Optimal Adjustment Sets for Nonparametric Estimation of Weighted Controlled Direct Effect

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Abstract

The *weighted controlled direct effect* (WCDE) generalizes the standard *controlled direct effect* (CDE) by averaging over the mediator distribution, providing a robust estimate when treatment effects vary across mediator levels. This makes the WCDE especially relevant in fairness analysis, where it isolates the direct effect of an exposure on an outcome, independent of mediating pathways. This work establishes three fundamental advances for WCDE in observational studies: First, we establish necessary and sufficient conditions for the identifiability of the WCDE, clarifying when it diverges from the CDE. Next, we consider nonparametric estimation of the WCDE and derive its influence function, focusing on the class of regular and asymptotically linear estimators. Lastly, we characterize the optimal covariate adjustment set that minimizes the asymptotic variance, demonstrating how mediator-confounder interactions introduce distinct requirements compared to *average treatment effect* (ATE) estimation. Using synthetic and real-world data, we validate our theory numerically, showing that the proposed optimal valid adjustment set yields the lowest variance at practical sample sizes. Our results offer a principled framework for efficient estimation of direct effects in complex causal systems, with practical applications in fairness and mediation analysis.

1 Introduction

The *controlled direct effect* quantifies the effect of an exposure when intervening to set the mediator to a fixed level, which may differ from its natural observed value [36, 39, 40]. While CDE has traditionally been used to analyze mediation effects, in this work, we employ the *weighted controlled direct effect* to detect the presence of a direct effect between treatment A and outcome Y given observed variables [31]. Specifically, we intervene on *all* observed mediators rather than an arbitrary subset. This approach is motivated by both pragmatic and theoretical considerations: e.g., from a fairness perspective, we seek to determine whether direct discrimination exists [31], while methodologically, 1) the identifiability of CDE becomes unnecessarily complex when some mediators are non-fixed [58], 2) CDE values can vary unreliably across mediator levels [53, 61], and 3) CDE estimates frequently vary across subpopulations defined by baseline covariates [23], complicating generalization.

Unlike *natural direct effects* (NDE) that require untestable *cross-world independence assumptions* (i.e., the potential outcome under one treatment level is independent of the potential mediator under another treatment level) [1, 2, 41], the CDE, and by extension WCDE, relies only on experimentally verifiable interventions (where the mediator is set to specific values through intervention)–a crucial advantage noted by [38] and [57]. The WCDE averages CDEs over the mediator distribution, yielding

a robust effect measure that automatically accounts for mediation interaction while maintaining the CDE's avoidance of cross-world assumptions. This provides critical insights into direct pathways across diverse domains, from evaluating medical treatments in epidemiology [11, 57] to assessing fairness [31, 62] and informing policy decisions [22].

WCDE raises two key statistical challenges in observational studies that should be resolved for valid inference: (1) What are the necessary and sufficient conditions on the adjustment set such that WCDE is uniquely identifiable from observed data? (2) Among all valid adjustment sets satisfying identifiability, which achieves the asymptotic efficiency bound for WCDE estimation? While optimal adjustment sets for ATEs are well-characterized [16, 34, 45], understanding WCDE introduces unique challenges due to the mediator's potential role as both a collider (between the exposure and the confounder) and an effect modifier (between the confounder and the outcome):

- Standard CDE adjustment criteria [36] only ensure identification at fixed mediator levels and fail to address the integration required for WCDE (Example B.1).
- ATE adjustment theory ignores the efficiency implications of mediator-confounder dependencies, which critically affect WCDE estimation variance through the weighting scheme.

This gap leads to potentially inefficient estimators and makes it unclear whether WCDE provides any theoretical advantages over conventional CDE in practice.

Contributions We study the identification and nonparametric estimation of WCDE given a known *directed acyclic graph* (DAG). Our contributions are *fourfold*. First, we establish necessary and sufficient conditions for *valid adjustment sets* (VASs) that guarantee WCDE's identifiability (Lemma 2.6), a prerequisite for meaningful estimation. Second, we derive the *influence function* (IF) for WCDE (Theorem 3.4). Third, we prove that its optimal adjustment set–defined as the covariate collection that minimizes asymptotic variance among all *regular and asymptotically linear* (RAL) estimates– necessarily differs from ATE adjustment sets due to the required integration over mediator distributions, and typically includes all parents of the outcome variable excluding the treatment (Theorem 4.3). These theoretical advances enable more precise direct effect estimation in settings with mediator-confounder dependence. Lastly, we provide an efficient estimator, *augmented inverse probability weighting* (AIPW), for WCDE, and numerically validate our theoretical results.

Outline Section 2 introduces notation, defines WCDE, and presents criteria for VASs. Section 3 reviews nonparametric estimation and RAL estimators, and derives the IF of WCDE. The asymptotic variance of any RAL estimator is then determined by the variance of its IF, which depends explicitly on the choice of the VAS. Section 4 presents the optimal VAS for WCDE. Section 5 contains AIPW for WCDE and numerical results. Section 6 discusses connections to the unknown-graph setting.

Related Works Causal effect estimation from observational data typically relies on covariate adjustment, with prior work examining variable selection through simulations [6, 28] and analyzing the minimum asymptotic variance [13, 43, 44]. Complete and sound graphical criteria for valid adjustment sets identifying total causal effects are well established [34, 37, 50], with later work optimizing asymptotic efficiency for ATEs under linear [16] and nonparametric [45] settings. Recent work has extended to individualized treatment rules and models with hidden variables [52]; establishing necessary and sufficient conditions for globally optimal adjustment sets [46]; and developing a polynomial-time algorithm for finding minimum-cost adjustment sets [51].

Beyond graphical criteria, data-driven methods for selecting sufficient adjustment sets include: 1) *CovSel* [8], which identifies minimal sets via conditional independence under correct model specification, 2) *Outcome-adaptive lasso* [48], which selects variables predictive of the outcome to improve efficiency, 3) *Double selection* [4], which combines variables selected from both treatment and outcome models to ensure robustness in high dimensions. Estimator-driven approaches, such as *Change-in-Estimate* [12] and *Focused Confounder Selection* [60], aim to minimize MSE but require parametric assumptions. These methods offer flexible and scalable alternatives when the causal structure is partially unknown, but their validity often hinges on model assumptions and sample size.

However, existing approaches remain centered on ATE and overlook key challenges posed by the WCDE. The mediator's dual role as a collider and effect modifier introduces unique obstacles to both identifiability and efficiency. While standard CDE-based criteria [59] do not guarantee identifiability of WCDE, ATE-oriented strategies treat mediators as forbidden variables [16, 45], precluding their integration as required by WCDE. We fill this gap by deriving graphical criteria and asymptotic efficiency results tailored to the mediator-weighted structure of the WCDE.

2 Problem Setup and Valid Adjustment Sets for WCDE

To formally define valid adjustment sets for the WCDE, we first introduce notation for graphical models. We then present the inferential framework and associated notation required to specify the estimation problem. Lastly, we classify variable types with respect to the exposure and outcome and provide a formal definition of VASs.

Structural Causal Model (SCM) An SCM is represented by a DAG, $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, a graph with directed edges and no directed cycles, where vertices $\mathbf{V} = \{X_1, \dots, X_d\}$ represent random variables and edges $\mathbf{E}$ encode direct causal relationships. An SCM is equipped with a probability distribution P over $\mathbf{V}$. The *parent set* of a vertex X_j, denoted $\mathrm{Pa}(X_j)$, consists of all vertices X_i for which $X_i \to X_j \in \mathbf{E}$, representing direct causes. The *children* $\mathrm{Ch}(X_i)$ are vertices X_j with $X_i \to X_j \in \mathbf{E}$, representing direct effects. *Ancestors* $\mathrm{An}(X_j)$ comprise all vertices connected to X_j via *directed paths* (Def. A.2), while *descendants* $\mathrm{De}(X_i)$ are all vertices reachable from X_i via *directed paths*.

The *d-separation* (Def. A.1) criterion formally characterizes conditional independence relationships implied by the graph structure. Under the assumption of *causal sufficiency* (no unmeasured confounding), d-separation perfectly captures the conditional independencies in the joint distribution through the *Markov property*: if $\mathbf{Z}$ d-separates $\mathbf{X}$ from $\mathbf{Y}$ in $\mathcal{G}$ (i.e., $\mathbf{X} \perp\!\!\!\perp_{\mathcal{G}} \mathbf{Y} \mid \mathbf{Z}$), then $\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z}$ in all distributions that are Markov with respect to $\mathcal{G}$. This Markov property implies two key consequences. First, the joint distribution factorizes as: $P(X_1, \dots, X_d) = \prod_{j=1}^{d} P(X_j \mid \mathrm{Pa}(X_j))$ where each component represents the conditional distribution of a variable given its direct causes. Second, it yields the *local Markov property* that each variable is conditionally independent of its non-descendant non-parents given its parents:

$$X_j \perp\!\!\!\perp \mathrm{ND}(X_j) \setminus \mathrm{Pa}(X_j) \mid \mathrm{Pa}(X_j),$$

where $\mathrm{ND}(X_j)$ denotes the non-descendants of X_j. These properties form the foundation for deriving identifiability conditions for causal effects in the presence of mediator-confounder relationships.

Weighted Controlled Direct Effect Consider a DAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ representing a structural causal model. We focus on estimating WCDE of a binary treatment variable $A \in \mathbf{V}$ with values $\{a, a^*\}$ (treatment vs. control) on an outcome $Y \in \mathbf{V}$.

Let $\mathbf{M} \subset \mathbf{V}$ be the set of all observed mediators between A and Y, defined through ancestral relations as $\mathbf{M} := \mathrm{De}(A) \cap \mathrm{An}(Y) \setminus A, Y$. The CDE measures the expected change in outcome as the exposure changes when mediators $\mathbf{M}$ are uniformly fixed to a constant value $\mathbf{m}$ through intervention:

Definition 2.1 (CDE, Pearl [35]). $\mathrm{CDE}(\mathbf{m}) := \mathbb{E}[Y \mid do(a, \mathbf{m})] - \mathbb{E}[Y \mid do(a^*, \mathbf{m})]$.

We provide the *identifiability condition* of CDE in Def. A.3. When mediator-exposure interactions are present, CDE becomes mediator-dependent, taking different values across $\mathbf{M}$ [35]. To obtain an unique population-level measure that admits valid adjustment, we define WCDE using the subset of mediators that are also direct parents of Y, ensuring that all mediator paths (Def. A.2) are blocked:

Definition 2.2 (WCDE, [31, 34]). *Given a DAG $\mathcal{G}$, let $\mathbf{M}' := \mathbf{M} \cap \mathrm{Pa}(Y)$ and let $\mathcal{M}'$ be the set of all possible joint values of the mediators in $\mathbf{M}'$. The WCDE can be expressed using do-probabilities as:*

$$\mathrm{WCDE} = \sum_{\mathbf{m}' \in \mathcal{M}'} \left(\mathbb{E}[Y \mid do(a, \mathbf{m}')] - \mathbb{E}[Y \mid do(a^*, \mathbf{m}')] \right) p(\mathbf{m}'). \tag{2.1}$$

Definition 2.2 builds upon prior work by Pearl [34, 35],[1] who established that fixing only the mediators that are parents of Y is sufficient for identifying direct effects.[2] As $\mathbf{M}'$ is uniquely determined given $\mathcal{G}$, for any given SCM, Eq. (2.1) admits an *unique value* assuming CDE's identifiability.

Remark 2.3 (WCDE definition). *Let $\mathcal{C}$ be the set of confounders between A and Y. In defining WCDE, we preserve the conceptual distinction between mediators and confounders, rather than treating mediators as additional confounders through joint conditioning. We weight the marginal distribution $P(\mathbf{m}')$ rather than the conditional $P(\mathbf{m}'|\mathcal{C})$ because the $\mathrm{CDE}(\mathbf{m}')$ in Eq. (2.1) is already causally*

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[1]In the *first edition* of *Causality* [34, Chapter 4.5.4, p. 131], Pearl defined the average direct effect as $\sum_{\mathrm{Pa}_Y \setminus X} \left(\mathbb{E}[Y \mid do(x), do(\mathrm{Pa}_Y \setminus X)] - \mathbb{E}[Y \mid do(x'), do(\mathrm{Pa}_Y \setminus X)] \right) P(\mathrm{Pa}_Y \setminus X)$. Subsequent work [35] clarified that only mediators need to be fixed. Accordingly, we can replace $\mathrm{Pa}_Y \setminus X$ with $\mathbf{M}' := \mathbf{M} \cap \mathrm{Pa}(Y)$.

[2]In this work, we do not assume that *all mediators* are observed. Rather, a nonzero WCDE value indicates that a direct effect between A and Y exists when accounting for all observed mediators.

standardized over confounders $\mathcal{C}$ through the do-operator, representing the population-average direct effect at mediator level $\mathbf{m}'$. Weighting by $P(\mathbf{m}'|\mathcal{C})$ would inappropriately treat the mediator as a confounder by reintroducing the mediator-confounder relationship after causal standardization. This creates conceptual redundancy by effectively conditioning twice on $\mathcal{C}$, yielding stratum-specific effects conditional on $\mathcal{C}$ rather than a coherent population-level measure.

The alternative factorization of the joint distribution, $P(\mathcal{C}|\mathbf{m}')P(\mathbf{m}')$, requires standardizing effects using $P(\mathcal{C}|\mathbf{m}')$, contradicting CDE's identification (Def. A.3). By using $P(\mathbf{m}')$, we maintain philosophical clarity by respecting the distinct roles of confounders (adjusted for in causal standardization) and mediators (the pathway of interest). This provides a clear population-level interpretation: the average direct effect under natural mediator variation.

While Def. 2.2 ensures WCDE identifiability, it does not specify how to compute it from observational data. We now characterize valid adjustment sets that recover the specific WCDE value in Eq. (2.1). While complete criteria exist for ATE adjustment sets [49, 50], no analogous solution currently exists for WCDE, a gap we now resolve. We define VAS for WCDE in Def. 2.4, and provide a set of adjustment criteria in Condition 2.5. We establish their sufficiency and necessity in Lemma 2.6.

Definition 2.4 (Valid Adjustment Set for WCDE). *Let $\mathcal{M}$ be an SCM that induces causal DAG $\mathcal{G}$ with pairwise disjoint node sets A, Y, and $\mathbf{M}$, representing the treatment, outcome, and the set of all mediators (Tab. A.1), respectively. Let $\mathbf{Z}_1 = \mathbf{Z} \cap \mathbf{M}$, $\mathbf{Z}_2 = \mathbf{Z} \setminus \mathbf{M}$, and*

$$\mathrm{WCDE}_{\mathbf{Z}} = \underbrace{\mathbb{E}_{\mathbf{Z}_1}\left\{\mathbb{E}_{\mathbf{Z}_2}\left[\mathbb{E}[Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2]\right]\right\}}_{T_a(\mathbf{Z})} - \underbrace{\mathbb{E}_{\mathbf{Z}_1}\left\{\mathbb{E}_{\mathbf{Z}_2}\left[\mathbb{E}[Y \mid A = a^*, \mathbf{Z}_1, \mathbf{Z}_2]\right]\right\}}_{T_{a^*}(\mathbf{Z})}. \qquad (2.2)$$

We say that $\mathbf{Z}$ is a valid adjustment set *for identifying the WCDE with respect to (A, Y) if and only if $\mathrm{WCDE}_{\mathbf{Z}}$ equals the interventional WCDE defined in Def. 2.2 when both are applied to the population defined by $\mathcal{M}$.*

Condition 2.5 (WCDE Adjustment Criteria). *Given the notation and SCM from Definition 2.4, a candidate adjustment set $\mathbf{Z}$ satisfies the* adjustment criterion *relative to (A, Y) for WCDE in $\mathcal{G}$ if:*

C1 All mediator paths (Def. A.2) for $\{A, Y\}$ are blocked by $\mathbf{Z}_1$;
C2 The subset $\mathbf{Z}_2$ satisfies the sufficient and necessary graphical conditions for identifying the CDE with mediator $\mathbf{Z}_1$, which we further discuss in Appendix A.1;
C3 $\{\mathbf{M}' \setminus \mathbf{Z_1}\} \perp\!\!\!\perp_{\mathcal{G}} A, \{Pa(Y) \setminus (\mathbf{M}' \cup \{A\})\} \mid \mathbf{Z_1}$, ensuring an unique weighting scheme across mediator levels (illustrated in Fig. B.3, Example B.1).

Lemma 2.6 (Sufficiency and Necessity of the Adjustment Criterion). *Let $\mathbf{Z}$ be any set that satisfies Condition 2.5 for WCDE with respect to (A, Y) in $\mathcal{G}$. Then $\mathbf{Z}$ is a VAS (Def. 2.4). Conversely, any adjustment set $\mathbf{Z}$ that identifies the WCDE in Eq. (2.1) must necessarily satisfy Condition 2.5.*

Lemma 2.6 establishes that the WCDE, which is uniquely defined by Def. 2.2, can be identified through any VAS satisfying Condition 2.5, with the same population value regardless of which valid set is chosen. However, in finite samples, estimates can vary across sets due to differences in statistical properties like bias and variance. A proof is provided in Appendix B.1, with a sketch below.

Necessity. We focus on the necessity of C3, as C1 is inherent to the definition of WCDE, and C2 follows from CDE's identification. For a given DAG $\mathcal{G}$ and a adjustment set $\mathbf{Z}$, suppose that $\mathbf{Z}$ satisfies C1 and C2, but violates C3, we show that there exists a joint distribution over the vertices $\mathbf{V}$ in $\mathcal{G}$ so that $\mathrm{WCDE}_{\mathbf{Z}}$ defined in Eq. (2.2) does not match the WCDE in (2.1). Specifically, let $\mathbf{W} = \mathbf{M}' \setminus \mathbf{Z_1}$, and $\mathbf{C} = Pa(Y) \setminus (\mathbf{M}' \cup A)$. Because C3 is violated, $\mathbf{W} \not\perp\!\!\!\perp_{\mathcal{G}} A, \mathbf{C} \mid \mathbf{Z_1}$. This conditional dependence results in a discrepancy between (2.2) and (2.1) due to a bias term involving $p(\mathbf{w} \mid \mathbf{z}_1) - p(\mathbf{w} \mid \mathbf{z}_1, a, \mathbf{c})$, which is non-zero whenever $\mathbf{W} \not\perp\!\!\!\perp_{\mathcal{G}} A, \mathbf{C} \mid \mathbf{Z_1}$ (see Example B.1).

Sufficiency. We show that if an adjustment set $\mathbf{Z}$ satisfies all three criteria in Condition 2.5, then Eq. (2.2) identifies the WCDE. We first decompose $\mathbf{Z}$ into mediators $\mathbf{Z}_1$ and non-mediators $\mathbf{Z}_2$, and then replace $\mathbf{Z}_2$ with $\mathbf{C}$, which continues to identify the CDE and therefore preserves the expectation. Next add $\mathbf{W} = \mathbf{M}' \setminus \mathbf{Z}_1$ to $\mathbf{Z}_1$, which likewise leaves the expectation unchanged. After removing non-parent variables, the remaining expectations are taken over $\mathbf{C}$ and $\mathbf{M}'$. Substituting $\mathbf{Z}_2 = \mathbf{C}$ and $\mathbf{Z}_1 = \mathbf{M}'$ into Eq. (2.2), and noting that $\mathbf{C}$ identifies the CDE for the mediator set $\mathbf{M}'$, we obtain the WCDE expression in Eq. (2.1).

Given the identification of all VASs, a natural question arises: when multiple such sets are available, which one should we choose? In this work, we address this question by comparing VASs through

the lens of *asymptotic variance*. Specifically, we characterize how the choice of VAS affects the asymptotic variance across a general class of estimators. Our objective is to identify the optimal VAS that minimizes asymptotic variance for any estimator within this class.

3 RAL Estimators and Influence Function of WCDE

To identify the optimal VAS, we begin by introducing *regular and asymptotically linear* estimators—a class of estimators with well-characterized asymptotic variance through their influence functions. These estimators provide the foundation for our optimality analysis, which we develop fully in Section 4 through a constructive variance-minimization procedure. We then specialize this framework to WCDE estimation, presenting concrete examples of RAL estimators. Theorem 3.4 establishes the explicit form of the WCDE influence function as a function of the chosen VAS.

RAL Estimators   We begin by introducing RAL estimators in a general statistical framework. Given n i.i.d. observations $O_1, \ldots, O_n$ drawn from an *unknown* distribution P on a sample space Ω. Our goal is to estimate a real-valued target parameter, $T(P)$, based on these observations. For example, in WCDE, for a given valid adjustment set $\mathbf{Z}$, the target parameter is given by Eq. (2.2). We consider a fully nonparametric model class $\mathcal{M}$, i.e., $P \in \mathcal{M}$, which contains all candidate distributions defined on a measurable space $(\Omega, \mathcal{F})$ dominated by a common σ-finite measure ν. Let p denote the corresponding density of P with respect to ν.

We first introduce the notion of regularity, which connects estimators to their stability under small model perturbations (Def. 3.1). Next, we define asymptotic linearity (Def. 3.2).

Definition 3.1 (Regular estimator, van der Vaart [56])**.** *Let $T(P) : \mathcal{M} \to \mathbb{R}^d$ be a target parameter of interest. An estimator $\widehat{T}_n$ is said to be **regular** in $\mathcal{M}$ at P if its convergence to $T(P)$ is **locally uniform** in a neighborhood of P in $\mathcal{M}$.*

Definition 3.2 (Asymptotically linear estimator, van der Vaart [56])**.** *Given $O_1, \ldots, O_n \overset{i.i.d.}{\sim} P$ and target parameter $T(P)$, an estimator $\widehat{T}_n = \widehat{T}_n(O_1, \ldots, O_n)$ of $T(P)$ is said to be* asymptotically linear *at P if there exists a function $\psi_P \in L_2^0(P)$ such that $\sqrt{n}(\widehat{T}_n - T(P)) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_P(O_i) + o_p(1)$, where $L_2^0(P)$ is the set of all mean-zero, square-integrable functions under P, i.e., $L_2^0(P) := \left\{ f : \mathbb{E}_P[f(O)^2] < \infty, \ \mathbb{E}_P[f(O)] = 0 \right\}$.*[3]

Common RAL estimators include the ordinary least squares estimator [33], inverse probability weighting [42], AIPW estimator [10], doubly robust estimators [3, 9], targeted maximum likelihood estimator [55], and double machine learning [7].

Influence Function   The function ψ_P in Definition 3.2 above is known as the *influence function* (IF) of $\widehat{T}_n$ at P. By the Central Limit Theorem, we have:

$$\sqrt{n}(\widehat{T}_n - T(P)) \overset{d}{\to} N\left(0, \mathrm{Var}_P\left[\psi_P(O)\right]\right). \tag{3.1}$$

In fully nonparametric models, this IF is unique for all RAL estimators and coincides with the influence function of the target parameter $T(P)$. At a high level, IF of the target parameter $T(P)$ measures the sensitivity of $T(P)$ under small perturbations of the distribution P [54]. We provide a formal definition in Appendix B.3.

Remark 3.3 (Influence Functions of WCDE$_\mathbf{Z}$)**.** *We emphasize that Eq. (2.2) defines distinct target parameters for each adjustment set $\mathbf{Z}$. While these parameters coincide in value at the true distribution P (yielding an unique causal estimand), different choices of the adjustment set $\mathbf{Z}$ would affect the sensitivity of the corresponding target parameter $\mathrm{WCDE}_\mathbf{Z}$ to local distributional perturbations (see concrete definitions of local distributional perturbation in Appendix B.3). Intuitively, $\mathrm{WCDE}_\mathbf{Z}$ is sensitive to perturbations in variables within $\mathbf{Z}$, so the choice of $\mathbf{Z}$ directly affects its sensitivity to different perturbations. This leads to distinct influence functions for $\mathrm{WCDE}_\mathbf{Z}$ across different $\mathbf{Z}$. We formalize this argument in Appendix B.3.*

Application to WCDE   Given a valid adjustment set $\mathbf{Z}$, let $\mathbf{Z_1} = \mathbf{Z} \cap \mathbf{M}$ and $\mathbf{Z_2} = \mathbf{Z} \setminus \mathbf{M}$. We first provide the IF corresponding to the first component of WCDE defined in Eq. (2.2), denoted by $T_a(\mathbf{Z})$. The IF of $T_{a^*}(\mathbf{Z})$ can be similarly obtained by replacing a with a^* in the expression.

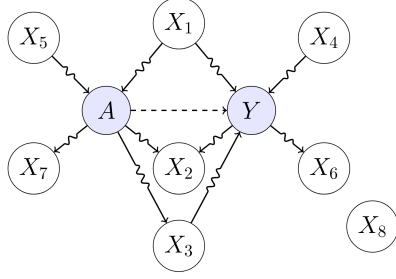[3]Here, we denote O as a generic random variable drawn from the distribution P.

Figure 1: The partition visualization follows Maasch et al. [32], with directed squiggly edges indicating the presence of a directed causal path (not necessarily direct parent-child relationships). The dashed edge represents potential parent-child relationships.

