, and conclude the proof by considering the telescoping sum, and noting that $\mathscr{L}(\pi^{\Phi(0)}) = \log|\mathbb{A}|$ since $F_t(\cdot; \Phi(0)) = 0$ by symmetric initialization. □

Proof of Theorem 7.2. We prove Theorem 7.2 by bounding the numbered terms in Prop. E.3.

Bounding ① in Prop. E.3. Recall that $p_T(\gamma) = \sum_{t<T} \gamma^t$. Then, by using Jensen's inequality,

$$\begin{aligned}&\mathbb{E}_\mu^{\pi^\star} \sum_{t=0}^{T-1} \gamma^t \left(\nabla^\top \ln \pi_t^{\Phi(n)}(A_t|Z_t)\omega_n - \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)\right) \\ &\qquad \leq \sqrt{p_T(\gamma)\mathbb{E}_\mu^{\pi^\star} \sum_{t=0}^{T-1} \gamma^t \left|\nabla^\top \ln \pi_t^{\Phi(n)}(A_t|Z_t)\omega_n - \mathcal{A}_t^{\pi^{\Phi(n)}}(\bar{Z}_t)\right|^2}, \\ &\qquad =: \sqrt{p_T(\gamma)}\sqrt{\kappa \varepsilon_{\mathsf{cfa}}^T(\Phi(n), \omega_n)},\end{aligned}$$

where κ yields a change-of-measure argment from $P_T^{\pi^\star,\mu}$ to $P_T^{\pi^{\Phi(n)},\mu}$.

Bounding ② in Prop. E.3. $\sup_{s,a}|r(s,a)| \leq r_\infty$, therefore $|\mathcal{A}_t^\pi(\bar{z}_t)| \leq \frac{2r_\infty}{1-\gamma}$ for any $t \in \mathbb{N}, \bar{z}_t \in (\mathbb{Y} \times \mathbb{A})^{t+1}$, and $\pi \in \Pi_{\mathsf{NM}}$.

Bounding ③ in Prop. E.3. For any $t \in \mathbb{N}$, Cauchy-Schwarz inequality implies

$$\left(\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\right)^\top \omega_n \leq \|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \|\rho\|_2.$$

Recall that

$$\begin{aligned}\nabla \ln \tilde{\pi}_t^{\Phi}(a_t|z_t) &= \nabla F_t(z_t, a_t; \Phi(0)) - \sum_{a'} \tilde{\pi}_t^{\Phi}(a'|z_t)\nabla F_t(z_t, a'; \Phi(0)), \\ \nabla \ln \pi_t^{\Phi}(a_t|z_t) &= \nabla F_t(z_t, a_t; \Phi) - \sum_{a'} \pi_t^{\Phi}(a'|z_t)\nabla F_t(z_t, a'; \Phi).\end{aligned}$$

First, from local β_t-Lipschitzness of $\Phi_i \mapsto \nabla H_t^{(i)}(\bar{z}_t; \Phi_i)$ for $\Phi \in \Omega_{\rho,m}$ by Lemma B.1, we have

$$\begin{aligned}\|\nabla_{\Phi_i} F_t(\bar{z}_t; \Phi(n)) - \nabla_{\Phi_i} F_t(\bar{z}_t; \Phi(0))\|_2 &= \frac{1}{\sqrt{m}}\|\nabla_{\Phi_i} H_t^{(i)}(\bar{z}_t; \Phi_i(n)) - \nabla_{\Phi_i} H_t^{(i)}(\bar{z}_t; \Phi_i(0))\|_2, \\ &\leq \frac{\beta_t \|\rho\|_2}{m},\end{aligned}$$

for any $n \in \mathbb{N}$ since $\max_i \|\Phi_i(n) - \Phi_i(0)\|_2 \leq \frac{\|\rho\|_2}{\sqrt{m}}$ by max-norm projection. Thus,

$$\|\nabla_\Phi F_t(\bar{z}_t; \Phi(n)) - \nabla_\Phi F_t(\bar{z}_t; \Phi(0))\|_2 \leq \frac{\beta_t \|\rho\|_2}{\sqrt{m}},\ t \in \mathbb{N}. \tag{66}$$

Thus,

$$\|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \leq \sum_a |\pi_t^{\Phi(n)}(a|z_t) - \tilde{\pi}_t^{\Phi(n)}(a|z_t)| \|\nabla F_t(\bar{z}_t; \Phi(0))\|_2$$
$$+ \frac{\beta_t \|\rho\|_2}{\sqrt{m}} + \sum_a \pi_t^{\Phi(n)}(a|z_t) \|\nabla F_t(z_t, a; \Phi(n)) - \nabla F_t(z_t, a; \Phi(0))\|_2.$$

