UNDERSTANDING THE STABILITY-BASED GENERALIZATION OF PERSONALIZED FEDERATED LEARNING

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ABSTRACT

Despite great achievements in algorithm design for Personalized Federated Learning (PFL), research on the theoretical analysis of generalization is still in its early stages. Some theoretical results have investigated the generalization performance of personalized models under the problem setting and hypothesis in the convex condition, which do not consider the real iteration performance during the non-convex training. To further understand the testing performance from the theoretical perspective, we propose the first [R2: algorithm-dependent] generalization analysis with uniform stability for the typical PFL method Partial Model Personalization on smooth and non-convex objectives. In an attempt to distinguish the shared and personalized errors, we decouple the shared aggregation and the local fine-tuning progress and illustrate the interaction mechanism between the shared and personalized variables. The [R2: algorithm-dependent] generalization bounds analyze the impact of the trivial hyperparameters like learning steps and stepsizes as well as the communication modes in both Centralized and Decentralized PFL (C-PFL and D-PFL), which also concludes that C-PFL generalizes better than D-PFL. Combined with the convergence errors, we then obtain the excess risk analysis and establish the recommended early stopping point for better population risk of PFL. Promising experiments on CIFAR datasets also corroborate our theoretical results.

1 INTRODUCTION

Modern Machine Learning (ML) increasingly deals with large-scale, distributed but privacy-concerned datasets, which urgently calls for effective model collaboration from the decentralized clients with Federated Learning (FL) technologies. However, due to the statistical heterogeneity among clients, the only consensus model can not meet the needs of all local data distributions. To tackle this problem, Personalized Federated Learning (PFL), aiming to customize the local optimal model for each client, effectively design the relationships to leverage global model collaboration and satisfy the unique needs of individual clients. Partial Model Personalization is one of the most significant strategies in PFL. It decouples the model into two variables, then satisfies the individual distribution with personalized variables and leverages the collective knowledge with shared variables.

Nowadays, most theoretical works in PFL primarily focus on the convergence capability of the training progress with Empirical Risk Minimization (ERM), but only convergence analysis can not access the real performance in the testing scenario. Due to the gap between the training and testing datasets, the well-converged training model may lead to the overfitting problem in the testing dataset. Therefore, it is necessary to conduct the generalization analysis and pursue both better convergence and generalization performance to obtain the expected risk for PFL. Currently, the existing generalization analysis for PFL is mainly obtained in three ways: 1) high-probability generalization bounds with concentration inequalities based on the PAC hypothesis complexity like VC dimension complexity (Deng et al., 2020; Marfoq et al., 2022; Xie et al., 2024), Rademacher complexity (Mansour et al., 2020); 2) information-theoretical distances between the output hypothesis and the prior from PAC-Bayes generalization (Achituve et al., 2021; Zhang et al., 2022); 3) the privacy-preserving ability of the change in output hypothesis when the algorithm is exposed to attacks (Dai et al., 2022b). Most upper bounds above only depend on the problem setting and hypothesis in the convex condition, which can not apply to the commonly used non-convex functions such as neural networks and can not reflect the real iteration performance during personalized training. In other words, they are weak in evaluating the effectiveness of the algorithm design and

Table 1: [All: Main results on the stability-based generalization bounds. G is G-Lipschitz of the loss function and L, L_u, L_v and L_{uv} are smoothness of the gradient. m denotes the total clients number, n is the partial selected clients number and S is each local data amount. N is the total sample size, so $N = mS$. σ_u^2 and σ_v^2 represent the local gradient variance. μ, μ_u, μ_v are specific constants associated with $1/L, 1/L_u, 1/L_v$. $U = sup_{u,v_i,z} f(u, v_i; z)$. C_λ, λ and κ_λ are the communication topologies variables in decentralized learning.]

Algorithm	Generalization Bound	Remark
SGM (Hardt et al., 2016)	$\mathcal{O}\left(\left[\frac{2G\mu(GL+1)}{L(N-1)}\right]^{\frac{1}{\mu L+1}} T^{\frac{\mu L}{\mu L+1}}\right)$	No multi local updates.
FedProx (Chen et al., 2021)	$\mathcal{O}\left(\frac{1}{N/m} \wedge \frac{R}{\sqrt{N/m}} + \frac{\sqrt{m}}{N}\right)$	Only in convex conditions, no local training analysis.
FedAvg (Sun et al., 2024b)	$\mathcal{O}\left(\frac{T}{n}(D_{\max} + \sigma)\right) + \mathcal{O}\left(\left(\frac{\Delta_0}{Km}\right)^{\frac{1}{4}} \frac{T^{\frac{3}{4}}}{n} + \left(\Delta_0^2 \tilde{D}\right)^{\frac{1}{6}} \frac{T^{\frac{2}{3}}}{n} + \sqrt{\Delta_0} \frac{T^{\frac{1}{2}}}{n}\right)$	No local learning rate.
D-SGD (Sun et al., 2021)	$\mathcal{O}\left(\left(\frac{1+C_\lambda}{N}\right) T^{\frac{\mu L}{\mu L+1}}\right)$	No multi local updates.
D-SGD (Zhu et al., 2022)	$\mathcal{O}\left(\frac{1}{N} + \left(\frac{\lambda^2}{\sqrt{m}} + \frac{1}{m}\right)\sqrt{N}\right)$	No multi local updates.
C-PFL (Our)	$\mathcal{O}\left(\frac{4}{N}\left[\frac{G(\sigma_u L_v + \sigma_v L_u)}{L_u L_v}\right]^{\frac{1}{1+\mu L}} (nUTK)^{\frac{\mu L}{1+\mu L}}\right)$	First algorithm dependent analysis for C-PFL, D-PFL with multi local update and hyperparameter analysis.
D-PFL (Our)	$\mathcal{O}\left(\frac{4}{S}\left[\frac{\sigma_u G}{L_u m}(1 + 6\sqrt{m}\kappa_\lambda) + \frac{\sigma_v G}{L_v}(1 + \frac{6\sqrt{m}\kappa_\lambda L_{uv}}{mL_v})\right]^{\frac{1}{1+\mu L}} (UTK)^{\frac{\mu L}{1+\mu L}}\right)$	

the hyperparameter selection while building the relationship between global collaboration and local fine-tuning. Moreover, the upper generalization bound of Decentralized PFL (D-PFL) without the central server is still unexplored, whose generalization performance is related to not only the trivial factors above but also the communication topologies. Therefore, the [R2: algorithm-dependent] generalization bounds can help us understand more about the optimization progress of C-PFL and D-PFL, and it is a powerful tool to promote personalized optimization design.

To advance the theoretical understanding and obtain further optimization guidance, we present the first stability-based generalization for the typical PFL method Partial Model Personalization in non-convex conditions and evaluate the excess risk for both C-PFL and D-PFL. Though there exist several works to study the stability bounds for SGD (Hardt et al., 2016; Sun et al., 2021; Zhou et al., 2021; Sun et al., 2024a), these results cannot be directly extended to PFL due to the biased gradient estimation from multiple updates and the personalized aims. Intuitively, each shared and personalized update may introduce a specific impact on the generalization errors. Therefore, we decompose the generalization errors into aggregation errors from shared variables and fine-tuning errors from both shared and personalized variables, then establish a generalization analysis framework corresponding to the gradient estimation process of the personalized training. We list the comparisons with other stability-based generalization bounds in both centralized and decentralized learning in Tabel 1 above and comparisons with other PFL generalization bounds in Table 2 in Appendix B. As we know, it is the first work to analyze the generalization impact from personalized variables to shared variables, which uncovers the interaction mechanism between these two updating processes and provides valuable guidance for alternating personalized optimization. Moreover, we conclude that the larger learning steps, larger learning rates and denser network connections will hurt the generalization performance for both C-PFL and D-PFL, meaning that better testing performance is the trade-off between communication cost and computational efficiency. Besides, with different aggregation modes in the shared variables, we demonstrate that C-PFL generalizes better than D-PFL, which aligns with the conclusion of the generalized FL (Sun et al., 2023). Combined with the convergence analysis, we obtain the excess risk and establish the recommended stop points to achieve better performance.

The findings in the stability-based generalization for PFL help us understand more about the nature of PFL and design better personalized methods. In summary, our main contributions are as follows:

- **First work on the algorithm-dependent generalization for both centralized and decentralized PFL under non-convex conditions.**[All: We build up the stability-based generalization analysis for PFL with the biased gradient from multi local updates. It decouples the global aggregation and the local fine-tuning corresponding to the training process and our analysis establishes the interaction mechanism between them.

We also extend this analysis to the decentralized scenarios with the consideration of different communication topologies.]

- **New theoretical results for upper generalization bounds and excess risks for PFL.** Our theoretical results reveal the impacts of the trivial factors on generalization performance. Also, we analyze how communication topologies influence the upper generalization bounds of D-PFL and demonstrate that C-PFL generalizes better than D-PFL. Combined with the convergence errors, we obtain the excess risk analysis and better early stopping points.
- **Massive experiments to verify the theoretical findings of PFL.** We evaluate important factors to verify our theoretical findings on CIFAR10/100 with different models under non-convex conditions. The empirical results strongly support our theoretical insights.

2 Related Work

Generalization for PFL. PFL is proposed to find the greatest personalized models for each client (related work in Appendix A). Generalization analysis represents the performance in the unseen data of a well-train model, which is defined as the difference between the population risk and empirical risk. Various statistical methods have been introduced into PFL, including methods based on PAC-based analysis, Differential Privacy analysis, and PAC-Bayes analysis. For PAC-based analysis, Deng et al. (2020) derives the VC dimension complexity bound of a mixture of local and global models, and finds the optimal mixing parameter. Mansour et al. (2020) derives the Rademacher complexity bound of the clusters, data interpolation, and model interpolation. Chen et al. (2021) analyzed the stability and excess risk of both FL and local SGD under different data heterogeneity, but failed to extend them to the non-convex condition. For Differential Privacy analysis, Dai et al. (2022a) assumes that the algorithm satisfies (ε, δ)-differentially private condition and proposes the lower generalization bound with the noisy perturbation. For PAC-Bayes analysis, Zhang et al. (2022) gives an upper bound of averaged generalization error on the Bayesian variational inference method and illustrates that the convergence rate of the generalization error is minimax optimal up to a logarithmic factor.

Stability for generalization. The stability-based methods measure the sensitivity of the data perturbation of an algorithm via uniform stability (Bousquet & Elisseeff, 2002; Hardt et al., 2016), Bayes stability (Li et al., 2019), model stability (Lei & Ying, 2020; Liu et al., 2017), on-average stability (Lei et al., 2023; Sun et al., 2024b; Kuzborskij & Lampert, 2018), and so on. More information can be seen from the introduction in Lei et al. (2023). For the generalization bounds in FL, Lei et al. (2023) develop the stability analysis for minibatch SGD and local SGD for convex, strongly convex and nonconvex problems. [R1: Sun et al. (2024b) show that the generalization performances of FedAvg, FedProx and Scaffold are closely related to the data heterogeneity and the convergence behaviors when training.] Sun et al. (2023) discuss the better generalization performance between the Central FL and Decentralized FL. In decentralized training, Zhu et al. (2022) extend the stability-based generalization to D-SGD and discuss the topology effect of it. Zhu et al. (2024) refine the stability analysis for the minimax problem in a decentralized manner.

Nowadays, almost all upper generalization bounds of PFL based on the complexity theory ignore the impact of algorithm design and the iteration nature. Therefore, we try to propose the stability-based generalization analysis to answer how algorithm design and hyperparameter selection impact the generalization capacity. Meanwhile, we extend the non-trivial analysis to D-PFL with various communication network topologies. Extensive experiments also corroborate our theoretical findings.

3 Problem Formulation

In this section, we first propose the problem setup for C-PFL and D-PFL. Then we present the uniform stability for generalization error and combine it with convergence error to obtain the excess risk.

3.1 Problem Setup

Personalized Federated Learning. Compared to the typical FL problem, PFL focuses on the average minimization with the personalized models rather than the consensus one. Partial Model Personalization is one of the most significant strategies for PFL, which decouples the model as the personalized variables to satisfy the individual requirements and the shared variables to leverage the collective knowledge. The optimization of Partial Model Personalization is defined as follows:

$$\min_{u,V} \quad f(u,V) := \frac{1}{m}\sum_{i=1}^{m} f_i\left(u, v_i\right), \quad where \quad f_i\left(u, v_i\right) = \mathbb{E}_{\xi \sim \mathcal{D}_i} F\left(u, v_i; \xi_i\right). \tag{1}$$

We consider the typical setting with m clients, where each client i owns the local training data ξ_i and it satisfies the data distribution $\mathcal{D}_i$. For each client, the machine learning model $w_i \in \mathbb{R}^d$ are partitioned into two parts: the *shared* parameters $u \in \mathbb{R}^{d_u}$ and the *personalized* parameters $v_i \in \mathbb{R}^{d_i}$ for $i = 1, \ldots, m$. To simplify the presentation, we denote $V = (v_1, \ldots, v_m) \in \mathbb{R}^{d_1+\ldots+d_m}$. So the full model on client i is denoted as $w_i = (u, v_i)$. f_i is the loss function for each client, and F denotes the loss function for each client trained on the specific data ξ_i.

From the perspective of engineering purposes, we set the feature extraction layers (close to the input) as the shared variables and the linear classification layers (close to the output) as the personalized variables as Arivazhagan et al. (2019); Collins et al. (2021); Pillutla et al. (2022). Meanwhile, we alternately update the shared and personalized variables to distinguish the generalization effects between them explicitly. Algorithm 1 illustrates the specific update process. We set ∇_u and ∇_v as stochastic gradients of the shared variables u and the personal variables v_i respectively. Personal variables v_i first perform the local updating with the shared variable u fixed in Line 2, then the shared variable u updates with the personal variables v_i fixed in Line 5.

Algorithm 1: Local updating for PFL.

Input: Local steps K, local learning rate η_u and η_v, initialize $u_{i,0}^t = u^t$, and $v_{i,0}^t = v_i^t$.
Output: For each client, locally update u_i^{t+1}, v_i^{t+1}.

1 **for** *local update round* $k = 0, 1, ..., K_v - 1$ **do**
2 $\quad v_{i,k+1}^t \leftarrow v_{i,k}^t - \eta_v \nabla_v F(u_{i,0}^t, v_{i,k}^t, \xi_{i,k}^t)$.
3 **end**
4 **for** *local update round* $k = 0, 1, ..., K_u - 1$ **do**
5 $\quad u_{i,k+1}^t \leftarrow u_{i,k}^t - \eta_u \nabla_u F(u_{i,k}^t, v_{i,K_v}^t, \xi_{i,k}^t)$.
6 **end**
7 $u_i^{t+1} \leftarrow u_{i,K_u}^t$, $v_i^{t+1} \leftarrow v_{i,K_v}^t$.

Algorithm 2: C-PFL and D-PFL.

Input: Total communication rounds T, number of selected clients n, initial the shared and personal variables u^0, $\mathbf{v}^0 = \{v_i^0\}_{i=0}^n$.
Output: Personal solution u^T and $\mathbf{v}^T = \{v_i^T\}_{i=0}^n$.

1 C-PFL:
2 **for** communication round $t = 0$ **to** $T-1$ **do**
3 $\quad$ Sample clients $|S^t| = n$ uniformly randomly and distribute the shared variables u^t.
4 $\quad$ **for** *client* $i \in S^t$ *in parallel* **do**
5 $\quad\quad u_i^{t+1}, v_i^{t+1} \leftarrow$ Local updating (u_i^t, v_i^t)
6 $\quad$ **end**
7 $\quad u^{t+1} \leftarrow$ [R3 : $\frac{1}{n}\sum_{i\in s^t} u_i^{t+1}$].
8 **end**
9 D-PFL:
10 **for** communication round $t = 0$ **to** $T-1$ **do**
11 $\quad$ **for** *client* $i \in [m]$ *in parallel* **do**
12 $\quad\quad u_i^{t+1}, v_i^{t+1} \leftarrow$ Local updating (u_i^t, v_i^t)
13 $\quad$ **end**
14 $\quad$ Receive shared variables u_i^{t+1} with matrix W: $u_{i,0}^{t+1} \leftarrow$ [R3 : $\sum_{l\in\mathcal{G}(i)} w_{i,l} u_l^{t+1}$] .
15 **end**

C-PFL and D-PFL. We consider both C-PFL and D-PFL in Algorithm 2. For C-PFL, the only central server first distributes the shared variables u^t to the n selected clients in Line 3, then aggregates the updated shared variables u_i^{t+1} to u^{t+1} from the selected clients in Line 7. Different from the general FL, partial model personalization only aggregates the shared variables u_i in the central server, while keeping the personal variables v_i on the client side. We focus on the case of the averaged aggregation, which means $\alpha_i = 1/n$. For D-PFL, it allows clients to communicate with their neighbors in a peer-to-peer manner without the central server. The communication can be modeled as an undirected connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{V}, \boldsymbol{W})$, where $\mathcal{N} = \{1, 2, \ldots, m\}$ is the set of all clients, $\mathcal{V} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of communication channels, and the gossip/mixing matrix $\boldsymbol{W}$ present as below records whether the communication connects or not between any two clients. [R3: Set $\mathcal{G}_i$ as the neighbors set for each client in the undirected graph.]