Group	Description
$\mathbf{X}_1$	Confounders and their proxies.
$\mathbf{X}_2$	Colliders and their proxies.
$\mathbf{X}_3$	Mediators and their proxies.
$\mathbf{X}_4$	Non-descendants of Y where $\mathbf{X}_4 \perp\!\!\!\perp A$ and $\mathbf{X}_4 \not\perp\!\!\!\perp A \mid Y$.
$\mathbf{X}_5$	Instruments and their proxies.
$\mathbf{X}_6$	Descendants of Y with A–Y paths mediated by Y.
$\mathbf{X}_7$	Descendants of A.
$\mathbf{X}_8$	Nodes with no active paths to A or Y.

Table 1: Exhaustive and Mutually Exclusive Causal Partitions [32]. Detailed definitions are included in Appendix A.2.

Theorem 3.4 (IF of WCDE$_{\mathbf{Z}}$). *Given any VAS $\mathbf{Z}$, the influence function of $T_a(\mathbf{Z})$ is given by*

$$\psi_a\left(Y, A, \mathbf{Z_1}, \mathbf{Z_2}; P\right) = \frac{I_a(A)p(\mathbf{Z_1})p(\mathbf{Z_2})}{p(\mathbf{Z_1}, \mathbf{Z_2}, a)} \cdot \left(Y - \mathbb{E}_Y[Y|A = a, \mathbf{Z_1}, \mathbf{Z_2}]\right)$$
$$+ \mathbb{E}_{\mathbf{Z_2}}\left[\mathbb{E}_Y[Y|A = a, \mathbf{Z_1}, \mathbf{Z_2}]\right] + \mathbb{E}_{\mathbf{Z_1}}\left[\mathbb{E}_Y[Y|A = a, \mathbf{Z_1}, \mathbf{Z_2}]\right] - 2T_a(\mathbf{Z}).$$

Here, $\mathbf{I}\{\cdot\}$ denotes the indicator function, and $T_a(\mathbf{Z})$ is evaluated at P.

The proof of Theorem 3.4 begins by substituting this functional $T_a(\mathbf{Z})$ into the definition of pathwise differentiability (see Appendix B.4 for details). Leveraging the fact that the influence function is unique in a fully nonparametric model, we directly obtain an unique influence function associated with each valid adjustment set.

4 Optimal Valid Adjustment Sets for WCDE

Given that distinct valid adjustment sets typically yield different IFs (and thus different asymptotic variances), we now leverage the adjustment criterion from Section 2 and the IF derived in Section 3 to identify the optimal VAS—the one minimizing asymptotic variance for all RAL estimators.

To facilitate the presentation of our results, we adopt the causal-partition taxonomy in Table 1 [32]. Given a causal DAG $\mathcal{G}$, let $(A, Y) \subseteq \mathbf{V}$ be an exposure-outcome pair, where Y is a nondescendant of A. This local taxonomy decomposes the rest variables in the DAG, $\mathbf{X} = \mathbf{V} \setminus \{A, Y\}$, into eight *exhaustive and mutually exclusive* subsets, each defined by the types of causal paths that it can share with A and Y. When the union of multiple partitions is needed, we write, for example, $\mathbf{X}_{1,3} := \mathbf{X}_1 \cup \mathbf{X}_3$. Moreover, for any $k \in \{1, \ldots, 8\}$, we denote by $\mathbf{X}_{k \in \mathrm{Pa}(Y)} := \mathbf{X}_k \cap \mathrm{Pa}(Y)$ the subset of $\mathbf{X}_k$ that also belongs to the set of parents of Y.

Under the causal partition taxonomy, we construct the set $\mathbf{O}$ as the union of all parents of Y belonging to relevant partition groups in Definition 4.1.

Definition 4.1 (O-set). *Let $\mathbf{O}(A, Y, \mathcal{G})$ denote the set of all non-treatment parents of Y in $\mathcal{G}$. Formally, under the causal partition taxonomy, these parents fall into three groups: confounders ($\mathbf{X}_1$), mediators ($\mathbf{X}_3$), and informative non-descendants ($\mathbf{X}_4$):*

$$\mathbf{O}(A, Y, \mathcal{G}) := \mathbf{X}_{1 \in Pa(Y)} \cup \mathbf{X}_{3 \in Pa(Y)} \cup \mathbf{X}_{4 \in Pa(Y)}.$$

Intuitively, $\mathbf{O}(A, Y, \mathcal{G})$ is constructed to maximize information on Y, while minimizing information on A and preserving validity. Next, we formally define the notion of optimal VAS in Definition 4.2. Theorem 4.3 establishes the optimality of $\mathbf{O}(A, Y, \mathcal{G})$.

Definition 4.2 (Optimal VAS). *Given a DAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and an exposure–outcome pair $(A, Y) \subseteq \mathbf{V}$, let $\mathcal{Z}$ denote the collection of all VASs for identifying the WCDE of A on Y. We say that $\mathbf{O}^* \in \mathcal{Z}$ is an* optimal valid adjustment set *if, for every $\mathbf{Z} \in \mathcal{Z}$, $\sigma_{\mathbf{Z}}^2(P) - \sigma_{\mathbf{O}^*}^2(P) \geq 0$, where $\sigma_{\mathbf{Z}}^2(P)$ denotes the asymptotic variance of any RAL estimator of WCDE$_{\mathbf{Z}}$ based on adjustment set $\mathbf{Z}$ under distribution P, as characterized by Equation (3.1).*
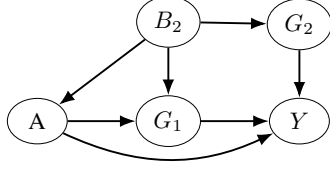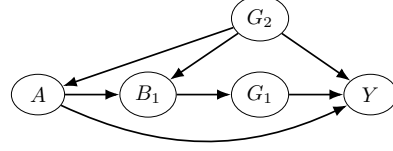
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Figure 2: A DAG illustrating Lemmas 4.4, 4.6.

Figure 3: A DAG illustrating Lemmas 4.5, 4.7.

Theorem 4.3 (O-set is Optimal). *Given a DAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and the exposure-outcome pair $(A, Y) \subseteq \mathbf{V}$, the set $\mathbf{O}(A, Y, \mathcal{G})$ in Definition 4.1 is the* optimal valid adjustment set *for the WCDE (2.1), satisfying Condition 2.5 and producing minimal asymptotic variance for all RAL estimators.*

Below, we sketch the main proof idea (full proof in Appendix C). Starting from an arbitrary VAS $\mathbf{Z}$ with $\mathbf{Z}_1 = \mathbf{Z} \cap \mathbf{M}$ and $\mathbf{Z}_2 = \mathbf{Z} \setminus \mathbf{M}$, we iteratively transform it into $\mathbf{O}(A, Y, \mathcal{G})$ through four variance-reducing operations: (1) adding non-mediating parents of Y, (2) including mediating parents of Y, (3) removing redundant backdoor variables that do not contribute to Y, and (4) pruning mediators that offer no further gain. Each step preserves the validity of the adjustment set and reduces the asymptotic variance, ultimately converging to the same optimal set $\mathbf{O}$ regardless of the starting VAS. While this sequence of variance-reducing operations resembles the pruning procedure proposed by [16] and the graphical criteria introduced by [45], our work extends these ideas to a new functional that involves mediator-confounder interactions, introducing additional structural complexity.

Step 1: Augmenting $\mathbf{Z}_2$ with Non-mediating Parents of Y. Lemma 4.4 (proof in Appendix C.1) establishes that including additional backdoor variables satisfying conditional independence with the treatment can reduce variance without losing validity, i.e., if we add $\mathbf{X}_{1 \in \text{Pa}(Y)}$ and $\mathbf{X}_{4 \in \text{Pa}(Y)}$ to $\mathbf{Z}_2$:

$$\sigma^2_{(\mathbf{Z}_1, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \text{Pa}(Y)} \cup \mathbf{X}_{4 \in \text{Pa}(Y)})}(P) \leq \sigma^2_{(\mathbf{Z}_1, \mathbf{Z}_2)}(P).$$

Lemma 4.4 (Supplement of backdoor variables). *Given a DAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and the exposure-outcome pair $(A, Y) \subseteq \mathbf{V}$, suppose $(\mathbf{G}_1, \mathbf{B}_2) \subset \mathbf{V} \setminus \{A, Y\}$ is a VAS, with $\mathbf{G}_2$ disjoint from both $\mathbf{G}_1$ and $\mathbf{B}_2$, satisfying $\mathbf{G}_2 \perp\!\!\!\perp_{\mathcal{G}} A, \mathbf{G}_1 \mid \mathbf{B}_2$. Then $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ is also a VAS, and for all $P \in \mathcal{M}$, $\sigma^2_{(\mathbf{G}_1, \mathbf{B}_2)}(P) - \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P) \geq 0$.*

Proof Sketch of Lemma 4.4. Figure 2 illustrates the conditions stated in Lemma 4.4. For clarity, we focus on the first component of the WCDE in Eq. (2.2), and let $\sigma^2_{a, \mathbf{Z}}(P)$ denote its asymptotic variance. The extension to the full WCDE is direct. We begin by decomposing the variance:

$$\sigma^2_{a, (\mathbf{G}_1, \mathbf{B}_2)}(P) = \text{Var}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) \right] = \text{Var}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \right]$$
$$+ \text{Var}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \right]$$
$$+ 2 \text{Cov}_P \left(\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P), \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \right).$$

To analyze the covariance term, consider the difference in IFs evaluated at $(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$:

$$\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P).$$

We construct a parametric submodel that perturbs only the conditional law $P(A, \mathbf{G}_1 \mid \mathbf{G}_2, \mathbf{B}_2)$, while leaving all other components unchanged. Under the assumption $\mathbf{G}_2 \perp\!\!\!\perp_{\mathcal{G}} A, \mathbf{G}_1 \mid \mathbf{B}_2$, the above difference is orthogonal to the IF of the larger adjustment set:

$$\mathbb{E}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \cdot (\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)) \right] = 0.$$

Hence, the covariance term vanishes. Substituting this result back into the law of total variance yields:

$$\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P)] = \text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)]$$
$$+ \text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)],$$

where the second term is nonnegative. This establishes $\sigma^2_{a, (\mathbf{G}_1, \mathbf{B}_2)}(P) \geq \sigma^2_{a, (\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P)$, showing that including $\mathbf{G}_2$ either improves or preserves estimation efficiency.

Step 2: Augmenting $\mathbf{Z}_1$ with Mediating Parents of Y. Next, Lemma 4.5 (proof in Appendix C.3) states that including mediators that are also parents of Y leads to improved efficiency. Therefore, we augment $\mathbf{Z}_1$ with $\mathbf{X}_{3 \in \text{Pa}(Y)}$:

$$\sigma^2_{(\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \text{Pa}(Y)}, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \text{Pa}(Y)} \cup \mathbf{X}_{4 \in \text{Pa}(Y)})}(P) \leq \sigma^2_{(\mathbf{Z}_1, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \text{Pa}(Y)} \cup \mathbf{X}_{4 \in \text{Pa}(Y)})}(P).$$

7

Lemma 4.5 (Supplement of mediator variables)*. Given a DAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and the exposure-outcome pair $(A, Y) \subseteq \mathbf{V}$, suppose $(\mathbf{B}_1, \mathbf{G}_2) \subset \mathbf{V} \backslash \{A, Y\}$ is a VAS, with $\mathbf{G}_1$ disjoint from both $\mathbf{B}_1$ and $\mathbf{G}_2$, satisfying $\mathbf{G}_1 \perp\!\!\!\perp_\mathcal{G} A, \mathbf{G}_2 \mid \mathbf{B}_1$. Then $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$ is also a VAS, and for all $P \in \mathcal{M}$, $\sigma^2_{(\mathbf{B}_1, \mathbf{G}_2)}(P) - \sigma^2_{(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)}(P) \geq 0$.*

Figure 3 provides a simplified illustration of the conditions stated in Lemma 4.5. The proof proceeds analogously to that of Lemma 4.4, with $\mathbf{G}_1$ playing the same role as $\mathbf{G}_2$.

Step 3: Pruning Redundant Backdoor Variables from $\mathbf{Z}_2$. Lemma 4.6 (proof in Appendix C.2) implies that unnecessary variables in $\mathbf{Z}_2$ (those not in $\mathbf{X}_{1 \in \text{Pa}(Y)}$ or $\mathbf{X}_{4 \in \text{Pa}(Y)}$) can be removed without increasing variance:

$$\sigma^2_{(\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \text{Pa}(Y)}, \mathbf{X}_{1 \in \text{Pa}(Y)} \cup \mathbf{X}_{4 \in \text{Pa}(Y)})}(P) \leq \sigma^2_{(\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \text{Pa}(Y)}, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \text{Pa}(Y)} \cup \mathbf{X}_{4 \in \text{Pa}(Y)})}(P).$$

Lemma 4.6 (Deletion of overadjustment backdoor variables)*. Given a DAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and the exposure-outcome pair $(A, Y) \subseteq \mathbf{V}$, and suppose $(\mathbf{G}_1, \mathbf{G}_2 \cup \mathbf{B}_2) \subset \mathbf{V} \backslash \{A, Y\}$ is a VAS, with $\mathbf{G}$ and $\mathbf{B}$ disjoint, and suppose $Y \perp\!\!\!\perp_\mathcal{G} \mathbf{B}_2 \mid \mathbf{G}_1, \mathbf{G}_2, A$. Then $(\mathbf{G}_1, \mathbf{G}_2)$ is also a VAS, and for all $P \in \mathcal{M}$, $\sigma^2_{(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P) - \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2)}(P) \geq 0$.*

Proof Sketch of Lemma 4.6. Figure 2 provides a simplified illustration of the conditions stated in Lemma 4.6. Again, we focus on comparing the asymptotic variance of $T_a(\mathbf{Z})$. We begin by applying the law of total variance to decompose the variance under the larger adjustment set $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$:

$$\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)] = \text{Var}[\mathbb{E}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2]] \\ + \mathbb{E}\left[\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2]\right].$$

Now, under the conditional independence assumption $Y \perp\!\!\!\perp_\mathcal{G} \mathbf{B}_2 \mid \mathbf{G}_1, \mathbf{G}_2, A$, the IF based on the adjustment set $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ satisfies $\mathbb{E}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2] = \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2; P)$. Substituting this into the variance decomposition yields:

$$\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)] \\ = \text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2; P)] + \mathbb{E}\left[\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2]\right].$$

Since the conditional variance term is nonnegative, we conclude that $\sigma^2_{a,(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P) \geq \sigma^2_{a,(\mathbf{G}_1, \mathbf{G}_2)}(P)$, implying that $\mathbf{B}_2$ can be safely excluded without increasing the asymptotic variance.

Step 4: Pruning Redundant Mediating Variables from $\mathbf{Z}_1$. Finally, Lemma 4.7 (proof in Appendix C.4) shows that variables in $\mathbf{Z}_1 \setminus \mathbf{X}_{3 \in \text{Pa}(Y)}$ can be removed:

$$\sigma^2_{(\mathbf{X}_{3 \in \text{Pa}(Y)}, \mathbf{X}_{1 \in \text{Pa}(Y)} \cup \mathbf{X}_{4 \in \text{Pa}(Y)})}(P) \leq \sigma^2_{(\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \text{Pa}(Y)}, \mathbf{X}_{1 \in \text{Pa}(Y)} \cup \mathbf{X}_{4 \in \text{Pa}(Y)})}(P).$$

Therefore, the final adjustment set minimizing the asymptotic variance is given by

$$\mathbf{O} := \mathbf{X}_{1 \in \text{Pa}(Y)} \cup \mathbf{X}_{3 \in \text{Pa}(Y)} \cup \mathbf{X}_{4 \in \text{Pa}(Y)}.$$

Lemma 4.7 (Deletion of overadjustment mediator variables)*. Given a DAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, and the exposure-outcome pair $(A, Y) \subseteq \mathbf{V}$, suppose $(\mathbf{G}_1 \cup \mathbf{B}_1, \mathbf{G}_2) \subset \mathbf{V} \backslash \{A, Y\}$ is a VAS with $\mathbf{G}$ and $\mathbf{B}$ disjoint, and suppose $Y \perp\!\!\!\perp_\mathcal{G} \mathbf{B}_1 \mid \mathbf{G}_1, \mathbf{G}_2, A$. Then $(\mathbf{G}_1, \mathbf{G}_2)$ is also a VAS, and for all $P \in \mathcal{M}$, $\sigma^2_{(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)}(P) - \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2)}(P) \geq 0$.*

Figure 3 provides a simplified illustration of the conditions stated in Lemma 4.7. The proof follows the same argument as Lemma 4.6, with $\mathbf{B}_1$ playing the same role as $\mathbf{B}_2$.

5 Experiments

To evaluate the robustness of our method under finite-sample conditions, we conduct synthetic experiments to verify Lemmas 4.4-4.7, corresponding to Figures 2 and 3, and further demonstrate the practical stability of the O-set using real-world data.[4]

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[4]Code available at https://github.com/Lin-Ruiyang/WCDE-Simulation

AIPW Estimator   In our experiments, we implemented the AIPW estimator, constructed using the plug-in components of the influence function:

$$
\widehat{\text{WCDE}} = \frac{1}{2n}\sum_{i=1}^{n}\Bigg\{ \underbrace{\frac{\mathbb{I}\{A_i = a\}\cdot \hat{p}(\mathbf{Z}_{1i})\cdot \hat{p}(\mathbf{Z}_{2i})}{\hat{p}(\mathbf{Z}_{1i},\mathbf{Z}_{2i},a)}\cdot(Y_i - \hat{\mu}(a,\mathbf{Z}_{1i},\mathbf{Z}_{2i}))}_{\text{IPW term at level } a} + \underbrace{\sum_{\mathbf{z}_2}\hat{\mu}(a,\mathbf{Z}_{1i},\mathbf{z}_2)\cdot\hat{p}(\mathbf{z}_2)}_{\text{Marginal over } \mathbf{Z}_2 \text{ at } a}
$$

$$
+ \underbrace{\sum_{\mathbf{z}_1}\hat{\mu}(a,\mathbf{z}_1,\mathbf{Z}_{2i})\cdot\hat{p}(\mathbf{z}_1)}_{\text{Marginal over } \mathbf{Z}_1 \text{ at } a}\Bigg\} - \frac{1}{2n}\sum_{i=1}^{n}\Bigg\{ \underbrace{\frac{\mathbb{I}\{A_i = a^*\}\cdot \hat{p}(\mathbf{Z}_{1i})\cdot \hat{p}(\mathbf{Z}_{2i})}{\hat{p}(\mathbf{Z}_{1i},\mathbf{Z}_{2i},a^*)}\cdot(Y_i - \hat{\mu}(a^*,\mathbf{Z}_{1i},\mathbf{Z}_{2i}))}_{\text{IPW term at level } a^*}
$$

$$
+ \underbrace{\sum_{\mathbf{z}_2}\hat{\mu}(a^*,\mathbf{Z}_{1i},\mathbf{z}_2)\cdot\hat{p}(\mathbf{z}_2)}_{\text{Marginal over } \mathbf{Z}_2 \text{ at } a^*} + \underbrace{\sum_{\mathbf{z}_1}\hat{\mu}(a^*,\mathbf{z}_1,\mathbf{Z}_{2i})\cdot\hat{p}(\mathbf{z}_1)}_{\text{Marginal over } \mathbf{Z}_1 \text{ at } a^*}\Bigg\}. \tag{5.1}
$$

To estimate the conditional expectations $\hat{\mu}(a,\mathbf{Z}_1,\mathbf{Z}_2)$, we fit linear regressions with spline-transformed features (degree 5, 10 knots) for continuous outcomes, and use multinomial logistic regression for discrete outcomes. The treatment probabilities $\hat{p}(A = a \mid \mathbf{Z}_1,\mathbf{Z}_2)$ are obtained from logistic regressions. The marginal and joint densities are estimated nonparametrically, using empirical frequencies for discrete variables and Gaussian kernel density estimation for continuous ones.

Synthetic Experiments   Under Figures 2 and 3, data are generated from a nonlinear structural equation model with random edge coefficients, nonlinear transformations, and additive Gaussian noise. Appendix D.1 contains full details of the data-generating process and estimator implementation. For each adjustment set, we report average variance and average MSE across 50 replications. Table 2 top and bottom respectively illustrate the results associated with Lemmas 4.4 and 4.6, and Lemmas 4.5 and 4.7. Our proposed O-set $\{G_1,G_2\}$ consistently achieves the lowest variance and MSE across all sample sizes, ranging from $n = 250$ to $n = 2000$, demonstrating its finite-sample robustness despite our theoretical guarantees being asymptotic.

Table 2: Average Variance and MSE for Figures 2 and 3. Optimal VAS are highlighted in bold.

		$n = 250$		$n = 500$		$n = 1000$		$n = 2000$	
Figure	**Adjust. Set**	Var	MSE	Var	MSE	Var	MSE	Var	MSE
	$\mathbf{G_1, G_2}$	**0.0029**	**0.0030**	**0.0016**	**0.0016**	**0.0009**	**0.0009**	**0.0006**	**0.0006**
Fig 2	G_1, B_2	0.0102	0.0103	0.0110	0.0112	0.0101	0.0102	0.0075	0.0076
	G_1, G_2, B_2	0.0050	0.0051	0.0028	0.0029	0.0021	0.0022	0.0016	0.0016
	$\mathbf{G_1, G_2}$	**0.0027**	**0.0028**	**0.0015**	**0.0015**	**0.0010**	**0.0010**	**0.0006**	**0.0006**
Fig 3	B_1, G_2	0.0100	0.0101	0.0072	0.0073	0.0066	0.0067	0.0050	0.0050
	G_1, B_1, G_2	0.0059	0.0059	0.0040	0.0041	0.0040	0.0040	0.0029	0.0029

Table 3: Variance and MSE of WCDE estimates on the ASIA network under different sample sizes. Variable names are abbreviated as follows: b: bronc, s: smoke, l: lung.

	$n = 250$		$n = 500$		$n = 1000$		$n = 4000$		$n = 10000$	
Adj. Set	Var	MSE	Var	MSE	Var	MSE	Var	MSE	Var	MSE
[b]	**0.0143**	**0.0158**	**0.0049**	**0.0052**	**0.00266**	**0.00292**	**0.00062**	**0.00067**	**0.00027**	**0.00029**
$[s]$	0.0210	0.0238	0.00862	0.00911	0.00408	0.00454	0.00101	0.00104	0.00046	0.00054
$[b, s]$	0.0176	0.0229	0.00628	0.00728	0.00419	0.00480	0.00087	0.00091	0.00044	0.00054
$[b, l, s]$	0.0296	0.0528	0.0174	0.0231	0.0141	0.0153	0.00333	0.00335	0.00177	0.00194
$[b, l]$	0.0360	0.0540	0.0207	0.0248	0.0133	0.0139	0.00340	0.00342	0.00170	0.00183
$[l]$	0.0532	0.0624	0.0205	0.0215	0.0133	0.0137	0.00346	0.00347	0.00184	0.00193
$[l, s]$	0.0458	0.0613	0.0218	0.0240	0.0151	0.0157	0.00367	0.00368	0.00189	0.00204

Real-World Experiments   We evaluate our method on three widely used semi-synthetic Bayesian networks from `bnlearn`: **ASIA**, **SIGNALING**, and **MILDEW**, which serve as standard benchmarks for causal inference due to their realistic structures and domain relevance [24, 27, 47]. For each dataset, we validated our theoretical results through a five-step process: (1) selecting a treatment and outcome variable (for multi-valued treatments, the two most frequent levels are used to define a binary treatment); (2) identifying all valid adjustment sets for small/medium DAGs and a sufficiently large and representative subset for large DAGs; (3) estimating the WCDE via the AIPW estimator

(Eq. (5.1)); (4) simulating data from the underlying causal model across sample sizes $n = 250, 500, 1000, 4000,$ and 10000; and (5) computing the empirical variance and MSE for each adjustment set.

The empirical validation across three distinct networks consistently affirms the efficacy of our identification criterion. In the **ASIA network** (an 8-node 8-edge Bayesian network for medical diagnosis), our criterion identified **bronc** as the optimal set (O-set) for estimating the effect of a single composite variable—indicating the presence of either tuberculosis or lung cancer—on **dysp** (shortness of breath). As shown in Table 3, **bronc** attains the lowest variance and MSE across all sample sizes, consistent with our asymptotic theory and demonstrating robust finite-sample performance. In the **SIGNALING network** (a 11-node 17-edge DAG on protein signaling cascades), our criterion identifies **{PKA, Erk}** as the O-set for estimating the effect of **PKC** on **Akt**. Table 4 illustrates a nuanced bias-variance trade-off. For larger sample sizes ($n \geq 1000$), our O-set achieved the smallest variance among all adjustment sets, though its MSE was not the absolute lowest. When the sample size was 500 or smaller, the O-set, while not achieving the very lowest variance or MSE, delivered performance that was highly comparable to the best-performing sets, demonstrating its practical utility and robustness even in finite-sample regimes where strict optimality is not guaranteed. Similar to the ASIA network, in **MILDEW** (a 35-node 46-edge DAG on crop disease management), the identified O-set (**{meldug_3, middel_3, mikro_3}** for estimating the effect of **mikro_1** on **meldug_4**) achieves the smallest variance and MSE when $n \geq 500$, and remains among the best-performing sets for smaller samples (Tables D.2 and D.3 in Appendix D.2).

