Broadcast Product: Redefining Shape-aligned Element-wise Multiplication and Beyond

Anonymous authors
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Abstract

Broadcast operations are widely used in scientific computing libraries, yet their mathematical formulation is often implicit and inconsistently represented in machine learning literature. This problem frequently leads to invalid equations when element-wise products are written despite mismatched tensor shapes. In this paper, we formalize such operations by introducing the *broadcast product* $\boxdot$, which explicitly extends the Hadamard product through shape-aligned element duplication. We provide a rigorous definition of the broadcast product, analyze its algebraic properties, and show how it can be expressed using standard linear algebra. Building on this framework, we formulate least-squares problems and propose a *broadcast decomposition*. Preliminary experiments on synthetic data suggest that the proposed decomposition captures structures that are challenging for conventional tensor decompositions. This work establishes a mathematical foundation for broadcast-aware tensor operations, connecting practical implementations with rigorous tensor analysis.

1 Introduction

Broadcast operations are a fundamental abstraction in scientific computing libraries for performing mathematical operations on tensors of different shapes. They automatically replicate elements as needed to align tensor shapes before applying the specified operation, enabling concise and expressive tensor computations. For example, in libraries such as `numpy` (Harris et al., 2020), adding a vector `x` to a matrix `A` implicitly replicates `x` to match the shape of `A`. Moreover, languages such as `MATLAB` and `Julia` (Bezanson et al., 2026) support broadcasting at the language level.

Although we cannot directly write broadcast operations into equations, many modern machine learning papers have attempted to do so by mimicking `numpy`'s descriptions. For example, consider selecting a region of a three-channel image $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{H\times W\times 3}$ using a binary mask $\boldsymbol{B} \in \{0,1\}^{H\times W}$ to obtain $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{H\times W\times 3}$, where H and W denote the height and width, respectively. One often describes this operation by mimicking a `numpy`-style expression, `Y = X * B`, and reformulating it using the Hadamard product $\odot$ as follows:

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \odot \boldsymbol{B}. \tag{1}$$

However, this expression is incorrect because the Hadamard product is only defined for tensors of the same shape. Therefore, this equation implicitly assumes the mask is broadcasted along the channel dimension. Such incorrect notation not only creates discrepancies between the source code and mathematical expressions but also leads to invalid mathematical manipulations and flawed discussions.

For example, for any two tensors with the same shape $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{N}}$, the L_0 norm of their Hadamard product must be smaller than that of the original tensor: $\|\boldsymbol{\mathcal{M}} \odot \boldsymbol{\mathcal{N}}\|_0 \leq \|\boldsymbol{\mathcal{N}}\|_0$. This is because the result of the Hadamard product can only increase the number of zero elements, never decrease it. However, substituting Equation 1 into this inequality gives $\|\boldsymbol{\mathcal{Y}}\|_0 = \|\boldsymbol{\mathcal{X}} \odot \boldsymbol{B}\|_0 \leq \|\boldsymbol{B}\|_0$, which is clearly incorrect[1]. In this way,

[1]For example, consider a 1×1 three-channel image $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{1\times 1\times 3}$ and a mask $\boldsymbol{B} \in \{0,1\}^{1\times 1}$, where the all elements of $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{B}$ are non-zero ($\|\boldsymbol{B}\|_0 = 1$). In this case, if we apply the incorrect Equation 1, all three elements of $\boldsymbol{\mathcal{Y}}$ also become non-zero, meaning that we have $\|\boldsymbol{\mathcal{Y}}\|_0 = 3$. Thus, the inequality $\|\boldsymbol{\mathcal{Y}}\|_0 \leq \|\boldsymbol{B}\|_0$, which should always hold, is no longer satisfied.

improper use of broadcast operations can lead to mathematically flawed reasoning. Many people tend to write this masking formula, and it has even appeared in award-nominated papers at top conferences (e.g., Figure 5 in Li et al. (2023) and Equation 2 in Lin et al. (2021))[2].

To address this issue, we redefine the *broadcast product* operator, denoted by $\boxdot$. This operator extends the Hadamard product $\odot$ by duplicating elements to align tensor shapes before multiplication. It is mathematically equivalent to broadcasting in `numpy`. The masking operation above can be expressed as:

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \boxdot \boldsymbol{B} = \boldsymbol{\mathcal{X}} \odot \boldsymbol{\mathcal{B}}^{\square}. \tag{2}$$

Here, $\boldsymbol{\mathcal{B}}^{\square} \in \{0,1\}^{H \times W \times 3}$ denotes the result of duplicating and concatenating the mask $\boldsymbol{B}$ along the channel dimension (as detailed later in Definition 2). Now, the inequality discussed above can be correctly written as: $\|\boldsymbol{\mathcal{Y}}\|_0 = \|\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{B}\|_0 = \|\boldsymbol{\mathcal{X}} \odot \boldsymbol{\mathcal{B}}^{\square}\|_0 \leq \|\boldsymbol{\mathcal{B}}^{\square}\|_0 = 3\|\boldsymbol{B}\|_0$. The proposed operator $\boxdot$ can accurately describe complex problems, potentially offering new interpretations of the problem and possibilities for optimization from alternative perspectives.

Furthermore, we analyze the mathematical properties of the broadcast product and position it within the context of tensor decomposition. We first represent the broadcast product in graphical notation widely used in tensor decomposition (Penrose, 1971). Next, we approximate an arbitrary tensor as the broadcast product of two tensors and minimize the reconstruction error, proposing a solution to the least squares problem based on the broadcast product. This least square formulation allows us to decompose an arbitrary tensor into multiple tensors via broadcast products, which we term *broadcast decomposition*. Finally, we demonstrate that this decomposition yields lower reconstruction error for specific synthetic data compared to other decomposition methods. While these results are experimental and preliminary, they indicate the potential of broadcast products for dimensionality reduction and other applications.

Our contributions are summarized as follows:

- We redefine the broadcast product operator $\boxdot$, providing a precise mathematical framework to represent broadcast operations and resolve inconsistencies between equations and source code. The proposed operator translates complex machine learning source code into the mathematical domain.
- We analyze the mathematical properties of the broadcast product and situate it within the context of tensor decomposition. We also formulate and solve a least squares problem using the broadcast product and propose a tensor decomposition method based on it. Although preliminary, our findings indicate that broadcast operations could serve as a new decomposition approach.

Following Kolda & Bader (2009), scalars, vectors, matrices, and tensors are denoted by $x \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^I$, $\boldsymbol{X} \in \mathbb{R}^{I \times J}$, and $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I \times J \times K}$, respectively. The element-wise (Hadamard) product is denoted by $\boldsymbol{\mathcal{X}} \odot \boldsymbol{\mathcal{Y}}$, and the element-wise division is denoted by $\boldsymbol{\mathcal{X}} \oslash \boldsymbol{\mathcal{Y}}$. An element of a tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I \times J \times K}$ is written as $x_{ijk} \in \mathbb{R}$. For a third-order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I \times J \times K}$, the frontal slice ($k$-th channel) is denoted by $\boldsymbol{X}_k \in \mathbb{R}^{I \times J}$. For an N-th order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ and a matrix $\boldsymbol{A} \in \mathbb{R}^{J \times I_n}$, we denote the n-th mode tensor-matrix product by $\boldsymbol{\mathcal{X}} \times_n \boldsymbol{A} \in \mathbb{R}^{I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N}$. We assume the domain of the numbers to be $\mathbb{R}$, although it could also be, for example, $\mathbb{C}$.

1.1 Examples

Before going into the detailed definition, let us introduce the broadcast product intuitively. If two input tensors have identical shapes, their broadcast product equals their Hadamard product, e.g., with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$:

$$\boldsymbol{x} \boxdot \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \boxdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \odot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}. \tag{3}$$

Next, we show an example of matrices with different shapes, $\mathbf{X} \in \mathbb{R}^{3 \times 2}$ and $\boldsymbol{y} \in \mathbb{R}^{1 \times 2}$, as follows:

$$\boldsymbol{X} \boxdot \boldsymbol{y} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \boxdot \begin{bmatrix} 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \odot \begin{bmatrix} 7 & 8 \\ 7 & 8 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 21 & 32 \\ 35 & 48 \end{bmatrix}. \tag{4}$$

[2]We emphasize that this point concerns notation only and does not affect the proposed algorithms.

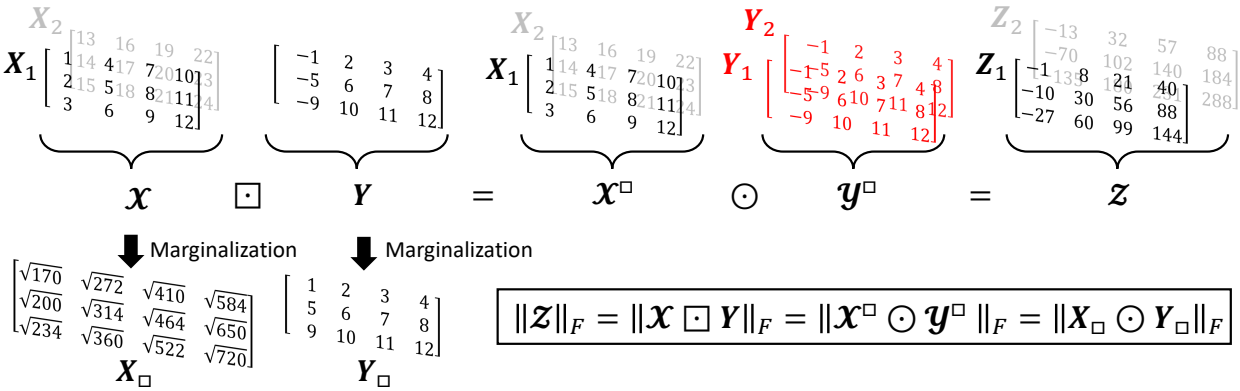


Figure 1: Example of the broadcast product of a third-order tensor $\mathcal{X} \in \mathbb{R}^{3\times4\times2}$ and a matrix $\boldsymbol{Y} \in \mathbb{R}^{3\times4}$. $\mathcal{X}^{\square}, \mathcal{Y}^{\square} \in \mathbb{R}^{3\times4\times2}$ represent their broadcasted forms. $\boldsymbol{X}_{\square}, \boldsymbol{Y}_{\square} \in \mathbb{R}^{3\times4}$ illustrate their marginalization.

Here, to match the shapes on both sides of $\boxdot$, $\boldsymbol{y}$ is duplicated along the first mode, producing a matrix with the same shape as $\boldsymbol{X}$ (shown in red). The Hadamard product with $\boldsymbol{X}$ is then computed. Note that although the result coincides with the Khatri–Rao product in this case, the broadcast product and the Khatri–Rao product are generally different. For instance, the Khatri–Rao product is defined for $\boldsymbol{X} \in \mathbb{R}^{3\times2}$ and $\boldsymbol{Y} \in \mathbb{R}^{2\times2}$, whereas the broadcast product is not (see Section 6.1).