Definition 1 (**The gossip/mixing matrix** (Nedic & Ozdaglar, 2009))**.** *The gossip matrix* $\mathbf{W} = [w_{i,j}] \in [0,1]^{m\times m}$ *is assumed to have these properties: (i) (Graph) If* $i \neq j$ *and* $(i,j) \notin \mathcal{V}$, $w_{i,j} = 0$, *otherwise,* $w_{i,j} > 0$*; (ii) (Symmetry)* $\mathbf{W} = \mathbf{W}^\top$*; (iii) (Null space property)* $\mathrm{null}\{\mathbf{I} - \mathbf{W}\} = \mathrm{span}\{\mathbf{1}\}$*; (iv) (Spectral property)* $\mathbf{I} \succeq \mathbf{W} \succ -\mathbf{I}$. Under these properties, the eigenvalues of $\mathbf{W}$ satisfies $1 = \lambda_1(\mathbf{W})) > \lambda_2(\mathbf{W})) \geq \cdots \geq \lambda_m(\mathbf{W})) > -1$. And $\lambda := \max\{|\lambda_2(\mathbf{W})|, |\lambda_m(\mathbf{W}))|\}$ and $1 - \lambda \in (0, 1]$ is the spectral gap of $\mathbf{W}$, which usually measures the degree of the network topology.

3.2 STABILITY AND EXCESS RISK

Generalization Stability. Recalling the unseen data distribution $\mathcal{D}_i$ in the population risk function in Formula (1), we select the sample ξ_i from the local datasets $\mathcal{S}_i$ and estimate the expectation to represent the real distribution. The training process is rewritten as the Empirical Risk Minimization:

$$\min_{u,V} \quad f(u,V) := \frac{1}{m}\sum_{i=1}^{m} f_i\left(u, v_i\right), \quad where \quad f_i\left(u, v_i\right) = \frac{1}{\mathcal{S}}\sum_{\xi_i \in \mathcal{S}_i}\left[F\left(u, v_i; \xi_i\right)\right]. \tag{2}$$

Assuming the joint datasets of local dataset $\mathcal{S}_i$ as $\mathcal{S}$, we consider a solution $\mathcal{A}(\mathcal{S})$ of a specific algorithm $\mathcal{A}$ trained on the joint dataset $\mathcal{S}$, the generalization error between the population risk in (1) and empirical risk in (2) can be defined as $\varepsilon_G = \mathbb{E}_{\mathcal{S},\mathcal{A}}[F(\mathcal{A}(\mathcal{S})) - f(\mathcal{A}(\mathcal{S}))]$. This joint impact caused by both the algorithm $\mathcal{A}$ and the datasets $\mathcal{S}$ may cause a bad performance from a well-trained model on the testing dataset, which is called overfitting. Motivated by the previous studies in Hardt et al. (2016), we use the uniform stability bound to explore the generalization performance of PFL.

Definition 2. *(**Uniform Stability**) Considering a new joint dataset $\widetilde{\mathcal{S}}$, which differs from the vanilla dataset $\mathcal{S}$ at most one data sample z. The ε-uniformly stability for algorithm $\mathcal{A}$ is defined as below:*

$$\sup_{z_j \sim \{\mathcal{D}_i\}} \mathbb{E}[f(u^T, V^T; z_j) - f(\widetilde{u}^T, \widetilde{V}^T; z_j)] \leq \epsilon. \tag{3}$$

The generalization error can be bound by $\epsilon_G \leq \epsilon$, if the algorithm $\mathcal{A}$ satisfies the ε-uniformly stability.

Excess Risk. Considering (u^*, V^*) as the optimal model that can be achieved by the algorithm $\mathcal{A}$ on the dataset $\mathcal{S}$, the real test performance $\mathbb{E}[F(\mathcal{A}(\mathcal{S}))]$ can be measured by the excess risk as follows:

$$\mathcal{E}_E = \mathbb{E}[F(\mathcal{A}(\mathcal{S}))] - \mathbb{E}[f(w^*)] = \underbrace{\mathbb{E}[F(\mathcal{A}(\mathcal{S})) - f(\mathcal{A}(\mathcal{S}))]}_{\mathcal{E}_G\text{: generalization error}} + \underbrace{\mathbb{E}[f(\mathcal{A}(\mathcal{S})) - f(u^*, V^*)]}_{\mathcal{E}_O\text{: optimization error}}. \tag{4}$$

Actually, if the optimal model (u^*, V^*) could fit the dataset well, the loss function $\mathbb{E}[f(u^*, V^*)]$ will tend to zero when the training time is large enough. Therefore, the real risk of the well-trained model (u, V) on the test dataset can be bounded by the generalization and optimization error. $\mathcal{E}_G$ represents the performance risk of (u, V) between the training dataset and testing dataset, while $\mathcal{E}_O$ means the empirical risk between the theoretical optimum (u^*, V^*) and the obtained one (u, V). Previous studies focus on the optimization error ε_O of general C-PFL and D-PFL, but there is little work to discuss the generalization nature for them. To further understand the optimization progress of the algorithm design and its iteration nature, we provide a comprehensive analysis of their excess risks.

3.3 BASIC ASSUMPTIONS

Assumption 1 (Smoothness). *For each client $i = \{1, \ldots, m\}$, the function F is continuously differentiable. There exist constants L_u, L_v, L_{uv}, L_{vu} such that for each client $i = \{1, \ldots, m\}$:*

- *$\nabla_u F(u_i, v_i)$ is L_u–Lipschitz with respect to u_i and L_{uv}–Lipschitz with respect to v_i*
- *$\nabla_v F(u_i, v_i)$ is L_v–Lipschitz with respect to v_i and L_{vu}–Lipschitz with respect to u_i.*

Assumption 2 (Bounded Variance). *The stochastic gradients in both C-PFL and D-PFL have bounded variance. That is to say, for all u_i and v_i, there exist constants σ_u and σ_v such that:*

$$\mathbb{E}\left[\left\|\nabla_u[\mathsf{R2: F(u_i, v_i; \xi_i) - \nabla_u F(u_i, v_i)}]\right\|^2\right] \leq \sigma_u^2, \mathbb{E}\left[\left\|\nabla_v[\mathsf{R2: F(u_i, v_i; \xi_i) - \nabla_v F(u_i, v_i)}]\right\|^2\right] \leq \sigma_v^2. \tag{5}$$

Assumption 3 (G-Lipschitz). *For $\mathcal{A}\left(\mathcal{S}\right), \mathcal{A}(\widetilde{\mathcal{S}}) \in \mathbb{R}^d$ which are well trained by an ϵ-uniformly stable algorithm $\mathcal{A}$ on dataset $\mathcal{S}$ and $\widetilde{\mathcal{S}}$, the personalized objective $f(u, V)$ satisfies G-Lipschitz continuity between them:*

$$\|f(\mathcal{A}(\mathcal{S})) - f(\mathcal{A}(\widetilde{\mathcal{S}}))\| \leq G\|\mathcal{A}(\mathcal{S}) - \mathcal{A}(\widetilde{\mathcal{S}})\|. \tag{6}$$

Assumptions 1 and 2 are mild and commonly used in the convergence analysis of FL and PFL (Liu et al., 2024; Chen et al., 2023; Shi et al., 2023; Li et al., 2023; Pillutla et al., 2022; Sun et al., 2022; Deng et al., 2020; Reddi et al., 2021). Assumption 3 is a variant of the vanilla Lipschitz continuity assumption, which is widely used in the uniform stability analysis (Elisseeff et al., 2005; Hardt et al., 2016; Zhou et al., 2021; Zhu et al., 2022; Sun et al., 2023; 2024a).

4 Theoretical Analysis

4.1 Stability and Excess Risk for Centralized Personalization

In this part, we first provide the stability analysis of PFL with the centralized server in the non-convex objectives. Then we combine its convergence performance to conduct the excess risk analysis.

Theorem 1 (**Stability of C-PFL**). *Under Assumption 1∼ 3, let the active ratio per communication round be n/m, and assume the learning rates satisfy $\eta_u = \mathcal{O}\left(\frac{1}{tK_u+k}\right) = \frac{\mu_u}{tK_u+k}$ and $\eta_v = \mathcal{O}\left(\frac{1}{tK_v+k}\right) = \frac{\mu_v}{tK_v+k}$. They decay per iteration $\tau = tK + k$, where μ_u and μ_v are the specific constants and satisfy $\mu_u \leq \frac{1}{L_u}$ and $\mu_v \leq \frac{1}{L_v}$.* [R4: Let $U = sup_{u,v_i,z} f(u, v_i; z)$,] *then the generalization bound of C-PFL satisfies:*

$$\begin{aligned}
&\mathbb{E}\left[\|f(u^T, V^T; z_j) - f(\widetilde{u}^T, \widetilde{V}^T; z_j)\|\right] \\
&\leq \frac{nU\tau_0}{mS} + \left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u} \frac{2G\sigma_u}{mSL_u} + \left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v} \left(1 + \frac{L_{uv}}{L_u}\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\right) \frac{2G\sigma_v}{mSL_v}.
\end{aligned} \tag{7}$$

To simplify subsequent analysis, we assume $\mu L = \max\{\mu_u L_u, \mu_v L_v\}$ and $K = \max\{K_u, K_v\}$. By selecting $\tau_0 = \left[\frac{2G(\sigma_u L_v + \sigma_v L_u)}{nUL_u L_v}\right]^{\frac{1}{1+\mu L}} (TK)^{\frac{\mu L}{1+\mu L}}$, we can minimize the bound with τ_0:

$$\mathbb{E}\left[\|f(u^T, V^T; z_j) - f(\widetilde{u}^T, \widetilde{V}^T; z_j)\|\right] \leq \frac{4}{mS}\left[\frac{G(\sigma_u L_v + \sigma_v L_u)}{L_u L_v}\right]^{\frac{1}{1+\mu L}} (nUTK)^{\frac{\mu L}{1+\mu L}}. \tag{8}$$

Remark 1 (**Influencal factors of C-PFL**). *From the stability-based generalization analysis above, the number of samples set S, the selected clients each round n, the total participated clients m as well as the total iterations TK_u and TK_v greatly influence the stability of C-PFL. More selected clients n and more local epochs K_u and K_v increase the time of training on only different samples, which leads to a larger generalization gap and worse generalization performance. In contrast, the generalization gap can be alleviated with more total clients m and the number of samples S involved.*

Remark 2 (**Special cases of C-PFL**). *If we remove all personal variables v_i, the problem (2) degenerates to the classical FL problem FedAvg. The stability reduces to $\mathcal{O}\left((nK_uT)^{\frac{\mu_u L_u}{1+\mu_u L_u}}/m\right)$ by removing the K_v and σ_v in the proof, which is compatible with the upper bound $\mathcal{O}\left((nKT)^{\frac{\mu L}{1+\mu L}}/m\right)$ of the stability of central FL algorithm FedAvg (Sun et al., 2023) with multiple local update. That is to say, the upper bound of the stability is only related to the training paradigm, no matter whether training for the consensus model or the personalized models. This finding builds the bridge between the stability of Federated Learning and Personalized Federated Learning.* [R1: If we remove all shared variables u, the stability of C-PFL in eq. (7) can be reduced to $\mathcal{O}\left((nK_vT)^{\frac{\mu_v L_v}{1+\mu_v L_v}}/mS\right)$, which is the stability bound of the whole FL system with partial participation ratio n/m and local updates K_v. For further degradation, we set full participation $n/m = 1$ and only one local update $K_v = 1$, our results can degrade to $\mathcal{O}\left(T^{\frac{\mu L}{1+\mu L}}/S\right)$ on each client, which are align with the vanilla SGD in (Hardt et al., 2016).]

Remark 3 (**Comparison with the other generalization of C-PFL**). *The generalization analysis compared in Table 2 in Appendix B calculates the complexity of the PAC problem in infinite space as the generalization error, which is mainly related to the total number of clients m in the PFL training. Although complexity-based generalization considers the nature of the learning problem, it cannot analyze the impact of algorithm design and its iterative nature on the generalization bound. Therefore, we highlight our contributions to the proposed stability-based generalization bound: 1) conduct the generalization analysis in the non-convex condition, which is based on the more realistic* *assumptions adapted to the neural networks; 2) analyze the impacts of the algorithm design and the hyperparameters selection of the number of samples S, the number of selected clients n, total clients m, total iterations TK_u and TK_v and the local learning rates η_u and η_v; 3) illustrate the error propagation process between model aggregation and local training with the iteration nature, which provides a reference for the choice of early stopping points when training.*

Corollary 1 (Excess risk of central partial model personalization). *Pillutla et al. (2022) provide the upper convergence bounds of $\varepsilon_O = \mathbb{E}\left[f(w^T) - f(w^\star)\right]$ on non-convex smooth objectives, which propose the convergence rates of C-PFL are dominated by $\mathcal{O}\left(1/\sqrt{T}\right)$ rate. Therefore, when the number of dataset samples S is fixed, the excess risks of C-PFL are dominated by $\mathcal{O}\left(1/\sqrt{T} + (nKT)^{\frac{\mu L}{1+\mu L}}/m\right)$. Both terms are caused by the stochastic variance σ_u and σ_v.*

Remark 4 ([R2: Influential] factors of the centralized excess risks). *Our analysis shows that the excess risk of the centralized partial personalization is decided by the number of active clients n, the local interval K_u, K_v, the total communication rounds T, the total clients m, the smoothness constants L and the gradient variance σ. Assume in an analysis of data distribution with the specific algorithm, the smoothness constants L, the gradient variance σ and the total client number m are fixed. Therefore, we can adjust the hyperparameters n, K_u, K_v and T to optimize the testing performance during training.* [R2: Though we can not present the optimal choice of hyperparameters due to the inability of a lower bound, we can recommend some strategies for better test performance. The good choice of the number of active clients n and the local interval K_u, K_v are the same as that in the stability analysis. But the time of the better theoretical performance is a trade-off of the total communication rounds T in central partial personalization. We meticulously provide the recommended training rounds T to achieve better efficiency. Increasing the communication rounds T leads to better convergence but worse generalization performance, which accounts for the training over-fitting. With a fixed local interval K_u, K_v and the number of the selected clients n, the better stopping point T of C-PFL satisfies $T^\star = \mathcal{O}\left(m^{\frac{1+\mu L}{1+2\mu L}}/nK\right)$, which could efficiently make a trade-off between the convergence error and generalization error to obtain the better excess risk.]

4.2 Stability and Excess Risk for Decentralized Personalization

In this section, we first provide the stability analysis of PFL with the peer-to-peer communication in the non-convex objectives. Then we combine its convergence performance to obtain the excess risk.

Theorem 2 (Stability for D-PFL). *Under Assumption 1∼ 3, let clients communicate with each other in a peer-to-peer manner, and assume the learning rates satisfy $\eta_u = \mathcal{O}\left(\frac{1}{tK_u+k}\right) = \frac{\mu_u}{tK_u+k}$ and $\eta_v = \mathcal{O}\left(\frac{1}{tK_v+k}\right) = \frac{\mu_v}{tK_v+k}$. They decay per iteration $\tau = tK + k$, where μ_u and μ_v are the specific constants and they satisfy $\mu_u \leq \frac{1}{L_u}$ and $\mu_v \leq \frac{1}{L_v}$.* [R4: Let $U = sup_{u,v_i,z} f(u, v_i; z)$,] *then the generalization bound of D-PFL satisfies:*

$$\begin{aligned}&\mathbb{E}\left[\|f(u^T, V^T; z_j) - f(\widetilde{u}^T, \widetilde{V}^T; z_j)\|\right]\\ &\leq \frac{U\tau_0}{S} + \frac{2\sigma_u G}{SL_u}\left(\frac{1+6\sqrt{m}\kappa_\lambda}{m}\right)\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u} + \frac{2\sigma_v G}{SL_v}\left(1+\frac{6\sqrt{m}\kappa_\lambda}{m}\left(\frac{L_{uv}}{L_v}\right)\right)\left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v}\end{aligned} \tag{9}$$

[R2: where $\kappa_\lambda = \left(\frac{\alpha}{e}\right)^\alpha \frac{1}{\lambda\left(\ln\frac{1}{\lambda}\right)^\alpha} + \frac{2^\alpha}{(1-\alpha)e\lambda\ln\frac{1}{\lambda}} + \frac{2^\alpha}{\lambda\ln\frac{1}{\lambda}}$ and λ are the widely used coefficient to measure different communication connections.]

