Human-Centric Perishable Inventory Management with AI-Assistance

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Abstract

Up to 20% of food purchased by restaurants is wasted before reaching customers, with commercial leftovers being the major contributor due to poor production planning. This study introduces two AI decision-making assistants that support perishable inventory management differently in human-centric commercial kitchens. We address the periodic review inventory control problem for perishable goods with a fixed shelf life and demand censoring in an offline, data-driven setting. One AI assistant leverages a data-driven prescriptive solution to the multi-period inventory control problem, directly telling how much a human decision maker should replenish for the upcoming season given past sales data. We justify this prescriptive assistant with associated performance guarantees. Building on the data-driven prescriptive solution, the second AI assistant is enhanced in terms of detecting potential human decision-making biases in managing perishable inventory. Using machine learning models, it identifies from past user behavior whether a human decision maker is biased in their inventory decision making, and (if so) what human bias likely accounts for. Through an online experiment with Prolific workers, we further investigate how human users react to the deployment of different forms of AI assistance and uncover factors influencing their effectiveness. Results show that both types of AI assistants, whether providing data-driven prescriptive solutions or bias detection, improve perishable inventory management performance for human decision-makers. Additionally, integrating bias detection with prescriptive solutions could foster greater human adherence to algorithmic recommendations.

1 Introduction

Efficient inventory management for perishable goods is crucial to improve both profitability and sustainability in food service establishments. These businesses are facing a unique set of challenges, including slim profit margins, product perishability, and seasonality, all while striving to maximize profits. Also, food production in restaurants contributes to 26% of global carbon emissions [1]. It is estimated that food waste costs the global hospitality sector around $100 billion annually [2], and that commercial kitchens often make poor production planning overproducing by 5%-15% (in a few

39th Conference on Neural Information Processing Systems (NeurIPS 2025) Workshop: MLxOR: Mathematical Foundations and Operational Integration of Machine Learning for Uncertainty-Aware Decision-Making.

cases up to 20%) of all the food they purchase [3]. The resources spent on procuring, preparing, and storing overproduced food, are resources that could have been invested elsewhere.

It has been challenging to manage perishable inventory systems both theoretically and practically. As a fundamental class of inventory systems, perishable inventory systems with a fixed shelf life of the product have been extensively studied since the 1960s [4][5][6][7][8][9][10][11][12][13]. The optimal replenishment policies for these systems are notoriously complex and computationally intractable [14] [15], as one needs to keep track of the inventory level of perishable items with various remaining lifetimes, leading to a multidimensional system state.

In addition, unlike more standardized, automated inventory systems found in industries such as online retail, the food production sector remains less digitalized, with limited technology use and a heavy reliance on human input, often leaving final decisions to individual kitchen managers. The lack of comprehensive operations management education and the service-focused incentives of these decision makers make it even harder to achieve an efficient inventory management system.

Recent advancements in artificial intelligence (AI) and digitalization have revolutionized the way human decisions are made. In this study, we propose a pioneering new model for managing perishable inventory systems considering the interaction between human decision-makers and the solution recommended by AI algorithms. We introduce two AI decision-making assistants designed to support perishable inventory management in different intelligent ways, making the traditional inventory management in commercial kitchens more data-driven, effective, and efficient.

One AI assistant leverages a data-driven solution to the multi-period perishable inventory control problem, prescribing how a human decision-maker should replenish for the upcoming period, given historical censored demand data. The second assistant builds on the prescriptive solution and incorporates human behavioral biases in managing perishable inventory systems. We consider scenarios where different kinds of human decision biases potentially lead to deviation from the target inventory solution. By employing machine learning models[16][17][18], the second assistant detects from the user's past behavior whether a human decision maker is biased in his/her inventory decisions, and (if so) what human bias likely accounts for. Through collaboration with Winnow Solutions[1], we apply our AI assistants to a real-world dataset, aiming to show how these assistants could be implemented and evaluate how much potential cost they could save in daily food production oversight by kitchen managers. To further explore how human users would react to the deployment of different forms of AI assistance and uncover factors influencing their effectiveness, we conduct an online experiment with Prolific workers. Our main findings and contributions are summarized as follows:

A data-driven base-stock policy with demand censoring and product perishability. In the context of a perishable inventory problem with censored demand, we develop a nonparametric estimation procedure for deriving a consistent optimal base-stock policy. We justify the estimated inventory policy by establishing its asymptotic consistency and providing asymptotic confidence intervals. To the best of our knowledge, this paper is the first to provide these theoretical results for this problem setting. In addition, we derive upper bounds on the gap between the cost of the proposed data-driven policy and the true optimal base-stock policy. The first type of prescriptive assistant functions by offering reliable algorithmic recommendations.

Tackling unique challenges from human-centric inventory systems. While existing literature primarily targets fully automated replenishment systems, we instead focus on decision-making challenges in less digitalized setting, where algorithm deployment is mediated by a human decision-maker rather than automated execution. In such context human decision-makers may exhibit aversion to algorithmic recommendations[19][20]. Given this, beyond the prescriptive assistant we build a second detective assistant that incorporates human behavioral factors. It operates as a machine learning-based bias detector that monitors human biases and guides users toward more optimal decision-making. We find the bias detector can serve as a standalone decision-support tool.

A case study using real-world data. We validate the empirical performance of our two AI decision-making assistants using real-world data. Using a sales and food waste dataset from our industry partners, Winnow Solutions, and its clients (e.g. restaurants, hotels, school dining halls), we infer parameters, conduct inventory model calibration, and demonstrate practical implementation of these

[1] A UK-based food tech company who has developed digital systems that record instances of wasted food in commercial kitchens.

assistants. Our study confirms the presence of human behavioral biases in food inventory management and underscores the potential cost savings that these AI assistants could generate.

Experimental evaluation on humans' reaction to different types of AI assistance. Our experimental results show that both types of AI assistants, either offering data-driven prescription or bias detection, improve human performance in managing perishable inventory. Although human users often exhibit algorithm aversion and do not always fully comply with algorithmic recommendations, the prescriptive assistant alone still outperforms the detective assistant. Direct instruction proves to be more effective than merely identifying biases. However, combining these two yields a greater improvement in human performance than either alone. Beyond serving as decision support, bias detection is an effective intervention to enhance compliance, as users may perceive the detected bias as informative feedback on their past behavior, thereby enhancing their trust in the prescribed actions. This effect is comparable to, or even exceeds, that of other existing solutions such as improving algorithm transparency[21][22][23].

2 AI Decision-Making Assistants

Multi-Period Perishable Inventory Management. We revisit the periodic review, stochastic inventory control problem for perishable goods with a fixed product shelf life. It has been established that base-stock policies, where if an order is placed, the total inventory level is raised to S, are asymptotic optimal ([13]). We approach the problem from a nonparametric perspective under demand censoring scenarios[24][25][26][27][28][29][30][31][32]. A decision maker does not know the demand distribution $F(\cdot)$ but has historical data on which to base her/his decision[33][34][35][36][37][38]. Assume that the firm has access to n periods of sales data, which are iid across periods, denoted as $\mathbf{D} = \{D_1, ..., D_n\}$. This is common in practice, e.g., commercial kitchens record how much food is sold, if not, they regularly track inventory levels whose change between the beginning and end of each period also indicates how much is sold.

2.1 Prescriptive Assistant: A Data-Driven Prescriptive Solution

Our first AI decision-making assistant is built leveraging a data-driven prescriptive solution to the multi-period perishable inventory control problem. We establish nonparametric estimation procedures for finding consistent estimators of the optimal inventory policy with censored data. Assume that for n_0, $1 \le n_0 \le n-1$ periods in the past, there is no censoring in the data as the firm could initially have stocked a large amount of its product for the first few periods to learn about the demand. The product has a fixed shelf life of m periods and there is zero lead time. Adapting the consistent use of censored demand data in [39], we estimate the long-run average cost $\widetilde{C}(S)$ (consisting of holding, penalty, and approximated outdating costs) by a de-biased empirical cost function

$$\overline{C}_n(S) := \frac{1}{n}\sum_{i=1}^{n}\overline{c}(S, D_i) = \frac{1}{n}\Big(\sum_{i=1}^{n_0}\widetilde{c}(S, D_i) + \sum_{i=n_0+1}^{n}\big[\widetilde{c}(S, D_i)\mathbb{I}(D_i < x_{i,m}) + \frac{\sum_{j=1}^{n_0}\widetilde{c}(S, D_j)\mathbb{I}(D_j \ge x_{j,m})}{\sum_{j=1}^{n_0}\mathbb{I}(D_j \ge x_{j,m})}\mathbb{I}(D_i \ge x_{i,m})\big]\Big)$$

where $(x_{1,m}, ..., x_{n,m})$ are past stocking levels (i.e., the total inventory of the items whose remaining shelf life is at most m periods immediately after an order is placed) that are available information to the firm assuming the system is initially empty, and

$$\widetilde{c}(S, D_i) := h(S - D_i)^+ + p(D_i - S)^+ + (S - \sum_{j=1}^{m} d_{i,j})^+$$

with $\{d_{i,j}\}_{j=1}^{m}$ being the demands re-sampled in each period i from the empirical uncensored demand distribution $F_n(\cdot)$ that is constructed from $D_1, ..., D_n$.

The firm thus estimates $\widetilde{S}$ (i.e., the optimal solution) by $\hat{S}_n$ using M-estimation theory, i.e.,

$$\hat{S}_n := \arg\min_{S \in [\underline{D}, \overline{D}]} \overline{C}_n(S).$$

We show the asymptotic consistency of estimated base-stock level $\hat{S}_n$ with its true value $\widetilde{S}$, i.e., as $n \to \infty$, $\hat{S}_n \xrightarrow{p} \widetilde{S}$. Besides, we examine what finite sample performance bounds the decision maker can assign to the estimated inventory policy. We prove the asymptotic normality of the estimated policy and derive its asymptotic confidence interval using Stein's method, as Theorem 1 states (see Appendix B for proof). For reasonably large n, the estimated $\hat{S}_n$ is approximately normally distributed with centers at the true value $\widetilde{S}$ with variance $\overline{\sigma}^2$. Confidence intervals are useful as they provide statistically meaningful error bars for estimators. We also provide the upper bounds on the cost gap induced by the proposed data-driven solution and the optimal solution. These results indicate that, for a sensible policy that converges to the truth asymptotically, the long-run average cost of the estimated policy should be remarkably close to that of the optimal base-stock policy.

Theorem 1 (Asymptotic Normality of $\hat{S}_n$)**.** *The estimated base-stock level $\hat{S}_n$ is asymptotically normally distributed, i.e., $\sqrt{n}(\hat{S}_n - \widetilde{S}) \xrightarrow{d} \mathcal{N}(0, \overline{\sigma}^2)$ where*

$$\overline{\sigma}^2 = \frac{\mathbb{E}[\widetilde{c}'(\widetilde{S}, D)]^2}{(\mathbb{E}[\widetilde{c}''(\widetilde{S}, D)])^2} + r\frac{\frac{(1+p_s)}{p_s}(\mathbb{E}[\widetilde{c}'(\widetilde{S}, D)\mathbb{I}(D \geq x_m)])^2 + \frac{1}{1-r}\mathit{Var}(\widetilde{c}'(\widetilde{S}, D)\mathbb{I}(D \geq x_m))}{(\mathbb{E}[\widetilde{c}''(\widetilde{S}, D)])^2}.$$

2.2 Detective Assistant: Machine Learning Based Bias Detection

Our second assistant builds upon the data-driven prescriptive solution but works differently from the first one, taking human-AI interaction into account. After justify the first prescriptive assistant with associated performance guarantees, we further incorporate various human mistakes or behavioral biases, e.g., static production plan, demand chasing, over-reaction or under-reaction to demand changes, into the data-driven solution, and collect synthesized sales and leftover inventory data under various scenarios. We then train machine/deep-learning based time-series classifiers such as random forests, long short-term memory networks for multivariate time-series classification, on the synthetic data that are labeled with different behavioral biases in inventory decision making. Our results show that via well-trained classifiers we are able to detect if there is deviation from the target data-driven solution and predict at high accuracy ($\approx 75\%$) what bias likely accounts for it.

We validate the empirical performance of two AI decision-making assistants using real-world data. Utilizing a sales and waste dataset provided by our industry partners, Winnow and its clients (restaurants, hotels, dining halls), we infer parameters, conduct inventory model calibration, and show that adopting these AI tools can achieve 16-22% cost savings in managing food production systems.

3 Online Experiment

Our main empirical question is whether these two AI-assisted decision-making systems can enhance human decision-making performance. We examine how people would react to the deployment of these AI systems in a virtual inventory management setting, using pre-registered experiments conducted with Prolific participants. See Appendix A for more design details.

Each participant completes a 42-period task simulating a periodic review inventory scenario for a perishable product with a fixed shelf life of 3 periods. In each period, participants choose an order quantity with the goal of minimizing the long-run total cost of managing the perishable inventory system. Participants complete 21 periods without any assistance (pre-intervention), then 21 periods with AI support (post-intervention). In the post-intervention phase, they are randomly assigned to one of six experiment conditions: five AI-assisted treatments (varying by type of assistance and algorithm transparency) or a no-assistance control group. We recruited 684 participants through Prolific and qualified 600 participants for analysis, who successfully passed a five-question comprehension test and completed the task (100 per condition). The average completion time was 38 minutes, and the average payment was \$10 (\$7.50 to the participant and \$2.50 as a platform fee).

Our results show that both types of AI assistance- prescriptive solutions and bias detection- enhance perishable inventory management performance, reducing the total cost by 19.7% and 10.4%, respectively. Directly instructing users, however, is not always the most effective approach, due to human aversion to AI algorithms, especially in human-centric, less-engineered environments. Incorporating bias detection with prescriptive solutions could promote greater adherence to algorithmic recommendations (e.g., reducing the average deviation in order quantity by 11 units) than either alone, as humans may interpret and appreciate the detected bias as feedback on their past behavior and enhance their trust in the algorithmic recommendations. This effect is comparable to, or even greater than that of other existing algorithm aversion mitigation solutions, such as increasing algorithm transparency.