Table 4: Variance and MSE of WCDE estimates on the SIGNALING network under different sample sizes. Variable names are abbreviated as follows: e: Erk, p: PKA, m: Mek, r: Raf.

Adj. Set	$n = 250$		$n = 500$		$n = 1000$		$n = 4000$		$n = 10000$	
	Var	MSE	Var	MSE	Var	MSE	Var	MSE	Var	MSE
[**e**, **p**]	0.00491	0.00618	0.00269	0.00328	**0.00125**	**0.00134**	**0.00033**	**0.00036**	**0.00012**	**0.00012**
[e, p, r]	0.00483	0.00569	0.00293	0.00344	0.00156	0.00164	0.00041	0.00044	0.00019	0.00019
[e, m, p]	**0.00429**	**0.00476**	**0.00238**	**0.00260**	0.00142	0.00146	0.00062	0.00064	0.00020	0.00020
[e, m, p, r]	0.00456	0.00490	0.00272	0.00289	0.00159	0.00160	0.00065	0.00067	0.00020	0.00020
[m, p, r]	0.00843	0.01024	0.00486	0.00586	0.00383	0.00407	0.00205	0.00217	0.00105	0.00105
[m, p]	0.00752	0.00941	0.00408	0.00534	0.00351	0.00386	0.00201	0.00213	0.00108	0.00108

6 Discussion

We provide the first comprehensive framework for identifying and estimating the WCDE in observational settings, where *all* observed mediators are fixed. We leave the analysis of more complex WCDE parameters, where only a subset of mediators is fixed, to future work. WCDE introduces unique challenges stemming from the integration over mediator distributions and the dual role of mediators as both colliders and effect modifiers. To address these challenges, we first establish a graphical criterion that characterizes VASs for WCDE. Second, we derive the unique IF of WCDE under nonparametric models, enabling principled comparison of valid adjustment strategies. Third, we identify the optimal VAS that minimizes asymptotic variance, showing that it systematically differs from those used for ATE estimation due to mediator–confounder interactions. Together, our identification results and efficiency analysis lay the foundation for extending WCDE estimation to more complex settings where careful handling of mediator–confounder structure is essential.

While we assume that the underlying DAG is known and accurate, in practice, the DAG must often be learned from data, a challenge actively studied in the causal discovery literature. Global discovery algorithms can, in principle, recover DAGs from observational data, but they often suffer from poor finite-sample performance or exponential runtime in the worst case [15, 25]. As a computationally efficient alternative, local discovery algorithms focus on the relative relationships between exposure and outcome, bypassing the need to recover the full DAG [31, 32]. Maasch et al. [31] proposes a local discovery method tailored for WCDE estimation under suitable assumptions, showing polynomial runtime while allowing for unobserved confounders that do not directly affect the outcome. Our results further establish that the set returned by their procedure is asymptotically optimal in their problem setting. Another promising approach is to assume an *additive noise model*, under which global discovery algorithms are provably consistent in polynomial time [18–20], potentially enabling the efficient identification of adjustment sets. While these alternatives offer practical avenues for learning the O-set from data, a key open challenge is to understand how errors from causal discovery propagate into WCDE estimation. Developing robust frameworks that integrate structure learning with optimal adjustment set selection remains an important direction for future work.

Acknowledgments

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Appendix

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A   Definition, Notation and Graph Terminology

Definition A.1 (d-separation). *For disjoint vertex sets* $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{Z}$, *we say* $\mathbf{Z}$ *d-separates* $\mathbf{X}$ *from* $\mathbf{Y}$ *if every path between* $\mathbf{X}$ *and* $\mathbf{Y}$ *is blocked by* $\mathbf{Z}$. *A path is blocked by* $\mathbf{Z}$ *if it contains either:*

   (1) A chain $X_i \to Z_k \to X_j$ *or fork* $X_i \leftarrow Z_k \to X_j$ *with* $Z_k \in \mathbf{Z}$, *or*

   (2) A collider $X_i \to Z_k \leftarrow X_j$ *where* $Z_k \notin \mathbf{Z}$ *and none of its descendants are in* $\mathbf{Z}$, *i.e.,* $De(Z_k) \cap \mathbf{Z} = \emptyset$.

Definition A.2 (Path types). *To ensure consistency with the standard causal inference literature [34, 37], we define seven types of paths in a directed acyclic graph (DAG)* $\mathcal{G} = (\mathcal{V}, \mathbf{E})$ *connecting treatment* A *and outcome* Y:

   *(1) **Directed path:** A path in which all arrows point in the same direction, from the starting node to the end node. Example:* $A \to B \to C \to Y$.

   *(2) **Backdoor path:** Any undirected path between* A *and* Y *that starts with an arrow pointing into* A. *Example:* $A \leftarrow Z \to Y$.

   *(3) **Mediator path:** Any directed path from* A *to* Y *that passes through at least one mediator variable* $M \in \mathbf{M}$. *Example:* $A \to M \to Y$.

   *(4) **Causal path:** A path from* A *to* Y *in which all arrows are oriented away from* A *and toward* Y.

(5) ***Proper causal path and causal nodes:*** *A causal path is* proper *(with respect to A) if only its first node lies in A. The set of all nodes lying on proper causal paths from A to Y, excluding A itself, is called the set of* causal nodes *and denoted by* $\operatorname{cn}(A, Y, \mathcal{G})$.

(6) ***Non-causal path:*** *Any path between A and Y that is not causal, i.e., that starts with an arrow pointing into A.*

Intuitively, directed and causal paths transmit causal influence from A to Y, whereas backdoor and other non-causal paths represent spurious associations that must be blocked for unbiased causal estimation.

Definition A.3 (Identification of the controlled direct effect). *Following the notation of VanderWeele and Vansteelandt [59], let $Y(a, m)$ denote the potential outcome that would have been observed had the treatment been set to $A = a$ and the mediator to $M = m$. Let $\mathbf{C}$ denote a (possibly multivariate) set of pre-exposure covariates. Identification of the CDE relies on the following counterfactual independence assumptions:*

(1) *No unmeasured confounding of the treatment–outcome relationship:*

$$Y(a, m) \perp\!\!\!\perp A \mid C,$$

(2) *No unmeasured confounding of the mediator–outcome relationship:*

$$Y(a, m) \perp\!\!\!\perp M \mid A, C.$$

Under these assumptions, the controlled direct effect of changing A from a to a^* while holding M fixed at m is identified as

$$\begin{aligned}
\operatorname{CDE}(a, a^*; m) &\equiv \mathbb{E}[Y(a, m) - Y(a^*, m)] \\
&= \int \Big(\mathbb{E}[Y \mid A = a, M = m, C = c] - \mathbb{E}[Y \mid A = a^*, M = m, C = c] \Big) \, dF_C(c).
\end{aligned}$$
(A.1)

A.1 Sufficient and Necessary Conditions for CDE identification C2

To characterize the identification of the controlled direct effect (CDE), we first revisit the general conditions under which a post-intervention density involving the do-operator can be expressed as a function of observable conditional densities. These general graphical concepts—including adjustment sets, forbidden sets, and the adjustment criterion—provide the foundational framework for identifying causal effects such as the CDE.

A.1.1 Necessary and Sufficient Adjustment Criteria for Identifying do-probability

Following Pearl [34] and Perković et al. [37], we recall the key definitions and the associated soundness and completeness theorem.

Definition A.4 (Adjustment set [34, 37, Def. 54]). *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a causal DAG $\mathcal{G}$. Then Z is an* adjustment set *relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if, for any density f consistent with $\mathcal{G}$,*

$$f(\mathbf{y} \mid do(\mathbf{x})) = \begin{cases} f(\mathbf{y} \mid \mathbf{x}), & \text{if } \mathbf{Z} = \emptyset, \\ \int f(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) \, f(\mathbf{z}) \, d\mathbf{z}, & \text{otherwise.} \end{cases}$$

Definition A.5 (Forbidden set [37]). *Let $\mathbf{X}$ and $\mathbf{Y}$ be disjoint node sets in a causal DAG $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. The* forbidden set *relative to $(\mathbf{X}, \mathbf{Y})$ is defined as*

$$\operatorname{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \big\{ W' \in \mathbf{V} : W' \in \operatorname{De}(W, \mathcal{G}) \text{ for some } W \in \operatorname{cn}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \big\}.$$

That is, $\operatorname{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ contains all descendants of nodes that lie on proper causal paths from $\mathbf{X}$ to $\mathbf{Y}$. Variables in this set must not *be adjusted for, since conditioning on them may introduce post-treatment bias.*

Definition A.6 (Adjustment criterion [37, Def. 55]). *Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be pairwise disjoint node sets in a DAG $\mathcal{G}$. Let $\operatorname{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ denote the set of all descendants in $\mathcal{G}$ of any $W \notin \mathbf{X}$ that lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$. Then $\mathbf{Z}$ satisfies the* adjustment criterion *relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if the following two conditions hold:*

(A1) **Forbidden set condition:** $\mathbf{Z} \cap \mathrm{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) = \emptyset$;

(A2) **Blocking condition:** *all proper non-causal paths from* $\mathbf{X}$ *to* $\mathbf{Y}$ *in* $\mathcal{G}$ *are blocked by* $\mathbf{Z}$.

These two conditions together ensure that adjusting for $\mathbf{Z}$ *yields an unbiased estimate of the causal effect of* $\mathbf{X}$ *on* $\mathbf{Y}$.

Theorem A.7 (Soundness and completeness of the adjustment criterion [37, Thm. 56])**.** *Let* $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{Z}$ *be pairwise disjoint node sets in a causal DAG* $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. *Then* $\mathbf{Z}$ *satisfies the adjustment criterion (Definition A.6) if and only if* $\mathbf{Z}$ *is an adjustment set (Definition A.4).*

These definitions and Theorem A.7 jointly provide a complete graphical characterization for when a post-intervention quantity such as $f(\mathbf{y} \mid do(\mathbf{x}))$ can be expressed through integration over observed densities. That is, the adjustment criterion is both *necessary and sufficient* for identifying a single interventional distribution in a causal DAG.

A.1.2 CDE Adjustment Criteria

In the context of the controlled direct effect, this criterion (Def. A.6) remains *sufficient* for identification. Specifically, by letting $\mathbf{X}$ in Definition A.4 denote the union of the treatment and mediator sets, $\mathbf{X} = (A, \mathbf{M})$, and by choosing $\mathbf{Z}$ that satisfies the conditions in Definition A.6, we can, according to Theorem A.7, identify the post-intervention distribution

$$f(y \mid do(a, \mathbf{m})) = \int f(y \mid a, \mathbf{m}, \mathbf{z}) \, f(\mathbf{z}) \, d\mathbf{z},$$

where the integration is performed over the adjustment variables $\mathbf{Z}$. Hence, each component $\mathbb{E}[Y(a, \mathbf{m})] = \int y \, f(y \mid do(a, \mathbf{m})) \, dy$ is identifiable, and consequently, the difference $\mathbb{E}[Y(a, \mathbf{m})] - \mathbb{E}[Y(a^*, \mathbf{m})]$ that defines the CDE is also identifiable.

Nevertheless, it is important to note that the CDE involves the *difference between two interventional expectations*, $\mathbb{E}[Y(a, \mathbf{m})] - \mathbb{E}[Y(a^*, \mathbf{m})]$. Because of this differencing structure, the adjustment criterion serves as a *sufficient but not necessary* condition: in certain causal graphs, neither $f(y \mid do(a, \mathbf{m}))$ nor $f(y \mid do(a^*, \mathbf{m}))$ is individually identifiable, yet their difference—the CDE—can still be identified because non-identifiable components cancel out when taking the contrast. We will illustrate such cases in Figures A.1 and A.2.

In parallel, VanderWeele [58] proposed another set of *sufficient* conditions specifically tailored to the identification of the CDE. Unlike the adjustment criterion, which applies to arbitrary causal effects defined by a single do-operator, these conditions explicitly target the treatment–mediator–outcome structure and are formulated in terms of conditional independence among potential outcomes:

(B1) The potential outcomes $Y(a, m)$ are conditionally independent of A given $\mathbf{Z}$:

$$Y(a, m) \perp\!\!\!\perp A \mid \mathbf{Z}, \quad \text{for all } a, m;$$

(B2) The potential outcomes $Y(a, m)$ are conditionally independent of M given $(A, \mathbf{Z})$:

$$Y(a, m) \perp\!\!\!\perp M \mid A, \mathbf{Z}, \quad \text{for all } a, m.$$

These conditions ensure that the CDE, defined as $\mathbb{E}[Y(a, m)] - \mathbb{E}[Y(a^*, m)]$, can be consistently estimated from observed data. Nevertheless, similar to the adjustment criterion, they only guarantee sufficiency rather than necessity.

Counterexamples to Necessity. Next, we provide counterexamples to illustrate that Conditions (A1), (A2), and (B2) are all *not necessary* for CDE's identification. Consider the DAGs in Figure A.1 and Figure A.2. In the first DAG, the empty set $\emptyset$ identifies $\mathrm{CDE}(m)$ even though it does not contain X_1, which lies on a backdoor path between M and Y, thus violating Conditions (A2) and (B2). Since there is no direct arrow from A to Y, the controlled direct effect is zero, i.e., $\mathrm{CDE}(m) = 0$. When the adjustment set is empty, Equation A.1 reduces to

$$E(Y \mid A = a, m) - E(Y \mid A = a^*, m) = E(Y \mid m) - E(Y \mid m) = 0,$$

which exactly matches the true value of $\mathrm{CDE}(m)$. Hence, the empty set is sufficient to identify $\mathrm{CDE}(m)$ in this graph.

In the second DAG, the set $\{X_2\}$ identifies $\mathrm{CDE}(m)$ even though $X_2 \in \mathrm{Forb}(A, Y, \mathcal{G})$, violating Condition (A1). Since there is again no direct arrow from A to Y, the controlled direct effect equals zero, i.e., $\mathrm{CDE}(m) = 0$. When adjusting for X_2, Equation A.1 becomes

$$E(Y \mid A = a, m, X_2) - E(Y \mid A = a^*, m, X_2) = E(Y \mid m, X_2) - E(Y \mid m, X_2) = 0,$$

which coincides with the true value of $\mathrm{CDE}(m)$. Therefore, the set $\{X_2\}$ is also sufficient to identify $\mathrm{CDE}(m)$ in this graph.

Figure A.1: Example illustrating Condition (A2) and (B2) is not necessary for CDE identification.
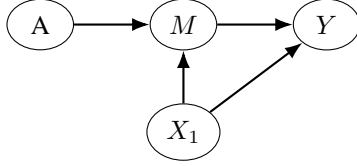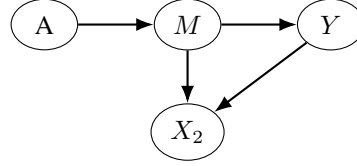
Figure A.2: Example illustrating Condition (A1) is not necessary for CDE identification.

These examples demonstrate that both frameworks—the adjustment criterion for the *do*-operator and VanderWeele's conditions for the CDE—are *sufficient but not necessary*. A complete graphical characterization that is both necessary and sufficient for CDE identification has yet to be established. Given that our primary focus is on the WCDE, we leave this theoretical development for future work.

A.2 Definitions of Causal Partition Introduced by Maasch et al. [32]

A complete description of the eight causal partitions is included in Table A.1.

EXHAUSTIVE AND MUTUALLY EXCLUSIVE PARTITIONS
$\mathbf{X}_1$ *Confounders and their proxies*: Non-descendants of X that lie on an active backdoor path between A and Y (Definition A.2), and their proxies (Definition A.8).
$\mathbf{X}_2$ *Colliders and their proxies*: Non-ancestors of $\{A, Y\}$ with at least one active path to X not mediated by Y and at least one active path to Y not mediated by A.
$\mathbf{X}_3$ *Mediators and their proxies*: Descendants of A that are ancestors of Y, and their proxies (Definition A.8).
$\mathbf{X}_4$ Non-descendants of Y that are marginally dependent on Y but marginally independent of A (Definition A.9).
$\mathbf{X}_5$ *Instruments and their proxies*: Non-descendants of A whose causal effect on Y is fully mediated by A, and that share no confounders with Y (Definition A.10).
$\mathbf{X}_6$ Descendants of Y where all active paths shared with A are mediated by Y.
$\mathbf{X}_7$ Descendants of A where all active paths shared with Y are mediated by A.
$\mathbf{X}_8$ All nodes that share no active paths with A nor Y.

Table A.1: Partitions are formally defined by the path combinations enumerated in Table 3 of Maasch et al. [32].

Definition A.8 (Proxy variables in $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, Maasch et al. 32). *A proxy variable for $\mathbf{X}_1$, $\mathbf{X}_2$, or $\mathbf{X}_3$ is a member of these partitions that is an ancestor or descendant of another member of its respective partition, such that the proxy is not strictly a confounder, mediator, or collider, but still satisfies the allowable path types for its respective partition (as defined in Maasch et al. [32, Table 3]). This includes $\mathbf{X}_3$ that are descended from $\mathbf{X}_3$ that lie on mediator chains, $\mathbf{X}_1$ that are ancestral to $\mathbf{X}_1$ on backdoor paths, etc. ([32, Figure B.3]).*

Definition A.9 (Partition $\mathbf{X}_4$, Maasch et al. [32]). *Partition $\mathbf{X}_4$ encompasses all non-descendants of Y that are marginally dependent on Y but marginally independent of A (Table A.1). Given this definition, we observe that any X_4 participates in a v-structure $A \cdots \to Y \leftarrow \cdots X_4$. This implies the following:*

1. *A cannot share active paths with any X_4. Thus, A can share no common causes with any X_4.*
2. *$\mathbf{X}_4$ is conditionally dependent on A given Y. This implicitly requires that A and Y are marginally dependent, though they may not be directly adjacent in $\mathcal{G}$.*

Definition A.10 (Instrumental variable, Lousdal [30]). *Any instrument X_5 meets the following criteria:*

1. *Relevance assumption: X_5 is causal for exposure A.*
2. *Exclusion restriction: The effect of instrument X_5 on outcome Y is fully mediated by A.*
3. *Exchangeability assumption: X_5 and Y do not share a common cause.*

B   Proofs and Extended Discussions for Sections 2 and 3

In this section, we provide the technical details and proofs omitted from Sections 2 and 3 of the main text. We begin by establishing the sufficiency and necessity of the graphical adjustment criteria for identifying the WCDE, as formalized in Lemma 2.6. We then proceed to elaborate on semiparametric foundations, including regular submodels, score functions, tangent spaces, and pathwise differentiability, which lay the groundwork for deriving the influence function of WCDE. The section concludes with a detailed proof of Theorem 3.4, which characterizes the influence function of WCDE in terms of its identifying functional.

B.1   Proof of Lemma 2.6

Proof of Lemma 2.6. **Sufficiency of the Graphical Criteria.** We demonstrate that any valid adjustment set $\mathbf{Z}$ satisfying Condition 2.5 identifies the same WCDE (Equation (2.1)) through two expectation-preserving transformations. The key insight is that the conditional independence in Criteria 2.5 allows us to safely incorporate the target mediators $\mathbf{M}'$ into the adjustment set without changing the identified effect. Let $\mathbf{Z_1} = \mathbf{Z} \cap \mathbf{M}, \mathbf{Z_2} = \mathbf{Z} \setminus \mathbf{M}, \mathbf{C} = \mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup \{A\})$.

Step 1: Replacing $\mathbf{Z}_2$ with C Including Non-Mediating Parents of Y   Since $\mathbf{C}$ satisfies the graphical conditions for identifying the CDE with mediator $\mathbf{Z}_1$ (Definition A.3), yielding:

$$\mathbb{E}_{\mathbf{Z}_1}\{\mathbb{E}_{\mathbf{Z}_2}[\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2)]\} - \mathbb{E}_{\mathbf{Z}_1}\{\mathbb{E}_{\mathbf{Z}_2}[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{Z}_2)]\}$$
$$= \mathbb{E}_{\mathbf{Z}_1}\Big(\mathbb{E}[Y \mid \mathrm{do}(A = a, \mathbf{Z}_1)] - \mathbb{E}[Y \mid \mathrm{do}(A = a^*, \mathbf{Z}_1)]\Big)$$
$$= \mathbb{E}_{\mathbf{Z}_1}\{\mathbb{E}_{\mathbf{C}}[\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{C})]\} - \mathbb{E}_{\mathbf{Z}_1}\{\mathbb{E}_{\mathbf{C}}[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{C})]\}.$$

Step 2: Incorporating Mediators in $\mathbf{M}'$ Define $\mathbf{Z}_1' = \mathbf{Z}_1 \cup \mathbf{M}'$. We now show that augmenting the conditioning set with $\mathbf{M}' \setminus \mathbf{Z}_1$ does not change the outer expectation over $\mathbf{Z}_1$. Let $\mathbf{W} = \mathbf{M}' \setminus \mathbf{Z}_1$. so we have:

$$E_{\mathbf{Z}_1}\{\mathbb{E}_{\mathbf{C}}[\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{C})]\} - \mathbb{E}_{\mathbf{Z}_1}\{\mathbb{E}_{\mathbf{C}}[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{C})]\}$$
$$= \mathbb{E}_{\mathbf{C}}\{\mathbb{E}_{\mathbf{Z}_1}(\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{C}))\} - \mathbb{E}_{\mathbf{C}}\{\mathbb{E}_{\mathbf{Z}_1}(\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{C}))\}$$
$$= \mathbb{E}_{\mathbf{C}}\{\mathbb{E}_{\mathbf{Z}_1}[\mathbb{E}_{\mathbf{W}|\mathbf{Z}_1, \mathbf{C}, A}(\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{W}, \mathbf{C}))]\} - \mathbb{E}_{\mathbf{C}}\{\mathbb{E}_{\mathbf{Z}_1}[\mathbb{E}_{\mathbf{W}|\mathbf{Z}_1, \mathbf{C}, A}(\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{W}, \mathbf{C}))]\}$$
$$= \mathbb{E}_{\mathbf{C}}\{\mathbb{E}_{\mathbf{Z}_1}[\mathbb{E}_{\mathbf{W}|\mathbf{Z}_1}(\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{W}, \mathbf{C}))]\} - \mathbb{E}_{\mathbf{C}}\{\mathbb{E}_{\mathbf{Z}_1}[\mathbb{E}_{\mathbf{W}|\mathbf{Z}_1}(\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{W}, \mathbf{C}))]\}$$
$$= \mathbb{E}_{\mathbf{C}}\{\mathbb{E}_{\mathbf{Z}_1, \mathbf{w}}[\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{W}, \mathbf{C})]\} - \mathbb{E}_{\mathbf{C}}\{\mathbb{E}_{\mathbf{Z}_1, \mathbf{w}}[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{W}, \mathbf{C})]\}$$
$$= \mathbb{E}_{\mathbf{Z}_1, \mathbf{w}}\{\mathbb{E}_{\mathbf{C}}[\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{W}, \mathbf{C})]\} - \mathbb{E}_{\mathbf{Z}_1, \mathbf{w}}\{\mathbb{E}_{\mathbf{C}}[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{W}, \mathbf{C})]\}$$
$$= \mathbb{E}_{\mathbf{Z}_1'}\{\mathbb{E}_{\mathbf{C}}[\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1', \mathbf{C})]\} - \mathbb{E}_{\mathbf{Z}_1'}\{\mathbb{E}_{\mathbf{C}}[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1', \mathbf{C})]\}.$$

The third equality follows from $\mathbf{W} \perp\!\!\!\perp_{\mathcal{G}} (\mathbf{C}, A) \mid \mathbf{Z}_1$ in Criterion C3 of Condition 2.5, which implies $p(\mathbf{w} \mid \mathbf{Z}_1, \mathbf{C}, A) = p(\mathbf{w} \mid \mathbf{Z}_1)$; the fourth uses the tower property, and the last merges $\mathbf{Z}_1$ and $\mathbf{W}$ into $\mathbf{Z}_1' = \mathbf{Z}_1 \cup \mathbf{W}$.