To simplify subsequent analysis, we assume $\mu L = \max\{\mu_u L_u, \mu_v L_v\}$ and $K = \max\{K_u, K_v\}$. By selecting $\tau_0 = \left[\frac{2G\sigma_u L_v^2(1+6\sqrt{m}\kappa_\lambda)+2G\sigma_v L_u L_{uv}(m+6\sqrt{m}\kappa_\lambda)}{UmL_u L_v^2}\right]^{\frac{1}{1+\mu L}}(TK)^{\frac{\mu L}{1+\mu L}}$, we can minimize the upper generalization bound:

$$\begin{aligned}&\mathbb{E}\left[\|f(u^T, V^T; z_j) - f(\widetilde{u}^T, \widetilde{V}^T; z_j)\|\right]\\ &\leq \frac{4}{S}\left[\frac{\sigma_u G}{L_u m}(1+6\sqrt{m}\kappa_\lambda) + \frac{\sigma_v G}{L_v}(1+\frac{6\sqrt{m}\kappa_\lambda L_{uv}}{mL_v})\right]^{\frac{1}{1+\mu L}}(UTK)^{\frac{\mu L}{1+\mu L}}.\end{aligned} \tag{10}$$

Remark 5 ([R2: Influential] **factors of the decentralized stability**). *The stability of D-PFL is impacted by the number of samples S, total clients m, and total iterations TK_u and TK_v. Besides, it is also decided by the communication topology. κ_λ is a widely used coefficient related to the λ that could measure different connections in the topology, significantly associated with the number of participation clients m. Figure 2 and Table 3 in Appendix C show the different topological diagrams and properties. For $\kappa_\lambda = \left(\frac{\alpha}{e}\right)^\alpha \frac{1}{\lambda\left(\ln\frac{1}{\lambda}\right)^\alpha} + \frac{2^\alpha}{(1-\alpha)e\lambda\ln\frac{1}{\lambda}} + \frac{2^\alpha}{\lambda\ln\frac{1}{\lambda}}$, when $\lambda \to 1$, the upper bound for κ_λ is mainly decided by $\mathcal{O}\left(1/(\lambda\left(\ln\frac{1}{\lambda}\right))\right)$, when $\lambda \to 0$, the upper bound for κ_λ is mainly decided by $\mathcal{O}\left(1/(\lambda\left(\ln\frac{1}{\lambda}\right)^\alpha)\right)$. We can clearly see that denser communication topology with a smaller κ_λ, leads to better generalization performance. Therefore, the fully connected topology achieves the best generalization performance of shared variables and is compatible with the central ones.*

Remark 6 (**Special cases of the decentralized partial personalization**). *If we remove all personal variables v_i, the problem 2 degenerates to the classical DFL algorithm DFedAvg. The stability reduces to $\mathcal{O}\left((1+6\sqrt{m}\kappa_\lambda/m)^{\frac{1}{1+\mu_u L_u}}(K_u T)^{\frac{\mu_u L_u}{1+\mu_u L_u}}\right)$ by removing all personal constants in the proof, which is compatible with the upper bound $\mathcal{O}\left((1+6\sqrt{m}\kappa_\lambda/m)^{\frac{1}{1+\mu L}}(KT)^{\frac{\mu L}{1+\mu L}}\right)$ of the stability of decentralized federated learning DFedAvg (Sun et al., 2023) with multiple local update.*

Remark 7 (**Comparison with the other generalization of D-PFL**). *The mere generalization analysis of D-PFL can be seen in Dis-PFL (Dai et al., 2022b), which acquires a generalization lower bound through the lens of differential privacy with the inspiration in He et al. (2021). It describes the relationship between the remaining model and generalization performance at each iteration point and suggests that a more sparse network leads to better generalization performance. However, the understanding of the algorithm design and the impacts of training parameters is still limited, especially lack of the analysis of the communication topologies in decentralized learning.*

Corollary 2 (**Excess risk of decentralized partial model personalization**). *Shi et al. (2023) provide the analysis of $\varepsilon_O = \mathbb{E}\left[f(w^T) - f(w^\star)\right]$ on non-convex smooth objectives. The convergence rates of decentralized partial model personalization are dominated by $\mathcal{O}\left(1/(1-\lambda)^2\sqrt{T}\right)$ rate. Therefore, when the number of dataset samples S is fixed, the excess risks of D-PFL are dominated by $\mathcal{O}\left(1/(1-\lambda)^2\sqrt{T} + (1+6\sqrt{m}\kappa_\lambda/m)^{\frac{1}{1+\mu L}}(KT)^{\frac{\mu L}{1+\mu L}}\right)$. Discussions are as follows.*

Remark 8 ([R2: Influential] **factors of the decentralized excess risks**). *Our analysis shows that the excess risk of D-PFL is highly influenced by the number of the local interval K_u, K_v, the total communication rounds T, the total clients m, the smoothness constants L and the gradient variance σ, and the communication topologies λ and κ_λ (More details about the communication topologies can be seen in Table 3). Assuming that the total client number m is fixed under the specific algorithm and data distribution (with the fixed smoothness constants L and the gradient variance σ), we can adjust the communication networks λ and k_λ, local interval K and the stopping point T to optimize the testing performance. A denser connection (smaller κ_λ and smaller $\frac{1}{1-\lambda}$) means better convergence performance and generalization performance, but it brings more communication cost. The better choice for local interval K_u, K_v is the same as that in stability analysis. However, the better theoretical stopping point T for the testing performance of D-PFL is a trade-off between the convergence error and the generalization error. Under a fixed local interval K_u, K_v and communication connections λ, the better stopping time T satisfies $T^\star = \mathcal{O}\left((1-\lambda)^{\frac{-2(1+\mu L)}{\mu L}}(1+6\sqrt{m}\kappa_\lambda/m)^{\frac{-1}{\mu L}})/K\right)$.*

Remark 9 (**Comparisions between the C-PFL and D-PFL**). *From the comparison between Theorem 1 and Theorem 2, we can clearly see that C-PFL always converges and generalizes better than D-PFL. The centralized mode largely reduces the propagations of the generalization error, which benefits from the regular averaging on a global server for the shared variables. That is to say, the global server helps C-PFL methods achieve a high level of shared consensus for better generalization performance throughout the training process. This conclusion is also consistent with the generalization analysis in the typical FL (Sun et al., 2023), which shares an identical aggregation process. However, to achieve a more reliable performance, the number of active clients n in C-PFL must satisfy at least a polynomial order of m. It means that the high communication costs are unavoidable when the whole federated system m gets larger. Also, the communication burden in the central server becomes a big challenge in the training progress. Therefore, the suitable choice between C-PFL and D-PFL or the choice of different communication topologies in real scenarios is a trade-off among communication ability, communication cost and personalized performance.*

5 Experiments

In this section, we conduct extensive experiments to verify the theoretical findings. We first introduce the typical setting for experiments, then present the empirical results and corresponding analysis.

5.1 Empirical Setup

We conduct the experiments on CIFAR-10 datasets (Krizhevsky et al., 2009) in the Dirichlet distribution (Non-IID $\alpha = 0.3$) (Hsu et al., 1909) with ResNet-18(He et al., 2016) and CIFAR-100 datasets (Krizhevsky et al., 2009) in the Pathological distribution (Non-IID $c = 20$) with VGG-11 (Simonyan & Zisserman, 2014) for both C-PFL and D-PFL. [All: Experiments on CIFAR-100 are placed in Appendix D.2]. To verify the impacts of the key hyperparameters, we follow (Hardt et al., 2016; Zhu et al., 2022; 2024) and study the parameter distance when disturbing only one data, the generalization gap of the difference between training and testing error, and testing performance during training. We explore the impact of the four factors: 1) Local Learning Epochs, 2) Local Learning Rates, 3) Client Fraction / Communication topology, 4) Total Client Number. In each study, we keep the same sets for the other factors for fairness. More implementation details can be seen in Appendix D.1.

5.2 Empirical Analysis

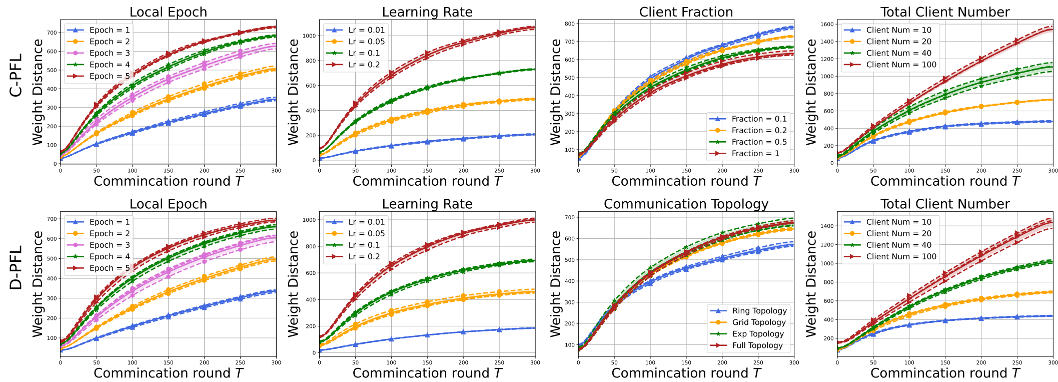
We show the changing trends of parameter distance in Figure 1a, generalization error between testing and training loss in Figure 1b, and real testing performance in Figure 1c for both C-PFL and D-PFL. From the empirical results, we present the conclusions as below:

Both less local learning epochs and lower learning rates lead to better generalization performance, but they affect the convergence speed more seriously. We discuss this phenomenon in the first two columns for Learning Epoch and Learning Rate in Figure 1a-1c. Increasing local learning epochs and learning rates means amplifying the model distance when learning on different samples. It brings about a larger generalization error and more severe fluctuation in the comparisons. However, less local learning epochs and lower local learning rates mean slower learning efficiency, which has a greater impact on model convergence while training. Combined with the testing accuracies in Figure 1c, larger local learning epochs and suitable learning rates help to achieve better performance.

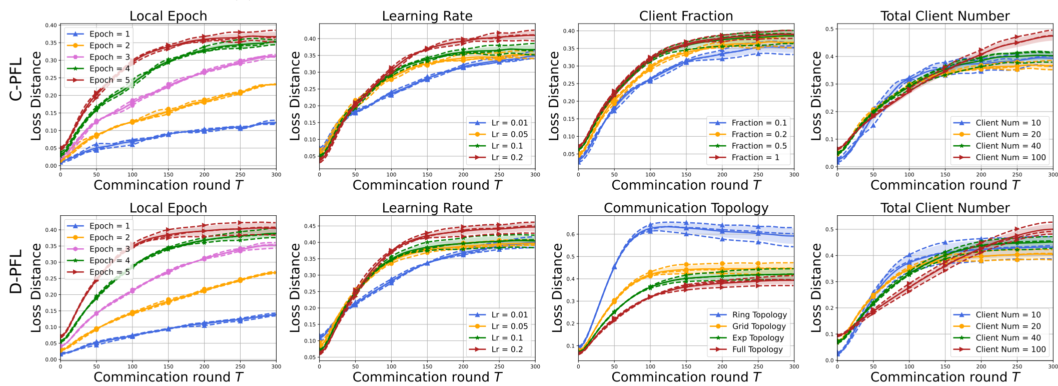
More client participation and denser network connection in each communication round enlarge the generalization gap, but they speed up the convergence rate to the same extent. We discuss this phenomenon in the third column for Client Selection and Communication Topology in Figure 1a-1c. From the weight distance and generalization errors in Figure 1a and 1b, increasing the fraction of client selection and choosing denser connection topologies means more frequency to learn unique samples, which enlarges the generalization gap between two models. This is aligned with our theoretical findings in Theorem 1. Moreover, from the testing performance in Figure 1c, the testing accuracies reach the same level despite the sparsest client participation condition, which means the convergence errors are affected almost to the same extent as the generalization errors. Therefore, it is a trade-off between the communication cost and the personalized performance in real-life scenarios.

A larger total participation clients and a smaller number of local training samples increase the generalization error and reduce the convergence speed simultaneously. We discuss this phenomenon in the fourth column for Total Client Number in Figure 1a-1c. Since the total data number on CIFAR-10 dataset remains the same, bigger participation clients mean fewer training samples per client. The generalization gaps get worse with the number of clients increasing in Figure 1b. Also, fewer training samples have an negative impact on the testing performance in Figure 1c.

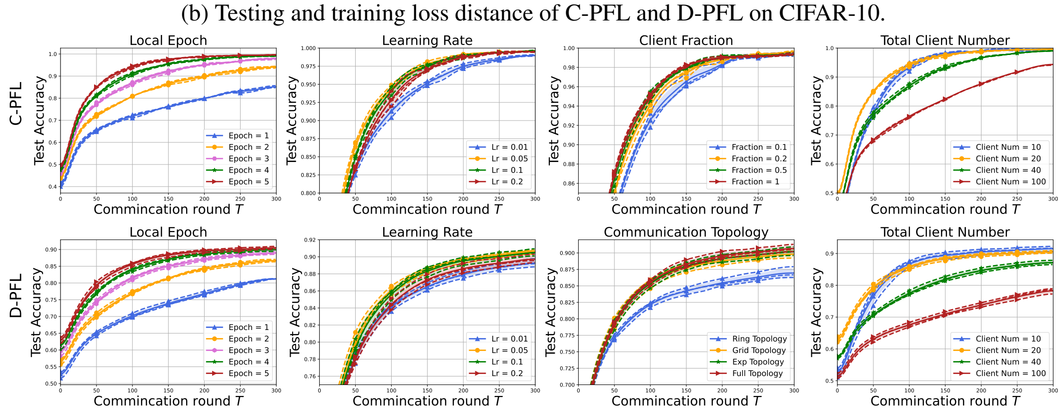
C-PFL outperforms D-PFL in both generalization performance and convergence performance when their upper communication bandwidths are at the same level. We discuss this difference in the comparisons of each line in Figure 1a-1c. Maintaining the maximum communication capacity of the busiest node, the central server in C-PFL helps mitigate the inconsistencies driven by the updates on different samples, which is consistent with the conclusions drawn from our theoretical analysis.



(a) Disturbed loss distance of C-PFL and D-PFL on CIFAR-10.



(b) Testing and training loss distance of C-PFL and D-PFL on CIFAR-10.



(c) Testing accuracies of C-PFL and D-PFL on CIFAR-10.

Figure 1: Empirical results of C-PFL (first line) and D-PFL (second line) on CIFAR-10.

6 Conclusion

In this paper, we develop the first stability-based generalization bounds and the corresponding excess risk analysis for PFL in centralized and decentralized scenarios under non-convex conditions. Compared with the previous works, the proposed analysis studies the impact of algorithm design and hyperparameter selection on each iteration point. Combined with the convergence errors, we obtain an early stopping point for better population risk. Various experiments verify our theoretical findings.

Limitation. Despite the contributions above, there are numerous avenues for future works: 1) improve the generalization bounds for C-PFL and D-PFL with the more advanced stability methods; 2) build up the bridge between the generalization bound and data heterogeneity analysis for PFL.

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Supplementary Material for " Understanding the Stability-based Generalization of the Personalized Federated Learning "

In this part, we provide the supplementary materials to prove the main theorem.

- **Appendix** A: Related Work about PFL.
- **Appendix** B: Detailed Comparisions of Generalization.
- **Appendix** C: Communication Network Topologies.
- **Appendix** D: Implementation Details and Results for Experiments.
- **Appendix** E: Generalization Bounds for C-PFL and D-PFL.

A Related Work about PFL.

Personalized Federated Learning. PFL aims to produce the optimal personalized models for each client via model decoupling (Arivazhagan et al., 2019; Collins et al., 2021), knowledge distillation (Li & Wang, 2019; Lin et al., 2020), multi-task learning (Huang et al., 2021; Shoham et al., 2019), model interpolation (Deng et al., 2020; Diao et al., 2020) and clustering (Ghosh et al., 2020; Sattler et al., 2020). More details can be referred to the PFL survey (Tan et al., 2022). Among them, the model decoupling method Partial Model Personalization, which divides the model into shared variables and personal variables, has proved to achieve better performance than full model personalization with fewer shared parameters. LG-FedAvg (Liang et al., 2020) relieves the data variance and device variance with jointly learning compact local representations on each device and a global model across all devices. FedPer (Arivazhagan et al., 2019), FedRep (Collins et al., 2021) and FedBABU (Oh et al., 2021) set the feature extractor as the shared variable and the linear classifiers as the personal variables. They are different from the optimization progress between the shared representation and the private linear parts. Fed-RoD (Chen & Chao, 2021) trains a global full model and many private classifiers with empirical risk minimization and balanced risk minimization. Most theoretical analyses for Partial Model Personalization mainly focus on their convergence performance. FedSim & FedAlt (Pillutla et al., 2022) provide the convergence analyses in the general non-convex setting, while FedAvg-P & Scaffold-P (Chen et al., 2023) achieve linear speedup respecting the number of the local steps. DFedPGP (Liu et al., 2024) presents the decentralized convergence bound in non-convex conditions under the directed graph, while DFedMDC & DFedSMDC (Shi et al., 2023) focus on the convergence with the undirected network.

B Detailed comparison of generalization.

Table 2: Main results on the upper generalization bounds of PFL.

Algorithm	Generalization Bound	T	K	η	n	m
APFL, (Deng et al., 2020)	$\mathcal{O}\left(2\left(1-\alpha_i\right)^2\left(\hat{\mathcal{L}}_{\overline{\mathcal{D}}}(\bar{h}^*)+B\left\|\overline{\mathcal{D}}-\mathcal{D}_i\right\|_1+C\sqrt{(d+\log(1/\delta))/N}\right)\right.$ $\left. +\mathcal{O}\left(2\alpha_i^2(\mathcal{L}_{\mathcal{D}_i}(h_i^*)+2C\sqrt{(d+\log(1/\delta))/S_i}+G\lambda_{\mathcal{H}}(\mathcal{S}_i))\right)\right)$	×	×	×	×	✓
MAPPER, (Mansour et al., 2020)	$\mathcal{O}\left(2L\left(\sqrt{\frac{d_c}{m}\log\frac{em}{d_c}}+\sqrt{\frac{d_l p}{m}\log\frac{em}{d_l}}\right)+2\sqrt{\frac{\log\frac{1}{\delta}}{m}}\right)$	×	×	×	×	✓
pFedBayes, (Zhang et al., 2022)	$\mathcal{O}\left(C_2 m^{-\frac{2\beta}{2\beta+d}}\log^{2\delta'}(m)\right)$	×	×	×	×	✓
FedAvg & LocalTraining, (Chen et al., 2021)	$\mathcal{O}\left(\frac{1}{N}+R^2\right) \quad \& \quad \mathcal{O}(m/N)$	×	×	×	×	✓
C-PFL (Ours)	$\mathcal{O}\left(\frac{4}{N}\left[\frac{G(\sigma_u L_v+\sigma_v L_u)}{L_u L_v}\right]^{\frac{1}{1+\mu L}}(nUTK)^{\frac{\mu L}{1+\mu L}}\right)$	✓	✓	✓	✓	✓

Compared to the above generalization bounds for PFL, the proposed analysis has made the following progress: 1) conduct the generalization analysis in the non-convex condition, which is based on the more realistic assumptions adapted to the neural networks; 2) analyze the impacts of the algorithm design and the hyperparameters selection of the number of samples S, the number of selected clients n, total clients m, total iterations TK_u and TK_v and the local learning rates η_u and η_v; 3) illustrate the error propagation process between model aggregation and local training with the iteration nature, which provides a reference for the choice of early stopping points when training.