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A Online Experiment

A.1 Experiment Flow and Interface

Step 1: Instructions and Comprehension Checks

Once participants give their content to participate in the experiment, they are first shown 7-page detailed instructions, including welcome, game overview, demand information and sampling, cost calculations, results table, demo video of 4-round apple pie inventory management task, and payment rule. After reviewing the instructions, participants are required to complete five comprehension check questions to ensure they fully understand the experiment task.

Step 2: Decision-Making without AI Assistance (Pre-intervention Phase)

Step 3: AI Assistance Introduction

Step 4: Decision-Making with AI Assistance (Post-intervention Phase)

Step 5: Exit Survey

Step 6: Bonus and Feedback

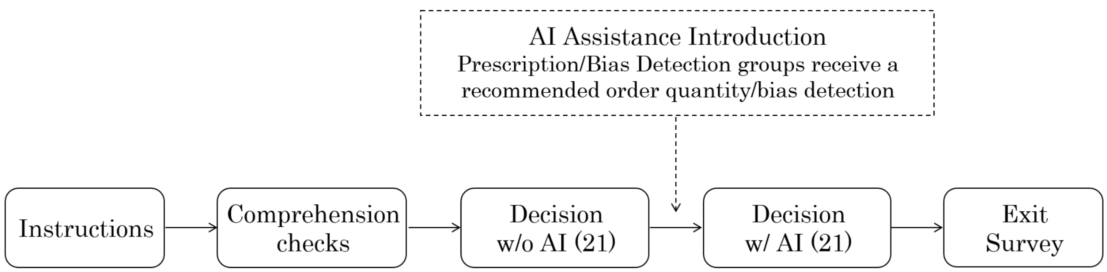


Figure 1: Experiment Flow (Numbers in Parentheses are the Number of Periods)

A.2 Experimental Results

We find that both of our AI-assisted decision-making systems significantly improve participant performance by either directly recommending optimal actions that are difficult for participants to determine on their own or providing guidance that helps them converge to better decisions. By contrast, directly instructing participants on what to do outperforms merely nudging them toward optimal actions. In addition, we document that participants do not fully adhere to our algorithmic recommendations when assisted by a data-driven prescriptive system. We therefore examine how participant compliance, a key ingredient to ultimately improving decisions, varies across different compliance interventions. Our results further show that bias detection can complement prescription by increasing participant compliance, though it can also function effectively on its own. Moreover, its effectiveness in improving compliance is comparable to that of providing additional information about the uncertainty in the prescription.

A.2.1 Performance: AI-Assisted Systems Substantially Improve Performance

We recruited 684 participants through Prolific platform for our main study. Following our pre-registered exclusion criteria, we qualified 300 participants for analysis, who successfully passed a five-question comprehension test and completed the experiment (100 per condition). The average completion time was 35 minutes, and the average payment was \$10 (\$7.50 to the participant and \$2.50 as a platform fee).

The main outcome variable is participants' performance in managing the inventory system. For each period, we record the initial stock, participants' order quantities, realized demand, inventory levels of fresh food items by age (e.g., inventory level of items with two periods of remaining shelf life), and inventory level of expired food items. We then calculate the corresponding total cost of managing the inventory system, which includes holding cost, stockout cost, and outdating cost.

	Total Cost	Holding Cost	Stockout Cost	Outdating Cost	Stockout Ratio
Prescription	329.95	106.69	196.12	27.14	0.16
No Assistance	410.87	114.89	257.06	38.92	0.21
Difference	-80.92 (-19.7%)	-8.2 (-7.1%)	-60.94 (-23.7%)	-11.78 (-30.3%)	-0.05 (-23.8%)
t-test p-value	0.008^{***}	0.248	0.035^{**}	0.092^{*}	0.013^{**}
Bias Detection	368.16	115.92	213.02	39.24	0.19
No Assistance	410.87	114.89	257.06	38.92	0.21
Difference	-42.71 (-10.4%)	1.03 (0.9%)	-44.04 (-17.1%)	0.32 (0.8%)	-0.02 (-9.5%)
t-test p-value	0.086^{*}	0.469	0.079^{*}	0.486	0.176

Notes. $^{*}p < 0.1;^{**} p < 0.05;^{***} p < 0.01$.

Table 1: Effectiveness of AI-Assisted Systems: t-test Results

To test Hypotheses 1 and 2, we first examine the performance of participants assisted by either a data-driven prescriptive system or a machine learning-based bias detection system, versus those in the benchmark condition without AI assistance, using a two-sample t-test. Table 1 reports the results of t-tests comparing five performance metrics: total cost, holding cost, stockout cost, outdating cost, and stockout ratio. Note that all five metrics here are averaged over the post-intervention phase, that is, the last 21 periods during which participants receive different interventions. The deployment of a data-driven prescriptive system significantly reduces four key metrics. In particular, it lowers the average stockout cost by 23.7% (p<0.05), the average stockout ratio by 23.8% (p<0.05), the average outdating cost by 30.3% (p<0.1), and the total cost by 19.7% (p<0.01), compared to the condition without AI assistance. By contrast, deploying a bias detection system significantly reduces only two of the five metrics: the average stockout cost decreases by 17.1% (p<0.1), and the total cost decreases by 10.4% (p<0.1), relative to the benchmark.

Difference-in-Differences Estimation. One may concern that the treatment and control groups may have exhibited slight differences in behavior even before the intervention, which could undermine the validity of t-tests that directly compare post-intervention performance. To further identify the effect of deploying AI inventory co-pilots on all five performance metrics, we apply a Difference-in-Differences (DiD) approach. We evaluate the performance of the treatment and control groups over the first 21 periods, defined as the pre-intervention phase, and then compare it with their performance during the post-intervention phase (i.e., the last 21 periods). In the pre-intervention phase, all participants make order decisions without any AI assistance. In the post-intervention phase, the treatment group receives support from either a data-driven prescriptive system or a bias detection system, while the control group continues without assistance. By comparing changes in performance before and after the intervention between the treatment and control groups, we estimate the effect of deploying AI inventory co-pilots.

The DiD estimation results for the prescriptive system and the bias detection system are presented in Table 2. The terms “Pre-inv” and “Post-inv” denote the pre-intervention and post-intervention phases, respectively. The results show that deploying our proposed inventory co-pilots- either a prescriptive system or a bias detection system- enhances human performance in managing a perishable inventory system, with the prescriptive system yielding greater improvements than bias detection, which are consistent with our predictions (Hypotheses 1 and 2). Specifically, the deployment of a prescriptive system reduces the average stockout cost by \$90.72 (p<0.05), the average stockout ratio by 0.08 (p<0.01), and the total cost by \$92.43 (p<0.01), compared with the condition without AI assistance. Providing bias detection improves human performance by reducing the average stockout cost by \$53.01 (p<0.1) and the total cost by \$76.50 (p<0.01). We find that for both deployments, improvements in inventory management performance are driven primarily by better control of stockout costs.

A.2.2 Compliance: Algorithm Aversion and Interventions to Improve Compliance

Although the proposed inventory co-pilots help human decision makers improve their perishable inventory management performance, there remains significant room for further improvement. In this section we further examine participant compliance, a key factor in ultimately enhancing human performance, and test interventions designed to improve compliance: two benchmark conditions adapted from existing approaches that mitigate algorithm aversion through increased transparency, and a focal intervention based on bias detection. We observe from the experiment that participants

	Treatment group			Control group			DiD Estimation	
	Pre-inv	Post-inv	Change	Pre-inv	Post-inv	Change	DiD	p-value
Total cost	415.05	329.95	-85.10	403.54	410.87	7.33	-92.43	0.006***
Holding cost	97.54	106.69	9.15	108.69	114.89	6.2	2.95	0.753
Stockout cost	288.89	196.12	-92.77	259.11	257.06	-2.05	-90.72	0.014**
Outdating cost	28.62	27.14	-1.48	35.75	38.92	3.17	-4.65	0.471
Stockout ratio	0.24	0.16	-0.08	0.21	0.21	0	-0.08	0.001***

(a) Effect of Data-Driven Prescriptive System

	Treatment group			Control group			DiD Estimation	
	Pre-inv	Post-inv	Change	Pre-inv	Post-inv	Change	DiD	p-value
Total cost	437.33	368.16	-69.17	403.54	410.87	7.33	-76.50	0.007***
Holding cost	121.68	115.92	-5.76	108.69	114.89	6.2	-11.96	0.234
Stockout cost	268.08	213.02	-55.06	259.11	257.06	-2.05	-53.01	0.066*
Outdating cost	47.57	39.24	-8.33	35.75	38.92	3.17	-11.50	0.114
Stockout ratio	0.22	0.19	-0.03	0.21	0.21	0	-0.03	0.258

(b) Effect of Bias Detection System

	Treatment group			Control group			DiD Estimation	
	Pre-inv	Post-inv	Change	Pre-inv	Post-inv	Change	DiD	p-value
Total cost	454.02	306.89	-147.13	403.54	410.87	7.33	-154.46	<0.001***
Holding cost	107.01	98.18	-8.83	108.69	114.89	6.2	-15.03	0.291
Stockout cost	310.10	184.93	-125.17	259.11	257.06	-2.05	-123.12	<0.001***
Outdating cost	36.90	23.77	-13.13	35.75	38.92	3.17	-16.30	0.052*
Stockout ratio	0.25	0.17	-0.08	0.21	0.21	0	-0.08	<0.001***

Notes. $^{*}p < 0.1;^{**} p < 0.05;^{***} p < 0.01$.

(c) Effect of Prescription + Bias Detection System

Table 2: Effectiveness of Inventory Co-Pilots: Difference-in-Differences Estimation

did not fully adhere to the recommended order quantities in the prescription condition; on average, only 27.4% of the prescriptions were followed, which could greatly undermine the effectiveness of deploying a prescriptive system.

The primary outcome of interest here is participants' compliance with the algorithmic recommendations. We measure this by calculating the deviation between the quantity chosen by participants and the quantity recommended by the algorithm. Larger deviations indicate lower compliance. Table 3 below reports the compliance metric, the average deviation from the recommended quantity, by condition. While human users did not 100% adhere to the AI algorithm's recommendations (e.g., on average there is 40.93 deviation from the recommended order quantities under prescription only condition), incorporating additional piece of information would reduce the deviation (e.g., an average deviation of 34.59 under "Prescription + HAC" condition, 25.75 under "Prescription + UI" condition, and 32.06 under "Prescription + BD" condition).

	Prescription	Prescription + BD	Prescription + HAC	Prescription + UI
Deviation	40.93 (4.28)	32.06 (3.29)	34.59 (3.83)	25.75 (2.66)

Notes. Standard errors, across participants, are reported in parentheses.

Table 3: Average Deviation by Condition

Linear Regression. One may concern that participants in different conditions may have had varying levels of performance prior to the interventions, which could bias the t-tests used to compare the effectiveness of different compliance interventions. Poorer pre-intervention performance suggests a greater deviation from the optimal actions, which may make it harder for those participants to comply with algorithmic prescriptions as their expectations would be far above or below the recommended quantities. To address this concern and facilitate a more accurate comparison of average compliance between treatment and control groups, we estimate the following linear regression model using the deviation metric as the dependent variable:

$$\text{Deviation} = \beta_0 + \beta_1 \text{Compliance-inv} + \beta_2 \text{Ave-pre-performance} + \epsilon,$$

where Compliance-inv is a binary variable that represents if a participant receives a compliance intervention (bias detection, or human-AI comparison, or uncertainty information). Ave-pre-performance denotes the participant's average inventory management performance prior to the intervention, measured by the total cost associated with managing the inventory system.

We report the coefficients and p-values for the treatment variable Compliance-inv in Table 4. Our results show that both bias detection and uncertainty information about the prescription significantly promote compliance, whereas the effect of adding a human-versus-AI performance comparison is not statistically significant. Specifically, incorporating bias detection with the prescription reduces the average deviation by 11 units (p<0.05) compared to the prescription alone, while additionally presenting uncertainty in the prescription reduces the average deviation by 15.69 units (p<0.01).

	Prescription + BD	Prescription + HAC	Prescription + UI
Coefficient	-11.00	-5.30	-15.69
p-value	0.036**	0.323	0.001***

Notes. $^{*}p < 0.1;^{**} p < 0.05;^{***} p < 0.01$.

Table 4: Effectiveness of Compliance Interventions: Linear Regression

Along with the increased compliance, we document a higher perceived value of the information provided among participants under "Prescription + BD" condition (4.14 out of 5 on average), compared to those under "Prescription only" condition (3.89 out of 5 on average). And via DiD, we estimate that bias detection increases participants' perceived usefulness of the provided information for decision-making by 11.45% (p-value<0.05). Human users may interpret and appreciate the detected bias as feedback on their past behavior, which could enhance their trust in the algorithmic recommendations and thus increase the likelihood to adopt.

Moreover, we estimate the effectiveness of compliance interventions on participants' inventory management performance and report the DiD estimation results in Table 5. We find that offering bias detection is comparable to presenting uncertainty information in improving both compliance and inventory management performance, and both of them outperform the intervention of adding a human-versus-AI performance comparison. Particularly, the bias detection intervention reduces the total cost by \$62.03 (p-value<0.05) while the uncertainty information intervention reduces it by 69.77 (p-value<0.1), compared to the prescription only. The performance improvement with uncertainty information is mostly driven by a reduction in the stockout cost, see an average reduction of \$90.54 (p-value<0.05). In contrast, bias detection enhances human performance through a different way. We document reductions in all kinds of inventory costs under bias detection intervention (i.e., 17.98 reduction in average holding cost, 32.40 in average stockout cost, 11.65 in average outdating cost), though only the reduction in outdating cost is statistically significant (p-value <0.1).

The above results support our conjecture that providing bias detection increases human adherence to algorithmic recommendations (Hypothesis 3) and improves performance in managing perishable inventory, suggesting that the two types of inventory co-pilots function as complementary tools. Additionally, bias detection turns to be a greener solution. Increasing algorithm transparency by presenting uncertainty in the prescription offers an alternative approach to boost compliance. In our study, we present uncertainty information as an interval indicating the range within which the optimal order quantity is expected to fall. This interval is constructed based on the asymptotic confidence interval of the estimated base-stock policy. Our findings underscore that the use of confidence intervals is a critical element in practical decision making. A key advantage of bias detection over uncertainty information is that bias detection is effective as a standalone intervention, whereas uncertainty information alone is not.