Step 3: Simplification to WCDE Then we remove elements in $\mathbf{Z}_1'$ that are not in $\mathrm{Pa}(Y)$ and show that it is equal to Equation (2.1).

$$\mathbb{E}_{\mathbf{Z}_1'} \left\{ \mathbb{E}_{\mathbf{C}} \left[\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1', \mathbf{C}) \right] \right\} - \mathbb{E}_{\mathbf{Z}_1'} \left\{ \mathbb{E}_{\mathbf{C}} \left[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1', \mathbf{C}) \right] \right\}$$

$$= \mathbb{E}_{\mathbf{C}} \left\{ \mathbb{E}_{\mathbf{Z}_1 \cup \mathbf{M}'} \left[\mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1 \cup \mathbf{M}', \mathbf{C}) \right] \right\} - \mathbb{E}_{\mathbf{C}} \left\{ \mathbb{E}_{\mathbf{Z}_1 \cup \mathbf{M}'} \left[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{Z}_1 \cup \mathbf{M}', \mathbf{C}) \right] \right\}$$

$$= \mathbb{E}_{\mathbf{C}} \left\{ \mathbb{E}_{\mathbf{Z}_1 \cup \mathbf{M}'} \left[\mathbb{E}_Y(Y \mid A = a, \mathbf{M}', \mathbf{C}) \right] \right\} - \mathbb{E}_{\mathbf{C}} \left\{ \mathbb{E}_{\mathbf{Z}_1 \cup \mathbf{M}'} \left[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{M}', \mathbf{C}) \right] \right\}$$

$$= \mathbb{E}_{\mathbf{C}} \left\{ \mathbb{E}_{\mathbf{M}'} \left[\mathbb{E}_Y(Y \mid A = a, \mathbf{M}', \mathbf{C}) \right] \right\} - \mathbb{E}_{\mathbf{C}} \left\{ \mathbb{E}_{\mathbf{M}'} \left[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{M}', \mathbf{C}) \right] \right\}$$

$$= \mathbb{E}_{\mathbf{M}'} \left\{ \mathbb{E}_{\mathbf{C}} \left[\mathbb{E}_Y(Y \mid A = a, \mathbf{M}', \mathbf{C}) \right] \right\} - \mathbb{E}_{\mathbf{M}'} \left\{ \mathbb{E}_{\mathbf{C}} \left[\mathbb{E}_Y(Y \mid A = a^*, \mathbf{M}', \mathbf{C}) \right] \right\}$$

$$= \mathbb{E}_{\mathbf{M}'} \Big(\mathbb{E}[Y \mid \mathrm{do}(A = a, \mathbf{M}')] - \mathbb{E}[Y \mid \mathrm{do}(A = a^*, \mathbf{M}')] \Big)$$

$$= \sum_{\mathbf{m}' \in \mathbf{M}'} \left(\mathbb{E}\left[Y \mid \mathrm{do}\,(a, \mathbf{m}') \right] - \mathbb{E}\left[Y \mid \mathrm{do}\,(a^*, \mathbf{m}') \right] \right) p\,(\mathbf{m}')$$

$$= \mathrm{WCDE}$$

The first equality expands the conditioning sets to include $\mathbf{M}'$; the second removes non-parent variables in $\mathbf{Z}_1'$ without changing the expectation (all parents of Y are already conditioned on); the third marginalizes out $\mathbf{Z}_1$; the fourth swaps the order of integration over $\mathbf{C}$ and $\mathbf{M}'$; and the fifth uses the CDE identification with mediator $\mathbf{M}'$: since $\mathbf{C}$ satisfies the graphical conditions for identifying the CDE with mediator $\mathbf{M}'$. This yields the WCDE in Equation (2.1).

Necessity of the Graphical Criteria We now show that the adjustment criteria are not only sufficient but also necessary. By definition, The necessity of Condition C1 follows directly from the definition of the CDE, which requires blocking all mediator paths between A and Y. By definition, Condition C2 is necessary (we discuss specific conditions that could lead to necessary and sufficient VAS for CDE in Section A.1). We therefore focus on demonstrating that the **third adjustment criterion** is also necessary for a given set $\mathbf{Z}$ to qualify as a valid adjustment set for the WCDE.

Assume we have an adjustment set $\mathbf{Z}$ that satisfies the first three criteria, but violates the third criterion with respect to (A, Y) in graph $\mathcal{G}$. Define the following subsets:

$$\mathbf{Z}_1 = \mathbf{Z} \cap \mathbf{M}, \quad \mathbf{Z}_2 = \mathbf{Z} \setminus \mathbf{M}, \quad \mathbf{W} = \mathbf{M}' \setminus \mathbf{Z}_1, \quad \mathbf{C} = \mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup \{A\}).$$

Note that $\mathrm{Pa}(Y) \setminus \{A\}$ is a valid adjustment set for identifying WCDE. We compare $\mathrm{WCDE}_{\mathrm{Pa}(Y) \setminus A}$ with $\mathrm{WCDE}_{\mathbf{Z}}$ and consider their difference:

$$\mathrm{WCDE}_{\mathrm{Pa}(Y) \setminus A} - \mathrm{WCDE}_{\mathbf{Z}} = \left(T_a(\mathrm{Pa}(Y) \setminus A) - T_a(\mathbf{Z}) \right) - \left(T_{a^*}(\mathrm{Pa}(Y) \setminus A) - T_{a^*}(\mathbf{Z}) \right).$$

We first analyze the term $T_a(\mathrm{Pa}(Y) \setminus \{A\}) - T_a(\mathbf{Z})$:

$$T_a(\mathrm{Pa}(Y) \setminus A) - T_a(\mathbf{Z})$$
$$= \mathbb{E}_{\mathbf{M}'} \{ \mathbb{E}_{\mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)} \{ \mathbb{E}_Y(Y \mid A = a, \mathbf{M}', \mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)) \} \}$$
$$\quad - \mathbb{E}_{\mathbf{Z}_1} \{ \mathbb{E}_{\mathbf{Z}_2} \{ \mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2) \} \}$$
$$= \mathbb{E}_{\mathbf{M}'} \{ \mathbb{E}_{\mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)} \{ \mathbb{E}_Y(Y \mid A = a, \mathbf{M}', \mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)) \} \}$$
$$\quad - \mathbb{E}_{\mathbf{Z}_1} \{ \mathbb{E}_{\mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)} \{ \mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)) \} \}$$
$$= \mathbb{E}_{\mathbf{M}' \cup \mathbf{Z}_1} \{ \mathbb{E}_{\mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)} \{ \mathbb{E}_Y(Y \mid A = a, \mathbf{M}' \cup \mathbf{Z}_1, \mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)) \} \}$$
$$\quad - \mathbb{E}_{\mathbf{Z}_1} \{ \mathbb{E}_{\mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)} \{ \mathbb{E}_Y(Y \mid A = a, \mathbf{Z}_1, \mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup A)) \} \}$$

The second equality holds because both $\mathrm{Pa}(Y) \setminus (\mathbf{M}' \cup \{A\})$ and $\mathbf{Z}_2$ satisfy the graphical conditions for identifying the CDE with mediator $\mathbf{M}'$. The third equality follows from the fact that all parents of Y are already included in the conditioning set; thus, adding elements from $\mathbf{Z}_1$ does not affect the conditional expectation of Y.

To simplify further, we translate these expectations into integrals:

$$\mathbb{E}_{\mathbf{M}' \cup \mathbf{Z_1}} \{ \mathbb{E}_{\mathrm{Pa}(Y) \backslash (\mathbf{M}' \cup A)} \{ \mathbb{E}_Y (Y \mid A = a, \mathbf{M}' \cup \mathbf{Z_1}, \mathrm{Pa}(Y) \backslash (\mathbf{M}' \cup A)) \} \}$$

$$- \mathbb{E}_{\mathbf{Z_1}} \{ \mathbb{E}_{\mathrm{Pa}(Y) \backslash (\mathbf{M}' \cup A)} \{ \mathbb{E}_Y (Y \mid A = a, \mathbf{Z_1}, \mathrm{Pa}(Y) \backslash (\mathbf{M}' \cup A)) \} \}$$

$$= \iint_{\mathbf{W}, \mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int y \, p(y \mid A = a, \mathbf{c}, \mathbf{w}, \mathbf{z}_1) \, dy \right) p(\mathbf{c}) d\mathbf{c} \right) p(\mathbf{w} \mid \mathbf{z_1}) p(\mathbf{z_1}) d\mathbf{w} d\mathbf{z_1}$$

$$- \int_{\mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int y \, p(y \mid A = a, \mathbf{c}, \mathbf{z}_1) \, dy \right) p(\mathbf{c}) d\mathbf{c} \right) p(\mathbf{z_1}) d\mathbf{z_1}$$

$$= \int_{\mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int y \left(\int p(y \mid A = a, \mathbf{c}, \mathbf{w}, \mathbf{z}_1) p(\mathbf{w} \mid \mathbf{z}_1) d\mathbf{w} \right) dy \right) p(\mathbf{c}) d\mathbf{c} \right) p(\mathbf{z_1}) d\mathbf{z_1}$$

$$- \int_{\mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int y \, p(y \mid A = a, \mathbf{c}, \mathbf{z}_1) \, dy \right) p(\mathbf{c}) d\mathbf{c} \right) p(\mathbf{z_1}) d\mathbf{z_1}$$

$$= \int_{\mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int y \left(\int p(y \mid A = a, \mathbf{c}, \mathbf{w}, \mathbf{z}_1) [p(\mathbf{w} \mid \mathbf{z}_1) - p(\mathbf{w} \mid A = a, \mathbf{c}, \mathbf{z}_1)] d\mathbf{w} \right) dy \right) p(\mathbf{c}) d\mathbf{c} \right) p(\mathbf{z_1}) d\mathbf{z_1}$$

$$= \int y \int_{\mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int p(y \mid A = a, \mathbf{c}, \mathbf{w}, \mathbf{z_1}) [p(\mathbf{w} \mid \mathbf{z}_1) - p(\mathbf{w} \mid A = a, \mathbf{z}_1, \mathbf{c})] p(\mathbf{c}) p(\mathbf{z_1}) d\mathbf{w} \right) d\mathbf{c} \right) d\mathbf{z_1} dy$$

The first equality expands the nested expectations using the law of total expectation, with integration over $p(\mathbf{w}|\mathbf{z}_1) p(\mathbf{z}_1)$ and $p(\mathbf{c})$. The second applies Fubini's theorem to switch the order of integration. The third rewrites the inner integral to highlight the difference. Similarly,

$$T_{a^*}(\mathrm{Pa}(Y) \backslash A) - T_{a^*}(\mathbf{Z})$$

$$= \int y \int_{\mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int p(y \mid A = a^*, \mathbf{c}, \mathbf{w}, \mathbf{z_1}) [p(\mathbf{w} \mid \mathbf{z_1}) - p(\mathbf{w} \mid A = a^*, \mathbf{z_1}, \mathbf{c})] p(\mathbf{c}) p(\mathbf{z_1}) d\mathbf{w} \right) d\mathbf{c} \right) d\mathbf{z_1} dy$$

Subtracting the two:

$$\mathrm{WCDE}_{\mathrm{Pa}(Y) \backslash A} - \mathrm{WCDE}_{\mathbf{Z}} = (T_a(\mathrm{Pa}(Y) \backslash A) - T_a(\mathbf{Z})) - (T_{a^*}(\mathrm{Pa}(Y) \backslash A) - T_{a^*}(\mathbf{Z}))$$

$$= \int y \int_{\mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int p(y \mid A = a, \mathbf{c}, \mathbf{w}, \mathbf{z_1}) [p(\mathbf{w} \mid \mathbf{z}_1) - p(\mathbf{w} \mid A = a, \mathbf{z}_1, \mathbf{c})] p(\mathbf{c}) p(\mathbf{z_1}) d\mathbf{w} \right) d\mathbf{c} \right) d\mathbf{z_1} dy$$

$$- \int y \int_{\mathbf{Z_1}} \left(\int_{\mathbf{C}} \left(\int p(y \mid A = a^*, \mathbf{c}, \mathbf{w}, \mathbf{z_1}) [p(\mathbf{w} \mid \mathbf{z}_1) - p(\mathbf{w} \mid A = a^*, \mathbf{z_1}, \mathbf{c})] p(\mathbf{c}) p(\mathbf{z_1}) d\mathbf{w} \right) d\mathbf{c} \right) d\mathbf{z_1} dy$$

We can view $\int p(y \mid A = a, \mathbf{c}, \mathbf{w}, \mathbf{z}_1) \, d\mathbf{w} \, d\mathbf{c} \, d\mathbf{z}_1$ and $\int p(y \mid A = a^*, \mathbf{c}, \mathbf{w}, \mathbf{z}_1) \, d\mathbf{w} \, d\mathbf{c} \, d\mathbf{z}_1$ as two operators that map probability distributions over $(\mathbf{C}, \mathbf{W}, \mathbf{Z}_1)$ onto probability distributions over Y. Arrange $p(y \mid A = a, \mathbf{c}, \mathbf{w}, \mathbf{z}_1)$ and $p(y \mid A = a^*, \mathbf{c}, \mathbf{w}, \mathbf{z}_1)$ so that this mapping is one-to-one. Then, to ensure equivalence between the induced distributions, we would require

$$p(\mathbf{w} \mid \mathbf{z}_1) p(\mathbf{c}) p(\mathbf{z}_1) = p(\mathbf{w} \mid A = a, \mathbf{z}_1, \mathbf{c}) p(\mathbf{c}) p(\mathbf{z}_1) = p(\mathbf{w} \mid A = a^*, \mathbf{z}_1, \mathbf{c}) p(\mathbf{c}) p(\mathbf{z}_1)$$

to hold for all models consistent with $\mathbf{G}$. However, this equality generally fails because $\mathbf{W} \not\perp\!\!\!\perp_{\mathcal{G}} (A, \mathbf{C}) \mid \mathbf{Z}_1$ when the third adjustment criterion is violated. Consequently, the integrand does not vanish in general, and the resulting functional WCDE_Z deviates from the true WCDE. We therefore conclude that the third adjustment criterion is necessary for a given $\mathbf{Z}$ to qualify as a valid adjustment set.

□

B.2 Illustration of Criterion C3

Example B.1 (Illustration of Criterion C3). *In Figure B.3, $\mathbf{M} = \{G_1, B_1\}$, $\mathrm{Pa}(Y) = \{A, G_1, G_2\}$, and $\mathbf{M}' = \{G_1\}$. One can verify that the sets $\{G_1, G_2\}$ and $\{G_1, B_1, G_2\}$ both satisfy Condition 2.5, and are therefore valid adjustment sets.*

In contrast, the set $\mathbf{Z} = \{B_1, G_2\}$ satisfies Criteria C1-C2, but violates the Criterion C3. In this case, $\mathbf{Z}_1 = \{B_1\}$, $\mathbf{Z}_2 = \{G_2\}$, $\mathbf{M}' \backslash \mathbf{Z}_1 = \{G_1\}$, and $\mathrm{Pa}(Y) \backslash \mathbf{M}' = \{A, G_2\}$. In Figure B.3, we
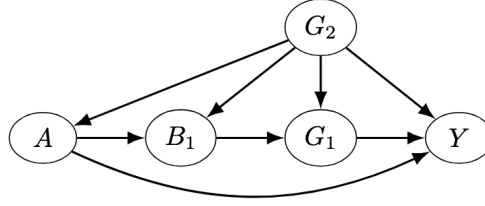
Figure B.3: A DAG illustrating Condition 2.5.

observe that when conditioned on G_1, B_1 is dependent on G_2 due to the direct edge between them; similarly for A. Thus $G_1 \not\perp\!\!\!\perp_\mathcal{G} G_2, A \mid B_1$, violating Criterion C3. One could further verify that $WCDE_{\{B_1, G_2\}} \neq WCDE^* = WCDE_{\{G_1, G_2\}}$ as defined by Equation (2.2). To see this, we compare the difference between the first term in $WCDE_{\{B_1, G_2\}}$ and that in $WCDE_{\{G_1, G_2\}}$:

$$T_a(\{B_1, G_2\}) - T_a(\{G_1, G_2\})$$
$$= \iint \left(\int y \left[\int p(y \mid A = a, g_1, b_1, g_2)(p(g_1|b_1, g_2, a) - p(g_1|b_1))dg_1 \right] dy \right) p(g_2)\, dg_2\, p(b_1)\, db_1.$$

Because $G_1 \not\perp\!\!\!\perp_\mathcal{G} G_2, A \mid B_1$, there exists a data generating process under which $p(g_1|b_1, g_2, a) \neq p(g_1|b_1)$ the integrand does not necessarily vanish (detailed construction in proof of Lemma 2.6), and the resulting functional based on $\{B_1, G_2\}$ may differ from the true WCDE as defined. A similar argument can be made on the difference of the second term $T_{a^*}(\{B_1, G_2\}) - T_{a^*}(\{G_1, G_2\})$. Therefore, the WCDE cannot be identified using $\mathbf{Z} = \{B_1, G_2\}$.

B.3 Discussions on Influence Functions

For a more detailed review of influence functions and their role in causal inference, we refer the reader to Hines et al. [17], Kennedy [26], and Tsiatis [54]. Here, we focus on definitions relevant to our derivations in Section 3.

To formally introduce nonparametric estimation, we begin by considering a general statistical model $\mathcal{M}$, which is a collection of candidate distributions defined on a measurable space $(\Omega, \mathcal{F})$. Each element $P \in \mathcal{M}$ admits a Radon–Nikodym derivative $p = dP/d\nu$ with respect to a dominating σ-finite measure ν, satisfying $p \geq 0$ ν-a.e. and $\int_\Omega p\, d\nu = 1$. With slight abuse of notation, we use P and p interchangeably to refer to the distribution and its density.

To prepare for the proof of Theorem 3.4, we begin by recalling several foundational concepts from semiparametric theory, including regular parametric submodels, score functions, tangent spaces, and pathwise differentiability. These concepts provide a principled framework for defining and deriving influence functions. To rigorously understand and formalize the sensitivity discussed in Remark 3.3, we also introduce the definition of influence functions via local distributional perturbations. This alternative characterization is particularly useful for interpreting the robustness and local sensitivity of statistical functionals to contamination at individual points.

Tangent Space, Pathwise Differentiability, and Influence Function.

We now introduce key concepts from the local geometry of the model $\mathcal{M}$ that play a central role in semiparametric estimation theory. We proceed step-by-step, starting with the notion of regular parametric submodels, which formalize how a nonparametric distribution can be locally perturbed in a smooth manner.

Definition B.2 (Regular One-dimensional Submodel, Bickel et al. 5). *Let $\mathcal{M}$ be a nonparametric model. A regular one-dimensional submodel $\{P_t\}_{t \in (-\varepsilon, \varepsilon)} \subset \mathcal{M}$ passing through P at $t = 0$ is a smooth parametric path such that $P_0 = P$ and the corresponding density p_t with respect to a dominating measure ν is differentiable in t at $t = 0$, with the derivative satisfying*

$$\left. \frac{\partial}{\partial t} \log p_t(O) \right|_{t=0} \in L_2(P).$$

The derivative of the log-density along a submodel is referred to as the *score function* and represents the direction of infinitesimal perturbation in the model:

Definition B.3 (Score Function, van der Vaart 56)**.** *For a regular one-dimensional submodel $\{P_t\}$ with density p_t, the* score function *at P is defined as*

$$S(O) := \left. \frac{\partial}{\partial t} \log p_t(O) \right|_{t=0},$$

which describes the direction of local perturbation of P along the submodel.

A commonly used example of a regular submodel is constructed by directly specifying a score function S and perturbing the baseline density p as follows:

$$p_t(o) = (1 + tS(o))p(o),$$

where the score function satisfies $\int S(o)\, p(o)\, d\nu(o) = 0$ and $\int S^2(o)\, p(o)\, d\nu(o) < \infty$. One can verify that this path satisfies $P_0 = P$ and that its score function is

$$\left. \frac{\partial}{\partial t} \log p_t(o) \right|_{t=0} = \left. \frac{\partial}{\partial t} \log(1 + tS(o)) \right|_{t=0} = S(o).$$

Such submodels are particularly useful for constructing influence functions and deriving efficiency bounds in semiparametric models.

The set of all such score functions defines the tangent space, which describes the directions in which the model $\mathcal{M}$ can be perturbed:

Definition B.4 (Tangent Space, van der Vaart 56)**.** *The* tangent space *at P is the closed linear subspace of $L_2(P)$ spanned by all score functions $S(O)$ generated by regular one-dimensional submodels through P. It characterizes the set of directions in which the distribution P can be locally perturbed.*

We now turn to a central concept in semiparametric theory: pathwise differentiability. This notion formalizes when a parameter of interest can be linearly approximated along such perturbations, and leads directly to the definition of the influence function:

Definition B.5 (Pathwise Differentiability and Influence Function, Tsiatis 54)**.** *A parameter T is said to be* pathwise differentiable *at P if there exists a function $\psi_P \in L_2(P)$ with $\mathbb{E}_P[\psi_P(O)] = 0$ such that, for every regular submodel $\{P_t\}$ with score function $S(O)$,*

$$\left. \frac{d}{dt} T(P_t) \right|_{t=0} = \mathbb{E}_P[\psi_P(O) \cdot S(O)] \tag{B.1}$$

Any such function ψ_P is called an influence function *(IF) of T at P.*

Equation B.1 provides a constructive method for deriving influence functions, which we elaborate upon in the next subsection.

In addition to submodel-based definitions, influence functions can be equivalently understood via local distributional perturbations. This perspective, which originates in robust statistics, is often more intuitive:

Definition B.6 (Influence Function via Local Distributional Perturbations, Hampel 14, Huber and Ronchetti 21)**.** *Let δ_o denote the Dirac measure at observation o, and consider a perturbed distribution*

$$P_\varepsilon = (1 - \varepsilon)P + \varepsilon\delta_o.$$

Then the influence function *of a statistical functional T at P evaluated at o is defined as*

$$\mathrm{IF}(o; T, P) := \left. \frac{d}{d\varepsilon} T(P_\varepsilon) \right|_{\varepsilon=0}.$$

This definition captures the first-order sensitivity of T to contamination at point o, and naturally connects to the notions of robustness and sensitivity analysis. It corroborates our observation in Remark 3.3 that $\mathrm{WCDE}_\mathbf{Z}$ is sensitive to perturbations in variables within $\mathbf{Z}$, implying that the choice of $\mathbf{Z}$ directly impacts the sensitivity of the functional.

A fundamental result in semiparametric theory links these definitions to statistical estimation. Specifically, if $\hat{T}$ is an asymptotically linear estimator of $T(P)$ at P with influence function ψ_P, then $\hat{T}$ is regular at P in model $\mathcal{M}$ if and only if $T(P)$ is pathwise differentiable and ψ_P is the influence function of T. See Theorem 2.2 of Newey [33]. This explains why the influence function defined in Definition 3.2 coincides with the influence function of the target parameter $T(P)$.

B.4 Proof of Theorem 3.4

Lemma B.7. *[29] Let $O = (O_1, O_2, \ldots, O_d)$ be a random vector with joint density $p_t(o)$ under a smooth parametric submodel $\{P_t\}$. Assume the density factorizes as*

$$p_t(o) = \prod_{i=1}^{d} p_{O_i,t}(o_i \mid \bar{o}_{i-1}),$$

where $\bar{o}_{i-1} = (o_1, \ldots, o_{i-1})$, and $p_{O_i,t}(o_i \mid \bar{o}_{i-1}) = p_{O_i}(o_i \mid \bar{o}_{i-1})$ at $t = 0$. Then, the pathwise derivative of each conditional density satisfies:

$$\frac{\partial}{\partial t} p_{O_i,t}(o_i \mid \bar{o}_{i-1})\Big|_{t=0} = p_{O_i}(o_i \mid \bar{o}_{i-1}) \cdot \frac{\partial}{\partial t} \log p_{O_i,t}(o_i \mid \bar{o}_{i-1})\Big|_{t=0}.$$

Define the conditional score function $S_{O_i}(\bar{o}_i)$ as the derivative of the log-density:

$$S_{O_i}(\bar{o}_i) := \frac{\partial}{\partial t} \log p_{O_i,t}(o_i \mid \bar{o}_{i-1})\Big|_{t=0}.$$

Then:

$$\frac{\partial}{\partial t} p_{O_i,t}(o_i \mid \bar{o}_{i-1})\Big|_{t=0} = S_{O_i}(\bar{o}_i) p_{O_i}(o_i \mid \bar{o}_{i-1}).$$

Moreover, under the chain rule for scores, this can be written in terms of the full-data score function $S(O)$ as:

$$\frac{\partial}{\partial t} p_{O_i,t}(o_i \mid \bar{o}_{i-1})\Big|_{t=0} = \left(\mathbb{E}[S(O) \mid \bar{O}_i = \bar{o}_i] - \mathbb{E}[S(O) \mid \bar{O}_{i-1} = \bar{o}_{i-1}] \right) p_{O_i}(o_i \mid \bar{o}_{i-1}).$$

This lemma allows for expressing the derivative of a marginal or conditional density in terms of conditional expectations of the full-data score function and is essential for decomposing pathwise derivatives in IF calculations.

Proof. (Proof of Theorem 3.4)

For the WCDE functional:

$$T_a(\mathbf{Z}) = \mathbb{E}_{\mathbf{Z}_1} \left\{ \mathbb{E}_{\mathbf{Z}_2} \left[\mathbb{E}[Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2] \right] \right\}$$

Recall the definition of the pathwise derivative:

$$\frac{\partial}{\partial t} T(P_t)\Big|_{t=0} = \int \psi(o; P) S(o) p(o) do.$$

where the score function $S(o)$ is defined as

$$S(o) = \frac{\partial}{\partial t} \log p_t(o)\Big|_{t=0}$$

for any smooth parametric submodel $\{P_t\}$.