APFL is a typical PFL method based on model interpolation, which aims to find the optimal combination of the global model and the local model with the adaptive parameter α_i to achieve a better client-specific model. It derives the generalization bound of a mixture of local and global models with the analysis of VC dimension complexity. $S_i, i = 1, 2, ..., n$ is the number of training data at ith user, $N = m_1 + ... + m_n$ is the total number of all data, $\mathcal{S}_i$ to be the local training set drawn from $\mathcal{D}_i$, $\left\|\overline{\mathcal{D}} - \mathcal{D}_i\right\|_1 = \int_\Xi \left|\mathbb{P}_{(\boldsymbol{x},y)\sim\overline{\mathcal{D}}} - \mathbb{P}_{(\boldsymbol{x},y)\sim\mathcal{D}_i}\right| d\boldsymbol{x}dy$, is the difference between distributions $\overline{\mathcal{D}} = (1/n)\sum_{i=1}^{n} \mathcal{D}_i$ and D_i, and $h_i^* = \arg\min_{h\in\mathcal{H}} \mathcal{L}_{\mathcal{D}_i}(h)$.

MAPPER is also a model interpolation method combining local and global models to pursue the better personalized results. It derives the generalization bound with the analysis of Rademacher complexity. $\mathcal{H}_c$ is the hypotheses class for the central model, and $\mathcal{H}_l$ is the hypotheses class for the local models. d_c is the pseudo-dimension of $\mathcal{H}_c$ and d_l is the pseudo-dimension of $\mathcal{H}_l$. This bound only depends on the average number of samples and not the minimum number of samples.

pFedBayes is a novel PFL method via Bayesian variational inference. Each client uses the aggregated global distribution as prior distribution and updates its personal distribution by balancing the construction error over its personal data and the KL divergence with aggregated global distribution. It derives the generalization bound with the PAC-Bayes analysis. $\delta' > \delta > 1$, and $C_1, C_2 > 0$ are constants related to Hölder smooth β, the intrinsic dimension of data d, the number of hidden layers L, the widths of neural network are equalwidth M, the balance parameter ζ between personalization and global aggregation, and sample size of each client n.

FedAvg and LocalTraining are the most typical methods for FL and PFL. Though the generalization analysis in (Chen et al., 2021) is not designed based on the PFL definition, it concludes a surprising theorem that there exists a threshold of data heterogeneity to decide whether FedAvg or LocalTraining could achieve the minimax optimal for PFL. It derives the generalization bound for LocalTraining with uniform stability and the generalization bound for FedAvg with federated stability under strongly convex conditions. m represenets the client index, and $N = n_1 + \ + n_m$ denotes the total number of training samples. $R^2 := \min_{\boldsymbol{w}\in\mathcal{W}} \sum_{i\in[m]} n_i \|\boldsymbol{w}_\star^{(i)} - \boldsymbol{w}\|^2/N$ measures the level of heterogeneity among clients (here $\|\|$ denotes the Euclidean distance).

C COMMUNICATION NETWORK TOPOLOGIES

We present various network topologies in DFL in Figure 2 and the corresponding spectral properties in Table 3.

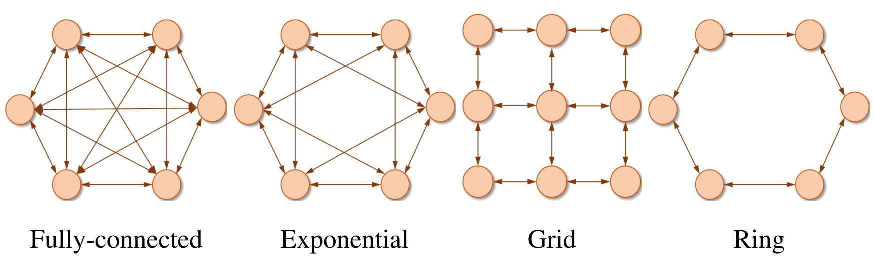


Figure 2: Illustration of various network topologies in DFL.

Table 3: κ_λ and Spectral Gap $1-\lambda$ of communication topologies (Sun et al., 2023; Zhu et al., 2024). m represents the number of total participating clients in DFL. $\kappa_\lambda = \left(\frac{\alpha}{e}\right)^\alpha \frac{1}{\lambda\left(\ln\frac{1}{\lambda}\right)^\alpha} + \frac{2^\alpha}{(1-\alpha)e\lambda\ln\frac{1}{\lambda}} + \frac{2^\alpha}{\lambda\ln\frac{1}{\lambda}}$ and λ are the widely used coefficient to measure different communication connections.

Network Topology	κ_λ	Spectral Gap $1-\lambda$
Fully-connected	0	1
Disconnected	1	0
Ring	$\mathcal{O}(m^2)$	$\approx 3m^2/16\pi^2$
Grid	$\mathcal{O}(mlnm)$	$\mathcal{O}(mlog\_2(m))$
Exponential	$\mathcal{O}(lnm)$	$\mathcal{O}\left(1+log\_2(m)\right)$

D Appendix for experiments.

D.1 Implementation Details for experiments.

According to Definition 2, we construct distributed neighboring dataset $\mathcal{S}=\{\mathcal{S}_1,...,\mathcal{S}_m\}$ and $\widetilde{\mathcal{S}}=\{\widetilde{\mathcal{S}}_1,...,\widetilde{\mathcal{S}}_m\}$, where each corresponding local dataset pair $(\mathcal{S}_i,\widetilde{\mathcal{S}}_i)$ only differs on one randomly selected data sample. Then we deploy the same initial model (u,V) with its local dataset pair $(\mathcal{S}_i,\widetilde{\mathcal{S}}_i)$ to the local client i. To focus on the effect of the essential factors, the regularization methods such as weight decay, data augmentations and dropout are ignored to prevent unnecessary impacts (Zhu et al., 2024; Lei et al., 2021). We keep the same experiment setting for all methods and perform 300 communication rounds. The number of client sizes is 20. The client sampling radio is 0.2 in C-PFL, while each client communicates with 4 neighbors in D-PFL accordingly. The batch size is 128 and the number of local epochs is 5. We set SGD (Robbins & Monro, 1951) as the base local optimizer with a learning rate $\eta = 0.1$. We ran each experiment 3 times with different random seeds and reported the mean accuracy with standard deviation for each method.

D.2 More Experiments Results on CIFAR-100.

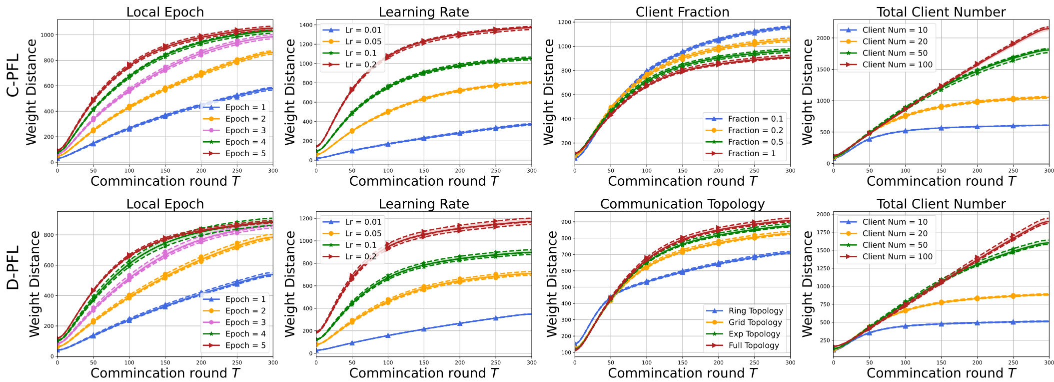
We explore the impact of the four factors on CIFAR-100 in Figure 3: 1) Local Learning Epochs, 2) Local Learning Rates, 3) Client Fraction / Communication Topology, and 4) Total Client Number. The empirical results on CIFAR-100 also verify that 1) Both less local learning epochs and lower learning rates lead to better generalization performance, but they affect the convergence speed more seriously; 2) More client participation and denser network connection in each communication round enlarge the generalization gap, but they speed up the convergence rate to the same extent; 3) A larger total participation clients and a smaller number of local training samples increase the generalization error and reduce the convergence speed simultaneously; 4) C-PFL outperforms D-PFL in both generalization performance and convergence performance when their upper communication bandwidths are at the same level.

E Generalization Bounds for C-PFL and D-PFL.

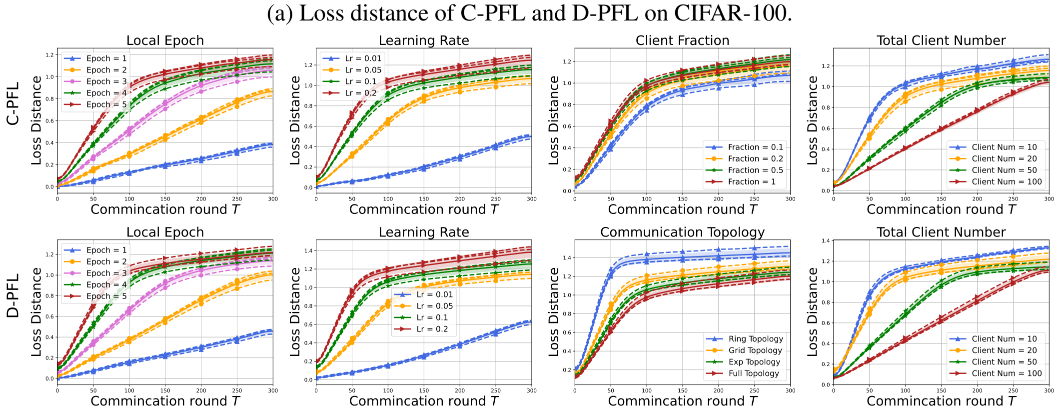
In this section, we introduce our proof of the generalization bounds in the main context. We first introduce the general lemmas for both C-PFL and D-PFL. Then we prove the uniform stability to measure the generalization error for them. At the beginning of our proof, we list the important variables used in the study as follows.

Table 4: Some abbreviations of the used terms in the proofs.

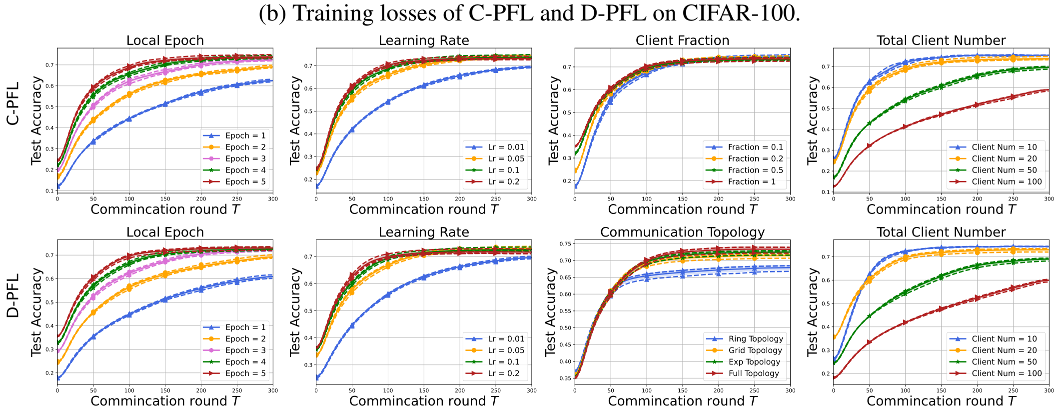
Notation	Description
$w_{i,k}^t = (u_{i,k}^t, v_{i,k}^t)$	parameters at k-th iteration
$w^t = (u^t, \mathbf{v}^t)$	parameters in round t with set $\mathcal{S}$
$\Delta_{u,k}^t = \sum_{i\in\,[m]} \mathbb{E}\Vert u^t - \widetilde{u}^t\Vert$	stability difference of variables u
$\Delta_{v,k}^t = \sum_{i\in\,[m]} \mathbb{E}\Vert v_i^t - \widetilde{v}_i^t\Vert$	stability difference of variables v_i
F	initial function value gap



(a) Loss distance of C-PFL and D-PFL on CIFAR-100.



(b) Training losses of C-PFL and D-PFL on CIFAR-100.



(c) Testing accuracies of C-PFL and D-PFL on CIFAR-100.

Figure 3: Empirical results of C-PFL (first line) and D-PFL (second line) on CIFAR-100.

E.1 Preliminary Lemmas

Lemma 1 (**Mixing Matrix for Decentralized FL**, Lemma 4, Lian et al. (2017))**.** *For any $t \in \mathbb{Z}^+$, the mixing matrix $\mathbf{W} \in \mathbb{R}^n$ satisfies $\|\mathbf{W}^t - \mathbf{P}\|_{\text{op}} \leq \lambda^t$, where $\lambda := \max\{|\lambda_2|, |\lambda_n(W)|\}$ and for a matrix $\mathbf{A}$, we denote its spectral norm as $\|\mathbf{A}\|_{\text{op}}$. Furthermore, $\mathbf{1} := [1, 1, \ldots, 1]^\top \in \mathbb{R}^m$ and*

$$\mathbf{P} := \frac{\mathbf{1}\mathbf{1}^\top}{n} \in \mathbb{R}^{n \times n}.$$

Lemma 2 (**Stability for C-PFL**)**.** *We follow the definition in (Hardt et al., 2016; Zhou et al., 2021) to upper bound the uniform stability term for the shared and personalized variables u and v_i after round T in the central FL paradigm. The updated progress of the shared variables u is like the vanilla FedAvg, where the local updates and server aggregation are conducted alternately. The* *updated progress of the personalized variables v_i is like the SGD with multiple local updates. Let function $f(w_i)$ satisfies Assumption 3, the models $w_i^T = \mathcal{A}(\mathcal{S})$ and $\widetilde{w}_i^T = \mathcal{A}(\widetilde{\mathcal{S}})$ are generated after T training rounds by the centralized method, we can bound their objective difference as:*

$$\begin{aligned} &\mathbb{E}\|f(w_i^T;z) - f(\widetilde{w}_i^T;z)\| \\ \leq\ &\frac{nU\tau_0}{mS} + G\mathbb{E}\left[\|w_i^T - \widetilde{w}_i^T\| \mid \xi\right] \\ \leq\ &\frac{nU\tau_0}{mS} + G\mathbb{E}\left[\|u^T - \widetilde{u}^T\| \mid \xi\right] + G\mathbb{E}\left[\|v_i^T - \widetilde{v}_i^T\| \mid \xi\right] \end{aligned} \tag{11}$$

[R3: where $U = \sup_{w_i,z} f(w_i;z) = \sup_{u,v_i,z} f(u,v_i;z) < +\infty$ is the upper bound of the loss and $\tau_0 = t_0K + k_0$ is a specific index of the total iterations.]

Proof. [R3: Let I represent the index of the first time to sample the perturbation sample $\widetilde{z}_{i^*,j^*}$ on the dataset $\widetilde{\mathcal{S}}_{i^*}$. When $t_0K + k_0 < I$, $\Delta_{k_0}^{t_0} = 0$. Then we define

$$P(\xi^c) = P(\Delta_{k_0}^{t_0} > 0) \leq P(I \leq t_0K + k_0).$$

Expanding the probability we have:]

$$\begin{aligned} &\mathbb{E}\|f(w_i^T;z) - f(\widetilde{w}_i^T;z)\| \\ =\ &P(\{\xi\})\,\mathbb{E}\left[\|f(w_i^T;z) - f(\widetilde{w}_i^T;z)\| \mid \xi\right] + P(\{\xi^c\})\,\mathbb{E}\left[\|f(w_i^T;z) - f(\widetilde{w}_i^T;z)\| \mid \xi^c\right] \\ \leq\ &\mathbb{E}\left[\|f(w_i^T;z) - f(\widetilde{w}_i^T;z)\| \mid \xi\right] + P(\{\xi^c\}) \sup_{w_i,z} f(w_i;z) \\ \leq\ &G\mathbb{E}\left[\|w_i^T - \widetilde{w}_i^T\| \mid \xi\right] + UP(\{\xi^c\}) \\ =\ &G\mathbb{E}\left[\|u^T - \widetilde{u}^T\| \mid \xi\right] + G\mathbb{E}\left[\|v_i^T - \widetilde{v}_i^T\| \mid \xi\right] + UP(\{\xi^c\}). \end{aligned}$$

Before the $j^\star$-th data on $i^\star$-th client is sampled, the iterative states are identical on both $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. [R3: When the dataset $\widetilde{\mathcal{S}}_{i^*}$ is selected, the perturbation sample $\widetilde{z}_{i^*,j^*}$ can be selected with probability $1/S$.] Define χ as the event sampling dataset $\mathcal{S}_{i^\star}$ and the observation moment $\tau_0 = t_0K + k_0$. Then we have:

$$\begin{aligned} &P(\{\xi^c\}) \leq P\left(I \leq t_0K + k_0\right) \\ \leq\ &\sum_{t=0}^{t_0-1}\sum_{k=0}^{K-1} P(I = tK + k; \chi) + \sum_{k=0}^{k_0} P\left(I = t_0K + k; \chi\right) \\ =\ &\sum_{t=0}^{t_0-1}\sum_{k=0}^{K-1}\sum_{\chi} P(I = tK + k \mid \chi)P(\chi) + \sum_{k=0}^{k_0}\sum_{\chi} P\left(I = t_0K + k \mid \chi\right)P(\chi) \\ =\ &\frac{n}{m}\left(\sum_{t=0}^{t_0-1}\sum_{k=0}^{K-1} P(I = tK + k) + \sum_{k=0}^{k_0} P\left(I = t_0K + k\right)\right) \\ =\ &\frac{nt_0K + k_0}{mS} \\ =\ &\frac{n\tau_0}{mS}. \end{aligned}$$