	Treatment group			Control group			DiD Estimation	
	Pre-inv	Post-inv	Change	Pre-inv	Post-inv	Change	DiD	p-value
Total cost	454.02	306.89	-147.13	415.05	329.95	-85.10	-62.03	0.043**
Holding cost	107.01	98.18	-10.49	97.54	106.69	9.15	-17.98	0.129
Stockout cost	310.10	184.93	-125.17	288.89	196.12	-92.77	-32.40	0.273
Outdating cost	36.90	23.77	-13.13	28.62	27.14	-1.48	-11.65	0.088*
Stockout rate	0.25	0.17	-0.08	0.24	0.16	-0.08	0	0.974

(a) Prescription + Bias Detection

	Treatment group			Control group			DiD Estimation	
	Pre-inv	Post-inv	Change	Pre-inv	Post-inv	Change	DiD	p-value
Total cost	398.38	318.01	-80.37	415.05	329.95	-85.10	4.73	0.897
Holding cost	91.72	100.06	8.34	97.54	106.69	9.15	-0.81	0.929
Stockout cost	281.05	195.84	-85.21	288.89	196.12	-92.77	7.56	0.849
Outdating cost	25.62	22.11	-3.51	28.62	27.14	-1.48	-2.03	0.721
Stockout rate	0.22	0.15	-0.07	0.24	0.16	-0.08	0.01	0.573

(b) Prescription + Human-versus-AI Comparison

	Treatment group			Control group			DiD Estimation	
	Pre-inv	Post-inv	Change	Pre-inv	Post-inv	Change	DiD	p-value
Total cost	427.23	272.36	-154.87	415.05	329.95	-85.10	-69.77	0.056*
Holding cost	81.01	104.33	23.32	97.54	106.69	9.15	14.17	0.125
Stockout cost	329.18	145.87	-183.31	288.89	196.12	-92.77	-90.54	0.033**
Outdating cost	17.04	22.16	5.12	28.62	27.14	-1.48	6.60	0.255
Stockout rate	0.25	0.13	-0.12	0.24	0.16	-0.08	-0.04	0.137

Notes. $^{*}p < 0.1;^{**} p < 0.05;^{***} p < 0.01$.

(c) Prescription + Uncertainty Information

Table 5: Effectiveness of Compliance Interventions: Difference-in-Differences Estimation

B Proof for Theorem 1

Proof. Let

$$W_1(S) = \sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^{n}\eta_1(S,D_i) - \widetilde{\mu}_1(S)}{\widetilde{\sigma}_1(S)},$$

where $\eta_1(S,D_i) = \frac{d}{dS}\overline{c}(S,D_i) = \overline{c}'(S,D_i)$, $\widetilde{\mu}_1(S) = \mathbb{E}[\eta_1(S,D)] = \mathbb{E}[\overline{c}'(S,D)]$, and

$$\begin{aligned}\widetilde{\sigma}_1^2(S) = &\lim_{n\to\infty}\text{Var}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta_1(S,D_i)\Big) = \lim_{n\to\infty}\frac{1}{n}\text{Var}\Big(\sum_{i=1}^{n}\overline{c}'(S,D_i)\Big)\\ =&\text{Var}(\widetilde{c}'(S,D)) + \frac{r(1+p)}{p}(\mathbb{E}[\widetilde{c}'(S,D)\mathbb{I}(D\geq x_m)])^2 + \frac{r}{1-r}\text{Var}(\widetilde{c}'(S,D)\mathbb{I}(D\geq x_m))\\ &- 2r\mathbb{E}[\widetilde{c}'(S,D)\mathbb{I}(D\geq x_m)]\mathbb{E}[\widetilde{c}'(S,D)].\end{aligned}$$

Lemma 1. *$W_1(S) \xrightarrow{d} \mathcal{N}(0,1)$ as $n\to\infty$.*

Note that

$$\sqrt{n}(\hat{S}_n - \widetilde{S}) = -V_1(\widetilde{S})^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta_1(\widetilde{S},d_i) + o_p(1) = -\frac{\widetilde{\sigma}_1(\widetilde{S})}{V_1(\widetilde{S})}W_1(\widetilde{S}) + o_p(1),$$

which implies $\sqrt{n}(\hat{S}_n - \widetilde{S}) \xrightarrow{d} \mathcal{N}(0, \overline{\sigma}^2)$ where $\overline{\sigma}^2 = \frac{\widetilde{\sigma}_1^2(\widetilde{S})}{V_1^2(\widetilde{S})}$, following Lemma 1.

Our main goal is thus to prove Lemma 1 which states asymptotic normality of normalized sums of dependent random variables in the presence of demand censoring.

Fix $S \in \mathcal{S}$, let $\boldsymbol{D} = \{D_1, ..., D_n\}$ be a vector of independent random variables. Let

$$W_{1,n}(S) = \sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^n \eta_1(S, D_i) - \widetilde{\mu}_n(S)}{\widetilde{\sigma}_n(S)},$$

where $\widetilde{\mu}_1(S)$ and $\widetilde{\sigma}_1(S)$ in $W_1(S)$ are replaced by the finite-sample versions

$$\widetilde{\mu}_n(S) = \frac{1}{n}\sum_{i=1}^n \eta_1(S, D_i) = \frac{1}{n}\sum_{i=1}^n \overline{c}'(S, D_i)$$

and

$$\widetilde{\sigma}_n^2(S) = \mathrm{Var}\big(\frac{1}{\sqrt{n}}\sum_{i=1}^n \eta_1(S, D_i)\big) = \frac{1}{n}\mathrm{Var}\big(\sum_{i=1}^n \overline{c}'(S, D_i)\big)$$

such that $W_{1,n}(S)$ is standardized.

Let $f : [\underline{D}, \overline{D}]^n \to \mathbb{R}$ be a measurable function such that $W_{1,n}(S) = f(\boldsymbol{D})$. Following the generalized perturbative approach by [40], we next construct an upper bound on the distance of $W_{1,n}(S)$ from the standard normal distribution using information about how f changes when one coordinate of $\boldsymbol{D}$ is perturbed.

Let $\boldsymbol{D'} = \{D'_1, ..., D'_n\}$ be an independent copy[2] of $\boldsymbol{D}$ and $[n] = \{1, ..., n\}$. For each $A \subset [n]$, define the random vector $\boldsymbol{D}^A$ as

$$D_i^A = \begin{cases} D'_i & \text{if } i \in A, \\ D_i & \text{if } i \notin A. \end{cases}$$

For simplicity, we write $\boldsymbol{D}^i$ if A is a singleton. Define a randomized derivative of f along the ith coordinate as $\Delta_i f := f(\boldsymbol{D}) - f(\boldsymbol{D}^i)$, and for each $A \subset [n]$ and $i \notin A$, let $\Delta_i f^A := f(\boldsymbol{D}^A) - f(\boldsymbol{D}^{A \cup i})$. Finally, let

$$T := \frac{1}{2}\sum_{i=1}^n \sum_{A \subset [n]\setminus\{i\}} \frac{1}{n\binom{n-1}{|A|}} \Delta_i f \Delta_i f^A.$$

Let Z be a standard normal random variable, then from Theorem 3.1 of [40],

$$\sup_{t \in \mathbb{R}} \big|\mathbb{P}(W_{1,n}(S) \le t) - \mathbb{P}(Z \le t)\big| \le 2\Big(\sqrt{\mathrm{Var}(\mathbb{E}[T|W_{1,n}(S)])} + \frac{1}{4}\sum_{i=1}^n \mathbb{E}|\Delta_i f|^3\Big)^{1/2}. \quad (1)$$

In our setup, we have

$$\frac{\widetilde{\sigma}_n(S)}{\sqrt{n}}\Delta_i f = \frac{1}{n}[\overline{c}'(S, D_i) - \overline{c}'(S, D'_i)] \Longrightarrow \Delta_i f = \frac{1}{\widetilde{\sigma}_n(S)\sqrt{n}}[\overline{c}'(S, D_i) - \overline{c}'(S, D'_i)],$$

and for all subset $A \subset [n] \setminus \{i\}$ we can show $\Delta_i f^A = \Delta_i f$. Hence

$$T = \frac{1}{2n}\sum_{i=1}^n (\Delta_i f)^2 = \frac{1}{2n^2\widetilde{\sigma}_n^2(S)}\sum_{i=1}^n [\overline{c}'(S, D_i) - \overline{c}'(S, D'_i)]^2,$$

which gives

$$\begin{aligned} \mathrm{Var}(\mathbb{E}[T|W_{1,n}(S)]) =& \mathrm{Var}(\mathbb{E}_{D_i, D'_i \sim F(\cdot), d_{ij} \sim F_n(\cdot), d'_{ij} \sim F'_n(\cdot)}[T|f(\boldsymbol{D})]) \\ \overset{(\mathrm{Var}(\mathbb{E}[T|\boldsymbol{D}]) = \mathrm{Var}(T) - \mathbb{E}[\mathrm{Var}(T|\boldsymbol{D})])}{\le}& \mathrm{Var}(\mathbb{E}[T|\boldsymbol{D}]) \end{aligned}$$

[2] $\boldsymbol{D'}$ shares the same joint distribution with $\boldsymbol{D}$ and is independent of $\boldsymbol{D}$.

$$
\begin{aligned}
\le & \mathrm{Var}\Big(\frac{1}{2n^2\widetilde{\sigma}_n^2(S)}\sum_{i=1}^{n}\big(\overline{c}'(S,D_i)^2+\mathbb{E}[\overline{c}'(S,D_i')^2]-2\overline{c}'(S,D_i)\mathbb{E}[\overline{c}'(S,D_i')]\big)\Big)\\
= & \frac{1}{4n^4\widetilde{\sigma}_n^4(S)}\mathrm{Var}\Big(\sum_{i=1}^{n}\big(\overline{c}'(S,D_i)^2-2\overline{c}'(S,D_i)\mathbb{E}[\overline{c}'(S,D_i')]\big)\Big)\\
= & \frac{1}{4n^4\widetilde{\sigma}_n^4(S)}\mathrm{Var}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\ge x_{im})+\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i< x_{im})+\sum_{i\in\mathcal{J}}\Gamma_i\Big)\\
= & \frac{1}{4n^4\widetilde{\sigma}_n^4(S)}\Big[\sum_{i\in\mathcal{J}^c}\mathrm{Var}\big(\Gamma_i\mathbb{I}(D_i\ge x_{im})\big)+\sum_{i\in\mathcal{J}^c}\mathrm{Var}\big(\Gamma_i\mathbb{I}(D_i< x_{im})\big)+\sum_{i\in\mathcal{J}}\mathrm{Var}(\Gamma_i)\\
& +2\mathrm{Cov}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\ge x_{im}),\sum_{j\in\mathcal{J}}\Gamma_j\Big)\Big] \qquad (2)
\end{aligned}
$$

where $\Gamma_i := \overline{c}'(S,D_i)^2-2\overline{c}'(S,D_i)\mathbb{E}[\overline{c}'(S,D_i')]$.

We next examine each term on the RHS of the above inequality. Note that for any $i\in\mathcal{J}^c$ we have

$$
\begin{aligned}
\mathrm{Var}\big(\Gamma_i\mathbb{I}(D_i\ge x_{im})\big) =& \mathbb{E}\big(\Gamma_i^2\mathbb{I}(D_i\ge x_{im})\big)-\mathbb{E}^2\big(\Gamma_i\mathbb{I}(D_i\ge x_{im})\big)\\
=& \mathbb{E}\big((\overline{c}'(S,D_i))^4\mathbb{I}(D_i\ge x_{im})\big)+4\mathbb{E}^2\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^2\mathbb{I}(D_i\ge x_{im})\big)\\
& -4\mathbb{E}\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^3\mathbb{I}(D_i\ge x_{im})\big)-\mathbb{E}^2\big((\overline{c}'(S,D_i))^2\mathbb{I}(D_i\ge x_{im})\big)\\
& +4\mathbb{E}\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^2\mathbb{I}(D_i\ge x_{im})\big)\mathbb{E}\big((\overline{c}'(S,D_i))\mathbb{I}(D_i\ge x_{im})\big)\\
& -4\mathbb{E}^2\big(\overline{c}'(S,D_i')\big)\mathbb{E}^2\big((\overline{c}'(S,D_i))\mathbb{I}(D_i\ge x_{im})\big)
\end{aligned}
$$

where

$$
w_j := \frac{\mathbb{I}(D_j\ge x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\ge x_{im})},
$$

$$
\begin{aligned}
& \mathbb{E}\big(\overline{c}'(S,D_i)\big)\\
=& \mathbb{E}\Big(\big[(h+p)\mathbb{I}(S\ge D_i)-p+\frac{\theta}{m}\mathbb{I}(S\ge\sum_{j=1}^{m}d_{ij})\big]\mathbb{I}(D_i<x_{im})+\sum_{j\in\mathcal{J}}w_j\big[(h+p)\mathbb{I}(S\ge D_j)-p+\frac{\theta}{m}\mathbb{I}(S\ge\sum_{k=1}^{m}d_{jk})\big]\mathbb{I}(D_i\ge x_{im})\Big)\\
=& \mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big[(h+p)\mathbb{I}(S\ge D_i)-p+\frac{\theta}{m}\mathbb{I}(S\ge\sum_{j=1}^{m}d_{ij})\big]\mathbb{I}(D_i<x_{im})\big|\boldsymbol{D}\Big)\Big)\\
& +\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{j\in\mathcal{J}}\frac{\mathbb{I}(D_j\ge x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\ge x_{im})}\big[(h+p)\mathbb{I}(S\ge D_j)-p+\frac{\theta}{m}\mathbb{I}(S\ge\sum_{k=1}^{m}d_{jk})\big]\mathbb{I}(D_i\ge x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=& \mathbb{E}_{\boldsymbol{D}}\Big(\big[(h+p)\mathbb{I}(S\ge D_i)-p\big]\mathbb{I}(D_i<x_{im})\Big)+\frac{\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\ge\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\Big)\Big)\\
& +\sum_{j\in\mathcal{J}}\mathbb{E}_{\boldsymbol{D}}\Big(\frac{\mathbb{I}(D_j\ge x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\ge x_{im})}\big[(h+p)\mathbb{I}(S\ge D_j)-p\big]\mathbb{I}(D_i\ge x_{im})\Big)\\
& +\frac{\theta}{m}\sum_{j\in\mathcal{J}}\mathbb{E}_{\boldsymbol{D}}\Big(\frac{\mathbb{I}(D_j\ge x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\ge x_{im})}\mathbb{I}(D_i\ge x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\ge\sum_{k=1}^{m}d_{jk})\big|\boldsymbol{D}\Big)\Big),
\end{aligned}
$$