From equation 61, we have

$$\|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \leq \frac{2\beta_t \|\rho\|_2}{\sqrt{m}} + 2L_t \mathscr{D}_{\mathsf{TV}}\left(\pi_t^{\Phi(n)}(\cdot|z_t) \| \tilde{\pi}_t^{\Phi(n)}(\cdot|z_t)\right),$$

where $\mathscr{D}_{\mathsf{TV}}$ denotes the total-variation distance between two probability measures. By Pinsker's inequality [11], we obtain

$$\|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \leq \frac{2\beta_t \|\rho\|_2}{\sqrt{m}} + \sqrt{2} L_t \sqrt{\mathscr{D}_{\mathsf{KL}}\left(\pi_t^{\Phi(n)}(\cdot|z_t) \| \tilde{\pi}_t^{\Phi(n)}(\cdot|z_t)\right)}. \tag{67}$$

By the log-linearization result in Prop. E.1, we have

$$\|\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\|_2 \leq \frac{2\beta_t \|\rho\|_2}{\sqrt{m}} + \sqrt{12} L_t \|\rho\|_2 \sqrt{\frac{\Lambda_t^2 \varrho_2 + \chi_t \varrho_1}{\sqrt{m}}}. \tag{68}$$

Thus, we have

$$\left(\nabla \ln \tilde{\pi}_t^{\Phi(n)}(a_t|z_t) - \nabla \ln \pi_t^{\Phi(n)}(a_t|z_t)\right)^\top \omega_n \leq \|\rho\|_2^2 \left(\frac{2\beta_t}{\sqrt{m}} + \sqrt{12} L_t \frac{\sqrt{\Lambda_t \varrho_2 + \chi_t \varrho_1}}{m^{1/4}}\right).$$

□

Proof of Lemma E.4. For any $y_0 \in \mathbb{Y}$, we have:

$$\begin{aligned}
\mathcal{V}^{\pi'}(y_0) - \mathcal{V}^{\pi}(y_0) &= \mathbb{E}_\mu^{\pi'}\Big[\sum_{t=0}^{\infty} \gamma^t r_t \Big| Z_0 = y_0\Big] - \mathcal{V}^{\pi}(y_0), \\
&= \mathbb{E}_\mu^{\pi'}\Big[\sum_{t=0}^{\infty} \gamma^t \Big(r_t + \mathcal{V}_t^{\pi}(Z_t) - \mathcal{V}_t^{\pi}(Z_t)\Big) \Big| Z_0 = y_0\Big] - \mathcal{V}^{\pi}(y_0), \\
&= \mathbb{E}_\mu^{\pi'}\Big[\sum_{t=0}^{\infty} \gamma^t (r_t + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1}) - \mathcal{V}_t^{\pi}(Z_t) \Big| Z_0 = y_0\Big],
\end{aligned}$$

where $r_t = r(S_t, A_t)$ and the last identity holds since

$$\sum_{t=0}^{\infty} \gamma^t \mathcal{V}_t^{\pi}(z_t) = \mathcal{V}_0^{\pi}(z_0) + \gamma \sum_{t=0}^{\infty} \gamma^t \mathcal{V}_{t+1}^{\pi}(z_{t+1}).$$

Then, letting $r_t = r(s_t, a_t)$ and by using law of iterated expectations,

$$\mathcal{V}^{\pi'}(y_0) - \mathcal{V}^{\pi}(y_0) = \mathbb{E}_\mu^{\pi'}\Big[\sum_{t=0}^{\infty} \gamma^t \Big(\mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}_{t+1}^{\pi}(Z_{t+1}) | \bar{Z}_t, S_t] - \mathcal{V}_t^{\pi}(Z_t)\Big) \Big| Z_0 = y_0\Big], \tag{69}$$

which holds because

$$\mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}^{\pi}(Z_{t+1}) | \bar{Z}_t] = \mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}^{\pi}(Z_{t+1}) | \bar{Z}_t, Z_0].$$