The random active clients with the probability of n/m in the second equality. ☐

Lemma 3 (Stability for D-PFL). *We follow the definition in (Hardt et al., 2016; Zhou et al., 2021) to upper bound the uniform stability term for the shared and personalized variables u and v_i after round T in the decentralized FL paradigm. Let function $f(w_i)$ satisfies Assumption 3, the models $w_i^T = \mathcal{A}(\mathcal{S})$ and $\widetilde{w}_i^T = \mathcal{A}(\widetilde{\mathcal{S}})$ are generated after T training rounds by the decentralized method, we can bound their objective difference as:*

$$\begin{aligned} &\mathbb{E}\|f(w_i^T;z) - f(\widetilde{w}_i^T;z)\| \\ \leq\ &\frac{U\tau_0}{S} + G\mathbb{E}\left[\|w_i^T - \widetilde{w}_i^T\| \mid \xi\right] \\ \leq\ &\frac{U\tau_0}{S} + G\mathbb{E}\left[\|u_i^T - \widetilde{u}_i^T\| \mid \xi\right] + G\mathbb{E}\left[\|v_i^T - \widetilde{v}_i^T\| \mid \xi\right] \end{aligned} \tag{12}$$

Proof. For the D-PFL, the most part is the same as the proof for the central algorithms except the probability $P(\chi) = 1$ in a Decentralized Federated Learning setup (because all clients will participate in the training). We bound their objective difference as:

$$\mathbb{E}\left[\|f(w^T; z) - f(\widetilde{w}^T; z)\|\right] \leq G \sum_{i\in[m]} \mathbb{E}\left[\|w_{i,K}^T - \widetilde{w}_{i,K}^T\| \mid \xi\right] + \frac{U\tau_0}{S}. \tag{13}$$

$\square$

Lemma 4 (**Upper Bound of Aggregation Gaps).** *According to Algorithm 2, the aggregation of C-PFL is $u_{i,0}^{t+1} = u^{t+1} = \frac{1}{n}\sum_{i\in S^t} u_{i,K_u}^t$, and the aggregation of D-PFL is $u_{i,0}^{t+1} = \sum_{j\in\mathcal{A}_i} a_{ij} u_{i,K_u}^t$. On both setups, we can upper bound the aggregation gaps by:*

$$\begin{aligned} \Delta_{u,0}^{t+1} &\leq \Delta_{u,K_u}^t, \\ \Delta_{v,0}^{t+1} &= \Delta_{v,K_v}^t. \end{aligned} \tag{14}$$

Proof. For the personal variable v_i, they are always kept locally without aggregation, which means that $v_{i,K}^t = v_{i,0}^{t+1}$. So it is obvious to see that $v_{i,K}^t - \widetilde{v}_{i,K}^t = v_{i,0}^{t+1} - \widetilde{v}_{i,0}^{t+1}$, which proves that $\Delta_{v,0}^{t+1} = \Delta_{v,K}^t$. Then we prove the inequation for the shared variables u. We discuss it in central and decentralized mode respectively.

(1) C-PFL setup (Acar et al., 2021).

In centralized federated learning, we select a subset S^t in each communication round t. Thus we have:

$$\begin{aligned} \Delta_{u,0}^{t+1} &= \sum_{i\in[m]} \mathbb{E}\|u_{i,0}^{t+1} - \widetilde{u}_{i,0}^{t+1}\| = \sum_{i\in[m]} \mathbb{E}\|u^{t+1} - \widetilde{u}^{t+1}\| \\ [\mathrm{R3} :&= \sum_{i\in[m]} \mathbb{E}\|\frac{1}{n}\sum_{i\in S^t}\left(u_i^{t+1} - \widetilde{u}_i^{t+1}\right)\| = \sum_{i\in[m]} \mathbb{E}\|\frac{1}{n}\sum_{i\in S^t}\left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right)\|] \\ &\leq \sum_{i\in[m]} \frac{1}{n}\mathbb{E}\left[\sum_{i\in S^t}\|u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\|\right] = \sum_{i\in[m]} \frac{1}{n}\frac{n}{m}\sum_{i\in[m]} \mathbb{E}\|u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\| \\ &= \sum_{i\in[m]} \frac{1}{m}\sum_{i\in[m]} \mathbb{E}\|u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\| = \sum_{i\in[m]} \mathbb{E}\|u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\| = \Delta_{u,K_u}^t. \end{aligned}$$

(2) D-PFL setup (Sun et al., 2023).

In decentralized federated learning, we aggregate the models in each neighborhood. Thus we have:

$$\begin{aligned} \Delta_{u,0}^{t+1} &= \sum_{i\in[m]} \mathbb{E}\|u_{i,0}^{t+1} - \widetilde{u}_{i,0}^{t+1}\| = \sum_{i\in[m]} \mathbb{E}\|\sum_{j\in\mathcal{A}_i} a_{ij}\left(u_{j,K_u}^t - \widetilde{u}_{j,K_u}^t\right)\| \\ &\leq \sum_{i\in[m]}\sum_{j\in\mathcal{A}_i} a_{ij}\mathbb{E}\|u_{j,K_u}^t - \widetilde{u}_{j,K_u}^t\| = \sum_{j\in[m]}\sum_{i\in\mathcal{A}_j} a_{ji}\mathbb{E}\|u_{j,K_u}^t - \widetilde{u}_{j,K_u}^t\| \\ &\leq \sum_{j\in[m]} \mathbb{E}\|u_{j,K_u}^t - \widetilde{u}_{j,K_u}^t\| = \Delta_{u,K_u}^t. \end{aligned}$$

The last equality adopts the symmetry of the adjacent matrix $\mathbf{A} = \mathbf{A}^\top$. $\square$

Lemma 5 (**Decentralized Topologies Bounds of λ).** *For $0 < \lambda < 1$ and $0 < \alpha < 1$, we have the following inequality:*

$$\sum_{s=0}^{t-1} \frac{\lambda^{t-s-1}}{(s+1)^\alpha} \leq \frac{\kappa_\lambda}{t^\alpha}, \tag{15}$$

where $\kappa_\lambda = \left(\frac{\alpha}{e}\right)^\alpha \frac{1}{\lambda\left(\ln\frac{1}{\lambda}\right)^\alpha} + \frac{2^\alpha}{(1-\alpha)e\lambda\ln\frac{1}{\lambda}} + \frac{2^\alpha}{\lambda\ln\frac{1}{\lambda}}$.

Proof. According to the accumulation, we have:

$$\begin{aligned}
\sum_{s=0}^{t-1}\frac{\lambda^{t-s-1}}{(s+1)^\alpha} &= \lambda^{t-1} + \sum_{s=1}^{t-1}\frac{\lambda^{t-s-1}}{(s+1)^\alpha} \leq \lambda^{t-1} + \int_{s=1}^{s=t}\frac{\lambda^{t-s-1}}{s^\alpha}ds \\
&= \lambda^{t-1} + \int_{s=1}^{s=\frac{t}{2}}\frac{\lambda^{t-s-1}}{s^\alpha}ds + \int_{s=\frac{t}{2}}^{s=t}\frac{\lambda^{t-s-1}}{s^\alpha}ds \\
&\leq \lambda^{t-1} + \lambda^{\frac{t}{2}-1}\int_{s=1}^{s=\frac{t}{2}}\frac{1}{s^\alpha}ds + \left(\frac{2}{t}\right)^\alpha\int_{s=\frac{t}{2}}^{s=t}\lambda^{t-s-1}ds \\
&\leq \lambda^{t-1} + \lambda^{\frac{t}{2}-1}\frac{1}{1-\alpha}\left(\frac{t}{2}\right)^{1-\alpha} + \left(\frac{2}{t}\right)^\alpha\frac{\lambda^{-1}}{\ln\frac{1}{\lambda}}.
\end{aligned}$$

Thus we have LHS $\leq \frac{1}{t^\alpha}\left(\lambda^{t-1}t^\alpha + \lambda^{\frac{t}{2}-1}\frac{t}{(1-\alpha)2^{1-\alpha}} + \frac{2^\alpha}{\lambda\ln\frac{1}{\lambda}}\right)$. The first term can be bounded as $\lambda^{t-1}t^\alpha \leq \left(\frac{\alpha}{e}\right)^\alpha\frac{1}{\lambda\left(\ln\frac{1}{\lambda}\right)^\alpha}$ and the second term can be bounded as $\lambda^{\frac{t}{2}-1}t \leq \frac{2}{e\lambda\ln\frac{1}{\lambda}}$, which indicates the selection of the constant $\kappa_\lambda = \left(\frac{\alpha}{e}\right)^\alpha\frac{1}{\lambda\left(\ln\frac{1}{\lambda}\right)^\alpha} + \frac{2^\alpha}{(1-\alpha)e\lambda\ln\frac{1}{\lambda}} + \frac{2^\alpha}{\lambda\ln\frac{1}{\lambda}}$. Furthermore, if $0 < \alpha \leq \frac{1}{2} < 1$, we have $\kappa_\lambda \leq \frac{1}{\lambda\left(\ln\frac{1}{\lambda}\right)^\alpha} + \frac{2\sqrt{2}}{e\lambda\ln\frac{1}{\lambda}} + \frac{\sqrt{2}}{\lambda\ln\frac{1}{\lambda}} \leq \max\left\{\frac{1}{\lambda}, \frac{1}{\lambda\sqrt{\ln\frac{1}{\lambda}}}\right\} + \frac{(2+e)\sqrt{2}}{e\lambda\ln\frac{1}{\lambda}} = \mathcal{O}\left(\max\left\{\frac{1}{\lambda}, \frac{1}{\lambda\sqrt{\ln\frac{1}{\lambda}}}\right\} + \frac{1}{\lambda\ln\frac{1}{\lambda}}\right)$ with respect to the constant λ. □

E.2 Generalization Bounds for C-PFL

Lemma 6 (Selecting the Same Sample). *Under the Assumption 1 and Assumption 3, the gradient for the shared and personalized variables satisfy* $g_{u,i,k}^t = \nabla_u F_i(u_{i,k}^t, v_{i,K_v}^t; z)$ *and* $g_{v,i,k}^t = \nabla_v F_i(u_i^t, v_{i,k}^t; z)$*, the local updates satisfy* $u_{i,k+1}^t = u_{i,k}^t - \gamma g_{u,i,k}^t$ *and* $v_{i,k+1}^t = v_{i,k}^t - \gamma g_{v,i,k}^t$*. We use* $\mathbb{E}[\nabla_u F_i(u_{i,k}^t, v_{i,K_v}^t; z)] = \nabla_u f_i(u_{i,k}^t, v_{i,K_v}^t; z)$ *and* $\mathbb{E}[\nabla_v F_i(u_i^t, v_{i,k}^t; z)] = \nabla_v f_i(u_i^t, v_{i,k}^t; z)$*. If we sample the same data* z *(not the* $z_{i^\star,j^\star}$*) in dataset* $\mathcal{S}$ *and* $\widetilde{\mathcal{S}}$ *at* k *iteration on round* t*, we have:*

$$\begin{aligned}
\mathbb{E}\|u_{i,k+1}^t - \widetilde{u}_{i,k+1}^t\| &\leq (1+\eta_u L_u)\mathbb{E}\|u_{i,k}^t - \widetilde{u}_{i,k}^t\| + \eta_u L_{uv}\mathbb{E}\|v_{i,K_v}^t - \widetilde{v}_{i,K_v}^t\|, \\
\mathbb{E}\|v_{i,k+1}^t - \widetilde{v}_{i,k+1}^t\| &\leq (1+\eta_v L_v)\mathbb{E}\|v_{i,k}^t - \widetilde{v}_{i,k}^t\| + \eta_v L_{vu}\mathbb{E}\|u_i^t - \widetilde{u}_i^t\|.
\end{aligned} \tag{16}$$

Proof. We first conduct the local update for the personalized variables. The update progress in each round t is as follows:

$$\begin{aligned}
&\mathbb{E}\|v_{i,k+1}^t - \widetilde{v}_{i,k+1}^t\| = \mathbb{E}\|v_{i,k}^t - \widetilde{v}_{i,k}^t - \eta_v(g_{v,i,k}^t - \widetilde{g}_{v,i,k}^t)\| \\
&\leq \mathbb{E}\|v_{i,k}^t - \widetilde{v}_{i,k}^t\| + \eta_v\mathbb{E}\|\nabla_v f_i(u_i^t, v_{i,k}^t; z) - \nabla_v f_i(\widetilde{u}_i^t, \widetilde{v}_{i,k}^t; z)\| \\
&\leq (1+\eta_v L_v)\mathbb{E}\|v_{i,k}^t - \widetilde{v}_{i,k}^t\| + \eta_v L_{vu}\mathbb{E}\|u_i^t - \widetilde{u}_i^t\|.
\end{aligned}$$

The alternative update progress for the shared variables is based on the updated $v_i^{t+1} = v_{i,K_v}^{t+1}$:

$$\begin{aligned}
&\mathbb{E}\|u_{i,k+1}^t - \widetilde{u}_{i,k+1}^t\| \\
&= \mathbb{E}\|u_{i,k}^t - \widetilde{u}_{i,k}^t - \eta_u(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t)\| \\
&\leq \mathbb{E}\|u_{i,k}^t - \widetilde{u}_{i,k}^t\| + \eta_u\mathbb{E}\|\nabla_u f_i(u_{i,k}^t, v_{i,K_v}^t; z) - \nabla_u f_i(\widetilde{u}_{i,k}^t, \widetilde{v}_{i,K}^t; z)\| \\
&\leq (1+\eta_u L_u)\mathbb{E}\|u_{i,k}^t - \widetilde{u}_{i,k}^t\| + \eta_u L_{uv}\mathbb{E}\|v_{i,K_v}^t - \widetilde{v}_{i,K_v}^t\|.
\end{aligned}$$

□

Lemma 7 (**Selecting the Different Sample**). *Assume $g^t_{u,i,k} = \nabla_u F_i(u^t_{i,k}, v^t_{i,K_v}; z)$ and $g^t_{v,i,k} = \nabla_v F_i(u^t_i, v^t_{i,k}; z)$, the local updates satisfy $u^t_{i,k+1} = u^t_{i,k} - \gamma g^t_{u,i,k}$ and $v^t_{i,k+1} = v^t_{i,k} - \gamma g^t_{v,i,k}$.If we sample the different data samples $z_{i^\star,j^\star}$ and $\widetilde{z}_{i^\star,j^\star}$ (simplified to z and $\widetilde{z}$), we have:*

$$\begin{aligned}
&\mathbb{E}\|u^t_{i,k+1} - \widetilde{u}^t_{i,k+1}\| \leq (1+\eta_u L_u)\mathbb{E}\|u^t_{i,k} - \widetilde{u}^t_{i,k}\| + \eta_u L_{uv}\mathbb{E}\|v^t_{i,K} - \widetilde{v}^t_{i,K}\| + 2\eta_u\sigma_u, \\
&\mathbb{E}\|v^t_{i,k+1} - \widetilde{v}^t_{i,k+1}\| \leq (1+\eta_v L_v)\mathbb{E}\|v^t_{i,k} - \widetilde{v}^t_{i,k}\| + \eta_v L_{vu}\mathbb{E}\|u^t_i - \widetilde{u}^t_i\| + 2\eta_v\sigma_v.
\end{aligned} \tag{17}$$

Proof. We first conduct the local update for the personalized variables. The update progress in each round t is as follows:

$$\begin{aligned}
&\mathbb{E}\|v^t_{i^\star,k+1} - \widetilde{v}^t_{i^\star,k+1}\| \\
&= \mathbb{E}\|v^t_{i^\star,k} - \widetilde{v}^t_{i^\star,k} - \eta_v(g^t_{v,i^\star,k} - \widetilde{g}^t_{v,i^\star,k})\| \\
&\leq \mathbb{E}\|v^t_{i^\star,k} - \widetilde{v}^t_{i^\star,k}\| + \eta_v\mathbb{E}\|\nabla_v F_{i^\star}(u^t_{i^\star}, v^t_{i^\star,k}, z) - \nabla_v F_{i^\star}(\widetilde{u}^t_{i^\star}, \widetilde{v}^t_{i^\star,k}\widetilde{z})\| \\
&\leq \mathbb{E}\|v^t_{i^\star,k} - \widetilde{v}^t_{i^\star,k}\| + \eta_v\mathbb{E}\|\nabla_v F_{i^\star}(u^t_{i^\star}, v^t_{i^\star,k}, z) - \nabla_v F_{i^\star}(\widetilde{u}^t_{i^\star}, \widetilde{v}^t_{i^\star,k}, z)\| \\
&\quad + \eta_v\mathbb{E}\|\nabla_v F_{i^\star}(\widetilde{u}^t_{i^\star}, \widetilde{v}^t_{i^\star,k}, z) - \nabla_v F_{i^\star}(\widetilde{u}^t_{i^\star}, \widetilde{v}^t_{i^\star,k}, \widetilde{z})\| \\
&\leq (1+\eta_v L_v)\mathbb{E}\|v^t_{i^\star,k} - \widetilde{v}^t_{i^\star,k}\| + \eta_v L_{vu}\mathbb{E}\|u^t_i - \widetilde{u}^t_i\| \\
&\quad + \eta_v\mathbb{E}\|\nabla_v F_{i^\star}(\widetilde{u}^t_{i^\star}, \widetilde{v}^t_{i^\star,k}, z) - \nabla_v f_{i^\star}(\widetilde{u}^t_{i^\star}, \widetilde{v}^t_{i^\star,k}) - \nabla_v f_{i^\star}(\widetilde{u}^t_{i^\star}, \widetilde{v}^t_{i^\star,k}, \widetilde{z}) + \nabla_v f_{i^\star}(\widetilde{u}^t_{i^\star}, \widetilde{v}^t_{i^\star,k})\| \\
&\leq (1+\eta_v L_v)\mathbb{E}\|v^t_{i,k} - \widetilde{v}^t_{i,k}\| + \eta_v L_{vu}\mathbb{E}\|u^t_i - \widetilde{u}^t_i\| + 2\eta_v\sigma_v.
\end{aligned}$$

The last inequality adopts $\mathbb{E}[x] = \sqrt{(\mathbb{E}[x])^2} = \sqrt{\mathbb{E}[x^2] - \mathbb{E}[x - \mathbb{E}[x]]^2} \leq \sqrt{\mathbb{E}[x^2]}$.

