Universal approximation and model compression for radial neural networks

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Abstract

We introduce a class of fully-connected neural networks whose activation functions, rather than being pointwise, rescale feature vectors by a function depending only on their norm. We call such networks *radial* neural networks, extending previous work on rotation equivariant networks that considers rescaling activations in less generality. We prove universal approximation theorems for radial neural networks, including in the more difficult cases of bounded widths and unbounded domains. Our proof techniques are novel, distinct from those in the pointwise case. Additionally, radial neural networks exhibit a rich group of orthogonal change-of-basis symmetries on the vector space of trainable parameters. Factoring out these symmetries leads to a practical lossless model compression algorithm. Optimization of the compressed model by gradient descent is equivalent to projected gradient descent for the full model.

1 Introduction

Inspired by biological neural networks, the theory of artificial neural networks has largely focused on pointwise (or "local") nonlinear layers [46, 14], in which the same function $\sigma\colon \mathbb{R} \to \mathbb{R}$ is applied to each coordinate independently:

$$\mathbb{R}^n \to \mathbb{R}^n, \qquad v = (v_1 \,, \, \dots \,, \, v_n) \;\mapsto\; (\sigma(v_1) \,, \, \sigma(v_2) \,, \, \dots \,, \, \sigma(v_n)). \tag{1.1}$$

In networks with pointwise nonlinearities, the standard basis vectors in $\mathbb{R}^n$ can be interpreted as "neurons" and the nonlinearity as a "neuron activation." Research has generally focused on finding functions σ which lead to more stable training, have less sensitivity to initialization, or are better adapted to certain applications [42, 38, 37, 10, 29]. Many σ have been considered, including sigmoid, ReLU, arctangent, ELU, Swish, and others.

However, by setting aside the biological metaphor, it is possible to consider a much broader class of nonlinearities, which are not necessarily pointwise, but instead depend simultaneously on many coordinates. Neural networks using such nonlinearities may yield expressive function classes with different advantages. One example is radial basis networks [4], which contain nonlinearities of the form $\|v - c\|$, which depend on all coordinates of v. However, each coordinate output is still independent.

In this paper, we introduce *radial* neural networks which employ non-pointwise nonlinearities called *radial rescaling* activations. Freedom from the pointwise assumption allows us to design activation functions that maximize symmetry in the parameter space of the

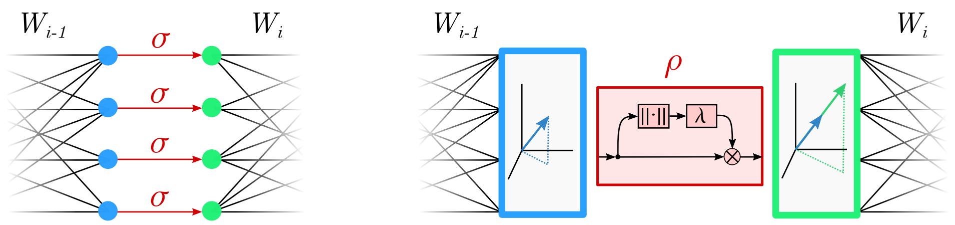


Figure 1: (Left) Pointwise activations distinguish a specific basis of each hidden layer and treat each coordinate independently, see equation 1.1. (Right) Radial rescaling activations rescale each feature vector by a function of the norm, see equation 1.2.

neural network. Such networks enjoy several provable properties including high model compressibility, symmetry in optimization, and universal approximation. These activations are defined by rescaling each vector by a scalar that depends only on the norm of the vector:

$$\rho : \mathbb{R}^n \to \mathbb{R}^n, \qquad v \ \mapsto \ \lambda(|v|)v, \tag{1.2}$$

where λ is a scalar-valued function of the norm. Whereas in the pointwise setting, only the linear layers mix information between different components of the latent features, for radial rescaling, all coordinates of the activation output vector are affected by all coordinates of the activation input vector. The inherent geometric symmetry of radial rescalings makes them particularly useful for designing equivariant neural networks [55, 47, 56, 57].

In our first set of main results, we prove that radial neural networks are in fact *universal approximators*. Specifically, we demonstrate that any asymptotically affine function can be approximated with a radial neural network, suggesting potentially good extrapolation behavior. Moreover, this approximation can be done with bounded width. Our approach to proving these results differs significantly from standard techniques used in the case of pointwise activations due to the fact that coordinates cannot be treated independently when dealing with non-pointwise activations.

In our second set of main results, we exploit parameter space symmetries of radial neural networks to achieve *model compression*. Using the fact that radial rescaling activations commute with orthogonal transformations, we develop a practical algorithm to systematically factor out orthogonal symmetries via iterated QR decompositions. This leads to another radial neural network with fewer neurons in each hidden layer. The resulting model compression algorithm is *lossless*: the compressed network and the original network both have the same value of the loss function on any batch of training data.

Furthermore, we prove that the loss of the compressed model after one step of gradient descent is equal to the loss of the original model after one step of *projected gradient descent*. As explained below, projected gradient descent involves zeroing out certain parameter values after each step of gradient descent. Although training the original network may result in a lower loss function after fewer epochs, in many cases the compressed network takes less time per epoch to train and is faster in reaching a local minimum.

