Transitions to synchronization in adaptive multilayer networks with higher-order interactions

synchronization, adaptive networks, multilayer networks, higher-order interactions, explosive transitions

Extended Abstract

Synchronization [1] is a widespread phenomenon observed in systems ranging from flashing fireflies to neuronal firing. Real-world networks often involve simultaneous, feedback-driven interactions among multiple agents, known as higher-order interactions [2]. In our study, the dynamics of an adaptive multilayer network with N Kuramoto oscillators undergoing both pairwise and higher-order interactions have been investigated. The equation of motion governing the dynamics of our model system on such a network of l layers with pairwise and higher-order interactions is represented as,

$$\dot{\theta}_{i,l} = \omega_{i,l} + \frac{\varepsilon_1 f_{p,l}(\vec{r}(t))}{N} \sum_{j=1}^{N} sin(\theta_{j,l} - \theta_{i,l} - \beta_l) + \frac{\varepsilon_2 f_{h,l}(\vec{r}(t))}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} sin(2\theta_{j,l} - \theta_{k,l} - \theta_{i,l} - \beta_l), \tag{1}$$

where, ε_1 and ε_2 are the coupling strengths corresponding to the pairwise and higher-order interactions, β_l is the phase lag, $f_{p,l}$ is the adaptation function associated with pairwise interactions, and $f_{h,l}$ is the adaptation function associated with higher-order interactions. ω_i is the natural frequency of the system where $i = 1,2,...,N$ and $l = 1,2,...,L$, with N being the number of oscillators in each layer and L being the number of layers. Using the Ott-Antonsen formalism, we reduce the N-dimensional Eq. 1 to a two-dimensional one:

$$\begin{aligned} \dot{r}_l &= -r_l \gamma_l + \frac{\cos(\beta_l)(1-r_l^2) r_l}{2} \left[\varepsilon_1 f_{p,l} + \varepsilon_2 f_{h,l} r_l^2 \right], \\ \dot{\phi}_l &= \omega_{0,l} - \frac{\sin(\beta_l)(1+r_l^2)}{2} \left[\varepsilon_1 f_{p,l} + \varepsilon_2 f_{h,l} r_l^2 \right]. \end{aligned} \tag{2}$$

r_l is defined as the global order parameter and phase ϕ_l for the l^{th} layer. For layers 1 and 2, the pairwise adaptation functions are $f_{p,i} = (Ar_j + B)^{p_i}$, and the higher-order functions are $f_{h,i} = (Ar_j + B)^{h_i}$, where $i \neq j$, i.e., the first layer is influenced by the global order parameter of the second layer and vice versa.

We first take $\beta_l = 0$ and explore the dynamics with a linear form of the adaptation function and discover a tiered transition to synchronization, along with continuous and abrupt routes to synchronization. For nonlinear adaptation forms, three different kinds of tiered transition to synchronization, viz., continuous tiered, discontinuous tiered, and tiered transition with a hysteretic region, are observed as illustrated in Fig.1. Our system also shows multiple routes to synchrony, as displayed in Fig.1(a). We also investigate the routes of the transitions to synchronization and discover that an extra saddle-node bifurcation occurs in the case of tiered transitions in adaptive multilayer networks with higher-order interactions.

Next, we consider a non-zero phase lag $\beta_l = \beta$ in Eq. 1. By appropriately tuning the phase frustration parameter, we demonstrate that phase lag effectively mitigates explosive behavior even in the presence of higher-order interactions in adaptive multilayer networks, converting the transition from first-order to second-order (illustrated in Fig. 2). As the phase lag increases, the hysteresis width progressively narrows, and the abrupt transition gives way to a smooth, continuous one. Thus, phase lag allows us to control the occurrence of discontinuous transitions to synchrony and transform them to smooth and gradual ones.

Our study shows that introducing order-parameter adaptations in higher-order multilayer networks alters system dynamics and reshapes the route to synchronization. We further demonstrate that incorporating phase lag can suppress explosive transitions in such adaptive multilayer systems.

References

[1] Arkady Pikovsky, Michael Rosenblum, and Jürgen Kurths. "Synchronization". In: *Cambridge university press* 12 (2001).

[2] Stefano Boccaletti et al. "The structure and dynamics of networks with higher order interactions". In: *Physics Reports* 1018 (2023), pp. 1–64.

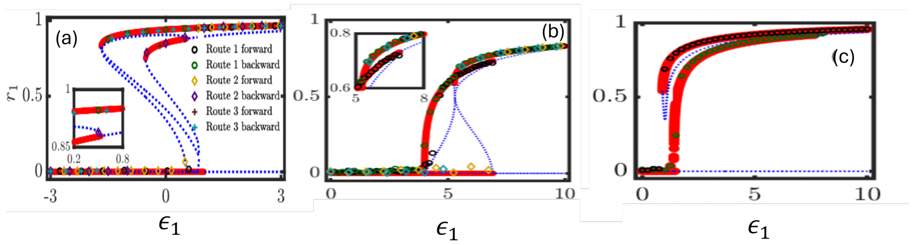

Figure 1: **Tiered transitions to synchronization in adaptive multilayer systems with higher-order interactions:** The stable analytical solutions are plotted in red while the unstable solutions are shown in dashed blue. r_1 denotes the global order parameter of layer-1. (a) Discontinuous tiered transition to synchronization (b) Continuous tiered transition (c) Tiered transition with a hysteretic region.

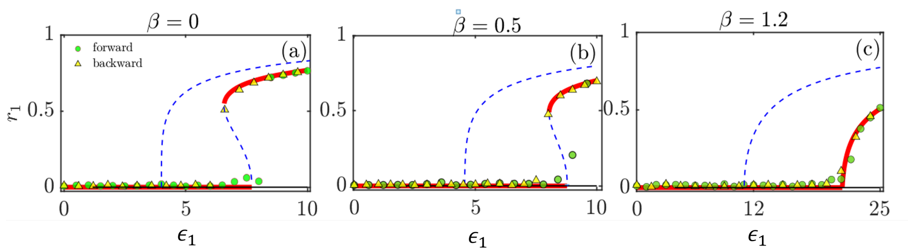

Figure 2: **Discontinuous transitions to synchronization changes to continuous ones in the presence of a phase lag in adaptive multilayer systems with higher-order interactions.** The stable analytical solutions are plotted in red while the unstable solutions are shown in dashed blue. The green markers signify the forward continuation and the yellow triangles show the backward continuation of ε_1. (a) The system shows a first-order transition to synchronization with $\beta = 0$. (b) With a low value of β, the system shows a first-order transition to synchrony. (c) The system shows a second-order transition to synchrony in both layers.