We start by expressing $T_a(P_t)$ as:

$$T_a(P_t) = \int y p_t(y \mid a, \mathbf{z}_1, \mathbf{z}_2) p_t(\mathbf{z}_1) p_t(\mathbf{z}_2) dy d\mathbf{z}_2 d\mathbf{z}_1$$

$$= \int y \frac{p_t(y, \mathbf{z}_1, \mathbf{z}_2, a) p_t(\mathbf{z}_1) p_t(\mathbf{z}_2)}{p_t(\mathbf{z}_1, \mathbf{z}_2, a)} dy d\mathbf{z}_2 d\mathbf{z}_1$$

Taking the derivative w.r.t. t at $t = 0$, and applying the product and chain rules, we get:

$$\frac{\partial T_a(P_t)}{\partial t}\Big|_{t=0}$$

$$= \int y \Bigg\{ \underbrace{\frac{p(\mathbf{z}_1) p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)} \frac{\partial}{\partial t} p_t(y, a, \mathbf{z}_1, \mathbf{z}_2)\Big|_{t=0}}_{\text{(i) first term}} + \underbrace{\frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2) p(\mathbf{z}_1)}{p(a, \mathbf{z}_1, \mathbf{z}_2)} \frac{\partial}{\partial t} p_t(\mathbf{z}_2)\Big|_{t=0}}_{\text{(ii) second term}}$$

$$+ \underbrace{\frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2) p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)} \frac{\partial}{\partial t} p_t(\mathbf{z}_1)\Big|_{t=0}}_{\text{(iii) third term}} - \underbrace{\frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2) p(\mathbf{z}_1) p(\mathbf{z}_2)}{p(a, \mathbf{z}_2, \mathbf{z}_1)^2} \frac{\partial}{\partial t} p_t(a, \mathbf{z}_1, \mathbf{z}_2)\Big|_{t=0}}_{\text{(iv) fourth term}} \Bigg\} d\nu(y) \, d\nu(\mathbf{z}_2) \, d\nu(\mathbf{z}_1)$$

Each term is handled individually. For the first term:

$$\int y\{\frac{p\left(\mathbf{z}_1\right)p\left(\mathbf{z}_2\right)}{p\left(a,\mathbf{z}_1,\mathbf{z}_2\right)}\frac{\partial}{\partial t}p_t\left(y,a,\mathbf{z}_1,\mathbf{z}_2\right)\Big|_{t=0}\}d\nu(y)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

$$=\int y\{\frac{p\left(\mathbf{z}_2\right)p\left(\mathbf{z}_1\right)p_t\left(y,a,\mathbf{z}_1,\mathbf{z}_2\right)}{p\left(a,\mathbf{z}_1,\mathbf{z}_2\right)}\frac{\partial}{\partial t}\log p_t\left(y,a,\mathbf{z}_1,\mathbf{z}_2\right)\Big|_{t=0}\}d\nu(y)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

$$=\int y\{\frac{p\left(\mathbf{z}_1\right)p\left(\mathbf{z}_2\right)p(o)}{p\left(a,\mathbf{z}_1,\mathbf{z}_2\right)}S(o)\}d\nu(y)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

$$=\int y\{\frac{p\left(\mathbf{z}_1\right)p\left(\mathbf{z}_2\right)p(o)}{p\left(a,\mathbf{z}_1,\mathbf{z}_2\right)}I_a(a')S(o)\}d\nu(y)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)d\nu(a')$$

$$=\int\{\frac{I_a(a')p\left(\mathbf{z}_1\right)p\left(\mathbf{z}_2\right)y}{p\left(a,\mathbf{z}_1,\mathbf{z}_2\right)}S(o)p(o)\}d\nu(o)$$

In the third equality, we extend the integration from the conditional domain over $(y,\mathbf{z}_1,\mathbf{z}_2)$ to the full domain $o=(y,a',\mathbf{z}_1,\mathbf{z}_2)$. To preserve the original conditioning on $A=a$, we introduce the indicator function $I_a(a'):=\mathbb{I}_{\{a'=a\}}$, where a denotes the fixed treatment level of interest, and a' serves as the input variable to the function, which may vary throughout the integration.

For the second term, we apply Lemma B.7. This result allows us to express the influence of the perturbation in $p(\mathbf{z}_2)$ on the target parameter through a conditional expectation of the score function.

$$\int y\left\{\frac{p\left(y,a,\mathbf{z}_1,\mathbf{z}_2\right)p\left(\mathbf{z}_1\right)}{p\left(a,\mathbf{z}_1,\mathbf{z}_2\right)}\frac{\partial}{\partial t}p_t\left(\mathbf{z}_2\right)\Big|_{t=0}\right\}d\nu(y)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

$$=\int y\left\{\frac{p\left(y,a,\mathbf{z}_1,\mathbf{z}_2\right)p\left(\mathbf{z}_1\right)}{p\left(a,\mathbf{z}_1,\mathbf{z}_2\right)}\left(\mathbb{E}[S(O)|\mathbf{z}_2]-\mathbb{E}[S(O)]\right)p(\mathbf{z}_2)\right\}d\nu(y)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

We now expand the conditional expectations explicitly. The first term inside the integrand is rewritten using the definition of conditional expectation and the joint density, while the second term represents the overall mean of the score function. By computing the expectation of Y given $(A=a,\mathbf{Z}_1,\mathbf{Z}_2)$ and rearranging the integration order, we ultimately express the entire term as an inner product of the score function $S(o)$ with a centered function depending on $\mathbf{Z}_1$ and $\mathbf{Z}_2$, which contributes to the influence function.

$$\int y\left\{\frac{p\left(y,a,\mathbf{z}_1,\mathbf{z}_2\right)p\left(\mathbf{z}_1\right)}{p\left(a,\mathbf{z}_1,\mathbf{z}_2\right)}\left(\mathbb{E}[S(O)|\mathbf{z}_2]-\mathbb{E}[S(O)]\right)p(\mathbf{z}_2)\right\}d\nu(y)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

$$=\int y\left\{\frac{p\left(y,a,\mathbf{z}_1,\mathbf{z}_2\right)p\left(\mathbf{z}_1\right)p(\mathbf{z}_2)}{p\left(a,\mathbf{z}_2,\mathbf{z}_1\right)}\left(\int S(o)p(y,a,\mathbf{z}_1|\mathbf{z}_2)d\nu(y)d\nu(a)d\nu(\mathbf{z}_1)\right.\right.$$

$$\left.\left.-\int S(o)p(o)d\nu(o)\right)\right\}d\nu(y)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

$$=\int\{p(\mathbf{z}_1)p(\mathbf{z}_2)\mathbb{E}_Y(Y|a,\mathbf{z}_1,\mathbf{z}_2)\}\left(\int S(o)p(y,a,\mathbf{z}_1|\mathbf{z}_2)d\nu(y)d\nu(a)d\nu(\mathbf{z}_1)\right.$$

$$\left.-\int S(o)p(o)d\nu(o)\right)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

$$=\int\{p(\mathbf{z}_1)\mathbb{E}_Y(Y|a,\mathbf{z}_1,\mathbf{z}_2)\}\left(\int S(o)p(y,a,\mathbf{z}_1,\mathbf{z}_2)d\nu(y)d\nu(a)d\nu(\mathbf{z}_1)\right)d\nu(\mathbf{z}_2)d\nu(\mathbf{z}_1)$$

$$-\int T_a(P)S(o)p(o)d\nu(o)$$

$$=\int\{\mathbb{E}_{\mathbf{Z}_1}\left[\mathbb{E}_Y[Y|A=a,\mathbf{Z}_1,\mathbf{Z}_2]\right]\}\left(\int S(o)p(y,a,\mathbf{z}_1,\mathbf{z}_2)d\nu(y)d\nu(a)d\nu(\mathbf{z}_1)\right)d\nu(\mathbf{z}_2)$$

$$-\int T_a(P)S(o)p(o)d\nu(o)$$

$$=\int\{\mathbb{E}_{\mathbf{Z}_1}\left[\mathbb{E}_Y[Y|A=a,\mathbf{Z}_1,\mathbf{Z}_2]\right]\}\left(S(o)p(o)\right)d\nu(o)-\int T_a(P)S(o)p(o)d\nu(o)$$

For the third term, we apply Lemma B.7. This result allows us to express the influence of the perturbation in $p(\mathbf{z}_1)$ on the target parameter through a conditional expectation of the score function.

$$\int y \left\{ \frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2)\, p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)} \frac{\partial}{\partial t} p_t(\mathbf{z}_1) \Big|_{t=0} \right\} d\nu(y) d\nu(\mathbf{z}_1) d\nu(\mathbf{z}_2)$$
$$= \int y \left\{ \frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2)\, p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)} \left(\mathbb{E}[S(O)|\mathbf{z}_1] - \mathbb{E}[S(O)] \right) p(\mathbf{z}_1) \right\} d\nu(y) d\nu(\mathbf{z}_1) d\nu(\mathbf{z}_2)$$

The rest of the computation follows by evaluating the conditional expectation of Y given $A = a, \mathbf{Z}_1, \mathbf{Z}_2$ and expressing the result in terms of an inner product between the score function and an identifying function.

$$\int y \left\{ \frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2)\, p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)} \left(\mathbb{E}[S(O)|\mathbf{z}_1] - \mathbb{E}[S(O)] \right) p(\mathbf{z}_1) \right\} d\nu(y) d\nu(\mathbf{z}_2) d\nu(\mathbf{z}_1)$$
$$= \int y \left\{ \frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2)\, p(\mathbf{z}_1)\, p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)} \left(\int S(o) p(y, a, \mathbf{z}_2|\mathbf{z}_1) d\nu(y) d\nu(a) d\nu(\mathbf{z}_2) \right. \right.$$
$$\left. \left. - \int S(o) p(o) d\nu(o) \right) \right\} d\nu(y) d\nu(\mathbf{z}_2) d\nu(\mathbf{z}_1)$$
$$= \int \left\{ p(\mathbf{z}_1) p(\mathbf{z}_2) \mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2) \right\} \left(\int S(o) p(y, a, \mathbf{z}_2|\mathbf{z}_1) d\nu(y) d\nu(a) d\nu(\mathbf{z}_2) \right.$$
$$\left. - \int S(o) p(o) d\nu(o) \right) d\nu(\mathbf{z}_2) d\nu(\mathbf{z}_1)$$
$$= \int \left\{ p(\mathbf{z}_2) \mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2) \right\} \left(\int S(o) p(y, a, \mathbf{z}_1, \mathbf{z}_2) d\nu(y) d\nu(a) d\nu(\mathbf{z}_2) \right) d\nu(\mathbf{z}_2) d\nu(\mathbf{z}_1)$$
$$- \int T_a(P) S(o) p(o) d\nu(o)$$
$$= \int \left\{ \mathbb{E}_{\mathbf{Z}_2} \left[\mathbb{E}_Y[Y|A = a, \mathbf{Z}_1, \mathbf{Z}_2] \right] \right\} \left(\int S(o) p(y, a, \mathbf{z}_1, \mathbf{z}_2) d\nu(y) d\nu(a) d\nu(\mathbf{z}_2) \right) d\nu(\mathbf{z}_1)$$
$$- \int T_a(P) S(o) p(o) d\nu(o)$$
$$= \int \left\{ \mathbb{E}_{\mathbf{Z}_2} \left[\mathbb{E}_Y[Y|A = a, \mathbf{Z}_1, \mathbf{Z}_2] \right] \right\} \left(S(o) p(o) \right) d\nu(o) - \int T_a(P) S(o) p(o) d\nu(o)$$

For the fourth term, we begin by applying Lemma B.7 to express the derivative of the joint probability $p_t(a, \mathbf{z}_1, \mathbf{z}_2)$ in terms of the score function $S(O)$.

$$\int y \left\{ \frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2)\, p(\mathbf{z}_1)\, p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)^2} \frac{\partial}{\partial t} p_t(a, \mathbf{z}_1, \mathbf{z}_2) \Big|_{t=0} \right\} d\nu(y) d\nu(\mathbf{z}_2) d\nu(\mathbf{z}_1)$$
$$= \int y \left\{ \frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2)\, p(\mathbf{z}_1)\, p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)^2} \left(\mathbb{E}[S(O)|a, \mathbf{z}_1, \mathbf{z}_2] - \mathbb{E}[S(O)] \right) p(a, \mathbf{z}_1, \mathbf{z}_2) \right\} d\nu(y) d\nu(\mathbf{z}_2) d\nu(\mathbf{z}_1)$$

Next, we expand the conditional expectations as integrals over the relevant conditional distributions. The term $\mathbb{E}[S(O)|a, \mathbf{z}_1, \mathbf{z}_2]$ becomes an integral over $p(y|a, \mathbf{z}_1, \mathbf{z}_2)$, while $\mathbb{E}[S(O)]$ integrates over

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the full joint distribution $p(o)$. We then factor out the conditional expectation of Y given $(a, \mathbf{z}_1, \mathbf{z}_2)$.

$$
= \int y \left\{ \frac{p\left(y, a, \mathbf{z}_1, \mathbf{z}_2\right) p\left(\mathbf{z}_1\right) p\left(\mathbf{z}_2\right)}{p\left(a, \mathbf{z}_1, \mathbf{z}_2\right)^2} \left(\mathbb{E}[S(O)|a, \mathbf{z}_1, \mathbf{z}_2] - \mathbb{E}[S(O)] \right) p(a, \mathbf{z}_1, \mathbf{z}_2) \right\} d\nu(y) d\nu(\mathbf{z}_2) d\nu(\mathbf{z}_1)
$$

$$
= \int y \left\{ \frac{p\left(y, a, \mathbf{z}_1, \mathbf{z}_2\right) p\left(\mathbf{z}_1\right) p\left(\mathbf{z}_2\right)}{p\left(a, \mathbf{z}_1, \mathbf{z}_2\right)} \left(\int S(o)p(y|a, \mathbf{z}_1, \mathbf{z}_2)d\nu(y) - \int S(o)p(o)d\nu(o) \right) \right\} d\nu(y) d\nu(\mathbf{z}_2) d\nu(\mathbf{z}_1)
$$

$$
= \int \left\{ p(\mathbf{z}_1)p(\mathbf{z}_2)\mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2) \right\} \left(\int S(o)p(y|a, \mathbf{z}_1, \mathbf{z}_2)d\nu(y) - \int S(o)p(o)d\nu(o) \right) d\nu(\mathbf{z}_1)d\nu(\mathbf{z}_2)
$$

$$
= \int \left\{ p(\mathbf{z}_1)p(\mathbf{z}_2)\mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2) \right\} \left(\int S(o)\frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)}d\nu(y) \right) d\nu(\mathbf{z}_1)d\nu(\mathbf{z}_2) - \int T_a(P)S(o)p(o)d\nu(o)
$$

Finally, we introduce the indicator function $I_a(a')$ and combine all integrals into a single expectation with respect to $p(o)$.

$$
\int \left\{ p(\mathbf{z}_1)p(\mathbf{z}_2)\mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2) \right\} \left(\int S(o)\frac{p(y, a, \mathbf{z}_1, \mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)}d\nu(y) \right) d\nu(\mathbf{z}_1)d\nu(\mathbf{z}_2) - \int T_a(P)S(o)p(o)d\nu(o)
$$

$$
= \int \left\{ \frac{p(\mathbf{z}_1)p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)}\mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2) \right\} \left(\int S(o)p(o)d\nu(y) \right) d\nu(\mathbf{z}_1)d\nu(\mathbf{z}_2) - \int T_a(P)S(o)p(o)d\nu(o)
$$

$$
= \int \left\{ \frac{I_a(a')p(\mathbf{z}_1)p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)}\mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2) \right\} S(o)p(o)d\nu(y)d\nu(\mathbf{z}_1)d\nu(\mathbf{z}_2)d\nu(a') - \int T_a(P)S(o)p(o)d\nu(o)
$$

$$
= \int \left\{ \frac{I_a(a')p(\mathbf{z}_1)p(\mathbf{z}_2)}{p(a, \mathbf{z}_1, \mathbf{z}_2)}\mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2) \right\} S(o)p(o)d\nu(o) - \int T_a(P)S(o)p(o)d\nu(o)
$$

Summing all terms together, we obtain:

$$
\left. \frac{\partial T_a\left(P_t\right)}{\partial t} \right|_{t=0}
$$
$$
= \int \{ [\frac{I_a(a')p\left(\mathbf{z}_1\right)p\left(\mathbf{z}_2\right)\left(y - \mathbb{E}_Y(Y|a, \mathbf{z}_1, \mathbf{z}_2)\right)}{p\left(a, \mathbf{z}_2, \mathbf{z}_1\right)} + \mathbb{E}_{\mathbf{Z}_1}\left[\mathbb{E}_Y[Y|A = a, \mathbf{z}_2, \mathbf{z}_1]\right]
$$
$$
+ \mathbb{E}_{\mathbf{Z}_2}\left[\mathbb{E}_Y[Y|A = a, \mathbf{z}_1, \mathbf{z}_2]\right] - 2T_a(P)]S(o)p(o)\}d\nu(o)
$$
$$
= \int \psi_a(o; P)S(o)p(o)d\nu(o)
$$

with

$$
\psi_a\left(Y, A, \mathbf{Z}_1, \mathbf{Z}_2; P\right) = \frac{I_a(A)p(\mathbf{Z}_1)p(\mathbf{Z}_2)}{p(\mathbf{Z}_1, \mathbf{Z}_2, a)} \cdot \left(Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2] \right)
$$
$$
+ \mathbb{E}_{\mathbf{Z}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2] \right] + \mathbb{E}_{\mathbf{Z}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2] \right] - 2T_a(P)
$$

as required. $\qquad \square$

C   Proofs of Results in Section 4

Theorem 4.3 follows from the four lemmas presented in Section 4. Accordingly, we first provide the proofs of Lemmas 4.4, 4.6, 4.5, and 4.7 individually, and then proceed to prove the theorem.

C.1   Proof of Lemma 4.4

Proof. (Proof of Lemma 4.4: Supplement of backdoor variables)

We begin by showing that the set $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ remains a VAS. We can verify this by comparing the identified WCDE functionals. Recall that for any adjustment set $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$, the WCDE is identified as

$$
\text{WCDE}_{\mathbf{Z}} = \mathbb{E}_{\mathbf{Z}_1}\Big\{ \mathbb{E}_{\mathbf{Z}_2}\big[\mathbb{E}(Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2) \big] - \mathbb{E}_{\mathbf{Z}_2}\big[\mathbb{E}(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{Z}_2) \big] \Big\}.
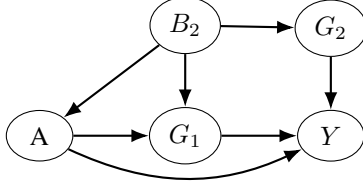$$

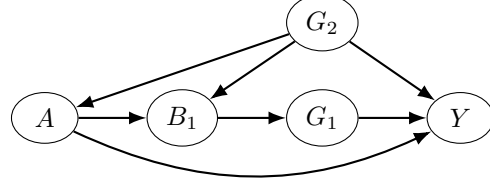Figure C.4: A DAG illustrating Lemmas 4.4, 4.6.

Figure C.5: A DAG illustrating Lemmas 4.5, 4.7.

For $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$, consider the term corresponding to a:

$$T_a(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) = \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_{\mathbf{G}_2, \mathbf{B}_2}\big[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)\big]\Big].$$

Under the conditional independence

$$\mathbf{G}_2 \perp\!\!\!\perp_{\mathcal{G}} A, \mathbf{G}_1 \mid \mathbf{B}_2 \tag{C.1}$$

we have $p(\mathbf{G}_2, \mathbf{B}_2) = p(\mathbf{G}_2 \mid \mathbf{B}_2)p(\mathbf{B}_2) = p(\mathbf{G}_2 \mid A = a, \mathbf{G}_1, \mathbf{B}_2)p(\mathbf{B}_2)$, and by the law of iterated expectations,

$$\mathbb{E}_{\mathbf{G}_2|\mathbf{B}_2}[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)] = \mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2).$$

Substituting this gives

$$T_a(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) = \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_{\mathbf{B}_2}\big[\mathbb{E}_{\mathbf{G}_2|\mathbf{B}_2}[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)]\big]\Big]$$

$$= \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_{\mathbf{B}_2}\big[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2)\big]\Big] = T_a(\mathbf{G}_1, \mathbf{B}_2),$$

and the same equality holds for a^*. Hence,

$$\mathrm{WCDE}_{(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)} = \mathrm{WCDE}_{(\mathbf{G}_1, \mathbf{B}_2)}.$$

By the uniqueness of the identified WCDE functional, any adjustment set that yields the same functional as a VAS must itself be a VAS. Therefore, $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ is also a VAS.

We now compare the asymptotic variances of two VASs, $(\mathbf{G}_1, \mathbf{B}_2)$ and $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$. For clarity, we first focus on the first component of the WCDE in Equation (2.2), and let $\sigma_{a,\mathbf{Z}}^2(P)$ denote its asymptotic variance. The extension to the full WCDE is straightforward. We begin by decomposing the variance:

$$\sigma_{a,(\mathbf{G}_1,\mathbf{B}_2)}^2(P) = \mathrm{Var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P)\right] = \mathrm{Var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)\right]$$

$$+ \mathrm{Var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)\right]$$

$$+ 2\,\mathrm{Cov}_P\left(\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P), \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)\right).$$

To analyze the covariance term, consider the difference in IFs evaluated at $(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$:

$$\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)$$

$$= \frac{I_a(A)p(\mathbf{G}_1)p(\mathbf{B}_2)}{p(\mathbf{G}_1, \mathbf{B}_2, a)} \cdot \left(Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right)$$

$$+ \mathbb{E}_{\mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right] + \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right]$$

$$- \frac{I_a(A)p(\mathbf{G}_1)p(\mathbf{G}_2, \mathbf{B}_2)}{p(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2, a)} \cdot \left(Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right)$$

$$- \mathbb{E}_{\mathbf{G}_2, \mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] - \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right]$$

$$= \frac{I_a(A) \cdot p(\mathbf{G}_1) \cdot p(\mathbf{B}_2)}{p(A = a, \mathbf{G}_1, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right)$$

$$+ \mathbb{E}_{\mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right] + \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right]$$

$$- \mathbb{E}_{\mathbf{G}_2, \mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] - \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right]$$

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The second equality above holds under the conditional independence assumption C.1, from which it follows that

$$p(\mathbf{G}_2 \mid \mathbf{B}_2) = p(\mathbf{G}_2 \mid \mathbf{G}_1, \mathbf{B}_2, A = a),$$

which implies:

$$\frac{p(\mathbf{G}_2, \mathbf{B}_2)p(\mathbf{G}_1)}{p(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2, A = a)} = \frac{p(\mathbf{B}_2)p(\mathbf{G}_1)}{p(\mathbf{G}_1, \mathbf{B}_2, A = a)} \cdot \frac{p(\mathbf{G}_2 \mid \mathbf{B}_2)}{p(\mathbf{G}_2 \mid \mathbf{B}_2, \mathbf{G}_1, A = a)} = \frac{p(\mathbf{B}_2)p(\mathbf{G}_1)}{p(A = a, \mathbf{G}_1, \mathbf{B}_2)}.$$

Therefore, the first terms in the two influence functions cancel out. To simplify the final term, note that:

$$\mathbb{E}_{\mathbf{G}_2, \mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] = \mathbb{E}_{\mathbf{B}_2}\left[\mathbb{E}_{\mathbf{G}_2 \mid \mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right]\right]$$

$$= \mathbb{E}_{\mathbf{B}_2}\left[\mathbb{E}_{\mathbf{G}_2 \mid \mathbf{G}_1, \mathbf{B}_2, A=a}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right]\right] = \mathbb{E}_{\mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right]$$

this equality follows from the conditional independence assumption C.1 and the law of iterated expectations. Substituting back, the difference simplifies to:

$$\frac{I_a(A) \cdot p(\mathbf{G}_1) \cdot p(\mathbf{B}_2)}{p(A = a, \mathbf{G}_1, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right)$$

$$+ \mathbb{E}_{\mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right] + \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right]$$

$$- \mathbb{E}_{\mathbf{G}_2, \mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] - \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right]$$

$$= \frac{I_a(A) \cdot p(\mathbf{G}_1) \cdot p(\mathbf{B}_2)}{p(A = a, \mathbf{G}_1, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right)$$

$$+ \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right] - \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right]$$

Therefore, we arrive at the simplified expression for their difference.