To summarize, our main contributions are:

- A formalization of radial neural networks, a new class of neural networks;
- Two universal approximations results for radial neural networks: a) approximation of asymptotically affine functions, and b) bounded width approximation;
- An implementation of a lossless model compression algorithm for radial neural networks;
- A theorem providing the precise relationship between gradient descent optimization of the original and compressed networks.

2 Related work

Radial rescaling activations. Radial rescaling functions have the symmetry property of preserving vector directions, and hence exhibit rotation equivariance. Consequently, examples of such functions, such as the squashing nonlinearity and Norm-ReLU, feature in the study of rotationally equivariant neural networks [55, 47, 56, 57, 26]. However, previous works apply the activation only along the channel dimension, and consider the orthogonal group $O(n)$ only for $n = 2, 3$. In contrast, we consider a radial rescaling activation across the entire hidden layer, and $O(n)$-equivariance where n is the hidden layer dimension. Our constructions echo the vector neurons formalism [15], in which the output of a nonlinearity is a vector rather than a scalar. For radial basis networks, each hidden neuron is a radial nonlinear function of the shifted input vector, but the outputs are independent, whereas for radial rescaling functions, the outputs are also tied together [4].

Universal approximation. Neural networks of arbitrary width and sigmoid activations have long been known to be universal approximators [14]. Universal approximation can also be achieved by bounded width networks with arbitrary depth [36]. Additional work has generalized to other activation functions and neural architectures [24, 60, 43, 50]. While most work has focused on compact domains, some recent work also considers non-compact domains [28, 54]. The techniques used for pointwise activation functions generalize to radial basis networks since the outputs of each RBF are independent [40], but do not easily generalize to the radial rescaling activations considered here, because all coordinates of the activation output vector are affected by all coordinates of the activation input vector. As a consequence, individual radial neural network approximators of two different functions cannot be easily combined to give an approximator of the sum of the functions.

Groups and symmetry. Appearances of symmetry in machine learning have generally focused on symmetric input and output spaces. Most prominently, equivariant neural networks incorporate symmetry as an inductive bias and feature weight-sharing constraints based on equivariance with respect to various symmetry groups. Examples of equivariant architectures include G-convolution, steerable CNN, and Clebsch-Gordon networks [13, 55, 11, 9, 30, 2, 58, 12, 57, 16, 31, 44]. By contrast, our approach to radial neural networks does not depend on symmetries of the input domain, output space, or feedforward mapping. Instead, we exploit parameter space symmetries and thus obtain more general results that apply to domains with no apparent symmetry.

Model compression. A major goal in machine learning is to find methods to reduce the number of trainable parameters, decrease memory usage, or accelerate inference and training [8, 61]. Our approach toward this goal differs significantly from most existing methods in that it is based on the inherent symmetry of network parameter spaces. One prior method is *weight pruning*, which removes redundant or small weights from a network with little loss in accuracy [20, 3, 27]. Pruning can be done during training [18] or at initialization [34, 53]. *Gradient-based pruning* identifies low saliency weights by estimating the increase in loss resulting from their removal [33, 22, 17, 39]. A complementary approach is *quantization*, which decreases the bit depth of weights [59, 25, 19]. *Knowledge distillation* identifies a small model mimicking the performance of a larger model or ensemble of models [5, 23, 1]. *Matrix Factorization* methods replace fully connected layers with lower rank or sparse factored tensors [6, 7, 52, 32, 45, 35] and can often be applied before training. Our method involves a type of matrix factorization based on the QR decomposition; however, rather than aim for a rank reduction of linear layers, we leverage this decomposition to reduce hidden widths via change-of-basis operations on the hidden representations. Close to our method are lossless compression methods which remove stable neurons in ReLU networks [49, 48] or exploit permutation parameter space symmetry to remove redundant neurons [51]; our compression instead follows from the symmetries of the radial rescaling activation. Finally, the model compression results of [26], while conceptually similar to ours, are weaker, as (1) the unitary group action is on disjoint layers instead of iteratively moving through all layers, and (2) the results are only stated for a version of the squashing nonlinearity.

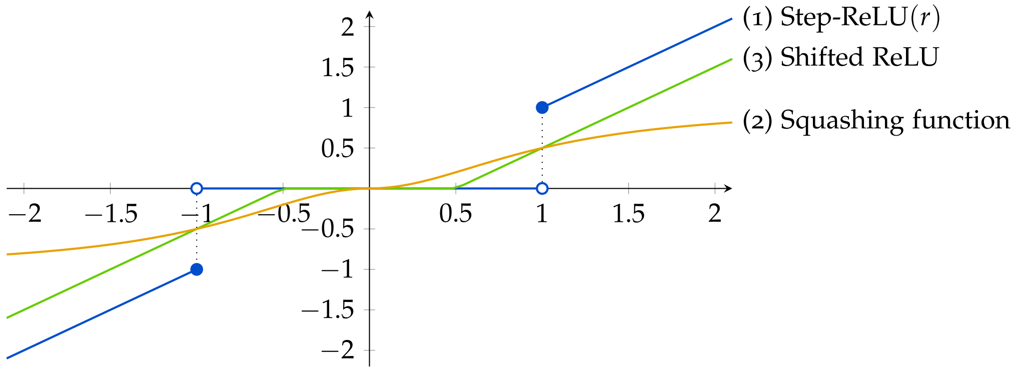


Figure 2: Examples of different radial rescaling functions in $\mathbb{R}^1$, see Example 1.