Next, let's look at an example of the product of a third-order tensor $\mathcal{X} \in \mathbb{R}^{3\times4\times2}$ and a matrix $\boldsymbol{Y} \in \mathbb{R}^{3\times4}$:

$$\boldsymbol{X}_1 = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \quad \boldsymbol{X}_2 = \begin{bmatrix} 13 & 16 & 19 & 22 \\ 14 & 17 & 20 & 23 \\ 15 & 18 & 21 & 24 \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} -1 & 2 & 3 & 4 \\ -5 & 6 & 7 & 8 \\ -9 & 10 & 11 & 12 \end{bmatrix}. \tag{5}$$

$\mathcal{X} \boxdot \boldsymbol{Y}$ is equivalent to duplicating $\boldsymbol{Y}$ along the third mode to match shapes and computing the Hadamard product. This example closely resembles the masking operation discussed in Equation (2), which is illustrated in the upper part of Figure 1, where we will define the notation of $\mathcal{X}^{\square}$ and $\mathcal{Y}^{\square}$ in Section 2.

2 Definition

Let us define the broadcast product in detail. First, we define the *broadcast condition*, determining whether one can apply the broadcast operation to two tensors.

Definition 1 (Broadcast Condition)**.** *Consider two N-th order tensors $\mathcal{X} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$ and $\mathcal{Y} \in \mathbb{R}^{J_1\times J_2\times\cdots\times J_N}$. Here, the pair $(\mathcal{X}, \mathcal{Y})$ satisfies the broadcast condition if the following holds: For any $n \in \{1, 2, \ldots, N\}$, one of the following conditions is satisfied: (1) $I_n = J_n$, (2) $I_n = 1$, or (3) $J_n = 1$.*

For example, the following meet the broadcast condition.

- Same shape: $\mathcal{X} \in \mathbb{R}^{3\times2}, \mathcal{Y} \in \mathbb{R}^{3\times2}$.
- The length of a mode (J_2) is one: $\mathcal{X} \in \mathbb{R}^{3\times2}, \mathcal{Y} \in \mathbb{R}^{3\times1}$.
- The lengths of some modes are one ($I_1 = J_2 = 1$): $\mathcal{X} \in \mathbb{R}^{1\times2\times5}, \mathcal{Y} \in \mathbb{R}^{3\times1\times5}$.
- Multiple ones: $\mathcal{X} \in \mathbb{R}^{1\times1\times5}, \mathcal{Y} \in \mathbb{R}^{3\times1\times5}$.

The following do not satisfy the broadcast condition.

- The length of a mode is different and not one ($I_2 = 2, J_2 = 3$): $\mathcal{X} \in \mathbb{R}^{3\times2}, \mathcal{Y} \in \mathbb{R}^{3\times3}$.
- Different orders: $\mathcal{X} \in \mathbb{R}^{3\times1\times5}, \mathcal{Y} \in \mathbb{R}^{4\times3\times1\times5}$.

Note that, when considering the broadcast condition, we equate $\mathbb{R}^I$ and $\mathbb{R}^{I\times 1}$ (both represent column vectors). Similarly, we equate all "ones added to the end of a shape". For example, we equate $\mathbb{R}^{3\times 4}$, $\mathbb{R}^{3\times 4\times 1}$, and $\mathbb{R}^{3\times 4\times 1\times 1}$. Therefore, $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{5\times 4\times 3}$ and $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{5\times 4}$ satisfy the broadcast condition because we consider that $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{5\times 4\times 1}$. See Appendix F for the detailed connection to the `numpy` description.

Next, we define the *broadcast product* as follows.

Definition 2 (Broadcast Product)**.** *When two N-th order tensors $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$ and $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{J_1\times J_2\times\cdots\times J_N}$ satisfy the broadcast condition, the broadcast product of these tensors, $\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}}$, is defined as follows, where $\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{X}}^{\square}, \boldsymbol{\mathcal{Y}}^{\square} \in \mathbb{R}^{\max(I_1,J_1)\times\max(I_2,J_2)\times\cdots\times\max(I_N,J_N)}$.*

$$\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}} := \mathrm{bc}(\boldsymbol{\mathcal{X}}, \mathrm{size}(\boldsymbol{\mathcal{Y}})) \odot \mathrm{bc}(\boldsymbol{\mathcal{Y}}, \mathrm{size}(\boldsymbol{\mathcal{X}})) = \boldsymbol{\mathcal{X}}^{\square} \odot \boldsymbol{\mathcal{Y}}^{\square}.$$

Here, "size" returns the input tensor's shape as a tuple, e.g., $\mathrm{size}(\boldsymbol{\mathcal{Y}}) = (J_1, J_2, \ldots, J_N)$, and "bc" is a function to duplicate a tensor. For $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ satisfying the broadcast condition, bc inputs $\boldsymbol{\mathcal{X}}$ itself and the shape of $\boldsymbol{\mathcal{Y}}$, and outputs $\boldsymbol{\mathcal{X}}^{\square}$, i.e., $\boldsymbol{\mathcal{X}}^{\square} = \mathrm{bc}(\boldsymbol{\mathcal{X}}, \mathrm{size}(\boldsymbol{\mathcal{Y}}))$. Here, $\boldsymbol{\mathcal{X}}^{\square}$ refers to the broadcasted version of $\boldsymbol{\mathcal{X}}$, which means the following. For all n, if $I_n = 1$, replicate all elements of $\boldsymbol{\mathcal{X}}$ along n-th mode J_n times. This operation is explicitly defined by focusing on its elements as follows. With $k_n \in \{1, 2, \ldots, \max(I_n, J_n)\}$ for all n, we write the $(k_1, k_2, \ldots, k_N)$-th element of $\boldsymbol{\mathcal{X}}^{\square}$ as $x^{\square}_{k_1 k_2 \ldots k_N} = x_{i'_1 i'_2 \ldots i'_N}$, where

$$i'_n = \begin{cases} 1 & (I_n = 1), \\ k_n & (I_n \neq 1). \end{cases} \tag{6}$$

The same applies to $\boldsymbol{\mathcal{Y}}^{\square}$. In the end, we obtain

$$z_{k_1,k_2,\ldots,k_N} = x^{\square}_{k_1 k_2 \ldots k_N} y^{\square}_{k_1 k_2 \ldots k_N} = x_{i'_1 i'_2 \ldots i'_N} y_{j'_1 j'_2 \ldots j'_N}, \quad \text{where} \quad j'_n = \begin{cases} 1 & (J_n = 1), \\ k_n & (J_n \neq 1). \end{cases} \tag{7}$$

In other words, placing $\square$ on the superscript means duplicating the elements appropriately so that the shapes of $\boldsymbol{\mathcal{X}}^{\square}$ and $\boldsymbol{\mathcal{Y}}^{\square}$ match, i.e., $\mathrm{size}(\boldsymbol{\mathcal{X}}^{\square}) = \mathrm{size}(\boldsymbol{\mathcal{Y}}^{\square})$. Note that $\boldsymbol{\mathcal{X}}^{\square}$ and $\boldsymbol{\mathcal{Y}}^{\square}$ are considered as a shorthand notation, and should only be used when the interpretation is unique and obvious from the context.

The result of the broadcast product is uniquely determined. In the example of Equation (4), we can write:

$$\boldsymbol{X}^{\square} = \boldsymbol{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \boldsymbol{Y}^{\square} = \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{y} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 7 & 8 \\ 7 & 8 \end{bmatrix}. \tag{8}$$

In the example of Equation (5) and Figure 1, we obtain

$$\boldsymbol{\mathcal{X}}^{\square} = \boldsymbol{\mathcal{X}}, \quad \boldsymbol{Y}_1^{\square} = \boldsymbol{Y}_2^{\square} = \boldsymbol{Y} = \begin{bmatrix} -1 & 2 & 3 & 4 \\ -5 & 6 & 7 & 8 \\ -9 & 10 & 11 & 12 \end{bmatrix}, \tag{9}$$

where stacking $\boldsymbol{Y}_1^{\square}$ and $\boldsymbol{Y}_2^{\square}$ along the third mode leads $\boldsymbol{\mathcal{Y}}^{\square}$. The following is an example of duplication for both $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$. Let us define $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{1\times 2\times 3}$ and $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{4\times 2\times 1}$:

$$\boldsymbol{X}_1 = [1, 2], \quad \boldsymbol{X}_2 = [3, 4], \quad \boldsymbol{X}_3 = [5, 6], \quad \boldsymbol{Y}_1 = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \\ 13 & 14 \end{bmatrix}. \tag{10}$$

In this case, $\boldsymbol{\mathcal{X}}^{\square}, \boldsymbol{\mathcal{Y}}^{\square} \in \mathbb{R}^{4\times 2\times 3}$ are written as

$$\boldsymbol{X}_1^{\square} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{X}_2^{\square} = \begin{bmatrix} 3 & 4 \\ 3 & 4 \\ 3 & 4 \\ 3 & 4 \end{bmatrix}, \quad \boldsymbol{X}_3^{\square} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \\ 5 & 6 \\ 5 & 6 \end{bmatrix}, \quad \boldsymbol{Y}_1^{\square} = \boldsymbol{Y}_2^{\square} = \boldsymbol{Y}_3^{\square} = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \\ 13 & 14 \end{bmatrix}. \tag{11}$$

3 Properties

We present various mathematical properties of the broadcast product. These properties help convert mathematical expressions using the broadcast product into the standard forms used in linear algebra.

3.1 Basic properties

We first show the basic properties of the broadcast product $\boxdot$.

Compatibility: When $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ have the same shapes, $\boxdot$ is equivalent to $\odot$.

$$\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \odot \boldsymbol{\mathcal{Y}}. \tag{12}$$

Commutativity: For $\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{Y}}$, and $\mathbf{0}$ satisfying the broadcast conditions, the following holds, where $k \in \mathbb{R}$.

$$\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{Y}} \boxdot \boldsymbol{\mathcal{X}}. \tag{13}$$

$$(k\boldsymbol{\mathcal{X}}) \boxdot \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \boxdot (k\boldsymbol{\mathcal{Y}}) = k(\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}}). \tag{14}$$

$$\boldsymbol{\mathcal{X}} \boxdot \mathbf{0} = \mathbf{0} \boxdot \boldsymbol{\mathcal{X}} = \mathbf{0}. \tag{15}$$

Associativity: When $\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{Y}}$, and $\boldsymbol{\mathcal{Z}}$ mutually satisfy the broadcast conditions, we obtain

$$(\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}}) \boxdot \boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{X}} \boxdot (\boldsymbol{\mathcal{Y}} \boxdot \boldsymbol{\mathcal{Z}}) = \boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}} \boxdot \boldsymbol{\mathcal{Z}}. \tag{16}$$

Be careful that, even if $(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}})$ and $(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Z}})$ satisfy the broadcast conditions, $(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}})$ do not necessarily satisfy the broadcast conditions, e.g., $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{3\times 1}$, $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{3\times 2}$, and $\boldsymbol{\mathcal{Z}} \in \mathbb{R}^{3\times 5}$.