We then construct a parametric submodel that perturbs only the conditional law $P(A, \mathbf{G}_1 \mid \mathbf{G}_2, \mathbf{B}_2)$, while keeping all other components of the distribution fixed. We show that under the conditional independence assumption C.1, the difference between the two IFs is orthogonal to the influence function of the larger adjustment set:

$$\mathbb{E}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \cdot (\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P))\right] = 0.$$

Hence, the covariance term vanishes. Substituting this into the variance decomposition yields:

$$\mathrm{Var}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P)]$$
$$= \mathrm{Var}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)] + \mathrm{Var}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)],$$

where the second term is nonnegative. To formalize this orthogonality, we appeal to two key results from semiparametric theory [54]:

1. If P_t is a smooth parametric submodel with $P_0 = P$ and score function g, then the influence function ψ_a satisfies:

$$\left.\frac{d}{dt}T_a(P_t)\right|_{t=0} = \mathbb{E}_P[\psi_a \cdot g].$$

2. In a nonparametric model, any function g such that

$$\mathbb{E}_P[g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) \mid \mathbf{G}_2, \mathbf{B}_2] = 0$$

is a valid score function for the conditional law $P(A, \mathbf{G}_1 \mid \mathbf{G}_2, \mathbf{B}_2)$.

Let us now consider any distribution P and define the function

$$g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) := \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P),$$

which satisfies the second condition above. Using this function, we define the following submodel for sufficiently small $t \geq 0$:

$$P_t(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) := P(Y \mid A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) \cdot P_t(A, \mathbf{G}_1 \mid \mathbf{G}_2, \mathbf{B}_2) \cdot P(\mathbf{G}_2, \mathbf{B}_2),$$

where

$$P_t(A, \mathbf{G}_1 \mid \mathbf{G}_2, \mathbf{B}_2) = P(A, \mathbf{G}_1 \mid \mathbf{G}_2, \mathbf{B}_2)(1 + t \cdot g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)).$$

This submodel perturbs only the conditional law of $(A, \mathbf{G}_1)$ given $(\mathbf{G}_2, \mathbf{B}_2)$, leaving the remaining components of the joint distribution fixed. The score function of this model at $t = 0$ is

$$\frac{\partial}{\partial t} \log p_t(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)\Big|_{t=0} = \frac{\partial}{\partial t} \log\{1 + t \cdot g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)\}\Big|_{t=0} = g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2).$$

Applying result (1), we obtain:

$$\frac{d}{dt} T_a(P_t)\Big|_{t=0} = \mathbb{E}_P[\psi_a \cdot g].$$

Now consider the functional:

$$T_a(P) = \sum_{\mathbf{G}_1} \sum_{\mathbf{G}_2, \mathbf{B}_2} \left[\mathbb{E}[Y \mid a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] p(\mathbf{G}_2, \mathbf{B}_2) p(\mathbf{G}_1).$$

This functional depends only on the joint distribution of $(\mathbf{G}_2, \mathbf{B}_2)$, $\mathbf{G}_1$, and the conditional distribution $P(Y \mid A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$. If the function g satisfies

$$\mathbb{E}_P[g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) \mid \mathbf{G}_2, \mathbf{B}_2] = 0 \quad \text{and} \quad \mathbb{E}_P[g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) \mid \mathbf{G}_1] = 0, \tag{C.2}$$

then perturbations in $P_t(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ do not affect those distributions. That is, the submodel is locally ancillary for the functional T_a, and therefore:

$$\frac{d}{dt} T_a(P_t)\Big|_{t=0} = 0.$$

Combining with the earlier result gives:

$$\mathbb{E}_P[\psi_a g] = 0$$

Then we obtain the orthogonality:

$$\mathbb{E}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)\right] = 0$$

We now verify that this function $g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ satisfies the two orthogonality conditions in Condition C.2. First, consider the conditional expectation given $\mathbf{G}_2, \mathbf{B}_2$:

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid \mathbf{G}_2, \mathbf{B}_2]$$

$$= \mathbb{E}_P\left\{ \frac{I_a(A) \cdot p(\mathbf{B}_2) \cdot p(\mathbf{G}_1)}{p(A = a, \mathbf{G}_1, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right) \right.$$

$$\left. + \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right] - \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] \mid \mathbf{G}_2, \mathbf{B}_2 \right\}$$

$$= \mathbb{E}_P\left\{ \frac{p(\mathbf{G}_2, \mathbf{B}_2) \cdot p(\mathbf{G}_1)}{p(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right) \right.$$

$$\left. + \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right] - \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] \mid \mathbf{G}_2, \mathbf{B}_2 \right\}$$

$$= \mathbb{E}_P\left\{ \frac{p(\mathbf{G}_2, \mathbf{B}_2) \cdot p(\mathbf{G}_1)}{p(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right) \mid \mathbf{G}_2, \mathbf{B}_2 \right\}$$

$$+ \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2]\right] - \mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right]$$

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In the second equality, we integrate over A conditional on $(\mathbf{G}_2, \mathbf{B}_2)$. Note that the last two expectation terms are fixed once $(\mathbf{G}_2, \mathbf{B}_2)$ are given, so we can move them outside the outer expectation accordingly.

We now rewrite the first expectation as an integral over $\mathbf{G}_1$ conditional on $(\mathbf{G}_2, \mathbf{B}_2)$:

$$\mathbb{E}_P \left\{ \frac{p(\mathbf{G}_2, \mathbf{B}_2) \cdot p(\mathbf{G}_1)}{p(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right) \Big| \mathbf{G}_2, \mathbf{B}_2 \right\}$$

$$= \int \frac{p(\mathbf{G}_2, \mathbf{B}_2) \cdot p(\mathbf{g}_1)}{p(\mathbf{g}_1, \mathbf{G}_2, \mathbf{B}_2)} \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{B}_2] \right) p(\mathbf{g}_1 \mid \mathbf{G}_2, \mathbf{B}_2) \, d\mathbf{g}_1$$

$$= \int \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{B}_2] \right) p(\mathbf{g}_1) \, d\mathbf{g}_1$$

$$= \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \right] - \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right]$$

Substituting this expression back into the original conditional expectation, we obtain:

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid \mathbf{G}_2, \mathbf{B}_2]$$

$$= \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \right] - \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right]$$

$$+ \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right] - \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \right]$$

$$= 0$$

This establishes that the function $g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ satisfies the first orthogonality condition in Condition C.2, $\mathbb{E}_P[g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) \mid \mathbf{G}_2, \mathbf{B}_2] = 0$

To verify that it satisfies the second orthogonality condition, we proceed similarly by conditioning on $\mathbf{G}_1$:

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid \mathbf{G}_1]$$

$$= \mathbb{E}_P \left\{ \frac{I_a(A) \cdot p(\mathbf{G}_1) \cdot p(\mathbf{B}_2)}{p(A = a, \mathbf{G}_1, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right) \right.$$

$$\left. + \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right] - \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \right] \Big| \mathbf{G}_1 \right\}$$

$$= \mathbb{E}_P \left\{ \frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{B}_2)}{p(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right) \right.$$

$$\left. + \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right] - \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \right] \Big| \mathbf{G}_1 \right\}$$

In the second equality, we integrate over A conditional on $\mathbf{G}_1$. We now rewrite the first term as an integral over $(\mathbf{G}_2, \mathbf{B}_2)$ with respect to $P(\mathbf{G}_2, \mathbf{B}_2 \mid \mathbf{G}_1)$, and then simplify.

$$\mathbb{E}_P\Bigg\{ \frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{B}_2)}{p(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)} \cdot \Big(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \Big)$$

$$+ \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \Big] - \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \Big] \mid \mathbf{G}_1 \Bigg\}$$

$$= \int \frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{B}_2)}{p(\mathbf{G}_1, \mathbf{g}_2, \mathbf{b}_2)} \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{g}_2, \mathbf{b}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{b}_2] \right) p(\mathbf{g}_2, \mathbf{b}_2 \mid \mathbf{G}_1) \, d\mathbf{g}_2 \, d\mathbf{b}_2$$

$$+ \mathbb{E}_P\Bigg\{ \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \Big] - \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \Big] \mid \mathbf{G}_1 \Bigg\}$$

$$= \int p(\mathbf{g}_2, \mathbf{b}_2) \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{g}_2, \mathbf{b}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{b}_2] \right) \, d\mathbf{g}_2 \, d\mathbf{b}_2$$

$$+ \mathbb{E}_P\Bigg\{ \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \Big] - \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \Big] \mid \mathbf{G}_1 \Bigg\}$$

Next, we cancel the two components of the expression by invoking the conditional independence assumption C.1.

$$\int p(\mathbf{g}_2, \mathbf{b}_2) \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{g}_2, \mathbf{b}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{b}_2] \right) \, d\mathbf{g}_2 \, d\mathbf{b}_2$$

$$+ \mathbb{E}_P\Bigg\{ \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \Big] - \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \Big] \mid \mathbf{G}_1 \Bigg\}$$

$$= \int \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{g}_2, \mathbf{b}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{b}_2] \right) p(\mathbf{g}_2 \mid \mathbf{b}_2, A = a, \mathbf{G}_1) p(\mathbf{b}_2) \, d\mathbf{g}_2 \, d\mathbf{b}_2$$

$$+ \mathbb{E}_P\Bigg\{ \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \Big] - \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \Big] \mid \mathbf{G}_1 \Bigg\}$$

$$= \mathbb{E}_P\Bigg\{ \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \Big] - \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \Big] \mid \mathbf{G}_1 \Bigg\}$$

The second equality holds because we have the conditional independence assumption C.1, so:

$$p(\mathbf{g}_2, \mathbf{b}_2) = p(\mathbf{g}_2 \mid \mathbf{b}_2) \cdot p(\mathbf{b}_2) = p(\mathbf{g}_2 \mid \mathbf{b}_2, A = a, \mathbf{G}_1) p(\mathbf{b}_2),$$

which justifies replacing the joint density in the integral. Furthermore, by the law of iterated expectations,

$$\int \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{g}_2, \mathbf{b}_2] \cdot p(\mathbf{g}_2 \mid \mathbf{b}_2, A = a, \mathbf{G}_1) \, d\mathbf{g}_2 = \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{b}_2].$$

As a result, the two integrals cancel out, yielding the final equality. To simplify this expression, we compute both terms as iterated integrals.

$$\mathbb{E}_P\Bigg\{ \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \Big] - \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \Big] \mid \mathbf{G}_1 \Bigg\}$$

$$= \iiint y \cdot [p(y \mid A = a, \mathbf{g}_1, \mathbf{b}_2)] \, dy \, p(\mathbf{g}_1) \, d\mathbf{g}_1 \cdot p(\mathbf{b}_2 \mid \mathbf{G}_1) \, d\mathbf{b}_2$$

$$- \iiiint y \cdot [p(y \mid A = a, \mathbf{g}_1, \mathbf{g}_2, \mathbf{b}_2)] \, dy \, p(\mathbf{g}_1) \, d\mathbf{g}_1 \cdot p(\mathbf{g}_2, \mathbf{b}_2 \mid \mathbf{G}_1) \, d\mathbf{g}_2 \, d\mathbf{b}_2$$

Based on Condition C.1, we can simplify the second expression through the following derivation. First, we apply Condition C.1 ($p(\mathbf{g}_2 \mid \mathbf{b}_2) = p(\mathbf{g}_2 \mid A = a, \mathbf{G}_1, \mathbf{b}_2)$) to expand the conditioning set:

$$\iiiint y \cdot [p(y \mid A = a, \mathbf{g}_1, \mathbf{g}_2, \mathbf{b}_2)] \, dy \, p(\mathbf{g}_1) \, d\mathbf{g}_1 \cdot p(\mathbf{g}_2, \mathbf{b}_2 \mid \mathbf{G}_1) \, d\mathbf{g}_2 \, d\mathbf{b}_2$$

$$= \iiiint y \cdot p(y \mid A = a, \mathbf{g}_1, \mathbf{g}_2, \mathbf{b}_2) \, dy \, p(\mathbf{g}_1) d\mathbf{g}_1 \cdot p(\mathbf{g}_2 \mid \mathbf{b}_2) d\mathbf{g}_2 \cdot p(\mathbf{b}_2 \mid \mathbf{G}_1) d\mathbf{b}_2$$

$$= \iiiint y \cdot p(y \mid A = a, \mathbf{g}_1, \mathbf{g}_2, \mathbf{b}_2) \, dy \, p(\mathbf{g}_1) d\mathbf{g}_1 \cdot p(\mathbf{g}_2 \mid A = a, \mathbf{g}_1, \mathbf{b}_2) d\mathbf{g}_2 \cdot p(\mathbf{b}_2 \mid \mathbf{G}_1) d\mathbf{b}_2$$

Next, we express the integrand as a joint probability distribution over y and $\mathbf{g}_2$:

$$= \iiiint y \cdot p(y, \mathbf{g}_2 \mid A = a, \mathbf{g}_1, \mathbf{b}_2) \, dy d\mathbf{g}_2 \, p(\mathbf{g}_1) d\mathbf{g}_1 \cdot p(\mathbf{b}_2 \mid \mathbf{G}_1) d\mathbf{b}_2$$

Using the chain rule of probability, we factorize the joint probability:

$$= \iiiint y \cdot p(\mathbf{g}_2 \mid y, A = a, \mathbf{g}_1, \mathbf{b}_2) p(y \mid A = a, \mathbf{g}_1, \mathbf{b}_2) \, d\mathbf{g}_2 dy \, p(\mathbf{g}_1) d\mathbf{g}_1 \cdot p(\mathbf{b}_2 \mid \mathbf{G}_1) d\mathbf{b}_2$$

Finally, we marginalize over $\mathbf{g}_2$ by integrating it out (since $\int p(\mathbf{g}_2 \mid y, A = a, \mathbf{g}_1, \mathbf{b}_2) d\mathbf{g}_2 = 1$), yielding the simplified expression:

$$= \iiint y \cdot p(y \mid A = a, \mathbf{g}_1, \mathbf{b}_2) \, dy \, p(\mathbf{g}_1) d\mathbf{g}_1 \cdot p(\mathbf{b}_2 \mid \mathbf{G}_1) d\mathbf{b}_2$$

This shows that the second term equals the first one. Thus, we conclude:

$$\mathbb{E}_P \left\{ \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_2] \right] - \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] \right] \mid \mathbf{G}_1 \right\} = 0$$

So far we have shown that the function $g(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ satisfies Condition C.2, so we obtain the orthogonality:

$$\mathbb{E}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \cdot (\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)) \right] = 0.$$

Therefore, we conclude that the variance decomposes as:

$$\sigma^2_{a,(\mathbf{G}_1, \mathbf{B}_2)}(P) = \operatorname{var}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) \right]$$
$$= \operatorname{var}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \right] + \operatorname{var}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \right]$$
$$\geq \sigma^2_{a,(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P)$$

Now extend this to the vectorized WCDE. Define: $\mathbf{c} \equiv (c_a)_{a \in \mathcal{A}}$, $\mathbf{Q} \equiv (Q_a)_{a \in \mathcal{A}}$ where

$$Q_a = \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)$$

For any $\mathbf{Z}$, define $\boldsymbol{\psi}(Y, A, \mathbf{Z}; P) \equiv (\psi_a(Y, A, \mathbf{Z}; P))_{a \in \mathcal{A}}$. Then, writing $\sum_{a \in \mathcal{A}} c_a \psi_a(Y, A, \mathbf{Z}; P) = \mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{Z}; P)$ We notice that:

$$\mathbb{E}_P[\mathbf{Q} \mid \mathbf{G}_2, \mathbf{B}_2] = \mathbf{0},$$
$$\mathbb{E}_P[\mathbf{Q} \mid \mathbf{G}_1] = \mathbf{0},$$

Then we obtain the orthogonality:

$$\mathbb{E}_P \left[\mathbf{Q}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \right] = \mathbf{0}$$

Therefore, we conclude that the variance decomposes as:

$$\sigma^2_{(\mathbf{G}_1, \mathbf{B}_2)}(P) = \operatorname{var}_P \left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_2; P) \right]$$
$$= \operatorname{var}_P \left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \right] + \operatorname{var}_P \left[\mathbf{c}^\top \mathbf{Q} \right] = \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P) + \mathbf{c}^\top \operatorname{var}_P(\mathbf{Q}) \mathbf{c}$$
$$\geq \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P)$$

This completes the proof. $\qquad\square$

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C.2 Proof of Lemma 4.6

Proof. (Proof of Lemma 4.6: Deletion of overadjustment backdoor variables)

We begin by showing that the set $(\mathbf{G}_1, \mathbf{G}_2)$ remains a VAS. We can verify this by comparing the identified WCDE functionals. Recall that for any adjustment set $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$, the WCDE is identified as

$$\text{WCDE}_\mathbf{Z} = \mathbb{E}_{\mathbf{Z}_1}\Big\{\mathbb{E}_{\mathbf{Z}_2}\big[\mathbb{E}(Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2)\big] - \mathbb{E}_{\mathbf{Z}_2}\big[\mathbb{E}(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{Z}_2)\big]\Big\}.$$

For $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$, consider the term corresponding to a:

$$T_a(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) = \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_{\mathbf{G}_2, \mathbf{B}_2}\big[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)\big]\Big].$$

Under the conditional independence

$$Y \per\!\!\!\perp_\mathcal{G} \mathbf{B}_2 \mid \mathbf{G}_1, \mathbf{G}_2, A, \tag{C.3}$$

the inner expectation simplifies as

$$\mathbb{E}[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2] = \mathbb{E}[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2],$$

Substituting this gives

$$T_a(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) = \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_{\mathbf{G}_2, \mathbf{B}_2}\big[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2)\big]\Big]$$
$$= \mathbb{E}_{\mathbf{G}_1}\Big[\mathbb{E}_{\mathbf{G}_2}\big[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2)\big]\Big] = T_a(\mathbf{G}_1, \mathbf{G}_2),$$

and the same equality holds for a^*. Hence,

$$\text{WCDE}_{(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)} = \text{WCDE}_{(\mathbf{G}_1, \mathbf{G}_2)}.$$

By the uniqueness of the identified WCDE functional, any adjustment set that yields the same functional as a VAS must itself be a VAS. Therefore, $(\mathbf{G}_1, \mathbf{G}_2)$ is also a VAS.

We now compare the asymptotic variances of two VASs, $(\mathbf{G}_1, \mathbf{G}_2)$ and $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$. For clarity, we first focus on the first component of the WCDE in Equation (2.2), and let $\sigma^2_{a,\mathbf{Z}}(P)$ denote its asymptotic variance. The extension to the full WCDE is straightforward.

We begin by applying the law of total variance to decompose the variance under the larger adjustment set $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$:

$$\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)] = \text{Var}[\mathbb{E}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2]]$$
$$+ \mathbb{E}\left[\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2]\right].$$

Next, we will show that under the conditional independence assumption C.3 the IF based on the adjustment set $(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)$ satisfies

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2] = \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2; P).$$

This equality allows us to relate the variances of the two IFs via the law of total variance. We now compute the conditional expectation of $\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)$:

$$\mathbb{E}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right] =$$

$$\underbrace{I_a(A)\{Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]\} \cdot \mathbb{E}_P\left[\frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{B}_2)}{p(A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}\middle| A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]}_{\text{(i) first term}}$$

$$+ \underbrace{\mathbb{E}_P\left[\mathbb{E}_{\mathbf{G}_2, \mathbf{B}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] \mid \mathbf{G}_1\right]}_{\text{(ii) second term}}$$

$$+ \underbrace{\mathbb{E}_P\left[\mathbb{E}_{\mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]\right] \mid \mathbf{G}_2\right]}_{\text{(iii) third term}} - 2T_a(P)$$

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This equality holds because, given the conditional independence relation C.3, it implies the key simplification:
$$\mathbb{E}_Y(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) = \mathbb{E}_Y(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2).$$

For the first conditional expectation term, we perform the detailed computation:

$$\mathbb{E}_P\left[\frac{p(\mathbf{G}_2, \mathbf{B}_2) \cdot p(\mathbf{G}_1)}{p(A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \mathbb{E}_P\left[\frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{B}_2)}{p(\mathbf{B}_2 \mid A = a, \mathbf{G}_1, \mathbf{G}_2) \cdot p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \mathbb{E}_P\left[\frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{B}_2)}{p(\mathbf{B}_2 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \mathbb{E}_P\left[\frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{B}_2)}{p(\mathbf{B}_2 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_2} \frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{b}_2)}{p(\mathbf{b}_2 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot p(\mathbf{b}_2 \mid A = a, Y, \mathbf{G}_1, \mathbf{G}_2)\, d\mathbf{b}_2$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_2} p(\mathbf{G}_1) \cdot p(\mathbf{G}_2, \mathbf{b}_2) \cdot \frac{p(\mathbf{b}_2 \mid A = a, Y, \mathbf{G}_1, \mathbf{G}_2)}{p(\mathbf{b}_2 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)}\, d\mathbf{b}_2$$

In the last line above, we expand the conditional expectation as an integral over $\mathbf{B}_2$. Now, under the conditional independence assumption C.2, it follows that

$$p(\mathbf{b}_2 \mid A = a, Y, \mathbf{G}_1, \mathbf{G}_2) = p(\mathbf{b}_2 \mid A = a, \mathbf{G}_1, \mathbf{G}_2).$$

Substituting this into the integrand, the ratio simplifies to 1:

$$\frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_2} p(\mathbf{G}_2, \mathbf{b}_2) \cdot p(\mathbf{G}_1) \cdot \frac{p(\mathbf{b}_2 \mid A = a, Y, \mathbf{G}_1, \mathbf{G}_2)}{p(\mathbf{b}_2 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)}\, d\mathbf{b}_2$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_2} p(\mathbf{G}_2, \mathbf{b}_2) \cdot p(\mathbf{G}_1)\, d\mathbf{b}_2$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_2} p(\mathbf{G}_2 \mid \mathbf{b}_2) \cdot p(\mathbf{G}_1) \cdot p(\mathbf{G}_2)\, d\mathbf{b}_2$$

$$= \frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2)}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_2} p(\mathbf{b}_2 \mid \mathbf{G}_2)\, d\mathbf{b}_2$$

$$= \frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2)}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)}$$

Since $\mathbb{E}_Y(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2) = \mathbb{E}_Y(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2)$, the second and third terms simplify as follows:

$$\mathbb{E}_P\left[E_{\mathbf{G}_2, \mathbf{B}_2}[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]] \mid \mathbf{G}_1\right]$$

$$= \mathbb{E}_P\left[E_{\mathbf{G}_2, \mathbf{B}_2}[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]] \mid \mathbf{G}_1\right]$$

$$= \mathbb{E}_{\mathbf{G}_2}\left[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]\right]$$

$$\mathbb{E}_P\left[E_{\mathbf{G}_1}[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2]] \mid \mathbf{G}_2\right]$$

$$= \mathbb{E}_P\left[E_{\mathbf{G}_1}[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]] \mid \mathbf{G}_2\right]$$

$$= \mathbb{E}_{\mathbf{G}_1}\left[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]\right]$$

Combining these results, we obtain:

$$\mathbb{E}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \frac{I_a(A) \cdot p(\mathbf{G}_1) \cdot p(\mathbf{G}_2)}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \left(Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]\right)$$

$$+ \mathbb{E}_{\mathbf{G}_2}\left[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]\right] + \mathbb{E}_{\mathbf{G}_1}\left[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]\right] - 2T_a(P)$$

$$= \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2; P)$$

That is, the conditional expectation of the influence function under the larger adjustment set equals the influence function under the smaller set.

Finally, applying the law of total variance, we establish the variance dominance relationship:

$$\sigma^2_{a,(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P) = \mathrm{var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)\right]$$

$$= \mathrm{var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2; P)\right] + \mathbb{E}_P\left[\mathrm{var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right]\right]$$

$$\geq \sigma^2_{a,(\mathbf{G}_1, \mathbf{G}_2)}(P)$$

The inequality follows because the conditional variance term is non-negative.

Now for the full WCDE estimator, define: $\mathbf{c} \equiv (c_a)_{a \in \mathcal{A}}$, for any $\mathbf{Z}$, define $\boldsymbol{\psi}(Y, A, \mathbf{Z}; P) \equiv (\psi_a(Y, A, \mathbf{Z}; P))_{a \in \mathcal{A}}$. Then, writing $\sum_{a \in \mathcal{A}} c_a \psi_a(Y, A, \mathbf{Z}; P) = \mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{Z}; P)$

note that:

$$\mathbb{E}_P\left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right] = \mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2; P)$$

So again by variance decomposition:

$$\sigma^2_{(\mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2)}(P) = \mathrm{var}_P\left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P)\right]$$

$$= \mathrm{var}_P[\mathbb{E}_P\left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right]] + \mathbb{E}_P\left[\mathrm{var}_P\left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right]\right]$$

$$= \mathrm{var}_P\left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2; P)\right] + \mathbf{c}^\top \mathbb{E}_P\left[\mathrm{var}_P\left[\boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right]\right]\mathbf{c}$$

$$= \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2)}(P) + \mathbf{c}^\top \mathbb{E}_P\left[\mathrm{var}_P\left[\boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2, \mathbf{B}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right]\right]\mathbf{c}$$

$$\geq \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2)}(P)$$

This completes the proof.