The alternative update progress for the shared variables is based on the updated $v^{t+1}_i = v^{t+1}_{i,K_v}$:

$$\begin{aligned}
&\mathbb{E}\|u^t_{i^\star,k+1} - \widetilde{u}^t_{i^\star,k+1}\| \\
&= \mathbb{E}\|u^t_{i^\star,k} - \widetilde{u}^t_{i^\star,k} - \eta_u(g^t_{u,i^\star,k} - \widetilde{g}^t_{u,i^\star,k})\| \\
&\leq \mathbb{E}\|u^t_{i^\star,k} - \widetilde{u}^t_{i^\star,k}\| + \eta_u\mathbb{E}\|\nabla_u F_{i^\star}(u^t_{i^\star,k}, v^t_{i^\star,K_v}, z) - \nabla_u F_{i^\star}(\widetilde{u}^t_{i^\star,k}, \widetilde{v}^t_{i^\star,K_v}, \widetilde{z})\| \\
&\leq \mathbb{E}\|u^t_{i^\star,k} - \widetilde{u}^t_{i^\star,k}\| + \eta_u\mathbb{E}\|\nabla_u F_{i^\star}(u^t_{i^\star,k}, v^t_{i^\star,K_v}, z) - \nabla_u F_{i^\star}(\widetilde{u}^t_{i^\star,k}, \widetilde{v}^t_{i^\star,K_v}, z)\| \\
&\quad + \eta_u\mathbb{E}\|\nabla_u F_{i^\star}(\widetilde{u}^t_{i^\star,k}, \widetilde{v}^t_{i^\star,K_v}, z) - \nabla_u F_{i^\star}(\widetilde{u}^t_{i^\star,k}, \widetilde{v}^t_{i^\star,K_v}, \widetilde{z})\| \\
&\leq (1+\eta_u L_u)\mathbb{E}\|u^t_{i^\star,k} - \widetilde{u}^t_{i^\star,k}\| + \eta_u L_{uv}\mathbb{E}\|v^t_{i,k} - \widetilde{v}^t_{i,K_v}\| \\
&\quad + \eta_u\mathbb{E}\|\nabla_u F_{i^\star}(\widetilde{u}^t_{i^\star,k}, \widetilde{v}^t_{i^\star,K_v}, z) - \nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k}, \widetilde{v}^t_{i^\star,K_v}) - \nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,K_v}, \widetilde{v}^t_{i^\star,K_v}, \widetilde{z}) + \nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k}, \widetilde{v}^t_{i^\star,K_v})\| \\
&\leq (1+\eta_u L_u)\mathbb{E}\|u^t_{i,k} - \widetilde{u}^t_{i,k}\| + \eta_u L_{uv}\mathbb{E}\|v^t_{i,K_v} - \widetilde{v}^t_{i,K_v}\| + 2\eta_u\sigma_u.
\end{aligned}$$

□

Lemma 8 (**Recursion in local update**). *Since $\Delta^t_k = \Delta^t_{u,k} + \Delta^t_{v,k}$, according to the Lemma 6 and 7, we can bound the recursion in the local training:*

$$\begin{aligned}
\Delta^t_{v,k+1} &\leq (1+\eta_v L_v)\,(\Delta^t_{v,k} + \frac{2\sigma_v}{SL_v} + \frac{L_{vu}\Delta^t_{u,0}}{L_v}). \\
\Delta^t_{u,k+1} &\leq (1+\eta_u L_u)\,(\Delta^t_{u,k} + \frac{2\sigma_u}{SL_u} + \frac{L_{uv}\Delta^t_{v,K_v}}{L_u}).
\end{aligned}$$

Proof. In each iteration, the specific $j^\star$-th data sample in the $\mathcal{S}_{i^\star}$ and $\widetilde{\mathcal{S}}_{i^\star}$ is uniformly selected with the probability of $1/S$. In other datasets $\mathcal{S}_i$, all the data samples are the same. Thus we have the recursion for the personalized variables:

$$
\begin{aligned}
&\Delta_{v,k+1}^{t} = \sum_{i\neq i^\star} \mathbb{E}\left[\|v_{i,k+1}^{t} - \widetilde{v}_{i,k+1}^{t}\|\right] + \mathbb{E}\left[\|v_{i^\star,k+1}^{t} - \widetilde{v}_{i^\star,k+1}^{t}\|\right] \\
&\leq (1+\eta_v L_v) \sum_{i\neq i^\star} \mathbb{E}\|v_{i,k}^{t} - \widetilde{v}_{i,k}^{t}\| + \eta_v L_{vu} \sum_{i\neq i^\star} \mathbb{E}\|u_{i}^{t} - \widetilde{u}_{i}^{t}\| \\
&\quad + \left(1-\frac{1}{S}\right)\left[(1+\eta_v L_v)\mathbb{E}\|v_{i,k}^{t} - \widetilde{v}_{i,k}^{t}\| + \eta_v L_{vu}\mathbb{E}\|u_{i}^{t} - \widetilde{u}_{i}^{t}\|\right] \\
&\quad + \frac{1}{S}\left[(1+\eta_v L_v)\mathbb{E}\|v_{i,k}^{t} - \widetilde{v}_{i,k}^{t}\| + \eta_v L_{vu}\mathbb{E}\|u_{i}^{t} - \widetilde{u}_{i}^{t}\| + 2\eta_v\sigma_v\right] \\
&= (1+\eta_v L_v)\,\Delta_{v,k}^{t} + \eta_v L_{vu}\Delta_{u,0}^{t} + \frac{2\eta_v\sigma_v}{S}.
\end{aligned}
$$

Similarly, for the shared variables, we have the progress in each round t:

$$
\begin{aligned}
&\Delta_{u,k+1}^{t} = \sum_{i\neq i^\star} \mathbb{E}\left[\|u_{i,k+1}^{t} - \widetilde{u}_{i,k+1}^{t}\|\right] + \mathbb{E}\left[\|u_{i^\star,k+1}^{t} - \widetilde{u}_{i^\star,k+1}^{t}\|\right] \\
&\leq (1+\eta_u L_u) \sum_{i\neq i^\star} \mathbb{E}\|u_{i,k}^{t} - \widetilde{u}_{i,k}^{t}\| + \eta_u L_{uv} \sum_{i\neq i^\star} \mathbb{E}\|v_{i,K_v}^{t} - \widetilde{v}_{i,K_v}^{t}\| \\
&\quad + \left(1-\frac{1}{S}\right)\left[(1+\eta_u L_u)\mathbb{E}\|u_{i,k}^{t} - \widetilde{u}_{i,k}^{t}\| + \eta_u L_{uv}\mathbb{E}\|v_{i,K_v}^{t} - \widetilde{v}_{i,K_v}^{t}\|\right] \\
&\quad + \frac{1}{S}\left[(1+\eta_u L_u)\mathbb{E}\|u_{i,k}^{t} - \widetilde{u}_{i,k}^{t}\| + \eta_u L_{uv}\mathbb{E}\|v_{i,K_v}^{t} - \widetilde{v}_{i,K_v}^{t}\| + 2\eta_u\sigma_u\right] \\
&= (1+\eta_u L_u)\,\Delta_{u,k}^{t} + \eta_u L_{uv}\Delta_{v,K_v}^{t} + \frac{2\eta_u\sigma_u}{S}.
\end{aligned}
$$

Then we can bound the recursion formulation as:

$$
\begin{aligned}
\Delta_{v,k+1}^{t} + \frac{2\sigma_v}{SL_v} + \frac{L_{vu}\Delta_{u}^{t}}{L_v} &\leq (1+\eta_v L_v)\,(\Delta_{v,k}^{t} + \frac{2\sigma_v}{SL_v} + \frac{L_{vu}\Delta_{u,0}^{t}}{L_v}), \\
\Delta_{u,k+1}^{t} + \frac{2\sigma_u}{SL_u} + \frac{L_{uv}\Delta_{v,K_v}^{t}}{L_u} &\leq (1+\eta_u L_u)\,(\Delta_{u,k}^{t} + \frac{2\sigma_u}{SL_u} + \frac{L_{uv}\Delta_{v,K_v}^{t}}{L_u}).
\end{aligned}
$$

Zoom out the variables on the left-hand side, then we finish the proof.

□

Main Proof for Theorem 1 According to the Lemma 4 and 8, it is easy to bound the local stability term. We still obverse it when the event ξ happens, and we have $\Delta_{k_0}^{t_0} = 0$. Therefore, we unwind the recurrence formulation from T, K to t_0, k_0. Let $\eta_u = \frac{\mu_u}{\tau} = \frac{\mu_u}{tK+k}$ and $\eta_v = \frac{\mu_v}{\tau} = \frac{\mu_v}{tK+k}$ are decayed as the communication round t and iteration k where $\mu_u \leq \frac{1}{L_u}$ and $\mu_v \leq \frac{1}{L_v}$ are specific constants, we have:

$$
\begin{aligned}
\Delta_{v,K_v}^T &\leq \left[\prod_{\tau=(T-1)K_v+1}^{TK_v}\left(1+\frac{\mu_v L_v}{\tau}\right)\right]\left(\Delta_{v,0}^T+\frac{2\sigma_v}{SL_v}+\frac{L_{vu}\Delta_{u,0}^T}{L_v}\right)\\
&\leq \left[\prod_{\tau=(T-1)K_v+1}^{TK_v}\left(1+\frac{\mu_v L_v}{\tau}\right)\right]\left(\Delta_{v,K_v}^{T-1}+\frac{2\sigma_v}{SL_v}+\frac{L_{vu}\Delta_{u,K_v}^{T-1}}{L_v}\right)\\
&\leq \left[\prod_{\tau=t_0K+k_0+1}^{TK_v}\left(1+\frac{\mu_v L_v}{\tau}\right)\right]\left(\Delta_{v,k_0}^{t_0}+\frac{2\sigma_v}{SL_v}+\frac{L_{vu}\Delta_{u,k_0}^{t_0}}{L_v}\right)\\
&\leq \left[\prod_{\tau=t_0K+k_0+1}^{TK_v}e^{\left(\frac{\mu_v L_v}{\tau}\right)}\right]\left(\frac{2\sigma_v}{SL_v}\right)\\
&= e^{\mu_v L_v\left(\sum_{\tau=t_0K+k_0+1}^{TK_v}\frac{1}{\tau}\right)}\frac{2\sigma_v}{SL_v}\\
&\leq e^{\mu_v L_v\ln\left(\frac{TK_v}{t_0K+k_0}\right)}\frac{2\sigma_v}{SL_v}\\
&\leq \left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v}\frac{2\sigma_v}{SL_v}.
\end{aligned}
\tag{18}
$$

Similarly, for the shared variables, we have the progress in round T:

$$
\begin{aligned}
\Delta_{u,K_u}^T &\leq \left[\prod_{\tau=(T-1)K_u+1}^{TK_u}\left(1+\frac{\mu_u L_u}{\tau}\right)\right]\left(\Delta_{u,0}^T+\frac{2\sigma_u}{SL_u}+\frac{L_{uv}\Delta_{v,K_v}^T}{L_u}\right)\\
&\leq \left[\prod_{\tau=(T-1)K_u+1}^{TK_u}\left(1+\frac{\mu_u L_u}{\tau}\right)\right]\left(\Delta_{u,K_u}^{T-1}+\frac{2\sigma_u}{SL_u}\right)+\left[\prod_{\tau=(T-1)K_u+1}^{TK_u}\left(1+\frac{\mu_u L_u}{\tau}\right)\right]\frac{L_{uv}\Delta_{v,K_v}^T}{L_u}\\
&\leq \left[\prod_{\tau=t_0K+k_0+1}^{TK_u}\left(1+\frac{\mu_u L_u}{\tau}\right)\right]\left(\Delta_{k_0}^{t_0}+\frac{2\sigma_u}{SL_u}\right)+\left[\prod_{\tau=t_0K+k_0+1}^{TK_u}\left(1+\frac{\mu_u L_u}{\tau}\right)\right]\frac{L_{uv}\Delta_{v,K_v}^T}{L_u}\\
&\leq \left[\prod_{\tau=t_0K+k_0+1}^{TK_u}e^{\left(\frac{\mu_u}{\tau}\right)}\right]\left(\frac{2\sigma_u}{SL_u}+\frac{L_{uv}\Delta_{v,K_v}^T}{L_u}\right)
\end{aligned}
$$

Expand the first item, then we have:

$$
\begin{aligned}
\Delta_{u,K_u}^T &\leq e^{\mu_u L_u\left(\sum_{\tau=t_0K+k_0+1}^{TK_u}\frac{1}{\tau}\right)}\left(\frac{2\sigma_u}{SL_u}+\frac{L_{uv}\Delta_{v,K_v}^T}{L_u}\right)\\
&\leq e^{\mu_u L_u\ln\left(\frac{TK_u}{t_0K+k_0}\right)}\left(\frac{2\sigma_u}{SL_u}+\frac{L_{uv}\Delta_{v,K_v}^T}{L_u}\right)\\
&\leq \left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\left(\frac{2\sigma_u}{SL_u}+\frac{L_{uv}}{L_u}\left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v}\frac{2\sigma_v}{SL_v}\right)\\
&\leq \left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\frac{2\sigma_u}{SL_u}+\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v}\frac{2L_{uv}\sigma_v}{SL_vL_u}.
\end{aligned}
$$

We can see that the bound of the local stability term for the shared variables in C-PFL has an extra term $\left(\frac{TK_u}{\tau_0}\right)^{\mu_u}\left(\frac{TK_v}{\tau_0}\right)^{\mu_v}\frac{2L_{uv}\sigma_v}{SL_vL_u}$. This is the alignment error caused by the alternative update for the personalized and shared variables, which is related to the smoothness of L_u, L_v, L_{uv}, the local epochs K_u, K_v and the variance bound σ_v.

Therefore, we get the combination of $\Delta_{u,K}^T$ and $\Delta_{v,K}^T$ as Δ_K^T:

$$\Delta_K^T = \Delta_{u,K_v}^T + \Delta_{v,K_u}^T \le \left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u} \frac{2\sigma_u}{SL_u} + \left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v} \left(1 + \frac{L_{uv}}{L_u}(\frac{TK_u}{\tau_0})^{\mu_u L_u}\right) \frac{2\sigma_v}{SL_v}.$$

According to the Lemma 2, the first term in the stability (condition is omitted for abbreviation) can be bound as:

$$\begin{aligned}
&\mathbb{E}\|w_i^{T+1} - \widetilde{w}_i^{T+1}\| \\
&= \mathbb{E}\|\frac{1}{n}\sum_{i\in S^t}\left(w_{i,K}^T - \widetilde{w}_{i,K}^T\right)\| = \frac{1}{n}\mathbb{E}\|\sum_{i\in S^t}\left(w_{i,K}^T - \widetilde{w}_{i,K}^T\right)\| \\
&\le \frac{1}{n}\mathbb{E}\sum_{i\in S^t}\|\left(w_{i,K}^T - \widetilde{w}_{i,K}^T\right)\| = \frac{1}{n}\frac{n}{m}\mathbb{E}\sum_{i\in[m]}\|\left(w_{i,K}^T - \widetilde{w}_{i,K}^T\right)\| \\
&= \frac{1}{m}\sum_{i\in[m]}\mathbb{E}\|\left(w_{i,K}^T - \widetilde{w}_{i,K}^T\right)\| = \frac{1}{m}\Delta_K^T \\
&\le \left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u} \frac{2\sigma_u}{mSL_u} + \left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v} \left(1 + \frac{L_{uv}}{L_u}(\frac{TK_u}{\tau_0})^{\mu_u L_u}\right) \frac{2\sigma_v}{mSL_v}.
\end{aligned}$$

Therefore, we can upper bound the stability in C-PFL as:

$$\begin{aligned}
&\mathbb{E}\|f(w_i^{T+1};z) - f(\widetilde{w}_i^{T+1};z)\| \\
&\le G\mathbb{E}\|w_i^{T+1} - \widetilde{w}_i^{T+1}\| + \frac{nU\tau_0}{mS} \\
&\le \left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u} \frac{2G\sigma_u}{mSL_u} + \frac{nU\tau_0}{mS} + \left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v} \left(1 + \frac{L_{uv}}{L_u}(\frac{TK_u}{\tau_0})^{\mu_u L_u}\right) \frac{2G\sigma_v}{mSL_v}.
\end{aligned}$$

Obviously, we can select a proper event ξ with a proper τ_0 to minimize the upper bound. For $\tau \in [1, TK]$, by selecting $\tau_0 = \left(\frac{2\sigma_l G}{nUL}\right)^{\frac{1}{1+\mu L}} (TK)^{\frac{\mu L}{1+\mu L}}$, we can minimize the bound as:

$$\begin{aligned}
&\mathbb{E}\|f(w^{T+1};z) - f(\widetilde{w}^{T+1};z)\| \\
&\le \frac{2nU\tau_0}{mS} = \frac{2nU}{mS}\left(\frac{2\sigma_l G}{nUL}\right)^{\frac{1}{1+\mu L}} (TK)^{\frac{\mu L}{1+\mu L}} \\
&\le \frac{4}{S}\left(\frac{\sigma_l G}{L}\right)^{\frac{1}{1+\mu L}} \left(\frac{n^{\frac{\mu L}{1+\mu L}}}{m}\right)(UTK)^{\frac{\mu L}{1+\mu L}}.
\end{aligned}$$