Distributivity: When $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{Z}}$ have identical shapes, and $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ satisfy the broadcast condition,

$$\boldsymbol{\mathcal{X}} \boxdot (\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Z}}) = \boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Z}}. \tag{17}$$

Even if $(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}})$ and $(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Z}})$ satisfy the broadcast conditions, differing shapes of $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{Z}}$ make the left-hand side uncomputable, though the right-hand side may be; e.g., $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{2\times 3}$, $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{2\times 1}$, and $\boldsymbol{\mathcal{Z}} \in \mathbb{R}^{1\times 3}$.

Generalization of the Hadamard Product: Whenever the Hadamard product is correctly defined (i.e., the operands have identical shapes), it can be replaced with the broadcast product. That is, the equation $\boldsymbol{\mathcal{A}} \odot \boldsymbol{\mathcal{B}}$ can always be rewritten as $\boldsymbol{\mathcal{A}} \boxdot \boldsymbol{\mathcal{B}}$. This is because the use of the Hadamard product implies that $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ have the same shape, allowing it to be rewritten as the broadcast product via Equation (12). Note that, however, the Hadamard product is often misused as discussed in Equation (1).

3.2 Marginalization and the Frobenius norm

We introduce marginalization for the broadcast product, enabling the efficient computation of the Frobenius norm and the derivation of the Cauchy–Schwarz inequality. We first define marginalization formally:

Definition 3 (Marginalization)**.** *When two N-th order tensors $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$ and $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{J_1\times J_2\times\cdots\times J_N}$ satisfy the broadcast condition, the marginalized tensors are written as*

$$\boldsymbol{\mathcal{X}}_{\square}, \boldsymbol{\mathcal{Y}}_{\square} \in \mathbb{R}^{\min(I_1,J_1)\times\min(I_2,J_2)\times\cdots\times\min(I_N,J_N)}.$$

These are defined by focusing on the elements. With $k_n \in \{1, 2, \ldots, \min(I_n, J_n)\}$ for all n,

$$x_{\square k_1 k_2 \ldots k_N} = \|\boldsymbol{\mathcal{X}}_{\bar{i}_1 \bar{i}_2 \ldots \bar{i}_N}\|_F, \quad \bar{i}_n = \begin{cases} : & (J_n = 1), \\ k_n & (J_n \neq 1). \end{cases}$$

That is, the marginalized tensor $\boldsymbol{\mathcal{X}}_{\square}$ is obtained by taking the Frobenius norm along each mode of $\boldsymbol{\mathcal{X}}$ if the mode's length is longer than that of $\boldsymbol{\mathcal{Y}}$. Here, $\boldsymbol{\mathcal{Y}}_{\square}$ is defined similarly. This marginalization can also be

regarded as a process that shrinks the shape of a tensor while preserving the Frobenius norm. As is clear from the construction, we have $\|\boldsymbol{\mathcal{X}}\|_F = \|\boldsymbol{\mathcal{X}}_\square\|_F \leq \|\boldsymbol{\mathcal{X}}^\square\|_F$ in general. Also, all elements of the marginalized tensor are non-negative: $\boldsymbol{\mathcal{X}}_\square \geq \mathbf{0}$. In the example of Equation (4), we obtain

$$\boldsymbol{x}_\square = \begin{bmatrix}\sqrt{35} & \sqrt{56}\end{bmatrix}, \quad \boldsymbol{y}_\square = \begin{bmatrix}7 & 8\end{bmatrix}. \tag{18}$$

Figure 1 shows the example of Equation (5). As this example of $\boldsymbol{Y}$ shows, negative elements must become positive even if the shape does not change. For Equation (10), we obtain

$$\boldsymbol{x}_\square = \begin{bmatrix}\sqrt{35} & \sqrt{56}\end{bmatrix}, \quad \boldsymbol{y}_\square = \begin{bmatrix}\sqrt{420} & \sqrt{504}\end{bmatrix}. \tag{19}$$

Using the marginalized tensors, we can write the Frobenius norm of the broadcast product as follows:

$$\|\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}}\|_F = \|\boldsymbol{\mathcal{X}}^\square \odot \boldsymbol{\mathcal{Y}}^\square\|_F = \|\boldsymbol{\mathcal{X}}_\square \odot \boldsymbol{\mathcal{Y}}_\square\|_F. \tag{20}$$

With this, one can compute the Frobenius norm efficiently. Computing the norm requires "enlarging" the tensors via $\boldsymbol{\mathcal{X}}^\square$, but one can compute it using smaller tensors by first "shrinking" them via $\boldsymbol{\mathcal{X}}_\square$ (see Figure 1).

From this, we can derive that the Cauchy–Schwarz inequality also holds for the broadcast product:

Theorem 3.1 (Cauchy–Schwarz Inequality for the Broadcast Product)**.** *Let $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ be tensors satisfying the broadcast conditions. Then:*

$$\|\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}}\|_F \leq \|\boldsymbol{\mathcal{X}}\|_F \cdot \|\boldsymbol{\mathcal{Y}}\|_F.$$

Proof. This follows directly from Equation (20) and the standard Cauchy–Schwarz inequality. See Appendix A for the complete proof. □

3.3 Properties of lower-order tensors

We describe the properties for lower-order tensors. By employing these transformations, one can express the description of the broadcast product using standard linear algebra notation.

Scalar: For any tensor $\boldsymbol{\mathcal{X}}$ and a scalar $y \in \mathbb{R}$, we obtain

$$\boldsymbol{\mathcal{X}} \boxdot y = y\boldsymbol{\mathcal{X}}. \tag{21}$$

Vector and vector: For vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^I$ of equal length, the following holds:

$$\boldsymbol{x} \boxdot \boldsymbol{y} = \boldsymbol{x} \odot \boldsymbol{y} = \mathrm{diag}(\boldsymbol{x})\boldsymbol{y} \in \mathbb{R}^I. \tag{22}$$

$$\boldsymbol{x} \boxdot \boldsymbol{y}^\top = \boldsymbol{x}\boldsymbol{y}^\top \in \mathbb{R}^{I \times I}. \tag{23}$$

For vectors $\boldsymbol{x} \in \mathbb{R}^I$ and $\boldsymbol{y} \in \mathbb{R}^J$ of different lengths, $\boldsymbol{x} \boxdot \boldsymbol{y}$ cannot be defined, but $\boldsymbol{x} \boxdot \boldsymbol{y}^\top$ can be defined:

$$\boldsymbol{x} \boxdot \boldsymbol{y}^\top = \boldsymbol{x}\boldsymbol{y}^\top \in \mathbb{R}^{I \times J}. \tag{24}$$

Matrix and vector: For a matrix $\boldsymbol{X} \in \mathbb{R}^{I \times J}$, vectors $\boldsymbol{y} \in \mathbb{R}^I$ and $\boldsymbol{z} \in \mathbb{R}^J$, the following holds:

$$\boldsymbol{X} \boxdot \boldsymbol{y} = \boldsymbol{X} \odot [\boldsymbol{y}, \boldsymbol{y}, \ldots, \boldsymbol{y}] = \boldsymbol{X} \odot (\boldsymbol{y}\mathbf{1}_J^\top) = \mathrm{diag}(\boldsymbol{y})\boldsymbol{X}. \tag{25}$$

$$\boldsymbol{X} \boxdot \boldsymbol{z}^\top = \boldsymbol{X} \odot \begin{bmatrix}\boldsymbol{z}^\top \\ \vdots \\ \boldsymbol{z}^\top\end{bmatrix} = \boldsymbol{X} \odot (\mathbf{1}_I \boldsymbol{z}^\top) = \boldsymbol{X}\mathrm{diag}(\boldsymbol{z}). \tag{26}$$

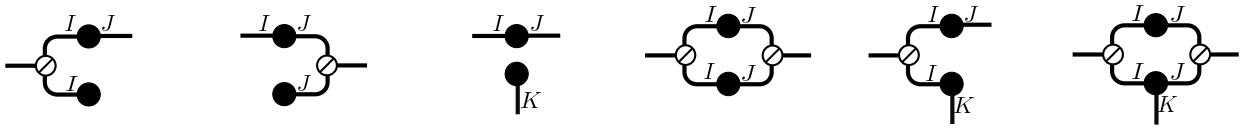
Additionally, the following holds for the Frobenius norm:

$$\|\boldsymbol{X} \boxdot \boldsymbol{y}\|_F^2 = \|\mathrm{diag}(\boldsymbol{y})\boldsymbol{X}\|_F^2 = \mathrm{tr}(\boldsymbol{X}^\top \mathrm{diag}^2(\boldsymbol{y})\boldsymbol{X}). \tag{27}$$

Third-order tensor and matrix: For a third-order tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I \times J \times K}$ and a matrix $\boldsymbol{Y} \in \mathbb{R}^{I \times J}$, the following holds. See Appendix B for details. Figure 1 is a visual example.

$$\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{Y} = \boldsymbol{\mathcal{X}} \odot \mathrm{fold}_1(\mathbf{1}_K^\top \otimes \boldsymbol{Y}) = \mathrm{fold}_3(\boldsymbol{X}_{(3)}\mathrm{diag}(\mathrm{vec}(\boldsymbol{Y}))). \tag{28}$$

This operation is typical in computer vision, e.g., $\boldsymbol{\mathcal{X}}$ represents an image and $\boldsymbol{Y}$ serves as a grayscale mask.



(a) (b) (c) (d) (e) (f)

Figure 2: Graphical notations of various patterns of broadcast product: Diagrams (a)-(f) represent broadcast products between (I, J)-matrix with (a) $(I, 1)$-vector, (b) $(1, J)$-vector, (c) $(1, 1, K)$-vector, (d) (I, J)-matrix, (e) $(I, 1, K)$-matrix, and (f) (I, J, K)-tensor, respectively. Especially, case (c) is equivalent to outer product, and case (d) equivalent to Hadamard product.

3.4 Graphical notations

Here, we show how to draw the broadcast product in graphical notations of tensors (Penrose, 1971; Taylor, 2024; Yokota, 2024). Here, we draw tensors as nodes with multiple edges, represented by black circles and lines. Additionally, we draw super-diagonal tensors using slashed circles, ⊘, instead of black circles for arbitrary tensors. For example, the third-order super-diagonal tensor $\boldsymbol{\mathcal{I}} \in \mathbb{R}^{3\times3\times3}$ is given by

$$\boldsymbol{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

Figure 2 shows the graphical notation of various patterns of the broadcast product. In this way, the broadcast product can be represented as a product (network) of tensors via super-diagonal tensors.