3 Radial neural networks

In this section, we define radial rescaling functions and radial neural networks. Let $h : \mathbb{R} \to \mathbb{R}$ be a function. For any $n \geq 1$, set:

$$h^{(n)} : \mathbb{R}^n \to \mathbb{R}^n \qquad\qquad h^{(n)}(v) = h(|v|)\frac{v}{|v|}$$

for $v \neq 0$, and $h^{(n)}(0) = 0$. A function $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is called a *radial rescaling* function if $\rho = h^{(n)}$ for some piecewise differentiable $h : \mathbb{R} \to \mathbb{R}$. Hence, ρ sends each input vector to a scalar multiple of itself, and that scalar depends only on the norm of the vector[1]. It is easy to show that radial rescaling functions commute with orthogonal transformations.

Example 1. (1) Step-ReLU, where $h(r) = r$ if $r \geq 1$ and 0 otherwise. In this case, the radial rescaling function is given by

$$\rho : \mathbb{R}^n \to \mathbb{R}^n, \qquad v \mapsto v \text{ if } |v| \geq 1; \qquad v \mapsto 0 \text{ if } |v| < 1 \tag{3.1}$$

(2) The squashing function, where $h(r) = r^2/(r^2+1)$. (3) Shifted ReLU, where $h(r) = \max(0, r-b)$ for $r > 0$ and b is a real number. See Figure 2. We refer to [55] and the references therein for more examples and discussion of radial functions.

A *radial neural network* with L layers consists of a positive integer n_i indicating the width of each layer $i = 0, 1, \dots, L$; the trainable parameters, comprising of a matrix $W_i \in \mathbb{R}^{n_i \times n_{i-1}}$ of weights and a bias vector $b_i \in \mathbb{R}^{n_i}$ for each $i = 1, \dots, L$; and a radial rescaling function $\rho_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ for each $i = 1, \dots, L$. We refer to the tuple $\mathbf{n} = (n_0, n_1, \dots, n_L)$ as the *widths vector* of the neural network. The hidden widths vector is $\mathbf{n}^{\text{hid}} = (n_1, n_2, \dots, n_{L-1})$. The feedforward function $F : \mathbb{R}^{n_0} \to \mathbb{R}^{n_L}$ of a radial neural network is defined in the usual way as an iterated composition of affine maps and activations. Explicitly, set $F_0 = \text{id}_{\mathbb{R}^{n_0}}$ and recursively define the partial feedforward functions:

$$F_i : \mathbb{R}^{n_0} \to \mathbb{R}^{n_i}, \qquad x \mapsto \rho_i\left(W_i \circ F_{i-1}(x) + b_i\right)$$

for $i = 1, \dots, L$. Then the feedforward function is $F = F_L$.

Remark 2. If $b_i = 0$ for all i, then the feedforward function takes the form $F(x) = W\left(\mu(x)x\right)$ where $\mu : \mathbb{R}^n \to \mathbb{R}$ is a scalar-valued function and $W = W_L W_{L-1} \cdots W_1 \in \mathbb{R}^{n_L \times n_0}$ is the product of the weight matrices. If any of the biases are non-zero, then the feedforward function lacks such a simple form.

[1]A function $\mathbb{R}^n \to \mathbb{R}$ that depends only on the norm of a vector is known as a *radial* function. Radial rescaling functions rescale each vector according to the radial function $v \mapsto \lambda(|v|) = \frac{h(|v|)}{|v|}$. This explains the connection to Equation 1.2.

4 Universal Approximation

In this section, we consider two universal approximation results. The first approximates asymptotically affine functions with a network of unbounded width. The second generalizes to bounded width networks. Proofs appear in Appendix B. Throughout, $B_r(c) = \{x \in \mathbb{R}^n : |x - c| < r\}$ denotes the r-ball around a point c, and an affine map $\mathbb{R}^n \to \mathbb{R}^m$ is one of the from $L(x) = Ax + b$ for a matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

4.1 Approximation of asymptotically affine functions

A continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *asymptotically affine* if there exists an affine map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that, for every $\epsilon > 0$, there is a compact subset K of $\mathbb{R}^n$ such that $|L(x) - f(x)| < \epsilon$ for all $x \in \mathbb{R}^n \setminus K$. In particular, continuous functions with compact support are asymptotically affine. The continuity of f and compactness of K imply that, for any $\epsilon > 0$, there exist $c_1, \dots, c_N \in K$ and $r_1, \dots, r_N \in (0,1)$ such that, first, the union of the balls $B_{r_i}(c_i)$ covers K and, second, for all i, we have $f\left(B_{r_i}(c_i) \cap K\right) \subseteq B_\epsilon(f(c_i))$. Let $N(f, K, \epsilon)$ be the minimal choice of N. In many cases, the constant $N(f, K, \epsilon)$ can be bounded explicitly[2].