Let $\mathbb{E}(d_ij)=\mathbb{E}(d)$ for any $i=1,...,n$, $j=1,...,m$. According to the Hoeffding's Inequality (providing upper bound on the probability that the sum of bounded independent random variables deviates from its expected value by more than a certain amount), we have

$$
\begin{aligned}
\mathbb{P}(\sum_{j=1}^{m}d_{ij}-\mathbb{E}[\sum_{j=1}^{m}d_{ij}]\ge S)=& \mathbb{P}\Big(\sum_{j=1}^{m}d_{ij}-m\mathbb{E}(d)\ge S\Big)\\
& \le e^{-\frac{2S^2}{\sum_{i=1}^{m}(\overline{D}-\underline{D})^2}}=e^{-\frac{2S^2}{m(\overline{D}-\underline{D})^2}}
\end{aligned}
$$

where $\underline{D} \leq d_{ij} \leq \overline{D}$ for any $j = 1, ..., m$ (by assumption). Thus,

$$\mathbb{P}(\sum_{j=1}^{m} d_{ij} \leq S) \leq \mathbb{P}(\sum_{j=1}^{m} d_{ij} - m\mathbb{E}(d) \leq S) = 1 - \mathbb{P}(\sum_{j=1}^{m} d_{ij} - m\mathbb{E}(d) \geq S) \in [1 - e^{-\frac{2S^2}{m(\overline{D}-\underline{D})^2}}, 1]$$

turns out to be of order $e^{-1/m}$.

$$\begin{aligned}
&\mathbb{E}\big(\bar{c}'(S, D_i)^2 \mathbb{I}(D_i \geq x_{im})\big) \\
=&\mathbb{E}\Big(\Big(\sum_{j \in \mathcal{J}} w_j \big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\big]\Big)^2 \mathbb{I}(D_i \geq x_{im})\Big) \\
=&\mathbb{E}\Big(\sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{J}} w_j w_l \big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\big]\big[(h+p)\mathbb{I}(S \geq D_l) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{lk})\big]\mathbb{I}(D_i \geq x_{im})\Big) \\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{J}} w_j w_l \big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\big] \\
&\qquad\qquad \big[(h+p)\mathbb{I}(S \geq D_l) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{lk})\big]\mathbb{I}(D_i \geq x_{im})\big|\boldsymbol{D}\Big)\Big) \\
=&\sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{J}} \mathbb{E}_{\boldsymbol{D}}\Big(\frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i \in \mathcal{J}} \mathbb{I}(D_i \geq x_{im})} \frac{\mathbb{I}(D_l \geq x_{lm})}{\sum_{i \in \mathcal{J}} \mathbb{I}(D_i \geq x_{im})} \big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\big[(h+p)\mathbb{I}(S \geq D_l) - p\big]\mathbb{I}(D_i \geq x_{im})\Big) \\
&+ \frac{\theta}{m} \sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{J}} \mathbb{E}_{\boldsymbol{D}}\Big(\frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i \in \mathcal{J}} \mathbb{I}(D_i \geq x_{im})} \frac{\mathbb{I}(D_l \geq x_{lm})}{\sum_{i \in \mathcal{J}} \mathbb{I}(D_i \geq x_{im})} \big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\mathbb{I}(D_i \geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S \geq \sum_{k=1}^{m} d_{lk})\big|\boldsymbol{D}\Big)\Big) \\
&+ \frac{\theta}{m} \sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{J}} \mathbb{E}_{\boldsymbol{D}}\Big(\frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i \in \mathcal{J}} \mathbb{I}(D_i \geq x_{im})} \frac{\mathbb{I}(D_l \geq x_{lm})}{\sum_{i \in \mathcal{J}} \mathbb{I}(D_i \geq x_{im})} \big[(h+p)\mathbb{I}(S \geq D_l) - p\big]\mathbb{I}(D_i \geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\big|\boldsymbol{D}\Big)\Big) \\
&+ \frac{\theta^2}{m^2} \sum_{j \in \mathcal{J}} \sum_{l \in \mathcal{J}} \mathbb{E}_{\boldsymbol{D}}\Big(\frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i \in \mathcal{J}} \mathbb{I}(D_i \geq x_{im})} \frac{\mathbb{I}(D_l \geq x_{lm})}{\sum_{i \in \mathcal{J}} \mathbb{I}(D_i \geq x_{im})}\mathbb{I}(D_i \geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\mathbb{I}(S \geq \sum_{k=1}^{m} d_{lk})\big|\boldsymbol{D}\Big)\Big),
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\big(\bar{c}'(S, D_i)^3 \mathbb{I}(D_i \geq x_{im})\big) \\
=&\mathbb{E}\Big(\Big(\sum_{j \in \mathcal{J}} w_j \big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\big]\Big)^3 \mathbb{I}(D_i \geq x_{im})\Big) \\
=&\mathbb{E}\Big(\sum_{\substack{I_1+I_2+...+I_{|J|}=3,\\ I_1, I_2, ..., I_{|J|} \geq 0}} \binom{3}{I_1, I_2, ..., I_{|J|}} \prod_{j=1}^{|J|} \big(w_j\big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\big]\big)^{I_j} \mathbb{I}(D_i \geq x_{im})\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by multinomial expansion theorem)} \\
=&\sum_{\substack{I_1+I_2+...+I_{|J|}=3,\\ I_1, I_2, ..., I_{|J|} \geq 0}} \frac{3!}{I_1! I_2! ... I_{|J|}!} \mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\prod_{j=1}^{|J|} \big(w_j\big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\big]\big)^{I_j} \mathbb{I}(D_i \geq x_{im})\big|\boldsymbol{D}\Big)\Big)
\end{aligned}$$

with each $j \in \mathcal{J}$ being symmetric and thus by setting $I_j = 3$ we have

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big(w_j\big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})\big]\big)^3 \mathbb{I}(D_i \geq x_{im})\big|\boldsymbol{D}\Big)\Big) \\
=&\mathbb{E}_{\boldsymbol{D}}\Big(w_j^3[(h+p)\mathbb{I}(S \geq D_j) - p]^3 \mathbb{I}(D_i \geq x_{im})\Big) + \frac{3\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(w_j^3[(h+p)\mathbb{I}(S \geq D_j) - p]^2 \mathbb{I}(D_i \geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})|\boldsymbol{D}\Big)\Big) \\
&+ \frac{3\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(w_j^3[(h+p)\mathbb{I}(S \geq D_j) - p]\mathbb{I}(D_i \geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})|\boldsymbol{D}\Big)\Big) \\
&+ \frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big(w_j^3 \mathbb{I}(D_i \geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S \geq \sum_{k=1}^{m} d_{jk})|\boldsymbol{D}\Big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\big(\overline{c}'(S,D_i)^4\mathbb{I}(D_i\geq x_{im})\big)\\
=&\mathbb{E}\Big(\Big(\sum_{j\in\mathcal{J}} w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big]\Big)^4\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}\Big(\sum_{\substack{I_1+I_2+\ldots+I_{|J|}=4,\\ I_1,I_2,\ldots,I_{|J|}\geq 0}}\binom{4}{I_1,I_2,...,I_{|J|}}\prod_{j=1}^{|J|}\big(w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big]\big)^{I_j}\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\sum_{\substack{I_1+I_2+\ldots+I_{|J|}=4,\\ I_1,I_2,\ldots,I_{|J|}\geq 0}}\frac{4!}{I_1!I_2!...I_{|J|}!}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\prod_{j=1}^{|J|}\big(w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big]\big)^{I_j}\mathbb{I}(D_i\geq x_{im})\big|\boldsymbol{D}\Big)\Big)
\end{aligned}
$$

with each $j\in\mathcal{J}$ being symmetric and thus by setting $I_j=4$ we have

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big(w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big]\big)^4\mathbb{I}(D_i\geq x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(w_j^4[(h+p)\mathbb{I}(S\geq D_j)-p]^4\mathbb{I}(D_i\geq x_{im})\Big)+\frac{4\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(w_j^4[(h+p)\mathbb{I}(S\geq D_j)-p]^3\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{6\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(w_j^4[(h+p)\mathbb{I}(S\geq D_j)-p]^2\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{4\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big(w_j^4[(h+p)\mathbb{I}(S\geq D_j)-p]\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{\theta^4}{m^4}\mathbb{E}_{\boldsymbol{D}}\Big(w_j^4\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)
\end{aligned}
$$

are all constants (being of order $m^{-1}e^{-\frac{1}{m}}$), and thus for any $i\in\mathcal{J}^c$, $\text{Var}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)$ is a constant (being of order $m^{-4}e^{-\frac{4}{m}}$). Then, $\sum_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)$ turns out to be of order one of n.

Similarly, for any $i\in\mathcal{J}^c$ we have

$$
\begin{aligned}
\text{Var}\big(\Gamma_i\mathbb{I}(D_i<x_{im})\big)=&\mathbb{E}\big(\Gamma_i^2\mathbb{I}(D_i<x_{im})\big)-\mathbb{E}^2\big(\Gamma_i\mathbb{I}(D_i<x_{im})\big)\\
=&\mathbb{E}\big((\overline{c}'(S,D_i))^4\mathbb{I}(D_i<x_{im})\big)+4\mathbb{E}^2\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^2\mathbb{I}(D_i<x_{im})\big)\\
&-4\mathbb{E}\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^3\mathbb{I}(D_i<x_{im})\big)-\mathbb{E}^2\big((\overline{c}'(S,D_i))^2\mathbb{I}(D_i<x_{im})\big)\\
&+4\mathbb{E}\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^2\mathbb{I}(D_i<x_{im})\big)\mathbb{E}\big((\overline{c}'(S,D_i))\mathbb{I}(D_i<x_{im})\big)\\
&-4\mathbb{E}^2\big(\overline{c}'(S,D_i')\big)\mathbb{E}^2\big((\overline{c}'(S,D_i))\mathbb{I}(D_i<x_{im})\big)
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbb{E}\big(\overline{c}'(S,D_i)^2\mathbb{I}(D_i<x_{im})\big)\\
=&\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^2\mathbb{I}(D_i<x_{im})\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^2\mathbb{I}(D_i<x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2\mathbb{I}(D_i<x_{im})\Big)+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big)\\
&+\frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\big(\overline{c}'(S,D_i)^3\mathbb{I}(D_i<x_{im})\big)\\
=&\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big)^3\mathbb{I}(D_i<x_{im})\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big)^3\mathbb{I}(D_i<x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^3\mathbb{I}(D_i<x_{im})\Big)+\frac{3\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\big)\Big)\\
&+\frac{3\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\big)\Big)+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\big(\overline{c}'(S,D_i)^4\mathbb{I}(D_i<x_{im})\big)\\
=&\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big)^4\mathbb{I}(D_i<x_{im})\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big)^4\mathbb{I}(D_i<x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^4\mathbb{I}(D_i<x_{im})\Big)+\frac{4\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^3\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\big)\Big)\\
&+\frac{6\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\big)\Big)\\
&+\frac{4\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\big)\Big)+\frac{\theta^4}{m^4}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\big)\Big)
\end{aligned}
$$

are all constants (being of order $m^{-1}e^{-\frac{1}{m}}$), and thus for any $i\in\mathcal{J}^c$, $\mathrm{Var}\big(\Gamma_i\mathbb{I}(D_i<x_{im})\big)$ is a constant (being of order $m^{-4}e^{-\frac{4}{m}}$). Then, $\sum_{i\in\mathcal{J}^c}\mathrm{Var}\big(\Gamma_i\mathbb{I}(D_i<x_{im})\big)$ turns out to be of order one of n.

Note that for any $i\in\mathcal{J}$ we have

$$
\begin{aligned}
\mathrm{Var}(\Gamma_i)=&\mathbb{E}(\Gamma_i^2)-\mathbb{E}^2(\Gamma_i)=\mathbb{E}\Big(\big(\overline{c}'(S,D_i)^2-2\overline{c}'(S,D_i)\mathbb{E}[\overline{c}'(S,D_i')]\big)^2\Big)-\mathbb{E}^2\Big(\overline{c}'(S,D_i)^2-2\overline{c}'(S,D_i)\mathbb{E}[\overline{c}'(S,D_i')]\Big)\\
=&\mathbb{E}\big((\overline{c}'(S,D_i))^4\big)+4\mathbb{E}^2\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^2\big)-4\mathbb{E}\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^3\big)-\mathbb{E}^2\big((\overline{c}'(S,D_i))^2\big)\\
&+4\mathbb{E}\big(\overline{c}'(S,D_i')\big)\mathbb{E}\big((\overline{c}'(S,D_i))^2\big)\mathbb{E}\big((\overline{c}'(S,D_i))\big)-4\mathbb{E}^2\big(\overline{c}'(S,D_i')\big)\mathbb{E}^2\big((\overline{c}'(S,D_i))\big)
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbb{E}\big(\overline{c}'(S,D_i)\big)=\mathbb{E}\Big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big((h+p)\mathbb{I}(S\geq D_i)-p\Big)+\frac{\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big|\boldsymbol{D}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\big(\overline{c}'(S,D_i)^2\big)=\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big)^2\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big)^2\big|\boldsymbol{D}\Big)\Big)
\end{aligned}
$$

$$=\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2\Big)+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big)$$
$$+\frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big),$$

$$\mathbb{E}\big(\overline{c}'(S,D_i)^3\big)=\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^3\Big)$$
$$=\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^3\big|\boldsymbol{D}\Big)\Big)$$
$$=\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^3\Big)+\frac{3\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big)$$
$$+\frac{3\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big)+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big),$$

$$\mathbb{E}\big(\overline{c}'(S,D_i)^4\big)=\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^4\Big)$$
$$=\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^4\big|\boldsymbol{D}\Big)\Big)$$
$$=\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^4\Big)+\frac{4\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^3\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big)$$
$$+\frac{6\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big)$$
$$+\frac{4\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big)+\frac{\theta^4}{m^4}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big|\boldsymbol{D}\big)\Big)$$

are all constants (being of order $m^{-1}e^{-\frac{1}{m}}$), and thus for any $i\in\mathcal{J}$, $\text{Var}\big(\Gamma_i\big)$ is a constant (being of order $m^{-4}e^{-\frac{4}{m}}$). Then, $\sum_{i\in\mathcal{J}}\text{Var}\big(\Gamma_i\big)$ turns out to be of order one of n.