Figure 2a represents a case of Equation (25). Since a tensor product of a super diagonal tensor with a vector is a diagonal matrix

$$\boldsymbol{\mathcal{I}} \times_3 \boldsymbol{y}^\top = \mathrm{diag}(\boldsymbol{y}), \tag{30}$$

the property Equation (25) can be immediately verified. The property Equation (26) can also be verified in a similar way. Figure 2f represents a case of Equation (28). For the proof, see Appendix B.

3.5 Broadcast sum, difference, and division

The broadcast sum (⊞), difference (⊟), and division (⊡) are similarly defined for $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ that meet the broadcast condition:

$$\boldsymbol{\mathcal{X}} \boxplus \boldsymbol{\mathcal{Y}} := \boldsymbol{\mathcal{X}}^{\square} + \boldsymbol{\mathcal{Y}}^{\square}, \qquad \boldsymbol{\mathcal{X}} \boxminus \boldsymbol{\mathcal{Y}} := \boldsymbol{\mathcal{X}}^{\square} - \boldsymbol{\mathcal{Y}}^{\square}, \qquad \boldsymbol{\mathcal{X}} \boxslash \boldsymbol{\mathcal{Y}} := \boldsymbol{\mathcal{X}}^{\square} \oslash \boldsymbol{\mathcal{Y}}^{\square}, \tag{31}$$

where $\boldsymbol{\mathcal{Y}}$ must not have zero elements for ⊡. See Appendix C for the basic properties of these operators.

4 Real-world Examples

This section demonstrates that the broadcast product naturally formalizes a wide range of operations commonly described using informal or ambiguous notation in machine learning. Through the broadcast operations, various expressions can be represented in a unified manner, offering new perspectives on them.

4.1 Blending

Similar to the masking operation in Equation (2), multi-channel blending can be expressed using the broadcast product. Consider two color images $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \in \mathbb{R}^{H\times W\times3}$ with height H and width W and a mask $\boldsymbol{A} \in [0, 1]^{H\times W}$ representing an alpha channel. The blending operation is written as

$$\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{A} + \boldsymbol{\mathcal{Y}} \boxdot (\boldsymbol{1} - \boldsymbol{A}), \tag{32}$$

where $\mathbf{1}$ denotes an all-ones matrix of the same shape as $\boldsymbol{A}$. This formulation naturally extends to batched inputs $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \in \mathbb{R}^{N \times H \times W \times C}$, where N denotes the batch size, by defining the mask as $\boldsymbol{\mathcal{A}} \in [0,1]^{1 \times H \times W}$ (or equivalently $[0,1]^{1 \times H \times W \times 1}$), and applying Equation (32) directly.

4.2 Batch Normalization

Batch Normalization (Ioffe & Szegedy, 2015), Layer Normalization (Ba et al., 2016), and Instance Normalization (Ulyanov et al., 2016) can all be described in a unified manner using broadcast operations. We first demonstrate that the following commonly used notation is not mathematically rigorous:

$$\mathrm{BN}(x) = \left(\frac{x - \mu}{\sigma}\right)\gamma + \beta. \tag{33}$$

Here, $x \in \mathbb{R}^{N \times C \times H \times W}$ denotes the input tensor, where N, C, H, and W are the batch size, number of channels, height, and width, respectively, following the channel-second convention of Wu & He (2018). Here, $\mu, \sigma \in \mathbb{R}^C$ denote the batch-wise mean and standard deviation, and $\gamma, \beta \in \mathbb{R}^C$ are learnable parameters.

This expression is mathematically incorrect, as tensor operations with different shapes rely on implicit broadcasting, e.g., we cannot divide a tensor by a vector (σ).

Let us now rewrite Batch Normalization explicitly by focusing on individual elements. Let the input and output tensors be $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \in \mathbb{R}^{N \times C \times H \times W}$. Batch Normalization is defined as follows. For each channel $c \in \{1, 2, \ldots, C\}$, let us define the mean $\mu_c \in \mathbb{R}$ and standard deviation $\sigma_c \in \mathbb{R}$ by:

$$\mu_c = \frac{1}{NHW}\sum_{n=1}^{N}\sum_{h=1}^{H}\sum_{w=1}^{W} X_{n,c,h,w}, \quad \sigma_c = \sqrt{\frac{1}{NHW}\sum_{n=1}^{N}\sum_{h=1}^{H}\sum_{w=1}^{W}\left(X_{n,c,h,w} - \mu_c\right)^2 + \varepsilon}. \tag{34}$$

Here, ε is a small constant introduced to prevent division by zero. Using the channel-wise learned parameters $\gamma_1, \ldots, \gamma_C \in \mathbb{R}$ and $\beta_1, \ldots, \beta_C \in \mathbb{R}$, Batch Normalization can be written element-wise as:

$$Y_{n,c,h,w} = \left(\frac{X_{n,c,h,w} - \mu_c}{\sigma_c}\right)\gamma_c + \beta_c. \tag{35}$$

This representation is rigorous. However, because it focuses on individual elements, the notation becomes cluttered with indices. The resulting tensor $\boldsymbol{\mathcal{Y}}$ cannot be easily substituted into other equations.

By contrast, using the broadcast operations, Equation (35) can be concisely expressed at the tensor level as:

$$\boldsymbol{\mathcal{Y}} = ((\boldsymbol{\mathcal{X}} \boxminus \boldsymbol{\mu}) \boxslash \boldsymbol{\sigma}) \boxdot \boldsymbol{\gamma} \boxplus \boldsymbol{\beta}. \tag{36}$$

Here, $\boldsymbol{\mu} \in \mathbb{R}^{1 \times C \times 1 \times 1}$ is the tensor containing $\mu_1, \ldots, \mu_C$, and $\boldsymbol{\sigma}, \boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathbb{R}^{1 \times C \times 1 \times 1}$ are defined similarly. Now, Batch Normalization is written in a mathematically rigorous manner using broadcast operations.

As modes of length one at the end of the shape can be omitted, $\boldsymbol{\mu} \in \mathbb{R}^{1 \times C \times 1 \times 1}$ can be written as $\boldsymbol{\mu} \in \mathbb{R}^{1 \times C}$. For clarity and consistency, we explicitly specify all shapes in the upcoming normalization examples.

4.3 Layer Normalization (for CNN)

Next, let us consider Layer Normalization (Ba et al., 2016). Here, we follow the notation convention used in Wu & He (2018). Let the input tensor be $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{N \times C \times H \times W}$. In Layer Normalization for CNNs, the mean over all elements of each instance $n \in \{1, ..., N\}$ is defined as $\mu_n = \frac{1}{CHW}\sum_{c=1}^{C}\sum_{h=1}^{H}\sum_{w=1}^{W} X_{n,c,h,w}$. The standard deviation is defined as $\sigma_n = \sqrt{\frac{1}{CHW}\sum_{c=1}^{C}\sum_{h=1}^{H}\sum_{w=1}^{W}\left(X_{n,c,h,w} - \mu_n\right)^2 + \varepsilon}$. Stacking these values gives $\boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathbb{R}^{N \times 1 \times 1 \times 1}$. As in Batch Normalization, the per-channel learnable parameters are $\boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathbb{R}^{1 \times C \times 1 \times 1}$. The output tensor is computed using the same expression as in Equation (36).

This formulation involves the input tensor $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{N \times C \times H \times W}$, the means $\boldsymbol{\mu} \in \mathbb{R}^{N \times 1 \times 1 \times 1}$, and the affine parameters $\boldsymbol{\gamma} \in \mathbb{R}^{1 \times C \times 1 \times 1}$. Despite these differing shapes, the entire computation can be written in a single equation using broadcast operations. This directly and rigorously reflects the actual implementation used in libraries such as `numpy` and `pytorch`.

4.4 Layer Normalization (for Transformer)

Layer Normalization is also widely used in Transformer models (Phuong & Hutter, 2022), but its usage differs slightly from the CNN case. Let the input tensor be $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{N\times L\times D}$, where N is the batch size, L is the number of tokens, and D is the embedding dimension. Unlike the CNN case, where normalization is performed over all elements of an instance, Transformer-style Layer Normalization computes statistics only over the embedding dimension. For each $n \in \{1, ..., N\}$ and $l \in \{1, ..., L\}$, the mean is defined as $\mu_{n,l} = \frac{1}{D}\sum_{d=1}^{D} X_{n,l,d}$. The standard deviation is defined as $\sigma_{n,l} = \sqrt{\frac{1}{D}\sum_{d=1}^{D}\left(X_{n,l,d} - \mu_{n,l}\right)^2 + \varepsilon}$. Stacking these values gives $\boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathbb{R}^{N\times L\times 1}$. In this setting, the learnable parameters are defined per embedding dimension as $\boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathbb{R}^{1\times 1\times D}$. The output tensor is again computed using the same expression in Equation (36).

4.5 Instance Normalization

Finally, let us consider Instance Normalization (Ulyanov et al., 2016). Let the input tensor be $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{N\times C\times H\times W}$. The mean over the spatial dimensions is defined as $\mu_{n,c} = \frac{1}{HW}\sum_{h=1}^{H}\sum_{w=1}^{W} X_{n,c,h,w}$. The standard deviation is defined as $\sigma_{n,c} = \sqrt{\frac{1}{HW}\sum_{h=1}^{H}\sum_{w=1}^{W}\left(X_{n,c,h,w} - \mu_{n,c}\right)^2 + \varepsilon}$. Stacking these values gives $\boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathbb{R}^{N\times C\times 1\times 1}$. With the learnable parameters $\boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathbb{R}^{1\times C\times 1\times 1}$, the output of Instance Normalization is computed using the same expression Equation (36) as for Batch Normalization and Layer Normalization.

In this way, Batch, Layer, and Instance Normalization can all be written in a unified and mathematically rigorous form by appropriately choosing the shapes of the variables. In contrast, the commonly used abbreviated notation in Equation (33) cannot precisely express these relationships and introduces ambiguity. For example, the fact that σ is a vector rather than a scalar is not apparent from that notation alone.

4.6 AdaIN

Adaptive Instance Normalization (AdaIN) (Huang & Belongie, 2017) is an extension of Instance Normalization that is widely used in style transfer. Using broadcast operations, AdaIN can be written in the same form as the normalization expression in Equation (36).

Let the input tensor be $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{N\times C\times H\times W}$. The transferred style information is represented by channel-wise affine parameters, which are given as $\boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathbb{R}^{1\times C\times 1\times 1}$. As in standard Instance Normalization, the mean and standard deviation of the input tensor are defined per instance and per channel, yielding $\boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathbb{R}^{N\times C\times 1\times 1}$. With these definitions, Adaptive Instance Normalization is computed using the same broadcast-based expression as in Equation (36).