The conditional expectation of $r_t + \gamma \mathcal{V}^\pi_{t+1}$ given $\{\bar{Z}_t = \bar{z}_t\}$ is independent of π':

$$\begin{aligned}\mathbb{E}^{\pi'}[r_t + \gamma \mathcal{V}^\pi(Z_{t+1})|\bar{Z}_t] &= \sum_{s\in\mathbb{S}} b_t(s)\mathbb{E}^{\pi'}[r_t + \gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})|\bar{Z}_t = \bar{z}_t, S_t = s],\\ &= \sum_{s_t, s_{t+1}\in\mathbb{S}}\sum_{y\in\mathbb{Y}} b_t(s_t)\left(r(s_t, A_t) + \gamma\mathcal{P}(s_{t+1}|s_t, A_t)\phi(y|s_{t+1})\mathcal{V}^\pi_{t+1}(Z_t, y_{t+1})\right),\\ &= \mathbb{E}[r_t + \gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})|\bar{Z}_t = \bar{z}_t],\end{aligned}$$

based on Prop. D.1. We also know from Prop. B.3 that

$$\mathbb{E}^{\pi'}[r_t + \gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})|\bar{Z}_t = \bar{z}_t] = \mathbb{E}[r_t + \gamma\mathcal{V}^\pi_{t+1}(Z_{t+1})|\bar{Z}_t = \bar{z}_t] = \mathcal{Q}^\pi_t(\bar{z}_t).$$

Using the above identity in equation 69, we obtain

$$\mathcal{V}^{\pi'}(y_0) - \mathcal{V}^\pi(y_0) = \mathbb{E}^{\pi'}_\mu\Big[\sum_{t=0}^\infty \gamma^t\Big(\mathcal{Q}^\pi_t(\bar{Z}_t) - \mathcal{V}^\pi(Z_t)\Big)\Big|Z_0 = y_0\Big], \tag{70}$$

which concludes the proof. □

Proof of Prop. 7.5. For any ω, we have

$$\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}}) \leq 2\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)}) + 2\sum_{t=0}^\infty \gamma^t(\mathcal{A}^{\pi^{\Phi(n)}}_t(Z_t, A_t) - \hat{\mathcal{A}}^{(n)}_t(Z_t, A_t))^2. \tag{71}$$

Let $\mathcal{G}_n := \sigma(\Phi(k), k \leq n)$ and $\mathcal{H}_n := \sigma(\bar{\Theta}^{(n)}, \Phi(k), k \leq n)$. Then, since

$$\varepsilon_{\mathsf{sgd},n} = \mathbb{E}[\ell_T(\omega_n; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n] - \inf_{\omega\in\mathcal{B}^{(m)}_{2,\infty}(0,\rho)} \mathbb{E}[\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n],$$

we obtain

$$\mathbb{E}[\ell_T(\omega_n; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] \leq 2\mathbb{E}\Big[\inf_\omega \mathbb{E}[\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n]\Big|\mathcal{G}_n\Big] + 2(\varepsilon_{\mathsf{td},n} + \varepsilon_{\mathsf{sgd},n}), \tag{72}$$

which uses the fact that $Var(X|\mathcal{G}_n) \leq \mathbb{E}[|X|^2|\mathcal{G}_n]$ for any square-integrable X. We also have

$$\begin{aligned}\inf_\omega \mathbb{E}[\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n] \leq 2\inf_\omega \mathbb{E}[\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n]&\\ + 2\sum_{t=0}^\infty \gamma^t(\mathcal{A}^{\pi^{\Phi(n)}}_t(Z_t, A_t) - \hat{\mathcal{A}}^{(n)}_t(Z_t, A_t))^2,&\end{aligned}$$

which further implies that

$$\mathbb{E}[\inf_\omega \mathbb{E}[\ell_T(\omega; \Phi(n), \hat{\mathcal{Q}}^{(n)})|\mathcal{H}_n]|\mathcal{G}_n] \leq 2\mathbb{E}[\inf_\omega \mathbb{E}[\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n]|\mathcal{G}_n] + 2\varepsilon_{\mathsf{td},n}.$$

Thus,

$$\mathbb{E}[\ell_T(\omega_n; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] \leq 4\mathbb{E}\Big[\inf_\omega \mathbb{E}[\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n]\Big|\mathcal{G}_n\Big] + 6\varepsilon_{\mathsf{td},n} + 2\varepsilon_{\mathsf{sgd},n}. \tag{73}$$