$\square$

C.3   Proof of Lemma 4.5

Proof. (Proof of Lemma 4.5: Supplement of mediator variables)

We begin by showing that the set $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$ remains a VAS. We can verify this by comparing the identified WCDE functionals. Recall that for any adjustment set $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$, the WCDE is identified as

$$\mathrm{WCDE}_{\mathbf{Z}} = \mathbb{E}_{\mathbf{Z}_1}\left\{\mathbb{E}_{\mathbf{Z}_2}\left[\mathbb{E}(Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2)\right] - \mathbb{E}_{\mathbf{Z}_2}\left[\mathbb{E}(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{Z}_2)\right]\right\}.$$

For $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$, consider the a-term:

$$T_a(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) = \mathbb{E}_{\mathbf{G}_1, \mathbf{B}_1}\left[\mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right]\right]$$

$$= \mathbb{E}_{\mathbf{B}_1}\left[\mathbb{E}_{\mathbf{G}_1 \mid \mathbf{B}_1}\left[\mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right]\right]\right].$$

By the conditional independence assumption:

$$\mathbf{G}_1 \perp\!\!\!\perp_{\mathcal{G}} A, \mathbf{G}_2 \mid \mathbf{B}_1 \tag{C.4}$$

we have $p(\mathbf{G}_1, \mathbf{B}_1) = p(\mathbf{G}_1 \mid \mathbf{B}_1)p(\mathbf{B}_1) = p(\mathbf{G}_1 \mid A = a, \mathbf{G}_2, \mathbf{B}_1)p(\mathbf{B}_1)$. Applying the law of iterated expectations

$$\mathbb{E}_{\mathbf{G}_1 \mid \mathbf{B}_1}\Big[\mathbb{E}_{\mathbf{G}_2}\big[\mathbb{E}[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\big]\Big] = \mathbb{E}_{\mathbf{G}_2}\Big[\mathbb{E}_{\mathbf{G}_1 \mid \mathbf{B}_1}[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)]\Big]$$

$$= \mathbb{E}_{\mathbf{G}_2}\Big[\mathbb{E}_{\mathbf{G}_1 \mid \mathbf{B}_1, \mathbf{G}_2, A=a}[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)]\Big]$$

$$= \mathbb{E}_{\mathbf{G}_2}[\mathbb{E}(Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2)].$$

Therefore

$$T_a(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) = \mathbb{E}_{\mathbf{B}_1}\Big[\mathbb{E}_{\mathbf{G}_2}\big[\mathbb{E}(Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2)\big]\Big] = T_a(\mathbf{B}_1, \mathbf{G}_2),$$

and the same equality holds for a^*. Hence,

$$\mathrm{WCDE}_{(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} = \mathrm{WCDE}_{(\mathbf{B}_1, \mathbf{G}_2)}.$$

By the uniqueness of the identified WCDE functional, any adjustment set that yields the same functional as a VAS must itself be a VAS. Therefore, $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$ is also a VAS.

We now compare the asymptotic variances of two VASs, $(\mathbf{B}_1, \mathbf{G}_2)$ and $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$. For clarity, we first focus on the first component of the WCDE in Equation (2.2), and let $\sigma_{a,\mathbf{Z}}^2(P)$ denote its asymptotic variance. The extension to the full WCDE is straightforward. We begin by decomposing the variance:

$$\sigma_{a,(\mathbf{B}_1, \mathbf{G}_2)}^2(P) = \mathrm{Var}_P\left[\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P)\right] = \mathrm{Var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)\right]$$

$$+ \mathrm{Var}_P\left[\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)\right]$$

$$+ 2\,\mathrm{Cov}_P\left(\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P), \psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)\right).$$

To analyze the covariance term, consider the difference in IFs evaluated at $(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$:

$$\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)$$

$$= \frac{I_a(A)p(\mathbf{B}_1)p(\mathbf{G}_2)}{p(\mathbf{B}_1, \mathbf{G}_2, a)} \cdot \left(Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right)$$

$$+ \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right] + \mathbb{E}_{\mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right]$$

$$- \frac{I_a(A)p(\mathbf{G}_1, \mathbf{B}_1)p(\mathbf{G}_2)}{p(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2, a)} \cdot \left(Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right)$$

$$- \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_1, \mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right]$$

$$= \frac{I_a(A) \cdot p(\mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(A = a, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right)$$

$$+ \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right] + \mathbb{E}_{\mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right]$$

$$- \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_1, \mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right]$$

The second equality above holds under the conditional independence assumption C.4, from which it follows that

$$p(\mathbf{G}_1 \mid \mathbf{B}_1) = p(\mathbf{G}_1 \mid \mathbf{B}_1, \mathbf{G}_2, A = a),$$

which implies:

$$\frac{p(\mathbf{G}_1, \mathbf{B}_1)p(\mathbf{G}_2)}{p(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2, A = a)} = \frac{p(\mathbf{B}_1)p(\mathbf{G}_2)}{p(\mathbf{B}_1, \mathbf{G}_2, A = a)} \cdot \frac{p(\mathbf{G}_1 \mid \mathbf{B}_1)}{p(\mathbf{G}_1 \mid \mathbf{B}_1, \mathbf{G}_2, A = a)} = \frac{p(\mathbf{B}_1)p(\mathbf{G}_2)}{p(A = a, \mathbf{B}_1, \mathbf{G}_2)}.$$

Therefore, the first terms in the two influence functions cancel out. To simplify the final term, note that:

$$\mathbb{E}_{\mathbf{G}_1, \mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] = \mathbb{E}_{\mathbf{B}_1}\left[\mathbb{E}_{\mathbf{G}_1 \mid \mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right]\right]$$

$$= \mathbb{E}_{\mathbf{B}_1}\left[\mathbb{E}_{\mathbf{G}_1 \mid \mathbf{B}_1, \mathbf{G}_2, A=a}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right]\right] = \mathbb{E}_{\mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right]$$

this equality follows from the conditional independence assumption C.4 and the law of iterated expectations. Substituting back, the difference simplifies to:

$$\frac{I_a(A) \cdot p(\mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(A = a, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right)$$

$$+ \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] + \mathbb{E}_{\mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right]$$

$$- \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{B}_1, \mathbf{G}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right]$$

$$= \frac{I_a(A) \cdot p(\mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(A = a, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right)$$

$$+ \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right]$$

Therefore, we arrive at the simplified expression for their difference.

We then construct a parametric submodel that perturbs only the conditional law $P(A, \mathbf{G}_2 \mid \mathbf{G}_1, \mathbf{B}_1)$, while keeping all other components of the distribution fixed. We show that under the assumption C.4, the difference between the two IFs is orthogonal to the influence function of the larger adjustment set:

$$\mathbb{E}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \cdot (\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)) \right] = 0.$$

Hence, the covariance term vanishes. Substituting this into the variance decomposition yields:

$$\mathrm{Var}_P[\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P)]$$
$$= \mathrm{Var}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)] + \mathrm{Var}_P[\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)],$$

where the second term is nonnegative. To formalize this orthogonality, we appeal to two key results from semiparametric theory [54]:

1. If P_t is a smooth parametric submodel with $P_0 = P$ and score function g, then the influence function ψ_a satisfies:
$$\left. \frac{d}{dt} T_a(P_t) \right|_{t=0} = \mathbb{E}_P[\psi_a \cdot g].$$

2. In a nonparametric model, any function g such that
$$\mathbb{E}_P[g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) \mid \mathbf{G}_1, \mathbf{B}_1] = 0$$

is a valid score function for the conditional law $P(A, \mathbf{G}_2 \mid \mathbf{G}_1, \mathbf{B}_1)$.

Let us now consider any distribution P and define the function

$$g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) := \psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P),$$

which satisfies the second condition above. Using this function, we define the following submodel for sufficiently small $t \geq 0$:

$$P_t(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) := P(Y \mid A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) \cdot P_t(A, \mathbf{G}_2 \mid \mathbf{G}_1, \mathbf{B}_1) \cdot P(\mathbf{G}_1, \mathbf{B}_1),$$

where
$$P_t(A, \mathbf{G}_2 \mid \mathbf{G}_1, \mathbf{B}_1) = P(A, \mathbf{G}_2 \mid \mathbf{G}_1, \mathbf{B}_1)(1 + t \cdot g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)).$$

This submodel perturbs only the conditional law of $(A, \mathbf{G}_2)$ given $(\mathbf{G}_1, \mathbf{B}_1)$, leaving the remaining components of the joint distribution fixed. The score function of this model at $t = 0$ is

$$\left. \frac{\partial}{\partial t} \log p_t(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) \right|_{t=0} = \left. \frac{\partial}{\partial t} \log\{1 + t \cdot g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)\} \right|_{t=0} = g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2).$$

Applying result (1), we obtain:

$$\left. \frac{d}{dt} T_a(P_t) \right|_{t=0} = \mathbb{E}_P[\psi_a \cdot g].$$

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Now consider the functional:

$$T_a(P) = \sum_{\mathbf{G}_1, \mathbf{B}_1} \sum_{\mathbf{G}_2} \left[\mathbb{E}[Y \mid a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] p(\mathbf{G}_2) p(\mathbf{G}_1, \mathbf{B}_1).$$

This functional depends only on the joint distribution of $(\mathbf{G}_1, \mathbf{B}_1)$, $\mathbf{G}_2$, and the conditional distribution $P(Y \mid A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$. If the function g satisfies

$$\mathbb{E}_P[g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) \mid \mathbf{G}_1, \mathbf{B}_1] = 0 \quad \text{and} \quad \mathbb{E}_P[g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) \mid \mathbf{G}_2] = 0, \tag{C.5}$$

then perturbations in $P_t(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$ do not affect those distributions. That is, the submodel is locally ancillary for the functional T_a, and therefore:

$$\left. \frac{d}{dt} T_a(P_t) \right|_{t=0} = 0.$$

Combining with the earlier result gives:

$$\mathbb{E}_P[\psi_a g] = 0$$

Then we obtain the orthogonality:

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)] = 0$$

We now verify that this function $g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$ satisfies the two orthogonality conditions in Condition C.5. First, consider the conditional expectation given $\mathbf{G}_1, \mathbf{B}_1$:

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid \mathbf{G}_1, \mathbf{B}_1]$$

$$= \mathbb{E}_P \Bigg\{ \frac{I_a(A) \cdot p(\mathbf{G}_2) \cdot p(\mathbf{B}_1)}{p(A = a, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right)$$

$$+ \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] \mid \mathbf{G}_1, \mathbf{B}_1 \Bigg\}$$

$$= \mathbb{E}_P \Bigg\{ \frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right)$$

$$+ \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] \mid \mathbf{G}_1, \mathbf{B}_1 \Bigg\}$$

$$= \mathbb{E}_P \Bigg\{ \frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right) \mid \mathbf{B}_1, \mathbf{G}_1 \Bigg\}$$

$$+ \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right]$$

In the second equality, we integrate over A conditional on $(\mathbf{G}_1, \mathbf{B}_1)$. Note that the last two expectation terms are fixed once $(\mathbf{G}_1, \mathbf{B}_1)$ are given, so we can move them outside the outer expectation accordingly.

We now rewrite the first expectation as an integral over $\mathbf{G}_2$ conditional on $(\mathbf{G}_1, \mathbf{B}_1)$:

$$\mathbb{E}_P \Bigg\{ \frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right) \mid \mathbf{G}_1, \mathbf{B}_1 \Bigg\}$$

$$= \int \frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{g}_2)}{p(\mathbf{G}_1, \mathbf{B}_1, \mathbf{g}_2)} \left(\mathbb{E}_Y[Y | A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{g}_2] - \mathbb{E}_Y[Y | A = a, \mathbf{B}_1, \mathbf{g}_2] \right) p(\mathbf{g}_2 | \mathbf{G}_1, \mathbf{B}_1) \, d\mathbf{g}_2$$

$$= \int \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{g}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{g}_2] \right) p(\mathbf{g}_2) \, d\mathbf{g}_2$$

$$= \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right]$$

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Substituting this expression back into the original conditional expectation, we obtain:

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid \mathbf{G}_1, \mathbf{B}_1]$$

$$= \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right]$$

$$+ \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right]$$

$$= 0$$

This establishes that the function $g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$ satisfies the first orthogonality condition in Condition C.2, $\mathbb{E}_P[g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) \mid \mathbf{G}_1, \mathbf{B}_1] = 0$.

To verify that it satisfies the second orthogonality condition, we proceed similarly by conditioning on $\mathbf{G}_2$:

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid \mathbf{G}_2]$$

$$= \mathbb{E}_P\left\{\frac{I_a(A) \cdot p(\mathbf{G}_2) \cdot p(\mathbf{B}_1)}{p(A = a, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right)\right.$$

$$\left. + \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] \mid \mathbf{G}_2\right\}$$

$$= \mathbb{E}_P\left\{\frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right)\right.$$

$$\left. + \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] \mid \mathbf{G}_2\right\}$$

In the second equality, we integrate over A conditional on $\mathbf{G}_2$. We now rewrite the first term as an integral over $(\mathbf{G}_1, \mathbf{B}_1)$ with respect to $P(\mathbf{G}_1, \mathbf{B}_1 \mid \mathbf{G}_2)$, and then simplify.

$$\mathbb{E}_P\left\{\frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} \cdot \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right)\right.$$

$$\left. + \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] \mid \mathbf{G}_2\right\}$$

$$= \int \frac{p(\mathbf{g}_1, \mathbf{b}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{g}_1, \mathbf{b}_1, \mathbf{G}_2)} \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{b}_1, \mathbf{G}_2]\right) p(\mathbf{g}_1, \mathbf{b}_1 \mid \mathbf{G}_2) \, d\mathbf{b}_1 \, d\mathbf{g}_1$$

$$+ \mathbb{E}_P\left\{\mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] \mid \mathbf{G}_2\right\}$$

$$= \int p(\mathbf{b}_1, \mathbf{g}_1) \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{b}_1, \mathbf{G}_2]\right) d\mathbf{b}_1 \, d\mathbf{g}_1$$

$$+ \mathbb{E}_P\left\{\mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2]\right] - \mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] \mid \mathbf{G}_2\right\}$$

Next, we cancel the two components of the expression by invoking the conditional independence assumption C.4.

$$\int p(\mathbf{g}_1, \mathbf{b}_1) \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{b}_1, \mathbf{G}_2] \right) d\mathbf{b}_1 \, d\mathbf{g}_1$$

$$+ \mathbb{E}_P \left\{ \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] \, \bigg| \, \mathbf{G}_2 \right\}$$

$$= \int \left(\mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{G}_2] - \mathbb{E}_Y[Y \mid A = a, \mathbf{b}_1, \mathbf{G}_2] \right) p(\mathbf{g}_1 \mid \mathbf{b}_1, A = a, \mathbf{G}_2) p(\mathbf{b}_1) \, d\mathbf{g}_1 \, d\mathbf{b}_1$$

$$+ \mathbb{E}_P \left\{ \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] \, \bigg| \, \mathbf{G}_2 \right\}$$

$$= \mathbb{E}_P \left\{ \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] \, \bigg| \, \mathbf{G}_2 \right\}$$

The second equality holds because we have the conditional independence assumption C.4, so:

$$p(\mathbf{g}_1, \mathbf{b}_1) = p(\mathbf{g}_1 \mid \mathbf{b}_1) \cdot p(\mathbf{b}_1) = p(\mathbf{g}_1 \mid \mathbf{b}_1, A = a, \mathbf{G}_2) p(\mathbf{b}_1),$$

which justifies replacing the joint density in the integral. Furthermore, by the law of iterated expectations,

$$\int \mathbb{E}_Y[Y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{G}_2] \cdot p(\mathbf{g}_1 \mid \mathbf{b}_1, A = a, \mathbf{G}_2) \, d\mathbf{g}_2 = \mathbb{E}_Y[Y \mid A = a, \mathbf{b}_1, \mathbf{G}_2].$$

As a result, the two integrals cancel out, yielding the final equality. To simplify this expression, we compute both terms as iterated integrals.

$$\mathbb{E}_P \left\{ \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] \, \bigg| \, \mathbf{G}_2 \right\}$$

$$= \iiint y \cdot [p(y \mid A = a, \mathbf{b}_1, \mathbf{g}_2)] \, dy \, p(\mathbf{g}_2) \, d\mathbf{g}_2 \cdot p(\mathbf{b}_1 \mid \mathbf{G}_2) \, d\mathbf{b}_1$$

$$- \iiiint y \cdot [p(y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{g}_2)] \, dy \, p(\mathbf{g}_2) \, d\mathbf{g}_2 \cdot p(\mathbf{g}_1, \mathbf{b}_1 \mid \mathbf{G}_2) \, d\mathbf{g}_1 d\mathbf{b}_1$$

Based on Condition C.4, we can simplify the second expression through the following derivation. First, we apply Condition C.4 ($p(\mathbf{g}_1 \mid \mathbf{b}_1) = p(\mathbf{g}_1 \mid A = a, \mathbf{b}_1, \mathbf{G}_2)$) to expand the conditioning set:

$$\iiiint y \cdot [p(y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{g}_2)] \, dy \, p(\mathbf{g}_2) \, d\mathbf{g}_2 \cdot p(\mathbf{g}_1, \mathbf{b}_1 \mid \mathbf{G}_2) \, d\mathbf{g}_1 d\mathbf{b}_1$$

$$= \iiiint y \cdot p(y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{g}_2) \, dy \, p(\mathbf{g}_2) d\mathbf{g}_2 \cdot p(\mathbf{g}_1 \mid \mathbf{b}_1) d\mathbf{g}_1 \cdot p(\mathbf{b}_1 \mid \mathbf{G}_2) d\mathbf{b}_1$$

$$= \iiiint y \cdot p(y \mid A = a, \mathbf{g}_1, \mathbf{b}_1, \mathbf{g}_2) \, dy \, p(\mathbf{g}_2) d\mathbf{g}_2 \cdot p(\mathbf{g}_1 \mid A = a, \mathbf{b}_1, \mathbf{g}_2) d\mathbf{g}_1 \cdot p(\mathbf{b}_1 \mid \mathbf{G}_2) d\mathbf{b}_1$$

Next, we express the integrand as a joint probability distribution over y and $\mathbf{g}_2$:

$$= \iiiint y \cdot p(y, \mathbf{g}_1 \mid A = a, \mathbf{b}_1, \mathbf{g}_2) \, dy d\mathbf{g}_1 \, p(\mathbf{g}_2) d\mathbf{g}_2 \cdot p(\mathbf{b}_1 \mid \mathbf{G}_2) d\mathbf{b}_1$$

Using the chain rule of probability, we factorize the joint probability:

$$= \iiiint y \cdot p(\mathbf{g}_1 \mid y, A = a, \mathbf{b}_1, \mathbf{g}_2) p(y \mid A = a, \mathbf{b}_1, \mathbf{g}_2) \, d\mathbf{g}_1 dy \, p(\mathbf{g}_2) d\mathbf{g}_2 \cdot p(\mathbf{b}_1 \mid \mathbf{G}_2) d\mathbf{b}_1$$

Finally, we marginalize over $\mathbf{g}_1$ by integrating it out (since $\int p(\mathbf{g}_1 \mid y, A = a, \mathbf{b}_1, \mathbf{g}_2) d\mathbf{g}_1 = 1$), yielding the simplified expression:

$$= \iiint y \cdot p(y \mid A = a, \mathbf{b}_1, \mathbf{g}_2) \, dy \, p(\mathbf{g}_2) d\mathbf{g}_2 \cdot p(\mathbf{b}_1 \mid \mathbf{G}_2) d\mathbf{b}_1$$

This shows that the second term equals the first one. Thus, we conclude:

$$\mathbb{E}_P \left\{ \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{B}_1, \mathbf{G}_2] \right] - \mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] \right] \,\middle|\, \mathbf{G}_2 \right\} = 0$$

So far we have shown that the function $g(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$ satisfies Condition C.5, so we obtain the orthogonality:

$$\mathbb{E}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \cdot (\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)) \right] = 0.$$

Therefore, we conclude that the variance decomposes as:

$$\sigma^2_{a,(\mathbf{B}_1, \mathbf{G}_2)}(P) = \mathrm{var}_P \left[\psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) \right]$$
$$= \mathrm{var}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \right] + \mathrm{var}_P \left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \right]$$
$$\geq \sigma^2_{a,(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)}(P)$$

Now extend this to the vectorized WCDE. Define: $\mathbf{c} \equiv (c_a)_{a \in \mathcal{A}}$, $\mathbf{Q} \equiv (Q_a)_{a \in \mathcal{A}}$ where

$$Q_a = \psi_a(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) - \psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)$$

For any $\mathbf{Z}$, define $\boldsymbol{\psi}(Y, A, \mathbf{Z}; P) \equiv (\psi_a(Y, A, \mathbf{Z}; P))_{a \in \mathcal{A}}$. Then, writing $\sum_{a \in \mathcal{A}} c_a \psi_a(Y, A, \mathbf{Z}; P) = \mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{Z}; P)$ We notice that:

$$\mathbb{E}_P[\mathbf{Q} \mid \mathbf{G}_1, \mathbf{B}_1] = \mathbf{0},$$
$$\mathbb{E}_P[\mathbf{Q} \mid \mathbf{G}_2] = \mathbf{0},$$

Then we obtain the orthogonality:

$$\mathbb{E}_P[\mathbf{Q}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)] = \mathbf{0}$$

Therefore, we conclude that the variance decomposes as:

$$\sigma^2_{(\mathbf{B}_1, \mathbf{G}_2)}(P) = \mathrm{var}_P \left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{B}_1, \mathbf{G}_2; P) \right]$$
$$= \mathrm{var}_P \left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \right] + \mathrm{var}_P \left[\mathbf{c}^\top \mathbf{Q} \right] = \sigma^2_{(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)}(P) + \mathbf{c}^\top \mathrm{var}_P(\mathbf{Q}) \mathbf{c}$$
$$\geq \sigma^2_{(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)}(P)$$

This completes the proof.

$\square$

C.4 Proof of Lemma 4.7

Proof. (Proof of Lemma 4.7: Deletion of overadjustment mediator variables)

We begin by showing that the set $(\mathbf{G}_1, \mathbf{G}_2)$ remains a valid adjustment set (VAS). We can verify this by comparing the identified WCDE functionals. For any adjustment set $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$, the WCDE is identified as

$$\mathrm{WCDE}_{\mathbf{Z}} = \mathbb{E}_{\mathbf{Z}_1} \left\{ \mathbb{E}_{\mathbf{Z}_2} \left[\mathbb{E}(Y \mid A = a, \mathbf{Z}_1, \mathbf{Z}_2) \right] - \mathbb{E}_{\mathbf{Z}_2} \left[\mathbb{E}(Y \mid A = a^*, \mathbf{Z}_1, \mathbf{Z}_2) \right] \right\}.$$

For $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$, consider the a-term:

$$T_a(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) = \mathbb{E}_{\mathbf{G}_1, \mathbf{B}_1} \left[\mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) \right] \right].$$

From the conditional independence

$$Y \perp\!\!\!\perp_{\mathcal{G}} \mathbf{B}_1 \mid \mathbf{G}_1, \mathbf{G}_2, A, \tag{C.6}$$

it follows that

$$\mathbb{E}[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2] = \mathbb{E}[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2],$$

Substituting this gives

$$T_a(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) = \mathbb{E}_{\mathbf{G}_1, \mathbf{B}_1} \left[\mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2) \right] \right]$$
$$= \mathbb{E}_{\mathbf{G}_1} \left[\mathbb{E}_{\mathbf{G}_2} \left[\mathbb{E}(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2) \right] \right] = T_a(\mathbf{G}_1, \mathbf{G}_2),$$

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and the same equality holds for a^*. Therefore

$$\text{WCDE}_{(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} = \text{WCDE}_{(\mathbf{G}_1, \mathbf{G}_2)}.$$

By the uniqueness of the identified WCDE functional, any adjustment set that yields the same functional as a VAS must itself be a VAS. Hence $(\mathbf{G}_1, \mathbf{G}_2)$ is a VAS.

We now compare the asymptotic variances of two VASs, $(\mathbf{G}_1, \mathbf{G}_2)$ and $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$. For clarity, we first focus on the first component of the WCDE in Equation (2.2), and let $\sigma^2_{a, \mathbf{Z}}(P)$ denote its asymptotic variance. The extension to the full WCDE is straightforward.