4.7 FiLM

The final example is Feature-wise Linear Modulation (FiLM) (Perez et al., 2018), widely used in image generation: $\mathrm{FiLM}(\boldsymbol{F}_{i,c} \mid \gamma_{i,c}, \beta_{i,c}) = \gamma_{i,c}\boldsymbol{F}_{i,c} + \beta_{i,c}$. Here, $\boldsymbol{F}_{i,c}$ denotes the c-th channel response of the i-th instance, and $\gamma_{i,c}$ and $\beta_{i,c}$ are scaling and shifting weights. The operation is difficult to express in matrix notation, making the expression cumbersome.

With the broadcast product, we can intuitively express FiLM. Let $\boldsymbol{\mathcal{F}} \in \mathbb{R}^{N\times C\times H\times W}$ be a feature volume with the number of instance N, C channels, height H, width W. The weights are represented as $\boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathbb{R}^{N\times C\times 1\times 1}$. FiLM can be expressed as:

$$\mathrm{FiLM}(\boldsymbol{\mathcal{F}} \mid \boldsymbol{\gamma}, \boldsymbol{\beta}) = \boldsymbol{\mathcal{F}} \boxdot \boldsymbol{\gamma} \boxplus \boldsymbol{\beta}. \tag{37}$$

This formulation removes explicit indexing and places FiLM in the same broadcast-based framework as normalization layers, revealing FiLM as an instance transformation without normalization. We can see the clear connection to Equation (36), where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are removed. FiLM thus belongs to the same class of broadcast-based transformations as normalization layers, differing in the absence of normalization statistics.

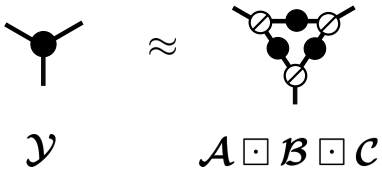


Figure 3: Broadcast decomposition

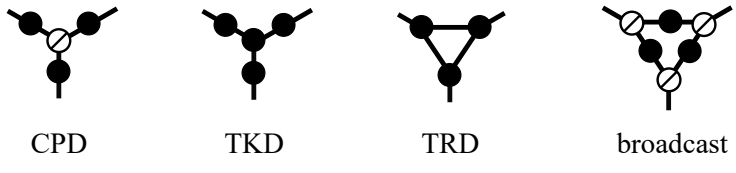


Figure 4: Graphical notation of canonical polyadic decomposition (CPD), Tucker decomposition (TKD), tensor ring decomposition (TRD), and broadcast decomposition (BD).

5 Optimizations

Up to this point, we have defined the broadcast product and its basic properties. In this section, we delve into the broadcast product in greater detail, particularly in the context of tensor decomposition, and address the new optimization challenges that arise from it, including matrix factorization techniques and their applications in dimensionality reduction.

5.1 Least squares

Let us consider three tensors $\mathcal{Y} \in \mathbb{R}^{I \times J \times K}$, $\mathcal{A} \in \mathbb{R}^{I \times J \times 1}$, $\mathcal{Z} \in \mathbb{R}^{1 \times J \times K}$ and the following least squares (LS) problem:

$$\underset{\mathcal{A}}{\text{minimize}} \, ||\mathcal{Y} - \mathcal{A} \boxdot \mathcal{Z}||_F^2, \tag{38}$$

then the solution can be given by

$$\widehat{\mathcal{A}} = \mathcal{P}_3(\mathcal{Y} \boxdot \mathcal{Z}) \boxslash \mathcal{P}_3(\mathcal{Z} \boxdot \mathcal{Z}), \tag{39}$$

where $\mathcal{P}_k(\mathcal{X}) := \mathcal{X} \times_k \mathbf{1}^\top$ performs a sum of the entries of an input tensor along the k-th mode. The proof is in Appendix D. This least-squares solution can be easily generalized to N-th order tensors by simply changing the modes of $\mathcal{P}$ to match the shape of $\mathcal{A}$ (i.e., summing along the modes with length 1 of $\mathcal{A}$).

5.2 Tensor decomposition

We propose a new tensor decomposition, called broadcast decomposition (BD), based on broadcast products:

$$\mathcal{Y} \approx \mathcal{A} \boxdot \mathcal{B} \boxdot \mathcal{C}, \tag{40}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ mutually satisfy the broadcast conditions. For minimizing squared errors $||\mathcal{Y} - \mathcal{A} \boxdot \mathcal{B} \boxdot \mathcal{C}||_F^2$, the alternating least squares (ALS) algorithm can be easily derived using Equation (39). For example, when updating $\mathcal{A}$, set $\mathcal{Z} = \mathcal{B} \boxdot \mathcal{C}$ and make $\mathcal{P}$ correspond to the shape of $\mathcal{A}$. Let us consider the sizes of tensors as $\mathcal{A} \in \mathbb{R}^{I \times J \times 1}$, $\mathcal{B} \in \mathbb{R}^{I \times 1 \times K}$, $\mathcal{C} \in \mathbb{R}^{1 \times J \times K}$, then these update rules[3] can be given by

$$\mathcal{A} \leftarrow \mathcal{P}_3(\mathcal{Y} \boxdot \mathcal{B} \boxdot \mathcal{C}) \oslash \mathcal{P}_3(\mathcal{B} \boxdot \mathcal{B} \boxdot \mathcal{C} \boxdot \mathcal{C}); \tag{41}$$

$$\mathcal{B} \leftarrow \mathcal{P}_2(\mathcal{Y} \boxdot \mathcal{A} \boxdot \mathcal{C}) \oslash \mathcal{P}_2(\mathcal{A} \boxdot \mathcal{A} \boxdot \mathcal{C} \boxdot \mathcal{C}); \tag{42}$$

$$\mathcal{C} \leftarrow \mathcal{P}_1(\mathcal{Y} \boxdot \mathcal{A} \boxdot \mathcal{B}) \oslash \mathcal{P}_1(\mathcal{A} \boxdot \mathcal{A} \boxdot \mathcal{B} \boxdot \mathcal{B}); \tag{43}$$

Here, the mode for sum operation $\mathcal{P}$ is determined according to the shape of the update tensor. It corresponds to the mode with length 1 of the update tensor. Figures 3 and 4 show tensor networks of BD and other tensor decompositions in graphical notation.

Furthermore, the expressive power of the model can be improved by considering the sum of BDs:

$$\mathcal{Y} \approx \sum_{r=1}^{R} \mathcal{A}^{(r)} \boxdot \mathcal{B}^{(r)} \boxdot \mathcal{C}^{(r)}, \tag{44}$$

[3]More generally $\oslash$ can be replaced with $\boxslash$ if the shapes are different.

where $\boldsymbol{\mathcal{A}}^{(r)} \in \mathbb{R}^{I\times J\times 1}$, $\boldsymbol{\mathcal{B}}^{(r)} \in \mathbb{R}^{I\times 1\times K}$, and $\boldsymbol{\mathcal{C}}^{(r)} \in \mathbb{R}^{1\times J\times K}$ mutually satisfy the broadcast conditions for each $r \in \{1, 2, ..., R\}$. The sum of BDs shown in Equation (44) is an extension of the outer product in CP decomposition (Hitchcock, 1927; Carroll & Chang, 1970; Harshman, 1970) to the broadcast product. For minimizing squared errors $||\boldsymbol{\mathcal{Y}} - \sum_{r=1}^{R} \boldsymbol{\mathcal{A}}^{(r)} \boxdot \boldsymbol{\mathcal{B}}^{(r)} \boxdot \boldsymbol{\mathcal{C}}^{(r)}||_F^2$, the hierarchical ALS (HALS) (Cichocki et al., 2007) can be adapted. Objective function of the sub-problem for k-th components is given by $||\boldsymbol{\mathcal{Y}}_k - \boldsymbol{\mathcal{A}}^{(k)} \boxdot \boldsymbol{\mathcal{B}}^{(k)} \boxdot \boldsymbol{\mathcal{C}}^{(k)}||_F^2$, where $\boldsymbol{\mathcal{Y}}_k := \boldsymbol{\mathcal{Y}} - \sum_{r\neq k} \boldsymbol{\mathcal{A}}^{(r)} \boxdot \boldsymbol{\mathcal{B}}^{(r)} \boxdot \boldsymbol{\mathcal{C}}^{(r)}$, then the update rules can be derived in the same way of Equation (41), Equation (42), and Equation (43).

5.3 Difference from conventional TDs

In this section, we experimentally demonstrate that the difference between the proposed BD and conventional tensor decompositions (TDs). Note that the tensors we want to approximate may represent some real data or they may represent parameters of a neural network in real applications, but the concrete case study is beyond the scope of this paper. Our goal here is to demonstrate, at least in part, that the proposed BD differs substantially from other decomposition models and to discuss its potential applications.

First, we constructed a tensor $\boldsymbol{\mathcal{W}} \in \mathbb{R}^{32\times 32\times 32}$ as follows:

$$\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{A}} \boxdot \boldsymbol{\mathcal{B}} \boxdot \boldsymbol{\mathcal{C}} + \sigma \boldsymbol{\mathcal{E}}, \tag{45}$$

where $\boldsymbol{\mathcal{A}} \in \mathbb{R}^{32\times 32\times 1}$, $\boldsymbol{\mathcal{B}} \in \mathbb{R}^{32\times 1\times 32}$, $\boldsymbol{\mathcal{C}} \in \mathbb{R}^{1\times 32\times 32}$ and noise $\boldsymbol{\mathcal{E}} \in \mathbb{R}^{32\times 32\times 32}$ are randomly generated, and $\sigma > 0$.

Next, we applied CP decomposition (CPD), Tucker decomposition (TKD) (Tucker, 1966; De Lathauwer et al., 2000), tensor-ring decomposition (TRD) (Zhao et al., 2016), and the proposed sum of BDs to a tensor $\boldsymbol{\mathcal{W}}$. The optimization algorithm was applied with various values of the rank parameters, and the signal-to-noise ratio (SNR) of the reconstructed tensors was evaluated by

$$\text{SNR} = 10 \log \frac{||\boldsymbol{\mathcal{W}}_0||_F^2}{||\boldsymbol{\mathcal{W}}_0 - \widehat{\boldsymbol{\mathcal{W}}}||_F^2}, \tag{46}$$

where $\boldsymbol{\mathcal{W}}_0 = \boldsymbol{\mathcal{A}} \boxdot \boldsymbol{\mathcal{B}} \boxdot \boldsymbol{\mathcal{C}}$ is a ground-truth tensor without noise and $\widehat{\boldsymbol{\mathcal{W}}}$ is a reconstructed tensor obtained by TD algorithms. The number of model parameters and SNRs are shown in Figure 5. It can be seen that while the proposed BD succeeds in achieving accurate approximation, other low-rank TD models have difficulty achieving efficient approximation. Although BD and TDs are similar in the sense that they are compact representations with few parameters, the properties of tensors are significantly different.