Theorem 3 (Universal approximation)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an asymptotically affine function. For any $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}^n$ and a function $F : \mathbb{R}^n \to \mathbb{R}^m$ such that:*

1. *F is the feedforward function of a radial neural network with $N = N(f, K, \epsilon)$ layers whose hidden widths are $(n+1, n+2, \dots, n+N)$.*
2. *For any $x \in \mathbb{R}^n$, we have $|F(x) - f(x)| < \epsilon$.*

We note that the approximation in Theorem 3 is valid on all of $\mathbb{R}^n$. To give an idea of the proof, first fix $c_1, \dots, c_N \in K$ and $r_1, \dots, r_N \in (0,1)$ as above. Let $e_1, \dots, e_N$ be orthonormal basis vectors extending $\mathbb{R}^n$ to $\mathbb{R}^{n+N}$. For $i = 1, \dots, N$ define the following affine maps:

$$T_i : \mathbb{R}^{n+i-1} \to \mathbb{R}^{n+i} \qquad\qquad S_i : \mathbb{R}^{n+i} \to \mathbb{R}^{n+i}$$
$$z \mapsto z - c_i + h_i e_i \qquad\qquad z \mapsto z - (1 + h_i^{-1})\langle e_i, z\rangle e_i + c_i + e_i$$

where $h_i = \sqrt{1 - r_i^2}$ and $\langle e_i, z\rangle$ is the coefficient of e_i in z. Setting ρ_i to be Step-ReLU (Equation 3.1) on $\mathbb{R}^{n+i}$, these maps are chosen so that the composition $S_i \circ \rho_i \circ T_i$ maps the points in $B_{r_i}(c_i)$ to $c_i + e_i$, while keeping points outside this ball the same. We now describe a radial neural network with widths $(n, n+1, \dots, n+N, m)$ whose feedforward function approximates f. For $i = 1, \dots, N$ the affine map from layer $i-1$ to layer i is given by $z \mapsto T_i \circ S_{i-1}(z)$, with $S_0 = \mathrm{id}_{\mathbb{R}^n}$. The activation at each hidden layer is Step-ReLU. Let L be the affine map such that $|L - f| < \epsilon$ on $\mathbb{R}^n \setminus K$. The affine map from layer N to the output layer is $\Phi \circ S_N$ where $\Phi : \mathbb{R}^{n+N} \to \mathbb{R}^m$ is the unique affine map determined by $x \mapsto L(x)$ if $x \in \mathbb{R}^n$, and $e_i \mapsto f(c_i) - L(c_i)$. This construction is illustrated in Figure 3.

4.2 Bounded width approximation

We now turn our attention to a bounded width universal approximation result.

Theorem 4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an asymptotically affine function. For any $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}^n$ and a function $F : \mathbb{R}^n \to \mathbb{R}^m$ such that:*

1. *F is the feedforward function of a radial neural network with $N = N(f, K, \epsilon)$ hidden layers whose widths are all $n + m + 1$.*
2. *For any $x \in \mathbb{R}^n$, we have $|F(x) - f(x)| < \epsilon$.*

The proof, which is more involved than that of Theorem 3, relies on using orthogonal dimensions to represent the domain and the range of f, together with an indicator

[2]For example, if K is the unit cube in $\mathbb{R}^n$ and f is Lipschitz continuous with Lipschitz constant R, then $N(f, K, \epsilon) \leq \left\lceil \frac{R\sqrt{n}}{2\epsilon} \right\rceil^n$.

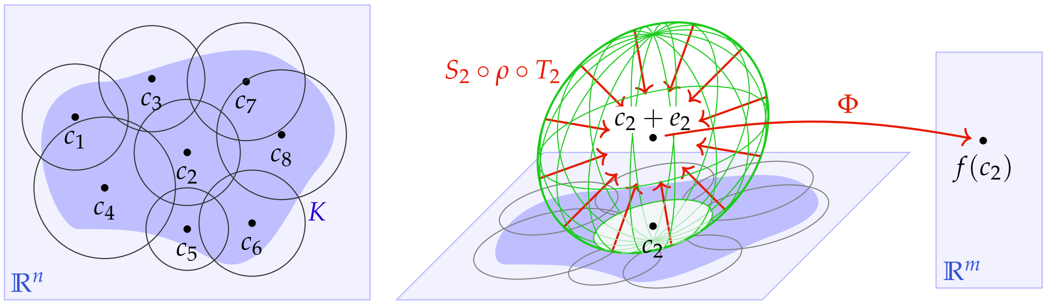


Figure 3: Two layers of the radial neural network used in the proof of Theorem 3. (Left) The compact set K is covered with open balls. (Middle) Points close to c_2 (green ball) are mapped to $c_2 + e_2$, all other points are kept the same. (Right) In the final layer, $c_2 + e_2$ is mapped to $f(c_2)$.

dimension to distinguish the two. We regard points in $\mathbb{R}^{n+m+1}$ as triples (x, y, θ) where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $\theta \in \mathbb{R}$. The proof of Theorem 4 parallels that of Theorem 3, but instead of mapping points in $B_{r_i}(c_i)$ to $c_i + e_i$, we map the points in $B_{r_i}((c_i, 0, 0))$ to $(0, \frac{f(c_i) - L(0)}{s}, 1)$, where s is chosen such that different balls do not interfere. The final layer then uses an affine map $(x, y, \theta) \mapsto L(x) + sy$, which takes $(x, 0, 0)$ to $L(x)$, and $(0, \frac{f(c_i) - L(0)}{s}, 1)$ to $f(c_i)$.