Putting together the first three terms in the RHS of (2), we can show there exists a constant $C_{1,1}$ such that $\sum_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)+\sum_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i<x_{im})\big)+\sum_{i\in\mathcal{J}}\text{Var}\big(\Gamma_i\big)\leq nC_{1,1}=n(c_{1,1}+k_{1,1}(m^{-4}e^{-\frac{4}{m}}))$ where $C_{1,1}:=\max\big\{\max_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big),\max_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i<x_{im})\big),\max_{i\in\mathcal{J}}\text{Var}\big(\Gamma_i\big)\big\}$, which can be split into $c_{1,1}$ and $k_{1,1}$ by the dependency on m.

Moreover, the fourth term in the RHS of (2) is

$$\text{Cov}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\geq x_{im}),\sum_{j\in\mathcal{J}}\Gamma_j\Big)=\mathbb{E}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\geq x_{im})\sum_{j\in\mathcal{J}}\Gamma_j\Big)-\mathbb{E}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\geq x_{im})\Big)\mathbb{E}\Big(\sum_{j\in\mathcal{J}}\Gamma_j\Big)$$
$$=\sum_{i\in\mathcal{J}^c}\sum_{j\in\mathcal{J}}\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\Gamma_j\big)-\sum_{i\in\mathcal{J}^c}\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)\sum_{j\in\mathcal{J}}\mathbb{E}\big(\Gamma_j\big)$$

where for any $i\in\mathcal{J}^c$, $j\in\mathcal{J}$

$$\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\Gamma_j\big)$$
$$=\mathbb{E}\Big(\big(\overline{c}'(S,D_i)^2-2\overline{c}'(S,D_i)\mathbb{E}[\overline{c}'(S,D_i')]\big)\mathbb{I}(D_i\geq x_{im})\big(\overline{c}'(S,D_j)^2-2\overline{c}'(S,D_j)\mathbb{E}[\overline{c}'(S,D_j')]\big)\Big)$$

$$
\begin{aligned}
=&\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_j)^2\mathbb{I}(D_i\geq x_{im})\Big)-2\mathbb{E}[\overline{c}'(S,D_j')]\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_j)\mathbb{I}(D_i\geq x_{im})\Big)\\
&-2\mathbb{E}[\overline{c}'(S,D_i')]\mathbb{E}\Big(\overline{c}'(S,D_i)\overline{c}'(S,D_j)^2\mathbb{I}(D_i\geq x_{im})\Big)+4\mathbb{E}[\overline{c}'(S,D_i')]\mathbb{E}[\overline{c}'(S,D_j')]\mathbb{E}\Big(\overline{c}'(S,D_i)\overline{c}'(S,D_j)\mathbb{I}(D_i\geq x_{im})\Big)
\end{aligned}
$$

with each term on the right-hand side being constants

$$
\begin{aligned}
&\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_j)^2\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}\Big(\Big(\sum_{l\in\mathcal{J}}w_l\big[(h+p)\mathbb{I}(S\geq D_l)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big]\Big)^2\Big((h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\Big)^2\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}w_l\big[(h+p)\mathbb{I}(S\geq D_l)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big]\Big)^2\\
&\qquad\Big((h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\Big)^2\mathbb{I}(D_i\geq x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\Big)^2\Big((h+p)\mathbb{I}(S\geq D_j)-p\Big)^2\mathbb{I}(D_i\geq x_{im})\Big)\\
&+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\Big)^2\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\Big)^2\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\Big((h+p)\mathbb{I}(S\geq D_j)-p\Big)^2\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\Big)^2\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{\theta^4}{m^4}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\Big)^2\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{2\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\Big)^2\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\Big)\Big((h+p)\mathbb{I}(S\geq D_j)-p\Big)^2\mathbb{I}(D_i\geq x_{im})\\
&\qquad\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{2\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\Big)\mathbb{I}(D_i\geq x_{im})\\
&\qquad\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{4\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\Big)\Big((h+p)\mathbb{I}(S\geq D_j)-p\Big)\mathbb{I}(D_i\geq x_{im})\\
&\qquad\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_j)\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}\Big(\Big(\sum_{l\in\mathcal{J}}w_l\big[(h+p)\mathbb{I}(S\geq D_l)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big]\Big)^2\big((h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big)\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}w_l\big[(h+p)\mathbb{I}(S\geq D_l)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big]\Big)^2\big((h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big)\mathbb{I}(D_i\geq x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\Big)^2\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\Big)^2\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big)\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\Big)^2\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\Big)^2\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\\
&\qquad\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{2\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\mathbb{I}(D_i\geq x_{im})\\
&\qquad\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Big(\overline{c}'(S,D_i)\overline{c}'(S,D_j)^2\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}\Big(\sum_{l\in\mathcal{J}}w_l\big[(h+p)\mathbb{I}(S\geq D_l)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big]\big((h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big)^2\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}w_l\big[(h+p)\mathbb{I}(S\geq D_l)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big]\big((h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big)^2\mathbb{I}(D_i\geq x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\big((h+p)\mathbb{I}(S\geq D_j)-p\big)^2\mathbb{I}(D_i\geq x_{im})\Big)\\
&+\frac{\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_j)-p\big)^2\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)\\
&+\frac{2\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Big(\overline{c}'(S,D_i)\overline{c}'(S,D_j)\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}\Big(\sum_{l\in\mathcal{J}}w_l\big[(h+p)\mathbb{I}(S\geq D_l)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big]\big((h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big)\mathbb{I}(D_i\geq x_{im})\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}w_l\big[(h+p)\mathbb{I}(S\geq D_l)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{lk})\big]\big((h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big)\mathbb{I}(D_i\geq x_{im})\big|\boldsymbol{D}\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\Big)\\
&+\frac{\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_l)-p\big]\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big|\boldsymbol{D}\Big)\Big)
\end{aligned}
$$

$$+\frac{\theta}{m}\mathbb{E}_{\boldsymbol{D}}\Big(\big((h+p)\mathbb{I}(S\geq D_j)-p\big)\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^{m}d_{lk})\big|\boldsymbol{D}\Big)\Big)$$

$$+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D}}\Big(\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d}|\boldsymbol{D}}\Big(\sum_{l\in\mathcal{J}}\frac{\mathbb{I}(D_l\geq x_{lm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\mathbb{I}(S\geq\sum_{k=1}^{m}d_{lk})\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\big|\boldsymbol{D}\Big)\Big).$$

And for $i\in\mathcal{J}^c$, $j\in\mathcal{J}$ we have already shown that $\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)$ and $\mathbb{E}(\Gamma_j)$ are constants (being of order $m^{-1}e^{-\frac{1}{m}}$) when we elaborate on $\text{Var}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)$ and $\text{Var}(\Gamma_j)$.

Therefore, $\text{Cov}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\geq x_{im}),\sum_{j\in\mathcal{J}}\Gamma_j\Big)$ turns out be of order two of n, that is, there exists a constant $C_{1,2}$ such that

$$\begin{aligned}\text{Cov}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\geq x_{im}),\sum_{j\in\mathcal{J}}\Gamma_j\Big)=&\mathbb{E}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\geq x_{im})\sum_{j\in\mathcal{J}}\Gamma_j\Big)-\mathbb{E}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\geq x_{im})\Big)\mathbb{E}\Big(\sum_{j\in\mathcal{J}}\Gamma_j\Big)\\=&\sum_{i\in\mathcal{J}^c}\sum_{j\in\mathcal{J}}\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\Gamma_j\big)-\sum_{i\in\mathcal{J}^c}\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)\sum_{j\in\mathcal{J}}\mathbb{E}\big(\Gamma_j\big)\\\leq&\Big|\sum_{i\in\mathcal{J}^c}\sum_{j\in\mathcal{J}}\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\Gamma_j\big)\Big|+\Big|\sum_{i\in\mathcal{J}^c}\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)\sum_{j\in\mathcal{J}}\mathbb{E}\big(\Gamma_j\big)\Big|\\\leq&n^2C_{1,2}=n^2(c_{1,2}+k_{1,2}(m^{-4}e^{-\frac{4}{m}}))\end{aligned}$$

where $C_{1,2}:=\max\big\{\max_{i\in\mathcal{J}^c,j\in\mathcal{J}}\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\Gamma_j\big),\max_{i\in\mathcal{J}^c,j\in\mathcal{J}}\mathbb{E}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)\mathbb{E}\big(\Gamma_j\big)\big\}$, which can be split into $c_{1,2}$ and $k_{1,2}$ by the dependency on m.

What remains is to check $\widetilde{\sigma}_n^2(S)$ in the RHS of (2). Note that

$$\begin{aligned}\widetilde{\sigma}_n^2(S)=&\frac{1}{n}\text{Var}\big(\sum_{i=1}^{n}\overline{c}'(S,D_i)\big)=\frac{1}{n}\text{Var}\Big(\sum_{i\in\mathcal{J}^c}\overline{c}'(S,D_i)\mathbb{I}(D_i\geq x_{im})+\sum_{i\in\mathcal{J}^c}\overline{c}'(S,D_i)\mathbb{I}(D_i<x_{im})+\sum_{i\in\mathcal{J}}\overline{c}'(S,D_i)\Big)\\=&\frac{1}{n}\Big[\sum_{i\in\mathcal{J}^c}\text{Var}\big(\overline{c}'(S,D_i)\mathbb{I}(D_i\geq x_{im})\big)+\sum_{i\in\mathcal{J}^c}\text{Var}\big(\overline{c}'(S,D_i)\mathbb{I}(D_i<x_{im})\big)+\sum_{i\in\mathcal{J}}\text{Var}\big(\overline{c}'(S,D_i)\big)\\&+2\text{Cov}\Big(\sum_{i\in\mathcal{J}^c}\overline{c}'(S,D_i)\mathbb{I}(D_i\geq x_{im}),\sum_{j\in\mathcal{J}}\overline{c}'(S,D_j)\Big)\Big]\end{aligned}$$

where the four terms within the brackets are a special case of

$$\sum_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)+\sum_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i<x_{im})\big)+\sum_{i\in\mathcal{J}}\text{Var}(\Gamma_i)+2\text{Cov}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i\geq x_{im}),\sum_{j\in\mathcal{J}}\Gamma_j\Big)$$

if we replace Γ_i with $\overline{c}'(S,D_i)$.

Thus, similarly there exist two appropriately matched constants $C_{1,3}$, $C_{1,4}$ such that

$$\begin{aligned}\widetilde{\sigma}_n^2(S)=&\frac{1}{n}\text{Var}\big(\sum_{i=1}^{n}\overline{c}'(S,D_i)\big)=\frac{1}{n}\text{Var}\Big(\sum_{i\in\mathcal{J}^c}\overline{c}'(S,D_i)\mathbb{I}(D_i\geq x_{im})+\sum_{i\in\mathcal{J}^c}\overline{c}'(S,D_i)\mathbb{I}(D_i<x_{im})+\sum_{i\in\mathcal{J}}\overline{c}'(S,D_i)\Big)\\=&\frac{1}{n}\Big[\sum_{i\in\mathcal{J}^c}\text{Var}\big(\overline{c}'(S,D_i)\mathbb{I}(D_i\geq x_{im})\big)+\sum_{i\in\mathcal{J}^c}\text{Var}\big(\overline{c}'(S,D_i)\mathbb{I}(D_i<x_{im})\big)+\sum_{i\in\mathcal{J}}\text{Var}\big(\overline{c}'(S,D_i)\big)\\&+2\text{Cov}\Big(\sum_{i\in\mathcal{J}^c}\overline{c}'(S,D_i)\mathbb{I}(D_i\geq x_{im}),\sum_{j\in\mathcal{J}}\overline{c}'(S,D_j)\Big)\Big]\\\leq&\frac{1}{n}\big(nC_{1,3}+n^2C_{1,4}\big)=C_{1,3}+nC_{1,4}=c_{1,3}+k_{1,3}(m^{-2}e^{-\frac{2}{m}})+n(c_{1,4}+k_{1,4}(m^{-2}e^{-\frac{2}{m}})).\end{aligned}$$

Overall, we show $\text{Var}(\mathbb{E}[T|W_{1,n}(S)])$ (LHS of 2) can be bounded from above, i.e.,

$$\text{Var}(\mathbb{E}[T|W_{1,n}(S)])\leq\frac{1}{4n^4\widetilde{\sigma}_n^4(S)}\Big[\sum_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i\geq x_{im})\big)+\sum_{i\in\mathcal{J}^c}\text{Var}\big(\Gamma_i\mathbb{I}(D_i<x_{im})\big)+\sum_{i\in\mathcal{J}}\text{Var}(\Gamma_i)$$

$$
\begin{aligned}
&\quad + 2\mathrm{Cov}\Big(\sum_{i\in\mathcal{J}^c}\Gamma_i\mathbb{I}(D_i \ge x_{im}), \sum_{j\in\mathcal{J}}\Gamma_j\Big)\Big] \\
&\le \frac{1}{n^4(C_{1,3}+nC_{1,4})^2}\left[nC_{1,1}+n^2C_{1,2}\right] \\
&= \frac{n(c_{1,1}+k_{1,1}(m^{-4}e^{-\frac{4}{m}}))+n^2(c_{1,2}+k_{1,2}(m^{-4}e^{-\frac{4}{m}}))}{n^4(c_{1,3}+k_{1,3}(m^{-2}e^{-\frac{2}{m}})+n(c_{1,4}+k_{1,4}(m^{-2}e^{-\frac{2}{m}})))^2}
\end{aligned}
$$