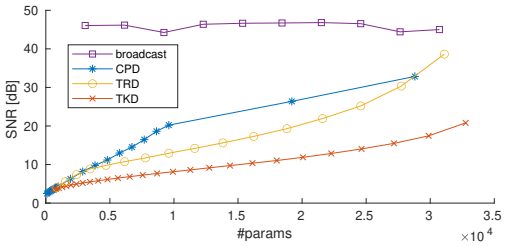


Figure 5: Dimensionality reduction of a synthetic tensor: This suggests the existence of a tensor structure that favors broadcast decomposition.

6 Related Work

6.1 Relationship with existing tensor products

The prototype of broadcast product has been proposed as the penetrating face (PF) product (Slyusar, 1999). The PF product is only defined for a matrix $\boldsymbol{X} \in \mathbb{R}^{I\times J}$ and a tensor $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{I\times J\times K}$ as

$$\boldsymbol{X} \odot_{\text{PF}} \boldsymbol{\mathcal{Y}} = \boldsymbol{X} \boxdot \boldsymbol{\mathcal{Y}}. \tag{47}$$

This can be regarded as a generalization of the Hadamard product between two matrices and a special case of the broadcast product at the same time. The broadcast product gives an integrated general definition of both the Hadamard product and the PF product.

From the perspective of graphical notation, broadcast product can be thought of as an operation that connects the corresponding modes of two tensors via third-order super-diagonal tensors (see Figure 2).

When the third-order super-diagonal tensors are $(1, 1, 1)$ in size, it is equivalent to a simple outer product. The broadcast product is notable for not including an inner-product-like operation (operation involving the $\sum$ symbol) that directly connects the edges of tensors. Its mathematical structure is completely different from that of ordinary tensor products, Einstein products Brazell et al. (2013), and t-products (Zhang et al., 2014; Zhang & Aeron, 2016).

The broadcast product is closely related to the Khatri-Rao (KR) product. The KR product is defined between two matrices with the same number of columns $\boldsymbol{A} = [\boldsymbol{a}_1, ..., \boldsymbol{a}_R] \in \mathbb{R}^{I \times R}$ and $\boldsymbol{B} = [\boldsymbol{b}_1, ..., \boldsymbol{b}_R] \in \mathbb{R}^{J \times R}$:

$$\boldsymbol{B} \odot_{\mathrm{KR}} \boldsymbol{A} = [\boldsymbol{b}_1 \otimes \boldsymbol{a}_1, ..., \boldsymbol{b}_R \otimes \boldsymbol{a}_R] \in \mathbb{R}^{IJ \times R}, \tag{48}$$

where $\otimes$ is the Kronecker product. The following relationship holds:

$$\boldsymbol{B} \odot_{\mathrm{KR}} \boldsymbol{A} = \mathrm{fold}_3(\tilde{\boldsymbol{\mathcal{A}}} \boxdot \tilde{\boldsymbol{\mathcal{B}}})^\top, \tag{49}$$

where $\tilde{\boldsymbol{\mathcal{A}}} \in \mathbb{R}^{I \times 1 \times R}$ and $\tilde{\boldsymbol{\mathcal{B}}} \in \mathbb{R}^{1 \times J \times R}$ are reshaped tensors of $\boldsymbol{A}$ and $\boldsymbol{B}$. In other words, the KR product and the broadcast product have the same mathematical structure because $\mathrm{fold}_3(\cdot)$ and $\cdot^\top$ are just tensor reshaping operations. Noting that the KR product is defined only for matrices, the broadcast product can therefore be regarded as a generalization of the KR product in a broader sense.

6.2 Notation for describing complex mathematical expressions

Some papers that aim for accurate descriptions have already introduced the concept of the broadcast product, such as Wang et al. (2022) and You et al. (2024). Unlike us, they have not discussed the mathematical properties. Also, Wang et al. (2022) used $\otimes$ as the symbol for the broadcast product, but $\otimes$ is generally used for the Kronecker product. This confusion can be avoided by using our $\boxdot$.

The Einops notation (Rogozhnikov, 2021) and the detailed Transformer description (Phuong & Hutter, 2022) are valuable references for writing clear mathematical descriptions. Looking at practical implementations, `TensorLy` is a modern library for tensor processing (Kossaifi et al., 2019).

6.3 Named tensor notation

Named Tensor Notation (Chiang et al., 2023) shares the most similar motivation with ours. By explicitly naming each mode, Named Tensor Notation allows complex tensor operations to be expressed concisely and code-like. For example, the masking operation discussed in Section 1 is described as follows:

$$Y = X \odot B, \quad \text{where } Y, X \in \mathbb{R}^{\mathtt{height} \times \mathtt{width} \times \mathtt{chans}},\ B \in \mathbb{R}^{\mathtt{height} \times \mathtt{width}} \tag{50}$$

Broadcasting is automatically performed when the modes do not match (Def. 6 in Chiang et al. (2023)), so broadcast products can be represented with the above simple expression. Named Tensor Notation has several valuable properties, such as omitting mode ordering and hiding mode lengths.

In contrast to Named Tensor Notation, we focus not on all complex tensor operations but on broadcast products, which are the most essential operations. We place particular emphasis on connecting our broadcast products with existing mathematical concepts. While Named Tensor Notation enables concise description of broadcast products, using this notation requires consistency throughout the entire paper. Also, connections with existing math, such as those introduced in Section 3 and Section 5, are no longer straightforward.

Named Tensor Notation and our broadcast product represent different approaches with the same motivation. Developing a unifying theory integrating these two notations would be a direction for future work.

7 Conclusion

We redefined the broadcast operation commonly used in scientific computing libraries. The broadcast product enhances the accuracy of mathematical expressions in machine learning and other fields. Additionally, we contextualized this product within tensor decomposition, highlighting its potential for developing new types of tensor decompositions.

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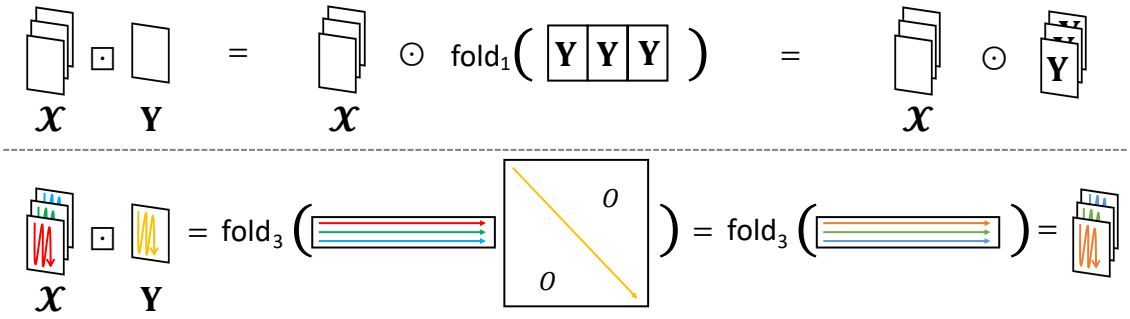


Figure 6: The broadcast product of a third-order tensor and a matrix Equation (28): $\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{Y} = \boldsymbol{\mathcal{X}} \odot \mathrm{fold}_1(\mathbf{1}_K^\top \otimes \boldsymbol{Y}) = \mathrm{fold}_3(\boldsymbol{X}_{(3)}\mathrm{diag}(\mathrm{vec}(\boldsymbol{Y})))$

A Proof of Theorem 3.1

We prove Theorem 3.1. For two tensors $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ that satisfy the broadcast condition,

$$\begin{aligned} \|\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{Y}}\|_F &= \|\boldsymbol{\mathcal{X}}_\square \odot \boldsymbol{\mathcal{Y}}_\square\|_F && \text{(Equation (20))} && (51)\\ &\leq \|\boldsymbol{\mathcal{X}}_\square\|_F \cdot \|\boldsymbol{\mathcal{Y}}_\square\|_F && \text{(Cauchy–Schwarz for the Hadamard product)} && (52)\\ &= \|\boldsymbol{\mathcal{X}}\|_F \cdot \|\boldsymbol{\mathcal{Y}}\|_F. && \text{(Definition of marginalization)} && (53) \end{aligned}$$

Thus, the claim follows.

Here, Cauchy–Schwarz for the Hadamard product (Horn & Johnson, 1994) means that for any tensors $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}$ of the same shape, $\|\boldsymbol{\mathcal{A}} \odot \boldsymbol{\mathcal{B}}\|_F \leq \|\boldsymbol{\mathcal{A}}\|_F \cdot \|\boldsymbol{\mathcal{B}}\|_F$. This is obtained by considering $\boldsymbol{A} = \mathrm{diag}(\boldsymbol{\mathcal{A}})$ and $\boldsymbol{B} = \mathrm{diag}(\boldsymbol{\mathcal{B}})$, and substituting them into $\|\boldsymbol{AB}\|_F \leq \|\boldsymbol{A}\|_F \cdot \|\boldsymbol{B}\|_F$ (the sub-multiplicativity of the Frobenius norm).

B Visualizing Third-order Tensors and Matrices

Figure 6 visualizes Equation (28), the broadcast product of a third-order tensor and a matrix. Here, the Kronecker product is denoted by $\boldsymbol{X} \otimes \boldsymbol{Y}$. The mode-1 unfolding is denoted by $\boldsymbol{X}_{(1)} \in \mathbb{R}^{I \times JK}$ as in (Kolda & Bader, 2009). With a slight abuse notation, the inverse operation of mode-1 unfolding is denoted by fold_1, meaning that $\boldsymbol{\mathcal{X}} = \mathrm{fold}_1(\boldsymbol{X}_{(1)})$. The same applies to other modes.

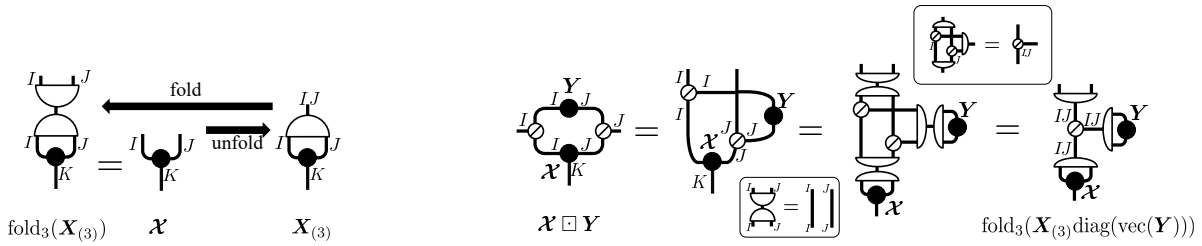
Here, $\mathbf{1}_K^\top \otimes \boldsymbol{Y} \in \mathbb{R}^{I \times JK}$ represents $\boldsymbol{Y}$ repeated K times horizontally. By fold_1, the shape is aligned with that of $\boldsymbol{\mathcal{X}}$. Also, one can express this computation by multiplying the matrix $\boldsymbol{X}_{(3)} \in \mathbb{R}^{K \times IJ}$ by the diagonal matrix $\mathrm{diag}(\mathrm{vec}(\boldsymbol{Y})) \in \mathbb{R}^{IJ \times IJ}$. Here, we first unfold each frontal slice of $\boldsymbol{\mathcal{X}}$ and arrange them to form a matrix. Then, for each row, we take the product with the unfolded $\boldsymbol{Y}$. Finally, the result is folded back into its original shape using fold_3.