We remark on several additional results; see Appendix B for full statements and proofs. The bound of Theorem 4 can be strengthened to $\max(n, m) + 1$ in the case of functions $f : K \to \mathbb{R}^m$ defined on a compact domain $K \subset \mathbb{R}^n$ (i.e., ignoring asymptotic behavior). Furthermore, with more layers, it is possible to reduce that bound to $\max(n, m)$.

5 Model compression

In this section, we prove a model compression result. Specifically, we provide an algorithm which, given any radial neural network, computes a different radial neural network with smaller widths. The resulting compressed network has the same feedforward function as the original network, and hence the same value of the loss function on any batch of training data. In other words, our model compression procedure is *lossless*. Although our algorithm is practical and explicit, it reflects more conceptual phenomena, namely, a change-of-basis action on network parameter spaces (Section 5.1).

5.1 The parameter space

Suppose a fully connected network has L layers and widths given by the tuple $\mathbf{n} = (n_0, n_1, n_2, \ldots, n_{L-1}, n_L)$. In other words, the i-th layer has input width n_{i-1} and output width n_i. The parameter space is defined as the vector space of all possible choices of parameter values. Hence, it is given by the following product of vector spaces:

$$\mathsf{Param}(\mathbf{n}) = \left(\mathbb{R}^{n_1 \times n_0} \times \mathbb{R}^{n_2 \times n_1} \times \cdots \times \mathbb{R}^{n_L \times n_{L-1}}\right) \times \left(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_L}\right)$$

We denote an element therein as a pair of tuples $(\mathbf{W}, \mathbf{b})$ where $\mathbf{W} = (W_i \in \mathbb{R}^{n_i \times n_{i-1}})_{i=1}^L$ are the weights and $\mathbf{b} = (b_i \in \mathbb{R}^{n_i})_{i=1}^L$ are the biases. To describe certain symmetries of the parameter space, consider the following product of orthogonal groups, with sizes corresponding to the widths of the hidden layers:

$$O(\mathbf{n}^{\text{hid}}) = O(n_1) \times O(n_2) \times \cdots \times O(n_{L-1})$$

There is a change-of-basis action of $O(\mathbf{n}^{\text{hid}})$ on the parameter space $\mathsf{Param}(\mathbf{n})$. Explicitly, the tuple of orthogonal matrices $\mathbf{Q} = (Q_i)_{i=1}^{L-1} \in O(\mathbf{n}^{\text{hid}})$ transforms the parameter values $(\mathbf{W}, \mathbf{b})$ to $\mathbf{Q} \cdot \mathbf{W} := \left(Q_i W_i Q_{i-1}^{-1}\right)_{i=1}^L$ and $\mathbf{Q} \cdot \mathbf{b} := (Q_i b_i)_{i=1}^L$, where $Q_0 = \text{id}_{n_0}$ and $Q_L = \text{id}_{n_L}$. We write $\mathbf{Q} \cdot (\mathbf{W}, \mathbf{b})$ for $(\mathbf{Q} \cdot \mathbf{W}, \mathbf{Q} \cdot \mathbf{b})$.

5.2 Model compression

In order to state the compression result, we first define the reduced widths. Namely, the reduction $\mathbf{n}^{\text{red}} = (n_0^{\text{red}}, n_1^{\text{red}}, \dots, n_L^{\text{red}})$ of a widths vector $\mathbf{n}$ is defined recursively by setting $n_0^{\text{red}} = n_0$, then $n_i^{\text{red}} = \min(n_i, n_{i-1}^{\text{red}} + 1)$ for $i = 1, \dots, L-1$, and finally $n_L^{\text{red}} = n_L$. For a tuple $\boldsymbol{\rho} = (\rho_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i})_{i=1}^{L}$ of radial rescaling functions, we write $\boldsymbol{\rho}^{\text{red}} = \left(\rho_i^{\text{red}} : \mathbb{R}^{n_i^{\text{red}}} \to \mathbb{R}^{n_i^{\text{red}}}\right)$ for the corresponding tuple of restrictions, which are all radial rescaling functions. The following result relies on Algorithm 1 below.

Theorem 5. *Let $(\mathbf{W}, \mathbf{b}, \boldsymbol{\rho})$ be a radial neural network with widths $\mathbf{n}$. Let $\mathbf{W}^{\text{red}}$ and $\mathbf{b}^{\text{red}}$ be the weights and biases of the compressed network produced by Algorithm 1. The feedforward function of the original network $(\mathbf{W}, \mathbf{b}, \boldsymbol{\rho})$ coincides with that of the compressed network $(\mathbf{W}^{\text{red}}, \mathbf{b}^{\text{red}}, \boldsymbol{\rho}^{\text{red}})$.*

Algorithm 1: QR Model Compression (`QR-compress`)