Note that

$$
\frac{1}{4}\sum_{i=1}^n \mathbb{E}|\Delta_i f|^3 = \frac{1}{4\widetilde{\sigma}_n^3(S)n^{\frac{3}{2}}}\sum_{i=1}^n \mathbb{E}\big|\overline{c}'(S,D_i)-\overline{c}'(S,D_i')\big|^3
$$

where for any i by the law of total expectation and symmetry

$$
\begin{aligned}
&\mathbb{E}\big|\overline{c}'(S,D_i)-\overline{c}'(S,D_i')\big|^3 \\
=&\mathbb{E}\Big(\big|\overline{c}'(S,D_i)-\overline{c}'(S,D_i')\big|^3\Big|\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\Big)\mathbb{P}\big(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\big) \\
&+\mathbb{E}\Big(\big|\overline{c}'(S,D_i)-\overline{c}'(S,D_i')\big|^3\Big|\overline{c}'(S,D_i)\le\overline{c}'(S,D_i')\Big)\mathbb{P}\big(\overline{c}'(S,D_i)\le\overline{c}'(S,D_i')\big) \\
=&\frac{1}{2}\mathbb{E}\Big(\big(\overline{c}'(S,D_i)-\overline{c}'(S,D_i')\big)^3\Big|\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\Big)+\frac{1}{2}\mathbb{E}\Big(\big(\overline{c}'(S,D_i')-\overline{c}'(S,D_i)\big)^3\Big|\overline{c}'(S,D_i)\le\overline{c}'(S,D_i')\Big) \\
=&\mathbb{E}\Big(\big(\overline{c}'(S,D_i)-\overline{c}'(S,D_i')\big)^3\Big|\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\Big) \\
=&\mathbb{E}\Big(\overline{c}'(S,D_i)^3\big|\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\Big)-3\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_i')\big|\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\Big) \\
&+3\mathbb{E}\Big(\overline{c}'(S,D_i)\overline{c}'(S,D_i')^2\big|\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\Big)-\mathbb{E}\Big(\overline{c}'(S,D_i')^3\big|\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\Big).
\end{aligned}
$$

We next examine each term on the RHS of the above equation. First, by the definition of conditional expectation, we have

$$
\mathbb{E}\Big(\overline{c}'(S,D_i)^3\big|\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\Big)=\frac{\mathbb{E}\Big(\overline{c}'(S,D_i)^3\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\Big)}{\mathbb{P}\big(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i')\big)}=2\mathbb{E}\Big(\overline{c}'(S,D_i)^3\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\Big)
$$

where for any $i\in\mathcal{J}$

$$
\begin{aligned}
&\mathbb{E}\Big(\overline{c}'(S,D_i)^3\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\Big) \\
=&\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\ge D_i)-p+\frac{\theta}{m}\mathbb{I}(S\ge\sum_{j=1}^m d_{ij})\big)^3\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\Big) \\
=&\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\big((h+p)\mathbb{I}(S\ge D_i)-p+\frac{\theta}{m}\mathbb{I}(S\ge\sum_{j=1}^m d_{ij})\big)^3\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
=&\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\big[(h+p)\mathbb{I}(S\ge D_i)-p\big]^3\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
&+\frac{3\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\big[(h+p)\mathbb{I}(S\ge D_i)-p\big]^2\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S\ge\sum_{j=1}^m d_{ij})\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
&+\frac{3\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\big[(h+p)\mathbb{I}(S\ge D_i)-p\big]\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S\ge\sum_{j=1}^m d_{ij})\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
&+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S\ge\sum_{j=1}^m d_{ij})\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big),
\end{aligned}
$$

and for any $i\in\mathcal{J}^c$

$$
\begin{aligned}
&\mathbb{E}\Big(\overline{c}'(S,D_i)^3\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\Big) \\
=&\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\ge D_i)-p+\frac{\theta}{m}\mathbb{I}(S\ge\sum_{j=1}^m d_{ij})\big)^3\mathbb{I}(D_i<x_{im})\mathbb{I}(\overline{c}'(S,D_i)\ge\overline{c}'(S,D_i'))\Big)
\end{aligned}
$$

$$+\mathbb{E}\Big(\Big(\sum_{j\in\mathcal{J}} w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\big]\Big)^3\mathbb{I}(D_i\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)$$

$$=\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big)^3\mathbb{I}(D_i<x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)+$$

$$\mathbb{E}\Big(\sum_{\substack{I_1+I_2+...+I_{|J|}=3,\\ I_1,...,I_{|J|}\geq 0}}\binom{3}{I_1,...,I_{|J|}}\prod_{j=1}^{|J|}\big(w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\big]\big)^{I_j}\mathbb{I}(D_i\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)$$

$$=\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\big)^3\mathbb{I}(D_i<x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$+\sum_{\substack{I_1+I_2+...+I_{|J|}=3,\\ I_1,I_2,...,I_{|J|}\geq 0}}\frac{3!}{I_1!I_2!...I_{|J|}!}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\prod_{j=1}^{|J|}\big(w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\big]\big)^{I_j}$$

$$\mathbb{I}(D_i\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$=\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^3\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$+\frac{3\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$+\frac{3\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(D_i<x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(S\geq\sum_{j=1}^{m}d_{ij})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$+\sum_{\substack{I_1+I_2+...+I_{|J|}=3,\\ I_1,I_2,...,I_{|J|}\geq 0}}\frac{3!}{I_1!I_2!...I_{|J|}!}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\prod_{j=1}^{|J|}\big(\frac{\mathbb{I}(D_j\geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\big]\big)^{I_j}$$

$$\mathbb{I}(D_i\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

with each $j\in\mathcal{J}$ being symmetric and thus by setting $I_j=3$ we have

$$\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\big(w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\big]\big)^3\mathbb{I}(D_i\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$=\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(w_j^3[(h+p)\mathbb{I}(S\geq D_j)-p]^3\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$+\frac{3\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(w_j^3[(h+p)\mathbb{I}(S\geq D_j)-p]^2\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$+\frac{3\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(w_j^3[(h+p)\mathbb{I}(S\geq D_j)-p]\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))|\boldsymbol{D},\boldsymbol{D'}\Big)\Big)$$

$$+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D'}}\Big(w_j^3\mathbb{I}(D_i\geq x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d'}|\boldsymbol{D},\boldsymbol{D'}}\Big(\mathbb{I}(S\geq\sum_{k=1}^{m}d_{jk})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))|\boldsymbol{D},\boldsymbol{D'}\Big)\Big).$$

Second, we have

$$\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_i')\Big|\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i')\Big)=\frac{\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_i')\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)}{\mathbb{P}\big(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i')\big)}$$

$$=2\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_i')\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)$$

where for any $i\in\mathcal{J}$

$$\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_i')\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)$$

$$
\begin{aligned}
=&\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^2\big((h+p)\mathbb{I}(S\geq D_i')-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij}')\big)\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)\\
=&\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^2\big((h+p)\mathbb{I}(S\geq D_i')-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij}')\big)\\
&\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2[(h+p)\mathbb{I}(S\geq D_i')-p]\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)\\
&+\frac{\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]^2\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij}')\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)\\
&+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S\geq D_i')-p]\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)\\
&+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\mathbb{I}(S\geq\sum_{j=1}^m d_{ij}')\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)\\
&+\frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S\geq D_i)-p][(h+p)\mathbb{I}(S\geq D_i')-p]\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)\\
&+\frac{2\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S\geq D_i)-p]\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\mathbb{I}(S\geq\sum_{j=1}^m d_{ij}')\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)
\end{aligned}
$$

and for any $i\in\mathcal{J}^c$

$$
\begin{aligned}
&\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D_i')\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)\\
=&\mathbb{E}\Big(\Big(\sum_{j\in\mathcal{J}}w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big]\Big)^2\Big(\sum_{j\in\mathcal{J}}w_j'\big[(h+p)\mathbb{I}(S\geq D_j')-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk}')\big]\Big)\\
&\mathbb{I}(D_i\geq x_{im})\mathbb{I}(D_i'\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)\\
&+\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^2\big((h+p)\mathbb{I}(S\geq D_i')-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij}')\big)\\
&\mathbb{I}(D_i<x_{im})\mathbb{I}(D_i'<x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)\\
&+\mathbb{E}\Big(\Big(\sum_{j\in\mathcal{J}}w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big]\Big)^2\big((h+p)\mathbb{I}(S\geq D_i')-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij}')\big)\\
&\mathbb{I}(D_i\geq x_{im})\mathbb{I}(D_i'<x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)\\
&+\mathbb{E}\Big(\big((h+p)\mathbb{I}(S\geq D_i)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{j=1}^m d_{ij})\big)^2\Big(\sum_{j\in\mathcal{J}}w_j'\big[(h+p)\mathbb{I}(S\geq D_j')-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk}')\big]\Big)\\
&\mathbb{I}(D_i<x_{im})\mathbb{I}(D_i'\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)
\end{aligned}
$$

with

$$
\begin{aligned}
&\mathbb{E}\Big(\Big(\sum_{j\in\mathcal{J}}w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big]\Big)^2\Big(\sum_{j\in\mathcal{J}}w_j'\big[(h+p)\mathbb{I}(S\geq D_j')-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk}')\big]\Big)\\
&\mathbb{I}(D_i\geq x_{im})\mathbb{I}(D_i'\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\Big)\\
=&\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}}w_j\big[(h+p)\mathbb{I}(S\geq D_j)-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk})\big]\Big)^2\Big(\sum_{j\in\mathcal{J}}w_j'\big[(h+p)\mathbb{I}(S\geq D_j')-p+\frac{\theta}{m}\mathbb{I}(S\geq\sum_{k=1}^m d_{jk}')\big]\\
&\mathbb{I}(D_i\geq x_{im})\mathbb{I}(D_i'\geq x_{im})\mathbb{I}(\overline{c}'(S,D_i)\geq\overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)\\
=&\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}}\frac{\mathbb{I}(D_j\geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_j)-p\big]\Big)^2\Big(\sum_{j\in\mathcal{J}}\frac{\mathbb{I}(D_j'\geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i'\geq x_{im})}\big[(h+p)\mathbb{I}(S\geq D_j')-p\big]\Big)
\end{aligned}
$$

$$\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D_i' \geq x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\Big)^2 \mathbb{I}(D_i \geq x_{im})\mathbb{I}(D_i' \geq x_{im})$$
$$\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j' \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i' \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk}')\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\big[(h+p)\mathbb{I}(S \geq D_j') - p\big]\Big)\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D_i' \geq x_{im})$$
$$\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\big)^2\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D_i' \geq x_{im})$$
$$\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\big)^2 \sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j' \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i' \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk}')\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\Big)\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j' \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i' \geq x_{im})}\big[(h+p)\mathbb{I}(S \geq D_j') - p\big]\Big)$$
$$\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D_i' \geq x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{2\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} w_j\big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\Big)\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D_i' \geq x_{im})$$
$$\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j' \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i' \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk}')\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big),$$

and

$$\mathbb{E}\Big(\big((h+p)\mathbb{I}(S \geq D_i) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{j=1}^m d_{ij})\big)^2\big((h+p)\mathbb{I}(S \geq D_i') - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{j=1}^m d_{ij}')\big)$$
$$\mathbb{I}(D_i < x_{im})\mathbb{I}(D_i' < x_{im})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\Big)$$
$$=\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\big((h+p)\mathbb{I}(S \geq D_i) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{j=1}^m d_{ij})\big)^2\big((h+p)\mathbb{I}(S \geq D_i') - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{j=1}^m d_{ij}')\big)$$
$$\mathbb{I}(D_i < x_{im})\mathbb{I}(D_i' < x_{im})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$=\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S \geq D_i) - p]^2[(h+p)\mathbb{I}(S \geq D_i') - p]\mathbb{I}(D_i < x_{im})\mathbb{I}(D_i' < x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S \geq D_i) - p]^2\mathbb{I}(D_i < x_{im})\mathbb{I}(D_i' < x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S \geq \sum_{j=1}^m d_{ij}')\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S \geq D_i') - p]\mathbb{I}(D_i < x_{im})\mathbb{I}(D_i' < x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S \geq \sum_{j=1}^m d_{ij})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(D_i < x_{im})\mathbb{I}(D_i' < x_{im})\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S \geq \sum_{j=1}^m d_{ij})\mathbb{I}(S \geq \sum_{j=1}^m d_{ij}')\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S \geq D_i) - p][(h+p)\mathbb{I}(S \geq D_i') - p]\mathbb{I}(D_i < x_{im})\mathbb{I}(D_i' < x_{im})$$
$$\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S \geq \sum_{j=1}^m d_{ij})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D_i'))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big)$$
$$+ \frac{2\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big([(h+p)\mathbb{I}(S \geq D_i) - p]\mathbb{I}(D_i < x_{im})\mathbb{I}(D_i' < x_{im})$$

$$\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S \geq \sum_{j=1}^m d_{ij})\mathbb{I}(S \geq \sum_{j=1}^m d'_{ij})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big),$$

and

$$\begin{aligned}
&\mathbb{E}\Big(\Big(\sum_{j\in\mathcal{J}} w_j\big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\big]\Big)^2\big((h+p)\mathbb{I}(S \geq D'_i) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{j=1}^m d'_{ij})\big) \\
&\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D'_i < x_{im})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\Big) \\
=&\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} w_j\big[(h+p)\mathbb{I}(S \geq D_j) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\big]\Big)^2\big((h+p)\mathbb{I}(S \geq D'_i) - p + \frac{\theta}{m}\mathbb{I}(S \geq \sum_{j=1}^m d'_{ij})\big) \\
&\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D'_i < x_{im})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
=&\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\Big)^2\big[(h+p)\mathbb{I}(S \geq D'_i) - p\big]\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D'_i < x_{im}) \\
&\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
&+\frac{\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\Big)^2\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D'_i < x_{im}) \\
&\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(S \geq \sum_{j=1}^m d'_{ij})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
&+\frac{\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\big[(h+p)\mathbb{I}(S \geq D'_i) - p\big]\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D'_i < x_{im}) \\
&\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\big)^2\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
&+\frac{\theta^3}{m^3}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D'_i < x_{im}) \\
&\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\big)^2\mathbb{I}(S \geq \sum_{i=1}^m d'_{ij})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
&+\frac{2\theta}{m}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\Big)\big[(h+p)\mathbb{I}(S \geq D'_i) - p\big]\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D'_i < x_{im}) \\
&\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big) \\
&+\frac{2\theta^2}{m^2}\mathbb{E}_{\boldsymbol{D},\boldsymbol{D}'}\Big(\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\big[(h+p)\mathbb{I}(S \geq D_j) - p\big]\Big)\mathbb{I}(D_i \geq x_{im})\mathbb{I}(D'_i < x_{im}) \\
&\mathbb{E}_{\boldsymbol{d},\boldsymbol{d}'|\boldsymbol{D},\boldsymbol{D}'}\Big(\sum_{j\in\mathcal{J}} \frac{\mathbb{I}(D_j \geq x_{jm})}{\sum_{i\in\mathcal{J}}\mathbb{I}(D_i \geq x_{im})}\mathbb{I}(S \geq \sum_{k=1}^m d_{jk})\mathbb{I}(S \geq \sum_{i=1}^m d'_{ij})\mathbb{I}(\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i))\big|\boldsymbol{D},\boldsymbol{D}'\Big)\Big).
\end{aligned}$$