In addition, the broadcast product of a third-order tensor and a matrix shown in Equation (28) can be analyzed using graphical notation in Figure 7. Here, unfolding and folding operators are denoted by half circles with three lines. The analysis is based on the topological equivalence of graphs and properties of operators of unfolding, folding with super-diagonal tensors. For examples, operations of unfolding-and-folding and folding-and-unfolding are identity mappings (see Figure 7a). The outer product of two super-diagonal tensors and their unfolding results in another super-diagonal tensor (see Figure 7b).

C Properties of broadcast sum, difference, and division

The broadcast difference can be expressed using the broadcast sum through the following trivial transformation. For tensors $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ satisfying the broadcast condition,

$$\boldsymbol{\mathcal{X}} \boxminus \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \boxplus (-\boldsymbol{\mathcal{Y}}). \tag{54}$$



(a) unfold and fold

(b) proof of main property in graphical notation

Figure 7: Proof of property Equation (28) using graphical notation. (a) Unfolding-and-folding is an identity mapping, and it can be applied to proof in (b). In addition, the outer product of two super-diagonal tensors and their unfolding results in another super-diagonal tensor. Then, we can obtain $\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{Y} = \mathrm{fold}_3(\boldsymbol{X}_{(3)}\mathrm{diag}(\mathrm{vec}(\boldsymbol{Y})))$ in graphical notation.

Similarly, the broadcast division can be expressed using the broadcast product through the following transformation. For tensors $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$ satisfying the broadcast condition (with $\boldsymbol{\mathcal{Y}}$ containing no zero elements),

$$\boldsymbol{\mathcal{X}} \boxslash \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \boxdot (\mathbf{1} \oslash \boldsymbol{\mathcal{Y}}), \tag{55}$$

where $\mathbf{1}$ is a tensor of the same shape as $\boldsymbol{Y}$, with all elements equal to 1. Therefore, focusing on the properties of the broadcast sum and product is sufficient.

Here, we introduce the basic properties of broadcast sum. First, Equations (12), (13) and (16) also hold if $\boxdot$ and $\odot$ are replaced with $\boxplus$ and $+$, respectively.

Scalar: For any tensor $\boldsymbol{\mathcal{X}}$ and a scalar y, we obtain

$$\boldsymbol{\mathcal{X}} \boxplus y = \boldsymbol{\mathcal{X}} + y\mathbf{1}, \tag{56}$$

where $\mathbf{1}$ is an all-one tensor having the same shape as $\boldsymbol{\mathcal{X}}$. This equation may seem extremely trivial initially, but it is important. Adding a constant to all tensor elements is the most basic form of broadcasting, and in `numpy` it is written as a simple addition like `X + y`. Unfortunately, many papers directly write this in mathematical notation as $X + y$. Such a description is incorrect and must be written as $\boldsymbol{\mathcal{X}} + y\mathbf{1}$ as shown above. In addition, there are frequent mistakes where this operation is misunderstood, and the identity matrix $\boldsymbol{I}$ is incorrectly used, resulting in expressions like $\boldsymbol{X} + y\boldsymbol{I}$.

Vector and vector: For vectors $\boldsymbol{x} \in \mathbb{R}^I$ and $\boldsymbol{y} \in \mathbb{R}^J$ of different lengths, we obtain

$$\boldsymbol{x} \boxplus \boldsymbol{y}^\top = \boldsymbol{x}\mathbf{1}_J^\top + \mathbf{1}_I\boldsymbol{y}^\top \in \mathbb{R}^{I \times J}. \tag{57}$$

Matrix and vector: For a matrix $\boldsymbol{X} \in \mathbb{R}^{I \times J}$, vectors $\boldsymbol{y} \in \mathbb{R}^I$ and $\boldsymbol{z} \in \mathbb{R}^J$, the following holds:

$$\boldsymbol{X} \boxplus \boldsymbol{y} = \boldsymbol{X} + [\boldsymbol{y}|\boldsymbol{y}|\ldots|\boldsymbol{y}] = \boldsymbol{X} + \boldsymbol{y}\mathbf{1}_J^\top. \tag{58}$$

$$\boldsymbol{X} \boxplus \boldsymbol{z}^\top = \boldsymbol{X} + \begin{bmatrix} \boldsymbol{z}^\top \\ \vdots \\ \boldsymbol{z}^\top \end{bmatrix} = \boldsymbol{X} + \mathbf{1}_I\boldsymbol{z}^\top. \tag{59}$$

D Proof of Least Squares Solution

D.1 Case of the third-order tensors

Here, we show the derivation of the LS solution Equation (39):

$$\widehat{\boldsymbol{\mathcal{A}}} = \underset{\boldsymbol{\mathcal{A}}}{\mathrm{argmin}} \, ||\boldsymbol{\mathcal{Y}} - \boldsymbol{\mathcal{A}} \boxdot \boldsymbol{\mathcal{Z}}||_F^2 \tag{60}$$

for $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{I\times J\times K}$, $\boldsymbol{\mathcal{A}} \in \mathbb{R}^{I\times J\times 1}$, and $\boldsymbol{\mathcal{Z}} \in \mathbb{R}^{1\times J\times K}$. Let us put $\boldsymbol{Y}_j := \boldsymbol{\mathcal{Y}}_{:j:} \in \mathbb{R}^{I\times K}$, $\boldsymbol{a}_j := \boldsymbol{\mathcal{A}}_{:j1} \in \mathbb{R}^{I}$ and $\boldsymbol{z}_j := \boldsymbol{\mathcal{Z}}_{1j:} \in \mathbb{R}^{K}$, the squared errors can be transformed as

$$||\boldsymbol{\mathcal{Y}} - \boldsymbol{\mathcal{A}} \boxdot \boldsymbol{\mathcal{Z}}||_F^2 = \sum_{j=1}^{J} ||\boldsymbol{Y}_j - \boldsymbol{a}_j \boldsymbol{z}_j^\top||_F^2. \tag{61}$$

Then the solution of $\boldsymbol{a}_j$ can be independently obtained by

$$\widehat{\boldsymbol{a}}_j = \underset{\boldsymbol{a}_j}{\operatorname{argmin}} ||\boldsymbol{Y}_j - \boldsymbol{a}_j \boldsymbol{z}_j^\top||_F^2 = \boldsymbol{Y}_j \boldsymbol{z}_j (\boldsymbol{z}_j^\top \boldsymbol{z}_j)^{-1} \tag{62}$$

for each $j \in \{1, 2, ..., J\}$. Since $(i, j, 1)$-th entry of $\widehat{\boldsymbol{\mathcal{A}}}$ corresponds to i-th entry of $\widehat{\boldsymbol{a}}_j$, we have

$$\widehat{a}_{ij1} = \widehat{\boldsymbol{a}}_j(i) = \left(\sum_{k=1}^{K} y_{ijk} z_{jk}\right) \left(\sum_{k=1}^{K} z_{jk}^2\right)^{-1} \iff \widehat{\boldsymbol{\mathcal{A}}} = \mathcal{P}_3(\boldsymbol{\mathcal{Y}} \boxdot \boldsymbol{\mathcal{Z}}) \boxslash \mathcal{P}_3(\boldsymbol{\mathcal{Z}} \boxdot \boldsymbol{\mathcal{Z}}). \tag{63}$$

D.2 Case of the N-th order tensors

Let us consider N-th order tensors $\boldsymbol{\mathcal{W}} \in \mathbb{R}^{D_1\times D_2\times\cdots\times D_N}$, $\boldsymbol{\mathcal{H}} \in \mathbb{R}^{F_1\times F_2\times\cdots\times F_N}$ and $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{\max(D_1,F_1)\times\max(D_2,F_2)\times\cdots\times\max(D_N,F_N)}$, then the problem can be written by

$$\widehat{\boldsymbol{\mathcal{W}}} = \underset{\boldsymbol{\mathcal{W}}}{\operatorname{argmin}} ||\boldsymbol{\mathcal{X}} - \boldsymbol{\mathcal{W}} \boxdot \boldsymbol{\mathcal{H}}||_F^2. \tag{64}$$

Without loss of generality, we can reduce the problem with N-th order tensors $(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{H}})$ in Equation (64) to the problem with third-order tensors $(\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{Z}})$ in Equation (60). Since $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{\mathcal{H}}$ satisfy the broadcast condition, the N modes can be divided into three categories:

$$\mathcal{L} = \{\ n \mid D_n > 1, F_n = 1\ \}, \tag{65}$$
$$\mathcal{S} = \{\ n \mid D_n = F_n\ \}, \tag{66}$$
$$\mathcal{R} = \{\ n \mid D_n = 1, F_n > 1\ \}. \tag{67}$$