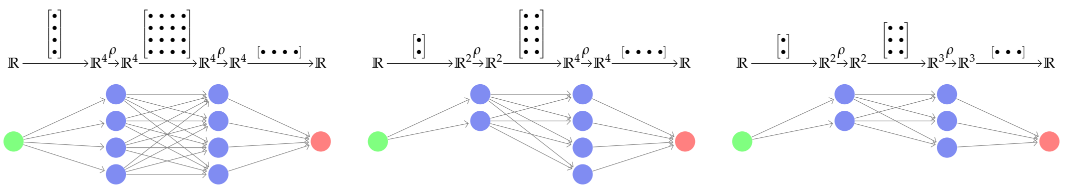
```
input   : $\mathbf{W}, \mathbf{b} \in \mathsf{Param}(\mathbf{n})$
output  : $\mathbf{Q} \in O(\mathbf{n}^{\text{hid}})$ and $\mathbf{W}^{\text{red}}, \mathbf{b}^{\text{red}} \in \mathsf{Param}(\mathbf{n}^{\text{red}})$

$\mathbf{Q}, \mathbf{W}^{\text{red}}, \mathbf{b}^{\text{red}} \leftarrow [\,], [\,], [\,]$        // initialize output lists
$A_1 \leftarrow [b_1 \quad W_1]$        // matrix of size $n_1 \times (n_0 + 1)$
for $i \leftarrow 1$ to $L - 1$ do        // iterate through layers
    $Q_i, R_i \leftarrow$ QR-decomp($A_i$ , mode = 'complete')        // $A_i = Q_i \mathrm{Inc}_i R_i$
    Append $Q_i$ to $\mathbf{Q}$
    Append first column of $R_i$ to $\mathbf{b}^{\text{red}}$        // reduced bias for layer $i$
    Append remainder of $R_i$ to $\mathbf{W}^{\text{red}}$        // reduced weights for layer $i$
    Set $A_{i+1} \leftarrow [b_{i+1} \quad W_{i+1} Q_i \mathrm{Inc}_i]$        // matrix of size $n_{i+1} \times (n_i^{\text{red}} + 1)$
end
Append the first column of $A_L$ to $\mathbf{b}^{\text{red}}$        // reduced bias for last layer
Append the remainder of $A_L$ to $\mathbf{W}^{\text{red}}$        // reduced weights for last layer
return $\mathbf{Q}$, $\mathbf{W}^{\text{red}}$, $\mathbf{b}^{\text{red}}$
```

We explain the notation of the algorithm. The inclusion matrix $\mathrm{Inc}_i \in \mathbb{R}^{n_i \times n_i^{\text{red}}}$ has ones along the main diagonal and zeros elsewhere. The method `QR-decomp` with `mode = 'complete'` computes the complete QR decomposition of the $n_i \times (1 + n_{i-1}^{\text{red}})$ matrix A_i as $Q_i \mathrm{Inc}_i R_i$ where $Q_i \in O(n_i)$ and R_i is upper-triangular of size $n_i^{\text{red}} \times (1 + n_{i-1}^{\text{red}})$. The definition of n_i^{red} implies that either $n_i^{\text{red}} = n_{i-1}^{\text{red}} + 1$ or $n_i^{\text{red}} = n_i$. The matrix R_i is of size $n_i^{\text{red}} \times n_i^{\text{red}}$ in the former case and of size $n_i \times (1 + n_{i-1}^{\text{red}})$ in the latter case.

Example 6. Suppose the widths of a radial neural network are $(1, 8, 16, 8, 1)$. Then it has $\sum_{i=1}^{4} (n_{i-1} + 1) n_i = 305$ trainable parameters. The reduced network has widths $(1, 2, 3, 4, 1)$ and $\sum_{i=1}^{4} (n_{i-1}^{\text{red}} + 1)(n_i^{\text{red}}) = 34$ trainable parameters. Another example appears in Figure 4.

We note that the tuple of matrices $\mathbf{Q}$ produced by Algorithm 1 does not feature in the statement of Theorem 5, but is important in the proof (which appears in Appendix C).



$\mathbb{R} \to \mathbb{R}^4 \xrightarrow{\rho} \mathbb{R}^4 \to \mathbb{R}^4 \xrightarrow{\rho} \mathbb{R}^4 \to \mathbb{R}$
$\mathbb{R} \to \mathbb{R}^2 \xrightarrow{\rho} \mathbb{R}^2 \to \mathbb{R}^4 \xrightarrow{\rho} \mathbb{R}^4 \to \mathbb{R}$
$\mathbb{R} \to \mathbb{R}^2 \xrightarrow{\rho} \mathbb{R}^2 \to \mathbb{R}^3 \xrightarrow{\rho} \mathbb{R}^3 \to \mathbb{R}$

Figure 4: Model compression in 3 steps. Layer widths can be iteratively reduced to 1 greater than the previous. The number of trainable parameters reduces from 33 to 17.

Namely, an induction argument shows that the i-th partial feedforward function of the original and reduced models are related via the matrices Q_i and Inc_i. A crucial ingredient in the proof is that radial rescaling activations commute with orthogonal transformations.

6 Projected gradient descent

The typical use case for model compression algorithms is to produce a smaller version of the fully trained model which can be deployed to make inference more efficient. It is also worth considering whether compression can be used to accelerate training. For example, for some compression algorithms, the compressed and full models have the same feedforward function after a step of gradient descent is applied to each, and so one can compress before training and still reach the same minimum. Unfortunately, in the context of radial neural networks, compression using Algorithm 1 and then training does not necessarily give the same result as training and then compression (see Appendix D.6 for a counterexample). However, `QR-compress` does lead to a precise mathematical relationship between optimization of the two models: the loss of the compressed model after one step of gradient descent is equivalent to the loss of (a transformed version of) the original model after one step of projected gradient descent. Proofs appear in Appendix D.