The above analyses show that $\mathbb{E}\Big(\overline{c}'(S,D_i)^3\big|\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i)\Big)$ and $\mathbb{E}\Big(\overline{c}'(S,D_i)^2\overline{c}'(S,D'_i)\Big|\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i)\Big)$ are constants (being of order $m^{-1}e^{-\frac{1}{m}}$), so are $\mathbb{E}\Big(\overline{c}'(S,D_i)\overline{c}'(S,D'_i)^2\big|\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i)\Big)$ and $\mathbb{E}\Big(\overline{c}'(S,D'_i)^3\big|\overline{c}'(S,D_i) \geq \overline{c}'(S,D'_i)\Big)$ by symmetry of D_i and D'_i. Thus, there exists an appropriately matched constant C_2 such that

$$\frac{1}{4}\sum_{i=1}^n \mathbb{E}|\Delta_i f|^3 = \frac{1}{4\widetilde{\sigma}_n^3(S)n^{\frac{3}{2}}}\sum_{i=1}^n \mathbb{E}\big|\overline{c}'(S,D_i) - \overline{c}'(S,D'_i)\big|^3 \leq \frac{C_2}{n^2} = \frac{c_2 + k_2(me^{\frac{1}{m}})}{n^2},$$

where the constant C_2 can be split into two parts by the dependency on m.

Putting all above together, we show $\sup_{t\in\mathbb{R}}\left|\mathbb{P}(W_{1,n}(S)\leq t)-\mathbb{P}(Z\leq t)\right|$ (LHS of 1) can be bounded from above, that is,

$$\begin{aligned}\sup_{t\in\mathbb{R}}\left|\mathbb{P}(W_{1,n}(S)\leq t)-\mathbb{P}(Z\leq t)\right| \leq& 2\Big(\sqrt{\text{Var}(\mathbb{E}[T|W_{1,n}(S)])}+\frac{1}{4}\sum_{i=1}^{n}\mathbb{E}|\Delta_i f|^3\Big)^{1/2}\\ \leq& 2\Big(\sqrt{\frac{nC_{1,1}+n^2C_{1,2}}{n^4(C_{1,3}+nC_{1,4})^2}}+\frac{C_2}{n^2}\Big)^{1/2}\\ =& 2\Big(\sqrt{\frac{n(c_{1,1}+k_{1,1}(m^{-4}e^{-\frac{4}{m}}))+n^2(c_{1,2}+k_{1,2}(m^{-4}e^{-\frac{4}{m}}))}{n^4(c_{1,3}+k_{1,3}(m^{-2}e^{-\frac{2}{m}})+n(c_{1,4}+k_{1,4}(m^{-2}e^{-\frac{2}{m}})))^2}}+\frac{c_2+k_2(me^{\frac{1}{m}})}{n^2}\Big)^{1/2}.\end{aligned}$$

Furthermore, $|W_1(S)-W_{1,n}(S)|=o_p(1)$ by construction. So it follows $W_1(S)\xrightarrow{d}\mathcal{N}(0,1)$ as $n\to\infty$, which proves Lemma 1. And thus

$$\sqrt{n}(\hat{S}_n-\widetilde{S})=-\frac{\widetilde{\sigma}_1(\widetilde{S})}{V_1(\widetilde{S})}W_1(\widetilde{S})+o_p(1)\xrightarrow{d}\mathcal{N}(0,\overline{\sigma}^2),$$

where

$$\overline{\sigma}^2=\frac{\widetilde{\sigma}_1^2(\widetilde{S})}{V_1^2(\widetilde{S})}=\frac{\text{Var}(\widetilde{c}'(\widetilde{S},D))}{V_1^2(\widetilde{S})}+r\frac{\left\{\frac{(1+p)}{p}(\mathbb{E}[\widetilde{c}'(\widetilde{S},D)\mathbb{I}(D\geq x_m)])^2+\frac{1}{1-r}\text{Var}(\widetilde{c}'(\widetilde{S},D)\mathbb{I}(D\geq x_m))\right\}}{V_1^2(\widetilde{S})},$$

with $V_1(S)$ defined as

$$V_1(S)=\frac{d^2}{dS^2}\mathbb{E}[\widetilde{c}(S,D)].$$

And note that

$$\mathbb{E}[\widetilde{c}(S,D)]=\mathbb{E}_D\Big(h(S-D)^+ +p(D-S)^+ +\frac{\theta}{m}(S-D_m)^+\Big)$$

where $D_m=mD\sim F_m(\cdot)$ and $D\sim F(\cdot)$. We have

$$\frac{d}{dS}\mathbb{E}[\widetilde{c}(S,D)]=hF(S)+p(1-F(S))+\frac{\theta}{m}F_m(S),$$

$$\frac{d^2}{dS^2}\mathbb{E}[\widetilde{c}(S,D)]=hf(S)+pf(S)+\frac{\theta}{m}f_m(S).$$

$\square$

Welcome | Game Overview | Demand Information | Cost Calculations | Results | Example | Payment

Instructions

Welcome!

You are about to participate in a **decision-making experiment**.

If you follow these instructions carefully and make good decisions, you will earn a **considerable amount of money** that will be paid to you at the end of the session. Your earnings will depend on your **decisions** and **chance**.

Please **do not** use any resources other than those provided to you during the experiment.

Next

Welcome | Game Overview | Demand Information | Cost Calculations | Results | Example | Payment

Instructions

Game Overview

Imagine you own a bakery store that sells **fresh pastries**. Each pastry can stay fresh on display for 3 rounds. You will play for **42 rounds**.

At the Start of Each Round:

- Check **how many pastries are in stock**, and **when they are going to expire**.
- Decide how many to order (**Order Quantity**), without knowing the exact customer demand.

Once your order is placed, it will be **immediately** added to your on-hand stock.
Customer demand arrives and will be fulfilled using the **oldest stock first**.

At the End of Each Round:

- **Unmet Demand**: If there are not enough pastries in stock to meet customer demand, any unfulfilled requests will be lost, incurring a penalty of **$19.00** per unit (**Unit Penalty Cost**) in *lab dollars*.
- **Leftover Stock**: The surplus of the pastries stock that exceeds actual demand, incurring a holding cost of **$1.00** per unit (**Unit Holding Cost**) in *lab dollars*.
- **Expired Stock**: Pastries that reach the end of their **3-round shelf life** will expire and be removed from the system, incurring an outdating cost of **$3.00** per unit (**Unit Outdating Cost**) in *lab dollars*.

Objective:

Decide the order quantity for each round to
minimize the total penalty, holding, and outdating costs over all 42 rounds.

Back

Next

Welcome | Game Overview | Demand Information | Cost Calculations | Results | Example | Payment

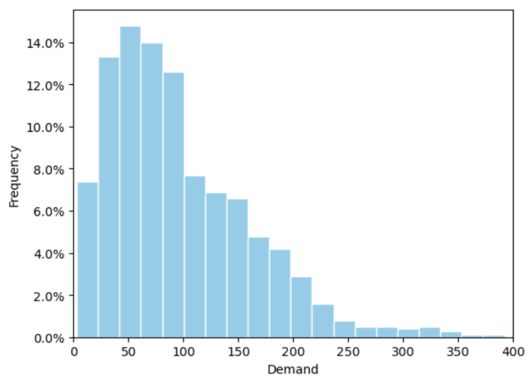
Instructions

Demand Information

Customer demand in each round is **uncertain** and does **not depend on any other rounds**. Say, a high demand in the last round does not imply high demand in the current round.

- The demand follows **a similar pattern or distribution**, but the exact pattern is **unknown**.
- **Data from past selling seasons** is available to help get a better idea of what to expect.

The column chart below shows the distribution of 1,000 past demand values, which are **integers** ranging from **0** to **400**, with an average value of **100**.



Minimum demand: **0** , Maximum demand: **400** , Average demand: **100**.

Here is a table that presents a sample of 100 rounds:

121	38	110	58	26	153	34	165	79	60
58	27	139	13	60	44	149	87	109	50
71	8	88	360	209	41	38	136	44	47
213	49	59	32	208	30	4	184	45	122
100	190	20	75	165	10	350	30	80	110
15	254	47	120	92	71	110	161	69	178
178	30	122	179	123	90	124	130	392	73
209	77	152	131	97	187	42	36	86	22
348	25	130	38	20	87	26	126	188	149
13	60	44	149	87	109	50	161	15	254

To help you understand the randomness in demand, you can click the **Sampling** button **up to 10 times**. Each time you click, a sample of 100 data points will be drawn.

Sampling

You haven't clicked the button yet.

Back

Next

Welcome | Game Overview | Demand Information | Cost Calculations | Results | Example | Payment

Instructions

Cost Calculations

After the demand is realized in current round, the penalty, holding, and outdating costs for the round are calculated as follows:

Your Costs = $19.00 * **Unmet Demand** + $1.00 * **Leftover Stock** + $3.00 * **Expired Stock**
(Penalty Cost) (Holding Cost) (Outdating Cost)

where

- **Unmet Demand** = Maximum{**Customer Demand** - (**Order Quantity** + *Initial Stock*), 0},
- **Leftover Stock** = Maximum{(**Order Quantity**+ *Initial Stock*) - **Customer Demand**, 0},
- **Expired Stock** = Maximum{*Near-Expiry Stock* - **Customer Demand**, 0}.

Definitions:

- Maximum{x, y} returns the larger value between x and y.
- "*Initial Stock*": Items with 2 or fewer rounds of shelf life remaining at the start of the round.
- "*Near-Expiry Stock*": Items with 1 round of shelf life remaining at the start of the round.

Back Next

Welcome | Game Overview | Demand Information | Cost Calculations | Results | Example | Payment

Instructions

Results

After the order decision is made and random demand is realized, you will see the following:

Round number	
Actual demand in this round	
Order quantity	
Sales (number of items sold)	
Expired stock (number of items expired)	
Stock that will expire in one more round	
Stock that will expire in two more rounds	
Total stock level at the end of this round	
Total costs incurred in this round	

This concludes one round.

After results are reported for each round, you will complete a **survey** to evaluate the value of the provided outcome information (**required** for rounds 1, 21, 22, & 42; **optional** for all others).

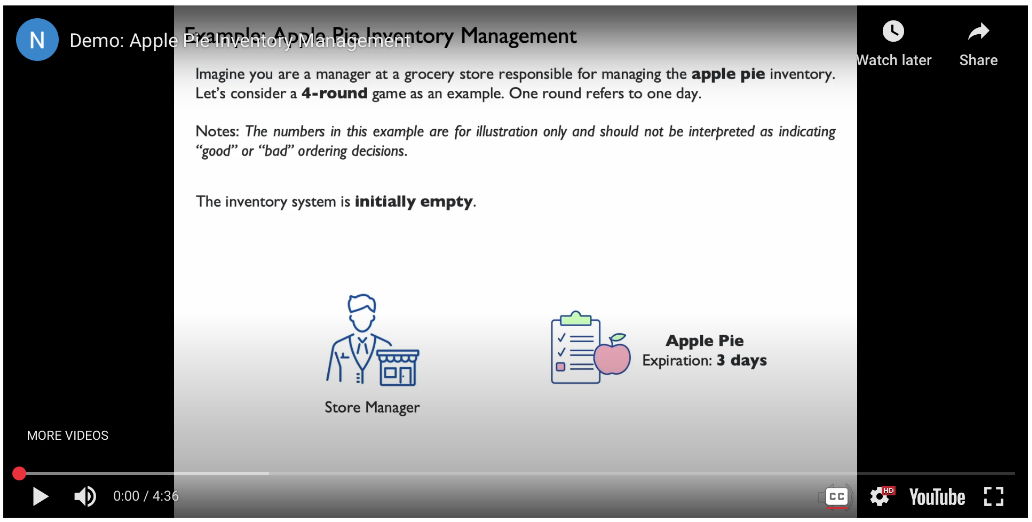
Back Next

Welcome | Game Overview | Demand Information | Cost Calculations | Results | Example | Payment

Instructions

Example: Managing Apple Pie Inventory

Imagine you are running a bakery store that sells fresh apple pies. You are responsible for managing the apple pie inventory.
Let's consider a **4-round** game as an example. One round refers to one day.



Back | Next

Welcome | Game Overview | Demand Information | Cost Calculations | Results | Example | Payment

Instructions

Payment

You will participate in one game, which consists of 42 decisions. Each player is given an *endowment* of 20,000 lab dollars at the beginning of the game. All rounds' costs will be added together to calculate the total inventory costs.