For any $\omega \in \mathcal{B}^{(m)}_{2,\infty}(0, \rho)$,

$$\begin{aligned}\mathbb{E}[\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] &\leq \mathbb{E}[\sum_{t<T}\gamma^t(\nabla^\top_\Phi F_t(\bar{Z}_t; \Phi(n))\omega - \mathcal{Q}^{\pi^{\Phi(n)}}_t(\bar{Z}_t))^2|\mathcal{H}_n],\\ &\leq 2\mathbb{E}[\sum_{t<T}\gamma^t(\nabla^\top_\Phi F_t(\bar{Z}_t; \Phi(0))\omega - \mathcal{Q}^{\pi^{\Phi(n)}}_t(\bar{Z}_t))^2 + (\nabla F_t(\bar{Z}_t; \Phi(n)) - \nabla F_t(\bar{Z}_t; \Phi(0))^\top\omega)^2|\mathcal{H}_n],\end{aligned}$$

which implies that

$$\begin{aligned}\inf_\omega \mathbb{E}[\ell_T(\omega; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] &\leq 2\varepsilon_{\mathsf{app},n} + 2\|\rho\|_2^2\mathbb{E}[\sum_{t<T}\gamma^t\|\nabla F_t(\bar{Z}_t; \Phi(n)) - \nabla F_t(\bar{Z}_t; \Phi(0)\|_2^2|\mathcal{H}_n],\\ &\leq 2\varepsilon_{\mathsf{app},n} + \frac{2\|\rho\|_2^4}{m}\sum_{t<T}\gamma^t\beta_t^2,\end{aligned}$$

using equation 66. Hence,

$$\mathbb{E}[\ell_T(\omega_n; \Phi(n), \mathcal{Q}^{\pi^{\Phi(n)}})|\mathcal{H}_n] \leq \frac{8\|\rho\|_2^4}{m} \sum_{t<T} \gamma^t \beta_t^2 + 8\varepsilon_{\mathsf{app},n} + 6\varepsilon_{\mathsf{td},n} + 2\varepsilon_{\mathsf{sgd},n},$$

concluding the proof. □

Proof of Prop. 7.7. Under Assumption 7.6, consider $f_t^{(j)}(\bar{z}_t) := \mathbb{E}[\psi_t^\top(\bar{z}_t; \phi_0)\boldsymbol{v}^{(j)}(\phi_0)]$ for $\boldsymbol{v}^{(j)} \in \mathscr{H}_{\mathcal{J},\nu}$. Let

$$\omega_i^{(j)} := \frac{1}{\sqrt{m}} c_i \boldsymbol{v}^{(j)}(\Phi_i(0)),\ i = 1, 2, \ldots, m, \tag{74}$$

for any $j \in \mathcal{J}$. Since $\|\omega^{(j)}\|_2 \leq \|\nu\|_2$ and $\rho \succeq \nu$, we have

$$\inf_{\omega \in \mathcal{B}_{2,\infty}^{(m)}(0,\rho)} \left|\nabla^\top F_t(\bar{z}_t; \Phi(0))\omega - f_t^{(j)}(\bar{z}_t)\right| \leq \left|\nabla^\top F_t(\bar{z}_t; \Phi(0))\omega^{(j)} - f_t^{(j)}(\bar{z}_t)\right|. \tag{75}$$

Thus, we aim to find a uniform upper bound for the second term over $j \in \mathcal{J}$. For each $\bar{z}_t$, we have

$$\nabla^\top F_t(\bar{z}_t; \Phi(0))\omega^{(j)} = \frac{1}{m} \sum_{i=1}^m \nabla_{\Phi_i}^\top H_t^{(i)}(\bar{z}_t; \Phi_i(0))\boldsymbol{v}^{(j)}(\Phi_i(0)),$$

thus $\mathbb{E}[\nabla^\top F_t(\bar{z}_t; \Phi(0))\omega^{(j)}] = f_t^{(j)}(\bar{z}_t)$. Furthermore, from Lemma B.1, since $\Phi(0) \in \Omega_{\rho,m}$ obviously, we have

$$\max_{1\leq i\leq m} \|\nabla_{\Phi_i}^\top H_t^{(i)}(\bar{z}_t; \Phi_i(0))\boldsymbol{v}^{(j)}(\Phi_i(0))\|_2 \leq L_t\|\nu\|_2 \leq L_t\|\rho\|_2,\ \text{a.s.}.$$