To state our results, fix a tuple of widths $\mathbf{n}$ and a tuple $\boldsymbol{\rho} = (\rho_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i})_{i=1}^{L}$ of radial rescaling functions. The loss function $\mathcal{L} : \mathsf{Param}(\mathbf{n}) \to \mathbb{R}$ associated to a batch of training data $\{(x_j, y_j)\} \subseteq \mathbb{R}^{n_0} \times \mathbb{R}^{n_L}$ is defined as taking parameter values $(\mathbf{W}, \mathbf{b})$ to the sum $\sum_j \mathcal{C}(F(x_j), y_j)$ where $\mathcal{C} : \mathbb{R}^{n_L} \times \mathbb{R}^{n_L} \to \mathbb{R}$ is a cost function on the output space, and $F = F_{(\mathbf{W},\mathbf{b},\boldsymbol{\rho})}$ is the feedforward of the radial neural network with parameters $(\mathbf{W}, \mathbf{b})$ and activations $\boldsymbol{\rho}$. Similarly, we have a loss function $\mathcal{L}_{\mathrm{red}}$ on the parameter space $\mathsf{Param}(\mathbf{n}^{\mathrm{red}})$ with reduced widths vector. For any learning rate $\eta > 0$, we obtain gradient descent maps:

$$\begin{aligned} \gamma : \mathsf{Param}(\mathbf{n}) &\to \mathsf{Param}(\mathbf{n}) & \gamma_{\mathrm{red}} : \mathsf{Param}(\mathbf{n}^{\mathrm{red}}) &\to \mathsf{Param}(\mathbf{n}^{\mathrm{red}}) \\ (\mathbf{W}, \mathbf{b}) &\mapsto (\mathbf{W}, \mathbf{b}) - \eta \nabla_{(\mathbf{W},\mathbf{b})} \mathcal{L} & (\mathbf{V}, \mathbf{c}) &\mapsto (\mathbf{V}, \mathbf{c}) - \eta \nabla_{(\mathbf{V},\mathbf{c})} \mathcal{L}_{\mathrm{red}} \end{aligned}$$

We will also consider, for $k \geq 0$, the k-fold composition $\gamma^k = \gamma \circ \gamma \circ \cdots \circ \gamma$ and similarly for γ_{red}. The *projected gradient descent* map on $\mathsf{Param}(\mathbf{n})$ is given by:

$$\gamma_{\mathrm{proj}} : \mathsf{Param}(\mathbf{n}) \to \mathsf{Param}(\mathbf{n}), \qquad (\mathbf{W}, \mathbf{b}) \mapsto \mathrm{Proj}\left(\gamma(\mathbf{W}, \mathbf{b})\right)$$

where the map Proj zeroes out all entries in the bottom left $(n_i - n_i^{\mathrm{red}}) \times n_{i-1}^{\mathrm{red}}$ submatrix of $W_i - \nabla_{W_i}\mathcal{L}$, and the bottom $(n_i - n_i^{\mathrm{red}})$ entries in $b_i - \nabla_{b_i}\mathcal{L}$, for each i. Schematically:

$$W_i - \nabla_{W_i}\mathcal{L} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \mapsto \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}, \qquad b_i - \nabla_{b_i}\mathcal{L} = \begin{bmatrix} * \\ * \end{bmatrix} \mapsto \begin{bmatrix} * \\ 0 \end{bmatrix}$$

To state the following theorem, let $\mathbf{W}^{\mathrm{red}}, \mathbf{b}^{\mathrm{red}}, \mathbf{Q} = \texttt{QR-compress}(\mathbf{W}, \mathbf{b})$ be the outputs of Algorithm 1 applied to $(\mathbf{W}, \mathbf{b}) \in \mathsf{Param}(\mathbf{n})$. Hence $(\mathbf{W}^{\mathrm{red}}, \mathbf{b}^{\mathrm{red}}) \in \mathsf{Param}(\mathbf{n}^{\mathrm{red}})$ are the parameters of the compressed model, and $\mathbf{Q} \in O(\mathbf{n}^{\mathrm{hid}})$ is an orthogonal parameter symmetry. We also consider the action (Section 5.1) of $\mathbf{Q}^{-1}$ applied to $(\mathbf{W}, \mathbf{b})$.

Theorem 7. *Let $\mathbf{W}^{\mathrm{red}}, \mathbf{b}^{\mathrm{red}}, \mathbf{Q} = \texttt{QR-compress}(\mathbf{W}, \mathbf{b})$ be the outputs of Algorithm 1 applied to $(\mathbf{W}, \mathbf{b}) \in \mathsf{Param}(\mathbf{n})$. Set $\mathbf{U} = \mathbf{Q}^{-1} \cdot (\mathbf{W}, \mathbf{b}) - (\mathbf{W}^{\mathrm{red}}, \mathbf{b}^{\mathrm{red}})$. For any $k \geq 0$, we have:*

$$\gamma^k(\mathbf{W}, \mathbf{b}) = \mathbf{Q} \cdot \gamma^k(\mathbf{Q}^{-1} \cdot (\mathbf{W}, \mathbf{b})) \qquad \gamma_{\mathrm{proj}}^k(\mathbf{Q}^{-1} \cdot (\mathbf{W}, \mathbf{b})) = \gamma_{\mathrm{red}}^k(\mathbf{W}^{\mathrm{red}}, \mathbf{b}^{\mathrm{red}}) + \mathbf{U}.$$