$\mathcal{L}$ is the set of broadcasting modes for $\boldsymbol{\mathcal{H}}$, corresponding to the first mode of $\boldsymbol{\mathcal{Z}}$ in Equation (60). $\mathcal{S}$ is the set of non-broadcasting modes, corresponding to the second mode in Equation (60). $\mathcal{R}$ is the set of broadcasting modes for $\boldsymbol{\mathcal{W}}$, corresponding to the third mode of $\boldsymbol{\mathcal{A}}$ in Equation (60). Then, we convert N-th order tensors to third-order tensors based on $(\mathcal{L}, \mathcal{S}, \mathcal{R})$ using mode permutation and tensor unfolding as follow:

$$\boldsymbol{\mathcal{Y}} = \text{unfold}_{(I,J,K)}\text{permute}_{(\mathcal{L},\mathcal{S},\mathcal{R})}(\boldsymbol{\mathcal{X}}) \in \mathbb{R}^{I\times J\times K}, \tag{68}$$
$$\boldsymbol{\mathcal{A}} = \text{unfold}_{(I,J,1)}\text{permute}_{(\mathcal{L},\mathcal{S},\mathcal{R})}(\boldsymbol{\mathcal{W}}) \in \mathbb{R}^{I\times J\times 1}, \tag{69}$$
$$\boldsymbol{\mathcal{Z}} = \text{unfold}_{(1,J,K)}\text{permute}_{(\mathcal{L},\mathcal{S},\mathcal{R})}(\boldsymbol{\mathcal{H}}) \in \mathbb{R}^{1\times J\times K}, \tag{70}$$

where $I = \prod_{n\in\mathcal{L}} D_n$, $J = \prod_{n\in\mathcal{S}} D_n$, and $K = \prod_{n\in\mathcal{R}} F_n$. The LS solution of third-order tensors $\widehat{\boldsymbol{\mathcal{A}}}$ can be obtained by Equation (63). By converting $\widehat{\boldsymbol{\mathcal{A}}}$ back to an N-th order tensor, the solution can be obtained as follows:

$$\widehat{\boldsymbol{\mathcal{W}}} = \text{permute}^{-1}_{(\mathcal{L},\mathcal{S},\mathcal{R})}\text{unfold}^{-1}_{(I,J,1)}\left(\widehat{\boldsymbol{\mathcal{A}}}\right) = \mathcal{P}_{\mathcal{R}}(\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{H}}) \boxslash \mathcal{P}_{\mathcal{R}}(\boldsymbol{\mathcal{H}} \boxdot \boldsymbol{\mathcal{H}}), \tag{71}$$

where $\text{permute}^{-1}_{(\mathcal{L},\mathcal{S},\mathcal{R})}$ and $\text{unfold}^{-1}_{(I,J,1)}$ are inverse of $\text{permute}_{(\mathcal{L},\mathcal{S},\mathcal{R})}$ and $\text{unfold}_{(I,J,1)}$, and $\mathcal{P}_{\mathcal{R}}(\cdot)$ is a sum operation along the modes in $\mathcal{R}$. For example, let be

$$\boldsymbol{\mathcal{X}} \in \mathbb{R}^{10\times 20\times 30\times 40\times 50\times 60},$$
$$\boldsymbol{\mathcal{W}} \in \mathbb{R}^{10\times 20\times 1\times 40\times 50\times 1},$$
$$\boldsymbol{\mathcal{H}} \in \mathbb{R}^{10\times 1\times 30\times 1\times 50\times 60},$$
$$\mathcal{L} = \{2, 4\}, \mathcal{S} = \{1, 5\}, \mathcal{R} = \{3, 6\},$$

Table 1: List of symbols used for element-wise multiplication

symbol	usage / conflict
$\circ$	Used in (Slyusar, 1999; Bernstein, 2009; Theodoridis, 2020). Conflicts with the outer product in (Cichocki et al., 2009; Kolda & Bader, 2009; Strang, 2019).
$\odot$	Used in (Markovsky, 2012; Goodfellow et al., 2016; Kochenderfer, 2019). Conflicts with the Khatri–Rao product in (Cichocki et al., 2009; Kolda & Bader, 2009; Strang, 2019).
$\circledast$	Used in (Cichocki et al., 2009). Conflicts with convolution in (Jain, 1989; Strang, 2019).
$*$	Used in (Kolda & Bader, 2009). Conflicts with convolution (Strang, 2019) and the t-product (Kernfeld et al., 2015).
$.*$	Used in (Golub & Van Loan, 2012; Cichocki et al., 2009; Strang, 2019). No conflicts reported.
$\boxdot$ (ours)	No conflicts reported (including Hadamard product).

then the permutation operation outputs

$$\begin{aligned}\tilde{\boldsymbol{\mathcal{X}}} &= \mathrm{permute}_{(\{2,4\},\{1,5\},\{3,6\})}(\boldsymbol{\mathcal{X}}) \in \mathbb{R}^{20\times 40\times 10\times 50\times 30\times 60},\\ \tilde{\boldsymbol{\mathcal{W}}} &= \mathrm{permute}_{(\{2,4\},\{1,5\},\{3,6\})}(\boldsymbol{\mathcal{W}}) \in \mathbb{R}^{20\times 40\times 10\times 50\times 1\times 1},\\ \tilde{\boldsymbol{\mathcal{H}}} &= \mathrm{permute}_{(\{2,4\},\{1,5\},\{3,6\})}(\boldsymbol{\mathcal{H}}) \in \mathbb{R}^{1\times 1\times 10\times 50\times 30\times 60},\end{aligned}$$

the unfolding operation outputs

$$\begin{aligned}\boldsymbol{\mathcal{Y}} &= \mathrm{unfold}_{(800,500,1800)}(\tilde{\boldsymbol{\mathcal{X}}}) \in \mathbb{R}^{800\times 500\times 1800},\\ \boldsymbol{\mathcal{A}} &= \mathrm{unfold}_{(800,500,1)}(\tilde{\boldsymbol{\mathcal{W}}}) \in \mathbb{R}^{800\times 500\times 1},\\ \boldsymbol{\mathcal{Z}} &= \mathrm{unfold}_{(1,500,1800)}(\tilde{\boldsymbol{\mathcal{H}}}) \in \mathbb{R}^{1\times 500\times 1800},\end{aligned}$$

and the sum operation outputs

$$\begin{aligned}\mathcal{P}_{\{3,6\}}(\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{H}}) &= (\boldsymbol{\mathcal{X}} \boxdot \boldsymbol{\mathcal{H}}) \times_3 \mathbf{1}^\top \times_6 \mathbf{1}^\top \in \mathbb{R}^{10\times 20\times 1\times 40\times 50\times 1},\\ \mathcal{P}_{\{3,6\}}(\boldsymbol{\mathcal{H}} \boxdot \boldsymbol{\mathcal{H}}) &= (\boldsymbol{\mathcal{H}} \boxdot \boldsymbol{\mathcal{H}}) \times_3 \mathbf{1}^\top \times_6 \mathbf{1}^\top \in \mathbb{R}^{10\times 1\times 1\times 1\times 50\times 1}.\end{aligned}$$

E Conflicts of Mathematical Symbols

Here we discuss the issue of symbols for element-wise multiplication. As shown in Table 1, the symbols used to represent element-wise multiplication (Hadamard product) are very diverse and most of them conflict with other mathematical operations.

Since the symbol $\boxdot$ for the broadcast product proposed in this paper does not conflict and can also be used for the Hadamard product, it may solve such problems. In addition, we can represent both the Hadamard product and the broadcast product using only one symbol $\boxdot$. A promising set of conflict-free notations would be given as follows.

- outer product $\circ$
- Kronecker product $\otimes$
- Khatri-Rao product $\odot$
- Hadamard product (with broadcast option) $\boxdot$
- element-wise division (with broadcast option) $\boxslash$
- convolution $\circledast$
- t-product $*$

Note that this paper does not follow the above notation set, but it was necessary because of the definition of the broadcast product using the Hadamard product and discussing their relationship.

F Translation from Numpy/Julia

The proposed broadcast product is almost identical to `numpy`'s broadcast operation, allowing `A * B` in `numpy` to be represented as $\mathcal{A} \boxdot \mathcal{B}$ in equations. However, there is a difference in how cases are handled when the tensor orders differ, but the broadcasting condition is satisfied.

In our definition, the broadcast condition considers two tensors equivalent only if a mode of length one is added at **the end** of the tensor's shape. In contrast, `numpy` applies this rule at **the beginning**. For instance, $\mathbb{R}^{2\times3}$ is considered equivalent to $\mathbb{R}^{2\times3\times1}$ in our framework, while `numpy` equates it to $\mathbb{R}^{1\times2\times3}$. This discrepancy stems from traditional mathematical notation being column-oriented, whereas `numpy` is row-oriented[4]. Below, we show some examples of Python codes.

As in the following example, if we explicitly specify the tensors' shapes including modes of length one, all computations work as expected. We can directly translate `A * B` in `numpy` to $\mathcal{A} \boxdot \mathcal{B}$ in equations.

```
A = np.random.rand(2, 3, 4)   # A.shape == (2, 3, 4), i.e., 𝒜 ∈ ℝ^{2×3×4}
B = np.random.rand(2, 3, 1)   # B.shape == (2, 3, 1), i.e., ℬ ∈ ℝ^{2×3×1}

# (ℝ^{2×3×4}, ℝ^{2×3×1}) satisfies the broadcast condition. Thus we can compute 𝒜 ⊡ ℬ
A * B   # OK
```

However, if we don't explicitly write down the mode of the length of one, `numpy`'s behavior is in the exact opposite way to our definition as follows.

```
C = np.random.rand(2, 3)   # C.shape == (2, 3), i.e., 𝒞 ∈ ℝ^{2×3}
D = np.random.rand(3, 4)   # D.shape == (3, 4), i.e., 𝒟 ∈ ℝ^{3×4}

# Our definition equates ℝ^{2×3} and ℝ^{2×3×1}, so 𝒜 ⊡ 𝒞 can be computed. However, numpy
# equates ℝ^{2×3} and ℝ^{1×2×3}, thus we cannot compute A * C as follows.
A * C  # ValueError: operands could not be broadcast together with shapes (2,3,4) (2,3)

# Our definition equates ℝ^{3×4} and ℝ^{3×4×1}, so 𝒜 ⊡ 𝒟 cannot be computed. However, numpy
# equates ℝ^{3×4} and ℝ^{1×3×4}, thus we can compute A * D as follows.
A * D  # OK
```

If confusion happens when writing the broadcast product, we recommend explicitly defining the shape even for the mode with the length of one. For example, an image with a single channel can be written as $\mathbb{R}^{H\times W\times 1}$.

In fact, since `Julia` is a column-oriented language consistent with mathematical notation, its behavior is exactly the same as our broadcast product. In `Julia`, the broadcast product is expressed by `.*` as follows.

```
A = rand(2, 3, 4)   # size(A) == (2, 3, 4), i.e., 𝒜 ∈ ℝ^{2×3×4}
B = rand(2, 3, 1)   # size(B) == (2, 3, 1), i.e., ℬ ∈ ℝ^{2×3×1}
# (ℝ^{2×3×4}, ℝ^{2×3×1}) satisfies the broadcast condition. Thus we can compute 𝒜 ⊡ ℬ
A .* B   # OK

C = rand(2, 3)   # size(C) == (2, 3), i.e., 𝒞 ∈ ℝ^{2×3}
D = rand(3, 4)   # size(D) == (3, 4), i.e., 𝒟 ∈ ℝ^{3×4}
# Both our definition and Julia equate ℝ^{2×3} and ℝ^{2×3×1}, so 𝒜 ⊡ 𝒞 can be computed.
A .* C  # OK
# Both our definition and Julia equate ℝ^{3×4} and ℝ^{3×4×1}, so 𝒜 ⊡ 𝒟 cannot be computed.
A .* D  # ERROR: DimensionMismatch("arrays could not be broadcast to a common size; ..
```

[4] `https://numpy.org/doc/stable/user/basics.broadcasting.html`