E.3 Generalization Bounds for D-PFL

Lemma 9 (Bounded the local gradients)**.** *When $(t,k) < (t_0,k_0)$, the sampled data is always the same between the different datasets, which shows $\Gamma_k^t = 0$. When $t = t_0$, only those updates at $k \ge k_0$ are different. When $t > t_0$, all the local gradients difference during local K iterations are non-zero. Thus we can first explore the upper bound of the stages with full K iterations when $t > t_0$. Let the data sample z be the same random data sample and $z/\widetilde{z}$ be a different sample pair for abbreviation, when $t \ge t_0$, we have: If we sample the same data z (not the $z_{i^\star,j^\star}$) in dataset $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ at k iteration on round t, we have:*

$$\mathbb{E}\|\eta_u \Gamma_{u,k}^t\| \le \left(\frac{\tau}{\tau_0}\right)^{\mu_u L_u} \frac{2\mu_u\sigma_u}{\tau S}. \tag{19}$$

Proof. According to the Lemma 4 and 8, we can also bound the local stability term for the personal variables. Let the learning rate $\eta_v = \frac{\mu_v}{\tau} = \frac{\mu_v}{tK_v+k}$ is decayed as the communication round t and iteration k where μ_v is a specific constant, we have:

$$\Delta_{v,k}^t + \frac{2\sigma_v}{SL_v} \le \left(\frac{\tau}{\tau_0}\right)^{\mu_v L_v} \frac{2\sigma_v}{SL_v}. \tag{20}$$

For the shared variables, we have:

$$
\begin{aligned}
&\mathbb{E}\|\eta_u\Gamma^t_{u,k}\| = \mathbb{E}\|\eta_u\left[g^t_{u,0,k}-\widetilde{g}^t_{u,0,k}, g^t_{u,1,k}-\widetilde{g}^t_{u,1,k},\cdots,g^t_{u,m,k}-\widetilde{g}^t_{u,m,k}\right]^\top\| \\
&\leq \eta_u\sum_{i\in[m]}\mathbb{E}\|g^t_{u,i,k}-\widetilde{g}^t_{u,i,k}\| \\
&\leq \eta_u\sum_{i\neq i^\star}\mathbb{E}\|\nabla_u f_i(u^t_{i,k},v^t_{i,K_v},z)-\nabla_u f_i(\widetilde{u}^t_{i,k},\widetilde{v}^t_{i,K_v},z)\| \\
&\quad+\frac{(S-1)\eta_u}{S}\mathbb{E}\|\nabla_u f_{i^\star}(u^t_{i^\star,k},v^t_{i^\star,K_v},z)-\nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k},\widetilde{v}^t_{i^\star,K_v},z)\| \\
&\quad+\frac{\eta_u}{S}\mathbb{E}\|\nabla_u f_{i^\star}(u^t_{i^\star,k},v^t_{i^\star,K_v},z)-\nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k},\widetilde{v}^t_{i^\star,K_v},z)+\nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k},\widetilde{v}^t_{i^\star,K_v},z)-\nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k},\widetilde{v}^t_{i^\star,K_v},\widetilde{z})\| \\
&\leq \eta_u\sum_{i\neq i^\star}\left(L_u\mathbb{E}\|u^t_{i,k}-\widetilde{u}^t_{i,k}\|+L_{uv}\mathbb{E}\|v^t_{i,K_v}-\widetilde{v}^t_{i,K_v}\|\right) \\
&\quad+\frac{(S-1)\eta_u}{S}(L_u\mathbb{E}\|u^t_{i,k}-\widetilde{u}^t_{i,k}\|+L_{uv}\mathbb{E}\|v^t_{i,K_v}-\widetilde{v}^t_{i,K_v}\|) \\
&\quad+\frac{\eta_u}{S}\left(L_u\mathbb{E}\|u^t_{i,k}-\widetilde{u}^t_{i,k}\|+L_{uv}\mathbb{E}\|v^t_{i,K_v}-\widetilde{v}^t_{i,K_v}\|\right) \\
&\quad+\frac{\eta_u}{S}\mathbb{E}\|(\nabla f_{i^\star}(\widetilde{u}^t_{i^\star,k},\widetilde{v}^t_{i^\star,K_v},z)-\nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k},\widetilde{v}^t_{i^\star,K_v}))-\left(\nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k},\widetilde{v}^t_{i^\star,K_v},\widetilde{z})-\nabla_u f_{i^\star}(\widetilde{u}^t_{i^\star,k},\widetilde{v}^t_{i^\star,K_v})\right)\| \\
&\leq \eta_u\sum_{i\in[m]}(L_u\mathbb{E}\|u^t_{i,k}-\widetilde{u}^t_{i,k}\|+L_{uv}\mathbb{E}\|v^t_{i,K}-\widetilde{v}^t_{i,K}\|)+\frac{2\eta_u\sigma_u}{S} \\
&=\eta_u L_u(\Delta^t_{u,k}+\frac{L_{uv}\Delta^t_{v,K}}{L_u}+\frac{2\sigma_u}{SL_u}).
\end{aligned}
$$

According to the Lemma 4, 8 and Eq.(20), we bound the gradient difference as:

$$
\begin{aligned}
\mathbb{E}\|\eta_u\Gamma^t_{u,k}\| &\leq \eta_u L_u\left(\Delta^t_{u,k}+\frac{L_{uv}\Delta^t_{v,K}}{L_u}+\frac{2\sigma_u}{SL_u}\right) \\
&\leq \left(\frac{\tau}{\tau_0}\right)^{\mu_u L_u}\frac{2\mu_u\sigma_u}{\tau S}+\left(\frac{L_{uv}}{L_v}\right)\left(\frac{\tau}{\tau_0}\right)^{\mu_v L_v}\frac{2\mu_u\sigma_v}{\tau S}.
\end{aligned}
$$

where $\tau = tK + k$.

□

Lemma 10 (Bounded the local gradients). *When $(t,k)<(t_0,k_0)$, the sampled data is always the same between the different datasets, which shows $\Gamma^t_k=0$. When $t=t_0$, only those updates at $k\geq k_0$ are different. When $t>t_0$, all the local gradients difference during local K iterations are non-zero. Thus we can first explore the upper bound of the stages with full K iterations when $t>t_0$. Let the data sample z be the same random data sample and $z/\widetilde{z}$ be a different sample pair for abbreviation, when $t\geq t_0$, we have: If we sample the same data z (not the $z_{i^\star,j^\star}$) in dataset $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ at k iteration on round t, we have:*

$$
\mathbb{E}\|\left(\mathbf{I}-\mathbf{P}\right)\Phi^t_{u,K_u}\| \leq \frac{4\mu_u\sigma_u\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\frac{1}{t^{1-\mu_u L_u}}+(\frac{L_{uv}}{L_v})\frac{4\mu_u\sigma_v\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\frac{1}{t^{1-\mu_v L_v}}, \tag{21}
$$

$$
\mathbb{E}\|\left(\mathbf{A}-\mathbf{P}\right)\Phi^t_{u,K_u}\| \leq \frac{2\mu_u\sigma_u\lambda\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\frac{1}{t^{1-\mu_u L_u}}+(\frac{L_{uv}}{L_v})\frac{2\mu_u\sigma_v\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\frac{1}{t^{1-\mu_v L_v}}. \tag{22}
$$

Proof. In the decentralized method, the aggregation performs after K local updates which demonstrates that the initial state of each round is $\mathbf{U}^t_0=\mathbf{A}\mathbf{U}^{t-1}_{K_u}$. It also works on their difference

$\Phi_{u,0}^t = \mathbf{A}\Phi_{u,K_u}^{t-1}$. Therefore, we have:

$$\Phi_{u,K_u}^t = \Phi_{u,0}^t - \sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t = \mathbf{A}\Phi_{u,K_u}^{t-1} - \sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t.$$

Then we prove the recurrence between adjacent rounds. Let $\mathbf{P} = \frac{1}{m}\mathbf{1}\mathbf{1}^\top \in \mathbb{R}^{m\times m}$ and $\mathbf{I} \in \mathbb{R}^{m\times m}$ is the identity matrix, due to the double stochastic property of the adjacent matrix $\mathbf{A}$, we have:

$$\mathbf{AP} = \mathbf{PA} = \mathbf{P}.$$

Thus,

$$\begin{aligned}
\left(\mathbf{I}-\mathbf{P}\right)\Phi_{u,K_u}^t &= \left(\mathbf{I}-\mathbf{P}\right)\mathbf{A}\Phi_{u,K_u}^{t-1} - \left(\mathbf{I}-\mathbf{P}\right)\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t \\
&= \left(\mathbf{A}\Phi_{u,K_u}^{t-1} - \sum_{k=0}^{K-1} \eta_u \Gamma_{u,k}^t\right) - \mathbf{PA}\Phi_{u,K_u}^{t-1} + \mathbf{PA}\Phi_{u,K_u}^{t-1} - \mathbf{P}\left(\mathbf{A}\Phi_{u,K_u}^{t-1} - \sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t\right).
\end{aligned}$$

By taking the expectation of the norm on both sides, we have:

$$\begin{aligned}
\mathbb{E}\| \left(\mathbf{I}-\mathbf{P}\right)\Phi_{u,K_u}^t\| &\leq \mathbb{E}\|\mathbf{A}\Phi_{u,K_u}^{t-1} - \sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t - \mathbf{PA}\Phi_{u,K_u}^{t-1}\| + \mathbb{E}\|\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t\| \\
&\leq \mathbb{E}\|\mathbf{A}\Phi_{u,K_u}^{t-1} - \mathbf{PA}\Phi_{u,K_u}^{t-1}\| + 2\mathbb{E}\|\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t\| \\
&= \mathbb{E}\|\left(\mathbf{A}-\mathbf{P}\right)\left(\mathbf{I}-\mathbf{P}\right)\Phi_{u,K_u}^{t-1}\| + 2\mathbb{E}\|\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t\| \\
&\leq \lambda\mathbb{E}\|\left(\mathbf{I}-\mathbf{P}\right)\Phi_{u,K_u}^{t-1}\| + 2\mathbb{E}\|\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t\|.
\end{aligned}$$

The equality adopts $\left(\mathbf{A}-\mathbf{P}\right)\left(\mathbf{I}-\mathbf{P}\right) = \mathbf{A}-\mathbf{P}-\mathbf{AP}+\mathbf{PP} = \mathbf{A}-\mathbf{PA}$. We know the fact that $\Phi_k^t = 0$ where $(t,k) \in (t_0,k_0)$. Thus unwinding the above inequality we have:

$$\begin{aligned}
\mathbb{E}\|\left(\mathbf{I}-\mathbf{P}\right)\Phi_{u,K_u}^t\| &\leq \lambda^{t-t_0+1}\mathbb{E}\|\left(\mathbf{I}-\mathbf{P}\right)\Phi_{u,K_u}^{t_0-1}\| + 2\sum_{s=t_0}^{t}\lambda^{t-s}\mathbb{E}\|\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^s\| \\
&= 2\sum_{s=t_0}^{t}\lambda^{t-s}\mathbb{E}\|\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^s\|.
\end{aligned}$$

To maintain the term of $\mathbf{A}$, we have:

$$\begin{aligned}
\left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^t &= \left(\mathbf{A}-\mathbf{P}\right)\mathbf{A}\Phi_{u,K_u}^{t-1} - \left(\mathbf{A}-\mathbf{P}\right)\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t \\
&= \left(\mathbf{A}-\mathbf{P}\right)\left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^{t-1} - \left(\mathbf{A}-\mathbf{P}\right)\sum_{k=0}^{K_u-1} \eta_u \Gamma_{u,k}^t.
\end{aligned}$$

The second equality adopts $\left(\mathbf{A}-\mathbf{P}\right)\left(\mathbf{A}-\mathbf{P}\right) = \left(\mathbf{A}-\mathbf{P}\right)\mathbf{A}-\mathbf{AP}+\mathbf{PP} = \left(\mathbf{A}-\mathbf{P}\right)\mathbf{A}$. Therefore we have the following recursive formula:

$$\begin{aligned}
\mathbb{E}\|\left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^t\| &\leq \mathbb{E}\|\left(\mathbf{A}-\mathbf{P}\right)\left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^{t-1}\| + \mathbb{E}\|\left(\mathbf{A}-\mathbf{P}\right)\sum_{k=0}^{K-1} \eta_u \Gamma_{u,k}^t\| \\
&\leq \lambda\mathbb{E}\|\left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^{t-1}\| + \lambda\mathbb{E}\|\sum_{k=0}^{K-1} \eta_u \Gamma_{u,k}^t\|.
\end{aligned}$$

The same as above, we can unwind this recurrence formulation from t to t_0 as:

$$\begin{aligned}\mathbb{E}\| \left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^{t}\| &\leq \lambda^{t-t_0+1}\mathbb{E}\| \left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^{t_0-1}\| + \sum_{s=t_0}^{t}\lambda^{t-s+1}\mathbb{E}\|\sum_{k=0}^{K_u-1}\eta_u\Gamma_{u,k}^{s}\| \\ &= \sum_{s=t_0}^{t}\lambda^{t-s+1}\mathbb{E}\|\sum_{k=0}^{K_u-1}\eta_u\Gamma_{u,k}^{s}\|.\end{aligned}$$

Unwinding the summation on k and adopting Lemma 5, we have:

$$\begin{aligned}&\sum_{s=t_0}^{t}\lambda^{t-s}\mathbb{E}\|\sum_{k=0}^{K_u-1}\eta_u\Gamma_{u,k}^{s}\| \\ &\leq \sum_{s=t_0}^{t}\lambda^{t-s}\sum_{k=0}^{K_u-1}\mathbb{E}\|\eta_u\Gamma_{u,k}^{s}\| \\ &\leq \frac{2\mu_u\sigma_u}{S\tau_0^{\mu_u L_u}}\sum_{s=t_0}^{t}\lambda^{t-s}\sum_{k=0}^{K_u-1}\frac{\tau^{\mu_u L_u}}{\tau} + (\frac{L_{uv}}{L_v})\frac{2\mu_u\sigma_v}{S\tau_0^{\mu_v L_v}}\sum_{s=t_0}^{t}\lambda^{t-s}\sum_{k=0}^{K_u-1}\frac{\tau^{\mu_v L_v}}{\tau} \\ &\leq \frac{2\mu_u\sigma_u}{S\tau_0^{\mu_u L_u}}\sum_{s=t_0}^{t}\lambda^{t-s}\sum_{k=0}^{K_u-1}\frac{(sK_u)^{\mu_u L_u}}{sK_u} + (\frac{L_{uv}}{L_v})\frac{2\mu_u\sigma_v}{S\tau_0^{\mu_v L_v}}\sum_{s=t_0}^{t}\lambda^{t-s}\sum_{k=0}^{K_u-1}\frac{(sK_u)^{\mu_v L_v}}{sK_u} \\ &= \frac{2\mu_u\sigma_u}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\sum_{s=t_0}^{t}\frac{\lambda^{t-s}}{s^{1-\mu_u L_u}} + (\frac{L_{uv}}{L_v})\frac{2\mu_u\sigma_v}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\sum_{s=t_0}^{t}\frac{\lambda^{t-s}}{s^{1-\mu_v L_v}} \\ &\leq \frac{2\mu_u\sigma_u}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\sum_{s=t_0-1}^{t-1}\frac{\lambda^{t-s-1}}{(s+1)^{1-\mu_u L_u}} + (\frac{L_{uv}}{L_v})\frac{2\mu_u\sigma_v}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\sum_{s=t_0-1}^{t-1}\frac{\lambda^{t-s-1}}{(s+1)^{1-\mu_v L_v}} \\ &\leq \frac{2\mu_u\sigma_u\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\frac{1}{t^{1-\mu_u L_u}} + (\frac{L_{uv}}{L_v})\frac{2\mu_u\sigma_v\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\frac{1}{t^{1-\mu_v L_v}}.\end{aligned}$$

Therefore, we get an upper bound on the aggregation gap which is related to the spectrum gap:

$$\begin{aligned}&\mathbb{E}\| \left(\mathbf{I}-\mathbf{P}\right)\Phi_{u,K_u}^{t}\| \leq 2\sum_{s=t_0}^{t}\lambda^{t-s}\mathbb{E}\|\sum_{k=0}^{K_u-1}\eta_u\Gamma_{u,k}^{s}\| \\ &\leq \frac{4\mu_u\sigma_u\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\frac{1}{t^{1-\mu_u L_u}} + (\frac{L_{uv}}{L_v})\frac{4\mu_u\sigma_v\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\frac{1}{t^{1-\mu_v L_v}}, \\ &\mathbb{E}\| \left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^{t}\| \leq \sum_{s=t_0}^{t}\lambda^{t-s+1}\mathbb{E}\|\sum_{k=0}^{K-1}\eta_u\Gamma_{u,k}^{s}\| \\ &\leq \frac{2\mu_u\sigma_u\lambda\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\frac{1}{t^{1-\mu_u L_u}} + (\frac{L_{uv}}{L_v})\frac{2\mu_u\sigma_v\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\frac{1}{t^{1-\mu_v L_v}}.\end{aligned}$$

The first inequality provides the upper bound between the difference between the averaged state and the vanilla state, and the second inequality provides the upper bound between the aggregated state and the averaged state. □