We conclude that gradient descent with initial values $(\mathbf{W}, \mathbf{b})$ is equivalent to gradient descent with initial values $\mathbf{Q}^{-1} \cdot (\mathbf{W}, \mathbf{b})$ since at any stage we can apply $\mathbf{Q}^{\pm 1}$ to move from one to the other. Furthermore, projected gradient descent with initial values $\mathbf{Q}^{-1} \cdot (\mathbf{W}, \mathbf{b})$ is equivalent to gradient descent on $\mathsf{Param}(\mathbf{n}^{\mathrm{red}})$ with initial values $(\mathbf{W}^{\mathrm{red}}, \mathbf{b}^{\mathrm{red}})$ since at any stage we can move from one to the other by $\pm\mathbf{U}$. Neither $\mathbf{Q}$ nor $\mathbf{U}$ depends on k.

7 Experiments

In addition to the theoretical results in this work, we provide an implementation of Algorithm 1, in order to validate the claims of Theorems 5 and 7 empirically, as well as to quantify real-world performance. Full experimental details are in Appendix E.

(1) Empirical verification of Theorem 5. We learn the function $f(x) = e^{-x^2}$ from samples using a radial neural network with widths $\mathbf{n} = (1, 6, 7, 1)$ and activation the radial shifted sigmoid $h(x) = 1/(1 + e^{-x+s})$. Applying `QR-compress` gives a compressed radial neural network with widths $\mathbf{n}^{\text{red}} = (1, 2, 3, 1)$. Theorem 5 implies that the respective neural functions F and F_{red} are equal. Over 10 random initializations, the mean absolute error is negligible up to machine precision: $(1/N) \sum_j |F(x_j) - F_{\text{red}}(x_j)| = 1.31 \cdot 10^{-8} \pm 4.45 \cdot 10^{-9}$.

(2) Empirical verification of Theorem 7. The claim is that training the transformed model with parameters $\mathbf{Q}^{-1} \cdot (\mathbf{W}, \mathbf{b})$ and objective $\mathcal{L}$ by projected gradient descent coincides with training the reduced model with parameters $(\mathbf{W}^{\text{red}}, \mathbf{b}^{\text{red}})$ and objective $\mathcal{L}_{\text{red}}$ by usual gradient descent. We verified this on synthetic data as above. Over 10 random initializations, the loss functions after training match: $|\mathcal{L} - \mathcal{L}_{\text{red}}| = 4.02 \cdot 10^{-9} \pm 7.01 \cdot 10^{-9}$.

(3) The compressed model trains faster. Our compression method may be applied before training to produce a smaller model class which *trains* faster without sacrificing accuracy. We demonstrate this in learning the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ sending (t_1, t_2) to $(e^{-t_1^2}, e^{-t_2^2})$ using a radial neural network with widths $\mathbf{n} = (2, 16, 64, 128, 16, 2)$ and activation the radial sigmoid $h(r) = 1/(1 + e^{-r})$. Applying `QR-compress` gives a compressed network with widths $\mathbf{n}^{\text{red}} = (2, 3, 4, 5, 6, 2)$. We trained both models until the training loss was ≤ 0.01. Over 10 random initializations on our system, the reduced network trained in 15.32 ± 2.53 seconds and the original network trained in 31.24 ± 4.55 seconds.

8 Conclusions and Discussion

This paper demonstrates that radial neural networks are universal approximators and that their parameter spaces exhibit a rich symmetry group, leading to a model compression algorithm. The results of this work combine to build a theoretical foundation for the use of radial neural networks, and suggest that radial neural networks hold promise for wider practical applicability. Furthermore, this work makes an argument for considering the advantages of non-pointwise nonlinearities in neural networks.

There are two main limitations of our results, each providing an opportunity for future work. First, our universal approximation constructions currently work only for Step-ReLU radial rescaling radial activations; it would be desirable to generalize to other activations. Additionally, Theorem 5 achieves compression only for networks whose widths satisfy $n_i > n_{i-1} + 1$ for some i. Neural networks which do not have increasing widths anywhere in their architecture, such as encoders, would not be compressible.

Further extensions of this work include: First, little is currently known about the stability properties of radial neural networks during training, as well as their sensitivity to initialization. Second, radial rescaling activations provide an extreme case of symmetry; there may be benefits to combining radial and pointwise activations within a single network, for example, through 'block' radial rescaling functions. Third, the parameter space symmetries may provide a key ingredient in analyzing the gradient flow dynamics of radial neural networks and computation of conserved quantities. Fourth, radial rescaling activations can be used within convolutional or group-equivariant NNs. Finally, based on the theoretical advantages laid out in this paper, future work will explore empirically applications in which we expect radial networks to outperform alternate methods. Such potential applications include data spaces with circular or distance-based class boundaries.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See Section 8.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] Our work is theoretical and does not hold specific risks of negative impacts.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
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3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]
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4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
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   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
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