We begin by applying the law of total variance to decompose the variance under the larger adjustment set $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$:

$$\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)] = \text{Var}[\mathbb{E}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2]]$$
$$+ \mathbb{E}\left[\text{Var}[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2]\right].$$

Next, we will show that under the conditional independence assumption C.6 , the IF based on the adjustment set $(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)$ satisfies

$$\mathbb{E}_P[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2] = \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2; P).$$

This equality allows us to relate the variances of the two IFs via the law of total variance. We now compute the conditional expectation of $\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P)$:

$$\mathbb{E}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2\right] =$$

$$\underbrace{I_a(A)\left\{Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]\right\} \cdot \mathbb{E}_P\left[\frac{p(\mathbf{G}_2) \cdot p(\mathbf{G}_1, \mathbf{B}_1)}{p(A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]}_{\text{(i) first term}}$$

$$+ \underbrace{\mathbb{E}_P\left[\mathbb{E}_{\mathbf{G}_2}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] \mid \mathbf{G}_1, \mathbf{B}_1\right]}_{\text{(ii) second term}}$$

$$+ \underbrace{\mathbb{E}_P\left[\mathbb{E}_{\mathbf{G}_1, \mathbf{B}_1}\left[\mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]\right] \mid \mathbf{G}_2\right]}_{\text{(iii) third term}} - 2T_a(P)$$

This equality holds because, given the conditional independence relation C.6, it implies the key simplification:

$$\mathbb{E}_Y(Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) = \mathbb{E}_Y(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2).$$

For the first conditional expectation term, we perform the detailed computation:

$$\mathbb{E}_P\left[\frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \mathbb{E}_P\left[\frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{B}_1 \mid A = a, \mathbf{G}_1, \mathbf{G}_2) \cdot p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \mathbb{E}_P\left[\frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{B}_1 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \mathbb{E}_P\left[\frac{p(\mathbf{G}_1, \mathbf{B}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{B}_1 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)} \,\middle|\, A = a, Y, \mathbf{G}_1, \mathbf{G}_2\right]$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_1} \frac{p(\mathbf{G}_1, \mathbf{b}_1) \cdot p(\mathbf{G}_2)}{p(\mathbf{b}_1 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot p(\mathbf{b}_1 \mid A = a, Y, \mathbf{G}_1, \mathbf{G}_2)\, d\mathbf{b}_1$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_1} p(\mathbf{G}_1, \mathbf{b}_1) \cdot p(\mathbf{G}_2) \cdot \frac{p(\mathbf{b}_1 \mid A = a, Y, \mathbf{G}_1, \mathbf{G}_2)}{p(\mathbf{b}_1 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)}\, d\mathbf{b}_1$$

In the last line above, we expand the conditional expectation as an integral over $\mathbf{B}_1$. Now, under the conditional independence assumption C.6, it follows that

$$p(\mathbf{b}_1 \mid A = a, Y, \mathbf{G}_1, \mathbf{G}_2) = p(\mathbf{b}_1 \mid A = a, \mathbf{G}_1, \mathbf{G}_2).$$

Substituting this into the integrand, the ratio simplifies to 1:

$$\frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_1} p(\mathbf{G}_2) \cdot p(\mathbf{G}_1, \mathbf{b}_1) \cdot \frac{p(\mathbf{b}_1 \mid A = a, Y, \mathbf{G}_1, \mathbf{G}_2)}{p(\mathbf{b}_1 \mid A = a, \mathbf{G}_1, \mathbf{G}_2)} \, d\mathbf{b}_1$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_1} p(\mathbf{G}_2) \cdot p(\mathbf{G}_1, \mathbf{b}_1) \, d\mathbf{b}_1$$

$$= \frac{1}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_1} p(\mathbf{b}_1 \mid \mathbf{G}_1) \cdot p(\mathbf{G}_1) \cdot p(\mathbf{G}_2) \, d\mathbf{b}_1$$

$$= \frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2)}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \int_{\mathbf{b}_1} p(\mathbf{b}_1 \mid \mathbf{G}_1) \, d\mathbf{b}_1$$

$$= \frac{p(\mathbf{G}_1) \cdot p(\mathbf{G}_2)}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)}$$

Since $\mathbb{E}_Y(Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2) = \mathbb{E}_Y(Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2)$, so the second and third terms simplify as follows:

$$\mathbb{E}_P\left[E_{\mathbf{G}_2}[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]] \mid \mathbf{G}_1, \mathbf{B}_1 \right]$$

$$= \mathbb{E}_P\left[E_{\mathbf{G}_2}[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]] \mid \mathbf{G}_1 \right]$$

$$= \mathbb{E}_{\mathbf{G}_2}\left[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2] \right]$$

$$\mathbb{E}_P\left[E_{\mathbf{B}_1, \mathbf{G}_1}[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2]] \mid \mathbf{G}_2 \right]$$

$$= \mathbb{E}_P\left[E_{\mathbf{B}_1, \mathbf{G}_1}[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2]] \mid \mathbf{G}_2 \right]$$

$$= \mathbb{E}_{\mathbf{G}_1}\left[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2] \right]$$

Combining these results, we obtain:

$$\mathbb{E}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2 \right]$$

$$= \frac{I_a(A) \cdot p(\mathbf{G}_2) \cdot p(\mathbf{G}_1)}{p(A = a, \mathbf{G}_1, \mathbf{G}_2)} \cdot \left(Y - \mathbb{E}_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2] \right)$$

$$+ \mathbb{E}_{\mathbf{G}_2}\left[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2] \right] + \mathbb{E}_{\mathbf{G}_1}\left[E_Y[Y \mid A = a, \mathbf{G}_1, \mathbf{G}_2] \right] - 2T_a(P)$$

$$= \psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2; P)$$

That is, the conditional expectation of the influence function under the larger adjustment set equals the influence function under the smaller set.

Finally, applying the law of total variance, we establish the variance dominance relationship:

$$\sigma^2_{a, (\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)}(P) = \text{var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \right]$$

$$= \text{var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{G}_2; P) \right] + \mathbb{E}_P\left[\text{var}_P\left[\psi_a(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2 \right] \right]$$

$$\geq \sigma^2_{a, (\mathbf{G}_1, \mathbf{G}_2)}(P)$$

The inequality follows because the conditional variance term is non-negative.

Now for the full WCDE estimator, define: $\mathbf{c} \equiv (c_a)_{a \in \mathcal{A}}$, for any $\mathbf{Z}$, define $\boldsymbol{\psi}(Y, A, \mathbf{Z}; P) \equiv (\psi_a(Y, A, \mathbf{Z}; P))_{a \in \mathcal{A}}$. Then, writing $\sum_{a \in \mathcal{A}} c_a \psi_a(Y, A, \mathbf{Z}; P) = \mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{Z}; P)$

note that:

$$\mathbb{E}_P\left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2 \right] = \mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2; P)$$

So again by variance decomposition:

$$\sigma^2_{(\mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2)}(P) = \mathrm{var}_P \left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \right]$$

$$= \mathrm{var}_P \left[\mathbb{E}_P \left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2 \right] \right] + \mathbb{E}_P \left[\mathrm{var}_P \left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2 \right] \right]$$

$$= \mathrm{var}_P \left[\mathbf{c}^\top \boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{G}_2; P) \right] + \mathbf{c}^\top \mathbb{E}_P \left[\mathrm{var}_P \left[\boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2 \right] \right] \mathbf{c}$$

$$= \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2)}(P) + \mathbf{c}^\top \mathbb{E}_P \left[\mathrm{var}_P \left[\boldsymbol{\psi}(Y, A, \mathbf{G}_1, \mathbf{B}_1, \mathbf{G}_2; P) \mid A, Y, \mathbf{G}_1, \mathbf{G}_2 \right] \right] \mathbf{c}$$

$$\geq \sigma^2_{(\mathbf{G}_1, \mathbf{G}_2)}(P)$$

This completes the proof. □

C.5 Proof of Theorem 4.3

Proof. (Proof of Theorem 4.3)

Let $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$ be a valid adjustment set, where $\mathbf{Z}_1 = \mathbf{Z} \cap \mathbf{M}$ contains mediating variables and $\mathbf{Z}_2 = \mathbf{Z} \setminus \mathbf{Z}_1$ contains non-mediators (typically backdoor covariates).

We aim to construct an adjustment set that minimizes the asymptotic variance of the WCDE estimator by augmenting and pruning $\mathbf{Z}_1$ and $\mathbf{Z}_2$ based on conditional independencies in the DAG.

We prove the result through a sequence of variance-reducing steps.

Step 1: Augmenting $\mathbf{Z}_2$ with Non-mediating Parents of Y

According to Condition C1 and Condition C2, all backdoor paths for $\{A, Y\}$ and $\{\mathbf{Z}_1, Y\}$ are blocked by $\mathbf{Z}_2$, so we have the conditional independence:

$$(\mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)} \setminus \mathbf{Z}_2) \perp\!\!\!\perp_{\mathcal{G}} A, \mathbf{Z}_1 \mid \mathbf{Z}_2.$$

Then by Lemma 4.4, augmenting $\mathbf{Z}_2$ with $\mathbf{X}_{1 \in \mathrm{Pa}(Y)}$ and $\mathbf{X}_{4 \in \mathrm{Pa}(Y)}$ reduces the variance:

$$\sigma^2_{(\mathbf{Z}_1, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)})}(P) \leq \sigma^2_{(\mathbf{Z}_1, \mathbf{Z}_2)}(P).$$

Step 2: Augmenting $\mathbf{Z}_1$ with Mediating Parents of Y

According to Condition C3, $\{\mathbf{M}' \setminus \mathbf{Z}_1\} \perp\!\!\!\perp_{\mathcal{G}} \{\mathrm{Pa}(Y) \setminus \mathbf{M}'\} \mid \mathbf{Z}_1$, so we have the conditional independence:

$$(\mathbf{X}_{3 \in \mathrm{Pa}(Y)} \setminus \mathbf{Z}_1) \perp\!\!\!\perp_{\mathcal{G}} A, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)} \mid \mathbf{Z}_1.$$

Then by Lemma 4.5, adding $\mathbf{X}_{3 \in \mathrm{Pa}(Y)}$ further reduces variance:

$$\sigma^2_{(\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \mathrm{Pa}(Y)}, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)})}(P) \leq \sigma^2_{(\mathbf{Z}_1, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)})}(P).$$

Step 3: Pruning Redundant Backdoor Variables from $\mathbf{Z}_2$

Since $\mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)}$, $\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \mathrm{Pa}(Y)}$, and A together form the full set of parents of Y, according to the *local Markov property*, Y is conditionally independent of all non-descendant non-parent variables given its parents. Therefore, we have the following conditional independence:

$$Y \perp\!\!\!\perp_{\mathcal{G}} \mathbf{Z}_2 \setminus (\mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)}) \mid \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)}, \mathbf{Z}_1 \cup \mathbf{X}_{3 \in \mathrm{Pa}(Y)}, A.$$

Then by Lemma 4.6, removing extraneous elements of $\mathbf{Z}_2$ does not increase variance:

$$\sigma^2_{(\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \mathrm{Pa}(Y)}, \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)})}(P) \leq \sigma^2_{(\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \mathrm{Pa}(Y)}, \mathbf{Z}_2 \cup \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)})}(P).$$

Step 4: Pruning Redundant Mediating Variables from $\mathbf{Z}_1$

Note that $\mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)}$, together with $\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \mathrm{Pa}(Y)}$ and A, constitute the full set of parents of Y. According to the *local Markov property*, Y is conditionally independent of all non-descendant non-parent variables given its parents, so we have the conditional independence:

$$Y \perp\!\!\!\perp_{\mathcal{G}} \mathbf{Z}_1 \setminus \mathbf{X}_{3 \in \mathrm{Pa}(Y)} \mid \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)}, \mathbf{X}_{3 \in \mathrm{Pa}(Y)}, A.$$

Then by Lemma 4.7, variables in $\mathbf{Z}_1$ outside $\mathbf{X}_{3 \in \mathrm{Pa}(Y)}$ can be excluded:

$$\sigma^2_{(\mathbf{X}_{3 \in \mathrm{Pa}(Y)}, \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)})}(P) \leq \sigma^2_{(\mathbf{Z}_1 \cup \mathbf{X}_{3 \in \mathrm{Pa}(Y)}, \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)})}(P).$$

Conclusion

Through these steps, we reduce the adjustment set to the minimal form that preserves validity and achieves minimal asymptotic variance. The final optimal adjustment set is:

$$\mathbf{O} := \mathbf{X}_{1 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{3 \in \mathrm{Pa}(Y)} \cup \mathbf{X}_{4 \in \mathrm{Pa}(Y)}.$$

□

D Experimental Details

D.1 Synthetic Experiments Data-Generating Process

Edge coefficients. For each DAG edge, coefficients are sampled independently: with 50% probability from $\mathrm{Uniform}[-1.5, -0.5]$, otherwise from $\mathrm{Uniform}[0.5, 1.5]$, ensuring non-negligible effects.

Structural equations. Each variable X_i in the DAG is generated recursively according to a structural equation model:

$$X_i = \sum_{j \in \mathrm{Pa}(i)} w_{ji}\, f_j(X_j) + \varepsilon_i,$$

where w_{ji} is the sampled edge coefficient from parent j to i. The function $f_j(\cdot)$ is applied independently to each parent variable X_j before combination and is selected uniformly at random from:

$$\text{Identity: } f(x) = x, \quad \text{Sine: } f(x) = \sin(x), \quad \text{Cosine: } f(x) = \cos(x).$$

Noise. The additive noise term ε_i is sampled independently from $\mathcal{N}(0, 1/4)$ for each node with parents. Root nodes (i.e., nodes with no parents) are sampled from $\mathcal{N}(0, 1)$.

Outcome transformation. For each simulation run, we randomly select one of the three nonlinear functions f to transform the outcome. The final outcome Y is generated by applying the chosen f to its parent variables.

Replications. We repeat this process for 50 randomly generated coefficient sets to assess the robustness of WCDE estimation. For each fixed DAG, adjustment set, and sample size, we additionally perform a Monte Carlo experiment with size 100 to estimate the empirical variance of the estimator across independent replications.

D.2 MILDEW Network

Table D.2: Variance and MSE of WCDE estimates on the MILDEW network (5 representative VASs) under different sample sizes. Variable names are abbreviated as follows: m_1: meldug_3, m_2: middel_3, k: mikro_3, k_2: mikro_2, l_1: lai_1, l_2: lai_2, l_3: lai_3.

	$n = 250$		$n = 500$		$n = 1000$		$n = 4000$		$n = 10000$	
Adj. Set	Var	MSE	Var	MSE	Var	MSE	Var	MSE	Var	MSE
$[\mathbf{m_1}, \mathbf{m_2}, \mathbf{k}]$	**0.00466**	**0.00470**	**0.00269**	**0.00269**	**0.00104**	**0.00106**	**0.00020**	**0.00021**	**0.00010**	**0.00011**
$[l_2, l_3, m_1, m_2, k]$	0.00512	0.00513	0.00301	0.00302	0.00114	0.00114	0.00027	0.00028	0.00012	0.00012
$[l_1, l_2, l_3, m, m_1, m_2, k_2, k]$	0.00609	0.00609	0.00422	0.00422	0.00248	0.00255	0.00057	0.00058	0.00026	0.00026
$[m_1, k]$	0.01445	0.01521	0.00810	0.00810	0.00451	0.00455	0.00084	0.00084	0.00042	0.00042
$[l_2, m_1]$	0.01292	0.01400	0.00873	0.00873	0.00450	0.00460	0.00082	0.00083	0.00046	0.00046
$[l_3, m_1]$	0.01409	0.01527	0.00888	0.00888	0.00463	0.00472	0.00081	0.00081	0.00047	0.00047

The MILDEW network exhibits a highly complex structure with a large number of nodes and edges, resulting in an enormous space of potential valid adjustment sets. To maintain both interpretability and representativeness, we adopted the following construction strategy. First, we identified all nodes that appear on the mediator paths between the treatment variable `mikro_1` and the outcome variable `meldug_4`. We note that this particular treatment outcome pair admits no backdoor path on MILDEW network (potentially explaining the close performance of top performing VASs.) Next, we selected relevant nodes from the parents of both the treatment and outcome. We then systematically enumerated all possible combinations of these nodes and filtered them using the adjustment criterion to ensure validity. This process yielded a total of **122** valid adjustment sets.

Table D.3: Variance and MSE of WCDE estimates on the MILDEW network (top-50 VAS ranked by $n = 10000$). **Abbreviations:** k (mikro_3), k_2 (mikro_2), l_1 (lai_1), l_2 (lai_2), l_3 (lai_3), m_1 (meldug_3), m_2 (middel_3), m (meldug_2).

	$n = 250$		$n = 500$		$n = 1000$		$n = 4000$		$n = 10000$	
Adj. Set	Var	MSE	Var	MSE	Var	MSE	Var	MSE	Var	MSE
$[m_1, m_2, k]$	0.00466	0.00470	0.00269	0.00269	0.00104	0.00106	0.00020	0.00021	0.00010	0.00011
$[l_2, m_1, m_2, k_2, k]$	0.00465	0.00465	0.00275	0.00276	0.00125	0.00125	0.00026	0.00026	0.00011	0.00011
$[l_3, m_1, m_2, k_2, k]$	0.00470	0.00471	0.00270	0.00271	0.00123	0.00124	0.00025	0.00025	0.00011	0.00011
$[l_2, m_1, m_2, k]$	0.00534	0.00534	0.00301	0.00302	0.00112	0.00112	0.00026	0.00027	0.00011	0.00012
$[l_3, m_1, m_2, k]$	0.00518	0.00520	0.00300	0.00301	0.00119	0.00119	0.00024	0.00024	0.00011	0.00012
$[l_2, l_3, m_1, m_2, k_2, k]$	0.00436	0.00436	0.00272	0.00272	0.00120	0.00120	0.00027	0.00027	0.00011	0.00012
$[m_1, m_2, k_2, k]$	0.00443	0.00453	0.00244	0.00245	0.00117	0.00121	0.00023	0.00023	0.00012	0.00012
$[l_2, m, m_1, m_2, k]$	0.00616	0.00622	0.00319	0.00320	0.00121	0.00121	0.00026	0.00027	0.00012	0.00012
$[l_2, l_3, m_1, m_2, k]$	0.00512	0.00513	0.00301	0.00302	0.00114	0.00114	0.00027	0.00028	0.00012	0.00012
$[l_3, m, m_1, m_2, k]$	0.00578	0.00587	0.00299	0.00299	0.00131	0.00131	0.00027	0.00028	0.00012	0.00012
$[m, m_1, m_2, k]$	0.00603	0.00608	0.00270	0.00271	0.00121	0.00121	0.00025	0.00027	0.00012	0.00013
$[l_2, l_3, m, m_1, m_2, k]$	0.00555	0.00563	0.00307	0.00308	0.00134	0.00134	0.00028	0.00029	0.00012	0.00013
$[l_3, m, m_1, m_2, k_2, k]$	0.00476	0.00484	0.00285	0.00285	0.00143	0.00143	0.00026	0.00026	0.00012	0.00013
$[l_2, l_3, m, m_1, m_2, k_2, k]$	0.00446	0.00449	0.00272	0.00273	0.00133	0.00134	0.00027	0.00027	0.00012	0.00013
$[m, m_1, m_2, k_2, k]$	0.00531	0.00548	0.00269	0.00271	0.00130	0.00130	0.00027	0.00028	0.00012	0.00013
$[l_2, m, m_1, m_2, k_2, k]$	0.00469	0.00474	0.00282	0.00282	0.00132	0.00132	0.00025	0.00026	0.00013	0.00013
$[l_1, m_1, m_2, k]$	0.00554	0.00555	0.00306	0.00307	0.00118	0.00119	0.00030	0.00030	0.00013	0.00013
$[l_2, m_1, m_2, k_2]$	0.00518	0.00523	0.00325	0.00326	0.00155	0.00159	0.00031	0.00031	0.00013	0.00013
$[l_2, l_3, m_1, m_2, k_2]$	0.00548	0.00552	0.00319	0.00321	0.00152	0.00154	0.00030	0.00030	0.00013	0.00013
$[l_3, m_1, m_2]$	0.00534	0.00540	0.00303	0.00304	0.00135	0.00138	0.00030	0.00030	0.00013	0.00013
$[l_2, l_3, m_1, m_2]$	0.00548	0.00550	0.00322	0.00322	0.00147	0.00149	0.00032	0.00032	0.00013	0.00014
$[l_2, m_1, m_2]$	0.00524	0.00525	0.00330	0.00330	0.00148	0.00151	0.00031	0.00031	0.00014	0.00014
$[l_3, m_1, m_2, k_2]$	0.00558	0.00566	0.00302	0.00305	0.00142	0.00145	0.00029	0.00029	0.00014	0.00014
$[l_1, m_1, m_2, k_2, k]$	0.00537	0.00538	0.00272	0.00275	0.00129	0.00131	0.00029	0.00029	0.00014	0.00014
$[l_1, m, m_1, m_2, k_2, k]$	0.00558	0.00559	0.00275	0.00276	0.00128	0.00128	0.00029	0.00031	0.00014	0.00015
$[l_3, m, m_1, m_2]$	0.00693	0.00705	0.00335	0.00335	0.00165	0.00165	0.00033	0.00033	0.00015	0.00015
$[l_1, m, m_1, m_2, k]$	0.00640	0.00640	0.00314	0.00314	0.00115	0.00117	0.00031	0.00033	0.00015	0.00016
$[l_2, m, m_1, m_2]$	0.00655	0.00667	0.00342	0.00343	0.00163	0.00163	0.00031	0.00032	0.00015	0.00015
$[l_2, l_3, m, m_1, m_2]$	0.00630	0.00642	0.00318	0.00319	0.00166	0.00166	0.00031	0.00032	0.00015	0.00016
$[l_3, m, m_1, m_2, k_2]$	0.00550	0.00566	0.00309	0.00309	0.00162	0.00163	0.00033	0.00034	0.00015	0.00016
$[l_2, m, m_1, m_2, k_2]$	0.00528	0.00543	0.00329	0.00329	0.00169	0.00169	0.00031	0.00031	0.00016	0.00016
$[l_2, l_3, m, m_1, m_2, k_2]$	0.00526	0.00538	0.00309	0.00309	0.00163	0.00163	0.00031	0.00032	0.00016	0.00016
$[l_1, l_3, m, m_1, m_2, k]$	0.00648	0.00649	0.00385	0.00385	0.00184	0.00190	0.00041	0.00042	0.00018	0.00018
$[l_1, l_3, m, m_1, m_2, k_2, k]$	0.00585	0.00585	0.00332	0.00332	0.00203	0.00207	0.00040	0.00041	0.00018	0.00018
$[l_1, l_2, m_1, m_2, k_2, k]$	0.00884	0.00885	0.00423	0.00425	0.00267	0.00275	0.00055	0.00056	0.00020	0.00020
$[l_1, l_3, m_1, m_2, k]$	0.00767	0.00767	0.00367	0.00368	0.00183	0.00190	0.00038	0.00038	0.00020	0.00020
$[l_1, l_3, m_1, m_2, k_2, k]$	0.00694	0.00694	0.00335	0.00335	0.00187	0.00194	0.00038	0.00039	0.00020	0.00020
$[l_1, l_2, m_1, m_2, k]$	0.00955	0.00955	0.00634	0.00640	0.00258	0.00267	0.00052	0.00053	0.00020	0.00020
$[l_1, l_2, l_3, m_1, m_2, k_2, k]$	0.00867	0.00870	0.00427	0.00428	0.00243	0.00255	0.00057	0.00057	0.00021	0.00021
$[l_1, l_2, l_3, m_1, m_2, k]$	0.01021	0.01021	0.00593	0.00597	0.00251	0.00265	0.00054	0.00054	0.00023	0.00023
$[l_1, l_3, m, m_1, m_2, k_2]$	0.00765	0.00766	0.00447	0.00448	0.00248	0.00249	0.00053	0.00054	0.00024	0.00025
$[l_1, l_3, m_1, m_2, k_2]$	0.00749	0.00751	0.00425	0.00425	0.00273	0.00275	0.00046	0.00046	0.00024	0.00024
$[l_1, l_3, m, m_1, m_2]$	0.00940	0.00940	0.00454	0.00455	0.00308	0.00310	0.00051	0.00052	0.00025	0.00026
$[l_1, m, m_1, m_2, k_2, k]$	0.00686	0.00686	0.00443	0.00444	0.00262	0.00269	0.00055	0.00057	0.00025	0.00025
$[l_1, l_2, l_3, m, m_1, m_2, k_2, k]$	0.00609	0.00609	0.00422	0.00422	0.00248	0.00255	0.00057	0.00058	0.00026	0.00026
$[l_1, l_3, m_1, m_2]$	0.00943	0.00944	0.00407	0.00409	0.00302	0.00304	0.00046	0.00046	0.00026	0.00026
$[l_1, l_2, l_3, m, m_1, m_2, k]$	0.00755	0.00757	0.00520	0.00520	0.00270	0.00276	0.00057	0.00058	0.00026	0.00027
$[l_1, l_2, m, m_1, m_2, k]$	0.00799	0.00799	0.00548	0.00548	0.00322	0.00335	0.00053	0.00055	0.00027	0.00027
$[l_1, l_2, m_1, m_2, k_2]$	0.01119	0.01125	0.00732	0.00739	0.00443	0.00446	0.00068	0.00069	0.00027	0.00027
$[l_1, l_2, l_3, m_1, m_2, k_2]$	0.01017	0.01036	0.00659	0.00660	0.00397	0.00401	0.00064	0.00066	0.00028	0.00028