Thus, by McDiarmid's inequality [31], we have with probability at least $1-\delta$,

$$\sup_{j\in\mathcal{J}} \left|\nabla^\top F_t(\bar{z}_t; \Phi(0))\omega^{(j)} - f_t^{(j)}(\bar{z}_t)\right| \leq 2\mathfrak{Rad}_m(G_t^{\bar{z}_t}) + L_t\|\rho\|_2\sqrt{\frac{\log(2/\delta)}{m}}, \tag{76}$$

for each $t < T$ and $\bar{z}_t$. By union bound,

$$\begin{aligned}\sup_{j\in\mathcal{J}} \max_{\bar{z}_t} \left|\nabla^\top F_t(\bar{z}_t; \Phi(0))\omega^{(j)} - f_t^{(j)}(\bar{z}_t)\right| &\leq 2\max_{\bar{z}_t} \mathfrak{Rad}_m(G_t^{\bar{z}_t}) + L_t\|\rho\|_2\sqrt{\frac{\log(2T|\mathbb{Y}\times\mathbb{A}|^{t+1}/\delta)}{m}}, \\ &\leq 2\max_{0\leq t<T} \max_{\bar{z}_t} \mathfrak{Rad}_m(G_t^{\bar{z}_t}) + L_T\|\rho\|_2\sqrt{\frac{\log(2T|\mathbb{Y}\times\mathbb{A}|^{T}/\delta)}{m}},\end{aligned}$$

simultaneously for all $t < T$ with probability $\geq 1-\delta$. Therefore,

$$\begin{aligned}\inf_\omega \mathbb{E}_\mu^{\pi^{\Phi(n)}} \sum_{t<T} \gamma^t |\nabla^\top F_t(\bar{Z}_t; \Phi(0))\omega - f_t^{(j)}|^2 &\leq \mathbb{E}_\mu^{\pi^{\Phi(n)}} \sum_{t<T} \gamma^t \sup_{j\in\mathcal{J}} |\nabla^\top F_t(\bar{Z}_t; \Phi(0))\omega^{(j)} - f_t^{(j)}|^2, \\ &\leq \frac{1}{1-\gamma}\left(2\max_{0\leq t<T} \max_{\bar{z}_t} \mathfrak{Rad}_m(G_t^{\bar{z}_t}) + L_T\|\rho\|_2\sqrt{\frac{\log(2T|\mathbb{Y}\times\mathbb{A}|^{T}/\delta)}{m}}\right)^2.\end{aligned}$$

□

E.3 Why do we consider uniform approximation error?

In a *static* problem (e.g., the regression problem in supervised learning or the policy evaluation problem in Section 6) with a target function $f \in \mathscr{F}$, the approximation error is easy to characterize:

$$\left|\nabla^\top F_t(\bar{z}_t; \Phi(0))\omega^\star - f_t(\bar{z}_t)\right| = \mathscr{O}\left(\sqrt{\frac{\ln(1/\delta)}{m}}\right), \tag{77}$$

by Hoeffding inequality with $\omega^\star := \left[\frac{1}{\sqrt{m}} c_i \boldsymbol{v}(\Phi_i(0))\right]_{i\in[m]}$.

In the *dynamical* policy optimization problem, the representational assumption $\mathcal{Q}^{\pi^{\Phi(n)}} \in \mathscr{F}$ does not imply an arbitrarily small approximation error as $m \rightarrow \infty$ since the target function $\mathcal{Q}^{\pi^{\Phi(n)}}$ also depends on $\Phi(0)$ [7]. Thus, an approximation

$$\nabla^\top F_t(\bar{z}_t; \Phi(0))\omega_n^\star = \sum_{i=1}^{m} \frac{\nabla^\top H_t^{(i)}(\bar{z}_t; \Phi(0))\boldsymbol{v}^{\Phi(n)}(\Phi_i(0))}{m},$$

with $\omega_n^\star := [\frac{1}{\sqrt{m}} c_i \boldsymbol{v}^{\Phi(n)}(\Phi_i(0))]_{i\in[m]}$ for the transportation mapping $\boldsymbol{v}^{\Phi(n)} \in \mathscr{H}$ may not converge to the target function $\mathcal{Q}^{\pi^{\Phi(n)}}$ because of the correlated $\nabla^\top H_t^{(i)}(\bar{z}_t; \Phi(0))\boldsymbol{v}^{\Phi(n)}(\Phi_i(0))$ across $i \in [m]$ as argued in [7]. To address this, we characterize the uniform approximation error as in Proposition 7.7 for the random features of the actor RNN in approximating all $\mathcal{Q}^{\pi^{\Phi(n)}}$ for all n based on Rademacher complexity.