Your final earnings will be based on the following formula:

$$Earnings = \substack{Completion\ Time\ Based \\ Compensation} + \underbrace{\frac{\left(\substack{Endowment \\ (20,000\ lab\ dollars)} - \substack{Total\ Inventory \\ Costs}\right)^{+}}{42} * Conversion\ Rate\ (0.01)}_{\text{Performance-based bonus}}$$

where

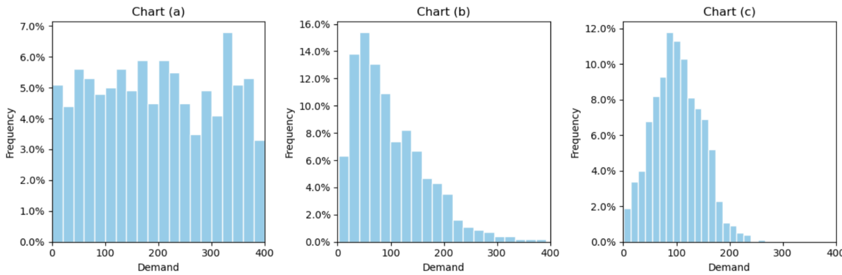
- *Completion Time Based Compensation* is granted at 8 US dollars per hour.
- *Conversion Rate* means 100 lab dollars convert to one US dollar.
- $(x)^{+}$ returns the greater value between x and 0.

Back | Next

Comprehension Checks

Hope the instructions are clear. Before the game begins, here are five questions to ensure you fully understand the task.

1. **Which of the three column charts below do you think best represents the distribution of past demands?**



- ◯ A. Chart (a)
- ◯ B. Chart (b)
- ◯ C. Chart (c)

2. **If demand was high in the previous round, it must be high as well in the current round.**

- ◯ True
- ◯ False

3. **What decision are you going to make in each round?**

- ◯ A. Selling Price
- ◯ B. Order Quantity

4. **What is a piece of information you will not have access to when making your decision?**

- ◯ A. Unit Penalty Cost
- ◯ B. Actual Demand in the Upcoming Round
- ◯ C. Initial Stock

5. **What is the goal when managing the inventory system?**

- ◯ A. To minimize the total penalty, holding, and outdating costs incurred over all 42 rounds
- ◯ B. To minimize unmet customer demand in each round

Submit

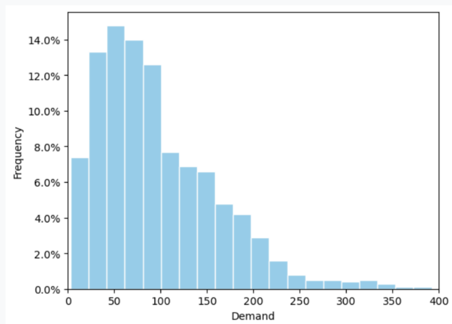
Read Instructions Again

Decision Making in Round 1

Great! You are now entering the Decision Making Phase.

Shelf Life: Each item can stay fresh on display for 3 rounds before it goes bad.
Penalty Cost: $19.00 per unit for unmet demand because of stockout.
Holding Cost: $1.00 per unit to keep leftovers in stock.
Outdating Cost: $3.00 per unit for processing those expired items.

How demands looked over the past 1000 selling seasons:



Minimum demand = 0, Maximum demand = 400, Average demand = 100.

Initial stock: You currently have 0 items in stock.

Your task is to decide how many items to order so that they are immediately added to your on-hand stock.

Your Decision

Choose the quantity you would like to order:

Submit Decision

Results in Round 1

Results are reported below:

Round number	1
Actual demand in this round	204
The order quantity you chose	200
Sales (number of items sold)	200
Expired inventory (number of items expired)	0
Stock that will expire in one more round	0
Stock that will expire in two more rounds	0
Total stock level at the end of this round	0
Total costs incurred in this round	$76.00

How would you rate the value of the information provided in this round?

○ Useless ○ Slightly useful ○ Moderately useful ○ Useful ○ Very useful

Next

Instructions for AI Assistance

You have successfully completed the first 21 rounds, the **Benchmark Decision-Making Phase**!

Next, you will progress to the **AI-Assisted Decision-Making Phase**!

> For each remaining round, alongside the provided cost and demand information, you will receive AI assistance when making your order decision. Specifically, our AI algorithm will recommend an order quantity.

After viewing the algorithm's recommendation, you will be prompted to submit your final order quantity decision.

Click the "Next" button to proceed.

Next

Read Instructions Again

Instructions for AI Assistance

You have successfully completed the first 21 rounds, the **Benchmark Decision-Making Phase**!

Next, you will progress to the **AI-Assisted Decision-Making Phase**!

> For each remaining round, alongside the provided cost and demand information, you will receive AI assistance when making your order decision. Specifically, our AI algorithm will alert you to any deviation from the target action and behavioral bias you are likely to have in managing the food inventory system, based on how you have acted in the first rounds.

After viewing the algorithm's detection, you will be prompted to submit your final order quantity decision.

Click the "Next" button to proceed.

Next

Note: *There will be about 10 seconds waiting time. Please do not refresh your page after clicking "Next".*

Read Instructions Again

Instructions for AI Assistance

You have successfully completed the first 21 rounds, the **Benchmark Decision-Making Phase**!

Next, you will progress to the **AI-Assisted Decision-Making Phase**!

> For each remaining round, alongside the provided cost and demand information, you will receive AI assistance when making your order decision. Specifically, our AI algorithm will:
>
> a. Alert you to any deviation from the target action and behavioral bias you are likely to have in managing the food inventory system, based on how you have acted in the first rounds;
> b. And recommend an order quantity.

After viewing the algorithm's detection and recommendation, you will be prompted to submit your final order quantity decision.

Click the "Next" button to proceed.

Next

Note: *There will be about 10 seconds waiting time. Please do not refresh your page after clicking "Next".*

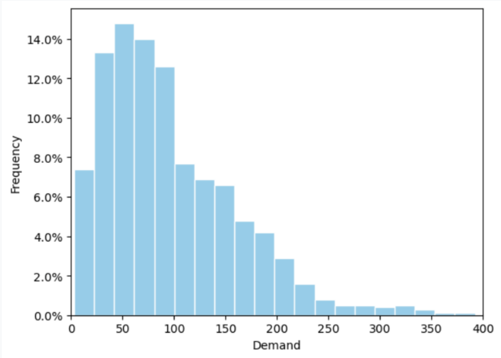
Read Instructions Again

Decision Making in Round 23

You are now entering the AI-Assisted Decision Making Phase!

Shelf Life: Each item can stay fresh on display for 3 rounds before it goes bad.
Penalty Cost: $19.00 per unit for unmet demand because of stockout.
Holding Cost: $1.00 per unit to keep leftovers in stock.
Outdating Cost: $3.00 per unit for processing those expired items.

How demands looked over the past 1000 selling seasons:



Minimum demand = 0, Maximum demand = 400, Average demand = 100.

Initial stock: You currently have 242 items in stock.

Your task is to decide how many items to order so that they are immediately added to your on-hand stock.

Our AI algorithm recommends ordering this amount: **0**.

Your Decision

Choose the quantity you would like to order:

Submit Decision

Results in Round 23

Results are reported below:

Round number	23
Actual demand in this round	90
The order quantity you chose	0
Sales (number of items sold)	90
Expired inventory (number of items expired)	17
Stock that will expire in one more round	135
Stock that will expire in two more rounds	0
Total stock level at the end of this round	135
Total costs incurred in this round	$203.00

How would you rate the value of the outcome information provided in the table above?

Feedback Value

○ Useless ○ Slightly useful ○ Moderately useful ○ Useful ○ Very useful

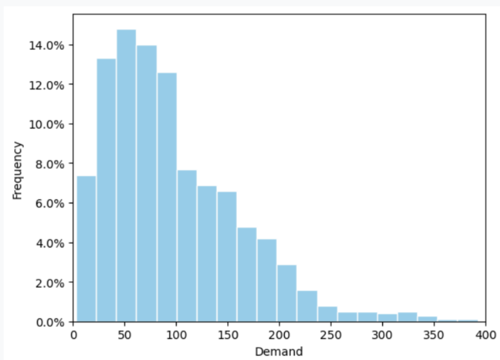
Next

Decision Making in Round 23

You are now entering the AI-Assisted Decision Making Phase!

Shelf Life: Each item can stay fresh on display for 3 rounds before it goes bad.
Penalty Cost: $19.00 per unit for unmet demand because of stockout.
Holding Cost: $1.00 per unit to keep leftovers in stock.
Outdating Cost: $3.00 per unit for processing those expired items.

How demands looked over the past 1000 selling seasons:



Minimum demand = 0, Maximum demand = 400, Average demand = 100.

Initial stock: You currently have 105 items in stock.

Your task is to decide how many items to order so that they are immediately added to your on-hand stock.

Based on how you have acted in the first 21 rounds, our AI algorithm thinks you are likely to adjust the order quantity towards demand in prior period. And for typical people who adjust the order quantity towards demand in prior period, they would order 52 for this round, which is lower than the optimal amount.

Your Decision

Choose the quantity you would like to order:

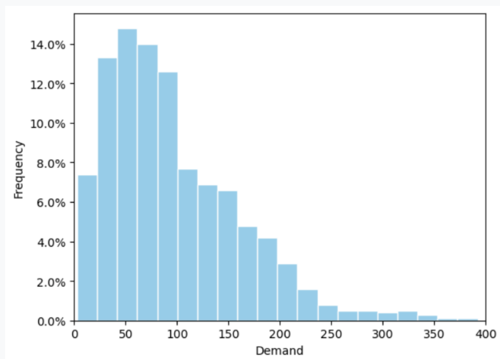
60

Submit Decision

Decision Making in Round 23

Shelf Life: Each item can stay fresh on display for 3 rounds before it goes bad.
Penalty Cost: $19.00 per unit for unmet demand because of stockout.
Holding Cost: $1.00 per unit to keep leftovers in stock.
Outdating Cost: $3.00 per unit for processing those expired items.

How demands looked over the past 1000 selling seasons:



Minimum demand = 0, Maximum demand = 400, Average demand = 100.

Initial stock: You currently have 170 items in stock.

Your task is to decide how many items to order so that they are immediately added to your on-hand stock.

Our AI algorithm recommends ordering this amount: **38**. In addition, based on how you have acted in the first 21 rounds, our AI algorithm thinks you are likely to put too much emphasis on the financial loss from under-ordering. For typical people who put too much emphasis on the financial loss from under-ordering, they would order 48 for this round, which is higher than the recommended quantity.

Your Decision

Choose the quantity you would like to order:

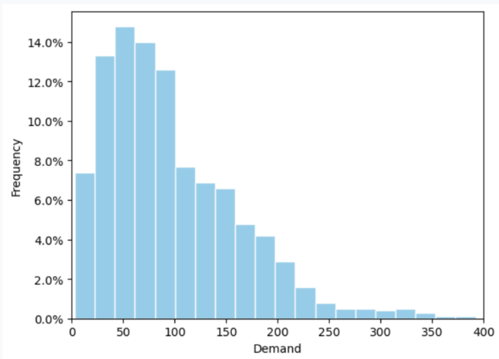
Submit Decision

Decision Making in Round 23

You are now entering the AI-Assisted Decision Making Phase!

Shelf Life: Each item can stay fresh on display for 3 rounds before it goes bad.
Penalty Cost: $19.00 per unit for unmet demand because of stockout.
Holding Cost: $1.00 per unit to keep leftovers in stock.
Outdating Cost: $3.00 per unit for processing those expired items.

How demands looked over the past 1000 selling seasons:



Minimum demand = 0, Maximum demand = 400, Average demand = 100.

Initial stock: You currently have 128 items in stock.

Your task is to decide how many items to order so that they are immediately added to your on-hand stock.

Our AI algorithm recommends ordering this amount: **80**.

In the last round, you ordered 210, which resulted in a total cost of $128.00. The AI had recommended 198, which would have led to a total cost of $116.00 if followed.

Your Decision

Choose the quantity you would like to order:

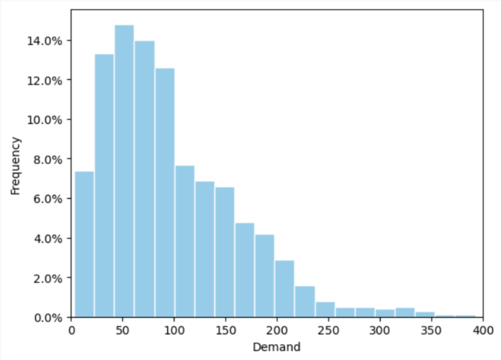
80

Submit Decision

Decision Making in Round 23

Shelf Life: Each item can stay fresh on display for 3 rounds before it goes bad.
Penalty Cost: $19.00 per unit for unmet demand because of stockout.
Holding Cost: $1.00 per unit to keep leftovers in stock.
Outdating Cost: $3.00 per unit for processing those expired items.

How demands looked over the past 1000 selling seasons:



Minimum demand = 0, Maximum demand = 400, Average demand = 100.

Initial stock: You currently have 128 items in stock.

Your task is to decide how many items to order so that they are immediately added to your on-hand stock.

Our AI algorithm recommends ordering this amount: **80**.

And our AI is 95% confident that the right amount is between 70 and 90 units.

Note: There is some uncertainty in the prescription because AI uses a limited data sample, making it not 100% confident.

Your Decision

Choose the quantity you would like to order:

85

Submit Decision

Questionnaire on Personal Background

Before ending the experiment, we have a few more questions.

Age	Age [ ]
Gender	Gender ◯ Female ◯ Male ◯ Other ◯ Prefer not to say
Education	Education ◯ Doctorate degree ◯ Master's degree ◯ Bachelor's degree ◯ High school graduate ◯ Other

How familiar are you with inventory management?

◯ No understanding or little understanding
◯ Moderately familiar
◯ Very familiar or worked with it many times before

How much do you know about artificial intelligence (AI)?

◯ No understanding or little understanding
◯ Moderately familiar
◯ Very familiar or worked with it many times before

Thank you very much for answering them!

Submit

Bonus

Congratulations on completing the multi-period food inventory control game!

The following are some results of the experiment:

Total rounds you played	42
Total penalty, holding, and outdating costs incurred over all rounds	$11559.00
Bonus based on your performance	$2.01

There are a few **optional** bonus questions. If you complete all of them, you will receive an additional bonus of $0.5 as compensation.

1. Please describe how you made the order quantity decisions:

   (At least 10 characters long and in English)

2. How useful were the recommended order quantities?

   Recommendation Feedback

   ◯ Useless ◯ Slightly useful ◯ Moderately useful ◯ Useful ◯ Very useful

3. Did you find the bias detection helpful?

   ◯ Not helpful ◯ Slightly helpful ◯ Moderately helpful ◯ Helpful ◯ Very helpful

Submit