Main Proof for Theorem 2 According to the Lemma 4 and 8, it is easy to bound the local stability. We obverse it when the event ξ happens, and we have $\Delta_{k_0}^{t_0}=0$. Therefore, we unwind the recurrence formulation from T, K to t_0, k_0. Let $\eta_u=\frac{\mu_u}{\tau}=\frac{\mu_u}{tK+k}$ and $\eta_v=\frac{\mu_v}{\tau}=\frac{\mu_v}{tK+k}$ are decayed as the communication round t and iteration k where $\mu_u \leq \frac{1}{L_u}$ and $\mu_v \leq \frac{1}{L_v}$ are specific constants, we have:

$$
\begin{aligned}
&\sum_{i\in[m]} \mathbb{E}\|u_{i,K_u}^{t+1} - \widetilde{u}_{i,K_u}^{t+1}\| \\
&= \sum_{i\in[m]} \mathbb{E}\| \left(u_{i,0}^{t+1} - \widetilde{u}_{i,0}^{t+1}\right) - \sum_{k=0}^{K-1} \eta_k^t \left(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t\right) \| \\
&= \sum_{i\in[m]} \mathbb{E}\| \left(u_{i,0}^{t+1} - \widetilde{u}_{i,0}^{t+1}\right) - \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) + \left(u_{i,K}^t - \widetilde{u}_{i,K_u}^t\right) - \sum_{k=0}^{K_u-1} \eta_u \left(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t\right) \| \\
&\leq \sum_{i\in[m]} \left[\mathbb{E}\| \left(u_{i,0}^{t+1} - \widetilde{u}_{i,0}^{t+1}\right) - \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \| + \mathbb{E}\| \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \| + \mathbb{E}\| \sum_{k=0}^{K_u-1} \eta_u \left(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t\right) \|\right] \\
&\leq \sum_{i\in[m]} \mathbb{E}\| \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \| + \sum_{i\in[m]} \mathbb{E}\| \sum_{k=0}^{K_u-1} \eta_u \left(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t\right) \| \\
&\quad + m\mathbb{E}\left[\frac{1}{m}\sum_{i\in[m]} \| \left(u_{i,0}^{t+1} - \widetilde{u}_{i,0}^{t+1}\right) - \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \|\right] \\
&\leq \sum_{i\in[m]} \mathbb{E}\| \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \| + \sum_{i\in[m]} \mathbb{E}\| \sum_{k=0}^{K_u-1} \eta_u \left(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t\right) \| \\
&\quad + m\mathbb{E}\sqrt{\frac{1}{m}\sum_{i\in[m]} \| \left(u_{i,0}^{t+1} - \widetilde{u}_{i,0}^{t+1}\right) - \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \|^2} \\
&= \sum_{i\in[m]} \mathbb{E}\| \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \| + \sqrt{m}\mathbb{E}\|\Phi_{u,0}^{t+1} - \Phi_{u,K_u}^t\| + \sum_{i\in[m]} \mathbb{E}\| \sum_{k=0}^{K_u-1} \eta_u \left(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t\right) \|
\end{aligned}
$$

Let $\Phi_{u,0}^{t+1} = \mathbf{A}\Phi_{u,K_u}^t$, we have:

$$
\begin{aligned}
&\sum_{i\in[m]} \mathbb{E}\|u_{i,K_u}^{t+1} - \widetilde{u}_{i,K_u}^{t+1}\| \\
&\leq \sum_{i\in[m]} \mathbb{E}\| \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \| + \sqrt{m}\mathbb{E}\|\mathbf{A}\Phi_{u,K_u}^t - \Phi_{u,K_u}^t\| + \sum_{i\in[m]} \mathbb{E}\| \sum_{k=0}^{K_u-1} \eta_u \left(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t\right) \| \\
&\leq \sum_{i\in[m]} \mathbb{E}\| \left(u_{i,K_u}^t - \widetilde{u}_{i,K_u}^t\right) \| + \sum_{i\in[m]} \mathbb{E}\| \sum_{k=0}^{K_u-1} \eta_u \left(g_{u,i,k}^t - \widetilde{g}_{u,i,k}^t\right) \| \\
&\quad + \sqrt{m}\mathbb{E}\| \left(\mathbf{A} - \mathbf{P}\right) \Phi_{u,K_u}^t\| + \sqrt{m}\mathbb{E}\| \left(\mathbf{P} - \mathbf{I}\right) \Phi_{u,K_u}^t\|.
\end{aligned}
$$

Since $v_{i,0}^{t+1} = v_{i,K_v}^t$ for the private variables, then we have the recursion:

$$
\begin{aligned}
\sum_{i\in[m]}\mathbb{E}\|v_{i,K_v}^{t+1}-\widetilde{v}_{i,K_v}^{t+1}\| &= \sum_{i\in[m]}\mathbb{E}\|\left(v_{i,0}^{t+1}-\widetilde{v}_{i,0}^{t+1}\right)-\sum_{k=0}^{K_v-1}\eta_k^t\left(g_{v,i,k}^t-\widetilde{g}_{v,i,k}^t\right)\| \\
&\leq \sum_{i\in[m]}\left[\mathbb{E}\|\left(v_{i,K_v}^t-\widetilde{v}_{i,K_v}^t\right)\|+\mathbb{E}\|\sum_{k=0}^{K_v-1}\eta_v\left(g_{v,i,k}^t-\widetilde{g}_{v,i,k}^t\right)\|\right] \\
&\leq \sum_{i\in[m]}\mathbb{E}\|\left(v_{i,K_v}^t-\widetilde{v}_{i,K_v}^t\right)\|+\sum_{i\in[m]}\mathbb{E}\|\sum_{k=0}^{K_v-1}\eta_u\left(g_{v,i,k}^t-\widetilde{g}_{v,i,k}^t\right)\|.
\end{aligned}
$$

Therefore, we can bound this by two terms in one complete communication round. One is the process of local multi-times SGD iterations, and the other is the aggregation step. For the local training process, we can continue to use Lemma 9, 7, and 8. Let $\tau = tK + k$ as above, we have:

$$
\begin{aligned}
&\Delta_{u,K_u}^t+\frac{2\sigma_u}{SL_u} \\
&\leq \left[\prod_{k=0}^{K_u-1}\left(1+\eta_u L_u\right)\right]\left(\Delta_{u,0}^t+\frac{2\sigma_u}{SL_u}\right)=\left[\prod_{k=0}^{K_u-1}\left(1+\frac{\mu_u L_u}{\tau}\right)\right]\left(\Delta_{u,0}^t+\frac{2\sigma_u}{SL_u}\right) \\
&\leq \left[\prod_{k=0}^{K-1}e^{\frac{\mu_u L_u}{\tau}}\right]\left(\Delta_{u,0}^t+\frac{2\sigma_u}{SL_u}\right)=e^{\mu L\sum_{k=0}^{K_u-1}\frac{1}{\tau}}\left(\Delta_{u,0}^t+\frac{2\sigma_u}{SL_u}\right) \\
&\leq e^{\mu_u L_u\ln\left(\frac{t+1}{t}\right)}\left(\Delta_{u,0}^t+\frac{2\sigma_u}{SL_u}\right)=\left(\frac{t+1}{t}\right)^{\mu_u L_u}\left(\Delta_{u,0}^t+\frac{2\sigma_u}{SL_u}\right) \\
&\leq \left(\frac{t+1}{t}\right)^{\mu_u L_u}\left[\Delta_{u,K_u}^{t-1}+\sqrt{m}(\mathbb{E}\|\left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^t\|+\mathbb{E}\|\left(\mathbf{P}-\mathbf{I}\right)\Phi_{u,K_u}^t\|)+\frac{2\sigma_u}{SL_u}\right] \\
&\leq \left(\frac{t+1}{t}\right)^{\mu_u L_u}\left(\Delta_{u,K_u}^{t-1}+\frac{2\sigma_u}{SL_u}\right)+\sqrt{m}\left(\frac{t+1}{t}\right)^{\mu_u L_u}\left(\mathbb{E}\|\left(\mathbf{A}-\mathbf{P}\right)\Phi_{u,K_u}^t\|+\mathbb{E}\|\left(\mathbf{P}-\mathbf{I}\right)\Phi_{u,K_u}^t\|\right) \\
&\leq \underbrace{\left(\frac{t+1}{t}\right)^{\mu_u L_u}\left(\Delta_{u,K_u}^{t-1}+\frac{2\sigma_u}{SL_u}\right)}_{\text{local updates}}+\underbrace{\frac{6\sqrt{m}\mu_u\sigma_u\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\left(\frac{t+1}{t}\right)^{\mu_u L_u}\frac{1}{t^{1-\mu_u L_u}}}_{\text{aggregation gaps}} \\
&\quad+\underbrace{(\frac{L_{uv}}{L_v})\frac{6\sqrt{m}\mu_u\sigma_v\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\left(\frac{t+1}{t}\right)^{\mu_v L_v}\frac{1}{t^{1-\mu_v L_v}}}_{\text{aggregation gaps}}.
\end{aligned}
$$

The last adopts the Eq.(21) and (22), and the fact $\lambda \leq 1$. Obviously, in the decentralized federated learning setup, the first term still comes from the updates of the local training. The second term comes from the aggregation gaps, which is related to the spectrum gap λ.

For the private variables, since we do not exchange them with neighbors, we have:

$$\begin{aligned}
\Delta_{v,K_v}^t + \frac{2\sigma_v}{SL_v} &\leq \left[\prod_{k=0}^{K_v-1}\left(1+\eta_v L_v\right)\right]\left(\Delta_{v,0}^t + \frac{2\sigma_v}{SL_v}\right) \\
&= \left[\prod_{k=0}^{K_v-1}\left(1+\frac{\mu_v L_v}{\tau}\right)\right]\left(\Delta_{v,0}^t + \frac{2\sigma_v}{SL_v}\right) \\
&\leq \left[\prod_{k=0}^{K_v-1} e^{\frac{\mu_v L_v}{\tau}}\right]\left(\Delta_{v,0}^t + \frac{2\sigma_v}{SL_v}\right) \\
&= e^{\mu_v L_v \sum_{k=0}^{K_v-1}\frac{1}{\tau}}\left(\Delta_{v,0}^t + \frac{2\sigma_v}{SL_v}\right) \\
&\leq e^{\mu_v L_v \ln\left(\frac{t+1}{t}\right)}\left(\Delta_{v,0}^t + \frac{2\sigma_v}{SL_v}\right) \\
&= \left(\frac{t+1}{t}\right)^{\mu_v L_v}\left(\Delta_{v,0}^t + \frac{2\sigma_v}{SL_v}\right).
\end{aligned}$$

Unwinding this from t_0 to T, we have:

$$\begin{aligned}
\Delta_{u,K_u}^T + \frac{2\sigma_u}{SL_u} \leq &\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\frac{2\sigma_u}{SL_u} + \frac{6\sqrt{m}\mu_u\sigma_u\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\sum_{t=t_0+1}^{T}\left(\frac{t+1}{t}\right)^{\mu_u L_u}\frac{1}{t^{1-\mu_u L_u}} \\
&+ \frac{6\sqrt{m}\mu_u\sigma_v\kappa_\lambda L_{uv}}{SL_v}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\sum_{t=t_0+1}^{T}\left(\frac{t+1}{t}\right)^{\mu_v L_v}\frac{1}{t^{1-\mu_v L_v}} \\
\leq &\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\frac{2\sigma_u}{SL_u} + \frac{12\sqrt{m}\mu_u\sigma_u\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\sum_{t=t_0+1}^{T}\frac{1}{t^{1-\mu_u L_u}} \\
&+ \left(\frac{L_{uv}}{L_v}\right)\frac{12\sqrt{m}\mu_u\sigma_v\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\sum_{t=t_0+1}^{T}\frac{1}{t^{1-\mu_v L_v}} \\
\leq &\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\frac{2\sigma_u}{SL_u} + \frac{12\sqrt{m}\mu_u\sigma_u\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_u L_u}\left.\frac{t^{\mu_u L_u}}{\mu_u L_u}\right|_{t=t_0+1}^{t=T} \\
&+ \left(\frac{L_{uv}}{L_v}\right)\frac{12\sqrt{m}\mu_u\sigma_v\kappa_\lambda}{S}\left(\frac{K_u}{\tau_0}\right)^{\mu_v L_v}\left.\frac{t^{\mu_v L_v}}{\mu_v L_v}\right|_{t=t_0+1}^{t=T} \\
\leq &\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\frac{2\left(1+6\sqrt{m}\kappa_\lambda\right)\sigma_u}{SL_u} + \left(\frac{L_{uv}}{L_v}\right)\left(\frac{TK_u}{\tau_0}\right)^{\mu_v L_v}\frac{12\sqrt{m}\kappa_\lambda\sigma_v}{SL_v}.
\end{aligned}$$

The second inequality adopts the fact that $1 < \frac{t+1}{t} \leq 2$ when $t > 1$ and the fact of $0 < \mu < \frac{1}{L}$.

For the personalized variables, unwinding this from t_0 to T, we have:

$$\Delta_{v,K_v}^T + \frac{2\sigma_v}{SL_v} \leq \left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v}\frac{2\sigma_v}{SL_v}.$$

Then the first term in the stability (conditions is omitted for abbreviation) can be bounded as:

$$\begin{aligned}
&\mathbb{E}\|u^{T+1} - \widetilde{u}^{T+1}\| \leq \frac{1}{m}\sum_{i\in[m]}\mathbb{E}\|\left(u_{i,K_u}^T - \widetilde{u}_{i,K_u}^T\right)\| \\
\leq &\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u}\frac{2\left(1+6\sqrt{m}\kappa_\lambda\right)\sigma_u}{SL_u} + \left(\frac{L_{uv}}{L_v}\right)\left(\frac{TK_u}{\tau_0}\right)^{\mu_v L_v}\frac{12\sqrt{m}\kappa_\lambda\sigma_v}{SL_v}, \\
&\mathbb{E}\|v^{T+1} - \widetilde{v}^{T+1}\| \leq \frac{1}{m}\sum_{i\in[m]}\mathbb{E}\|\left(v_{i,K_u}^T - \widetilde{v}_{i,K_u}^T\right)\| \leq \left(\frac{TK_u}{\tau_0}\right)^{\mu_v L_v}\frac{2\sigma_v}{SL_v}.
\end{aligned}$$

Therefore, we can upper bound the stability in decentralized federated learning as:

$$
\begin{aligned}
& \mathbb{E}\left[\|f(w_i^{T+1}; z) - f(\widetilde{w}_i^{T+1}; z)\|\right] \\
\leq\ & G\mathbb{E}\left[\|w_i^{T+1} - \widetilde{w}_i^{T+1}\| \mid \xi\right] + \frac{U\tau_0}{S} \\
\leq\ & G\mathbb{E}\left[\|u^{T+1} - \widetilde{u}^{T+1}\| \mid \xi\right] + G\mathbb{E}\left[\|v^{T+1} - \widetilde{v}^{T+1}\| \mid \xi\right] + \frac{U\tau_0}{S} \\
\leq\ & \frac{2\sigma_u G}{SL_u}\left(\frac{1+6\sqrt{m}\kappa_\lambda}{m}\right)\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u} + \frac{U\tau_0}{S} \\
& + \left(\frac{L_{uv}}{L_v}\right)\left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v}\frac{12\sqrt{m}\kappa_\lambda\sigma_v G}{SL_v} + \frac{2\sigma_v G}{SL_v}\left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v} \\
\leq\ & \frac{2\sigma_u G}{SL_u}\left(\frac{1+6\sqrt{m}\kappa_\lambda}{m}\right)\left(\frac{TK_u}{\tau_0}\right)^{\mu_u L_u} + \frac{2\sigma_v G}{SL_v}\left(1+\frac{6\sqrt{m}\kappa_\lambda}{m}\left(\frac{L_{uv}}{L_v}\right)\right)\left(\frac{TK_v}{\tau_0}\right)^{\mu_v L_v} + \frac{U\tau_0}{S}.
\end{aligned}
$$

The same as the centralized setup, we can select a proper event ξ with a proper τ_0 to minimize the error of the stability. To simplify subsequent analysis, we assume $\mu L = \max\{\mu_u L_u, \mu_v L_v\}$ and $K = \max\{K_u, K_v\}$. For $\tau \in [1, TK]$, by selecting $\tau_0 = \left[\frac{2G\sigma_u L_v^2(1+6\sqrt{m}\kappa_\lambda)+2G\sigma_v L_u L_{uv}(m+6\sqrt{m}\kappa_\lambda)}{UmL_uL_v^2}\right]^{\frac{1}{1+\mu L}}(TK)^{\frac{\mu L}{1+\mu L}}$, we get the minimal generalization bound for D-PFL:

$$
\begin{aligned}
& \mathbb{E}\left[\|f(w^{T+1}; z) - f(\widetilde{w}^{T+1}; z)\|\right] \leq \frac{2U\tau_0}{S} \\
=\ & \frac{2U}{S}\left[\frac{2G\sigma_u L_v^2(1+6\sqrt{m}\kappa_\lambda) + 2G\sigma_v L_u L_{uv}(m+6\sqrt{m}\kappa_\lambda)}{UmL_uL_v^2}\right]^{\frac{1}{1+\mu L}}(TK)^{\frac{\mu L}{1+\mu L}} \\
=\ & \frac{4}{S}\left[\frac{\sigma_u G}{L_u m}(1+6\sqrt{m}\kappa_\lambda) + \frac{\sigma_v G}{L_v}(1+\frac{6\sqrt{m}\kappa_\lambda L_{uv}}{mL_v})\right]^{\frac{1}{1+\mu L}}(UTK)^{\frac{\mu L}{1+\mu L}}.
\end{aligned}
$$