Fast Approximate Dynamic Programming for Infinite-Horizon Markov Decision Processes

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Abstract

In this study, we consider the infinite-horizon, discounted cost, optimal control of stochastic nonlinear systems with separable cost and constraints in the state and input variables. Using the linear-time Legendre transform, we propose a novel numerical scheme for implementation of the corresponding value iteration (VI) algorithm in the conjugate domain. Detailed analyses of the convergence, time complexity, and error of the proposed algorithm are provided. In particular, with a discretization of size X and U for the state and input spaces, respectively, the proposed approach reduces the time complexity of each iteration in the VI algorithm from $\mathcal{O}(XU)$ to $\mathcal{O}(X+U)$, by replacing the minimization operation in the primal domain with a simple addition in the conjugate domain.

1 Introduction

Value iteration (VI) is one of the most basic and wide-spread algorithms employed for tackling problems in reinforcement learning and optimal control [8, 24] formulated as Markov decision processes (MDPs). The VI algorithm simply involves the consecutive applications of the dynamic programming (DP) operator $\mathcal{T}J(x_t) = \min_{u_t} \left\{C(x_t, u_t) + \gamma \mathbb{E} J(x_{t+1})\right\}$, where $C(x_t, u_t)$ is the cost of taking the control action u_t at the state x_t. This fixed point iteration is known to converge to the optimal value function for discount factors $\gamma \in (0, 1)$. However, this algorithm suffers from a high computational cost for large scale finite state spaces. For problems with a continuous state space, the DP operation becomes an infinite-dimensional optimization problem, rendering the exact implementation of VI impossible in most cases. A common approach is to incorporate function approximation techniques and and compute the output of the DP operator for a finite sample (i.e., a discretization) of the underlying continuous state space. This approximation again suffers from a high computational cost for fine discretizations of the state space, particularly in high-dimensional problems. We refer the reader to [8, 22] for various approximation schemes for VI.

For some problems, however, it is possible to partially address this issue by using duality theory, i.e., approaching the minimization problem in the conjugate domain. In particular, as we will see in Section 3, the minimization in the primal domain in the DP operator can be transformed to a simple addition in the dual domain, at the expense of three conjugate transforms. However, proper application of this transformation relies on efficient numerical algorithms for conjugation. Fortunately, such an algorithm, known as linear-time Legendre transform (LLT) has been developed in late 90s [19]. Other than the classical application of LLT (and other fast algorithms for conjugate transform) in solving Hamilton-Jacobi equation [1, 12, 13], these algorithms are used in image processing [20], thermodynamics [11], and optimal transport [16].

The application of conjugate duality for the DP problem is not new and actually goes back to Bellman [4]. Further applications of this idea for reducing the computational complexity were later explored in [14, 17]. However, surprisingly, the application of LLT for solving discrete-time optimal

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control problems, has been limited. In particular, in [10], the authors propose the "fast value iteration" algorithm (without a rigorous analysis of the complexity and error of the proposed algorithm) for a particular class of infinite-horizon problems with state-independent stage cost $C(x,u) = C(u)$ and deterministic linear dynamics $x_{t+1} = Ax_t + Bu_t$, where A is a non-negative, monotone, invertible matrix. More recently, in [18], the authors also considered the application of LLT for solving the DP operation in finite-horizon, optimal control of input-affine dynamics. In particular, they introduced the "discrete conjugate DP" (d-CDP) operator, and provided a detailed analysis of its complexity and error. As we will discuss shortly, the current study is an extension of the corresponding d-CDP algorithm that, among other things, considers infinite horizon, discounted cost problems. We note that the algorithms developed in [15, 20] for distance transform can also potentially tackle the optimal control problems similar to the ones of interest in the current study. In particular, these algorithms require the stage cost to be reformulated as a convex function of the "distance" between the current and next states. While this property might arise naturally, it can generally be restrictive, as it is in the problem class considered in this study. Another line of work that is closely related to ours invloves utilizing max-plus algebra in solving deterministic, continuous-state, continuous-time, optimal control problems; see, e.g., [2, 21]. These works exploit the compatibility of the DP operation with max-plus operations, and approximate the value function as a max-plus linear combination. Recently, in [3, 5], the authors used this idea to propose an approximate VI algorithm for continuous-state, deterministic MDPs. In this regard, we note that the proposed approach in the current study also involves max-plus linear approximation of the value function as the maximum of affine functions. The key difference is however that by choosing a grid-like (factorized) set of slopes for the linear terms (the basis of the max-plus linear combination), we take advantage of linear time complexity of LLT in computing the constant terms (the coefficients of the max-plus linear combination).

Main contribution. We focus on an approximate implementation of VI involving discretization of the state-input space for solving the optimal control problem of discrete-time systems with continuous state-input space. Building upon the earlier work [18], we employ conjugate duality to speed-up VI for problems with separable cost (in state and input) and input-affine dynamics. We propose the conjugate VI (CVI) algorithm based on a modified version of the d-CDP operator introduced in [18], and extend the existing results for *infinite-horizon, discounted cost* problems, while taking into account *stochastic dynamics* and *numerical approximation of the conjugate of input cost* in our analysis. In particular, **(i)** we provide sufficient conditions for the convergence of CVI (Theorem 3.9); **(ii)** we show that CVI can achieve a linear time complexity of $\mathcal{O}(X+U)$ in each iteration (Theorem 3.10), where X (U) is the cardinality of the discrete state (input) space; **(iii)** we analyze the error of CVI (Theorem 3.11), and use that result to provide specific guidelines on the construction of the discrete dual domain (Section 3.4); **(iv)** we provide a MATLAB package for the implementation of the proposed algorithm.

Notations. The standard inner product in $\mathbb{R}^n$ and the corresponding induced 2-norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_2$, respectively. $\|\cdot\|_\infty$ denotes the infinity norm. We use the superscript d to denote *finite (discrete)* sets (as in $\mathbb{X}^{\mathrm{d}}$) and *discrete* functions (as in $h^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$). We use the superscript g to denote *grid-like* finite sets (as in $\mathbb{X}^{\mathrm{g}} = \Pi_{i=1}^n \mathbb{X}_i^{\mathrm{g}}$ where $\mathbb{X}_i^{\mathrm{g}} \subset \mathbb{R}$). We also use $\mathbb{X}^{\mathrm{g}}_{\mathrm{sub}}$ to denote the *sub-grid* of $\mathbb{X}^{\mathrm{g}}$ derived by omitting the smallest and the largest elements of $\mathbb{X}^{\mathrm{g}}$ in each dimension. The cardinality of the finite set $\mathbb{X}^{\mathrm{d}}$ or $\mathbb{X}^{\mathrm{g}}$ is denoted by X. We use $\widetilde{h^{\mathrm{d}}} : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ to denote a *generic extension* of a discrete function h^{d}, and $\overline{h^{\mathrm{d}}} : \mathbb{R}^n \to \overline{\mathbb{R}}$ to denote *multilinear interpolation and extrapolation (LERP)* extension of a discrete function with a *grid-like* domain. Let $\mathbb{X}, \mathbb{Y}$ be two arbitrary sets in $\mathbb{R}^n$. $\mathrm{co}(\mathbb{X})$ is the convex hull of $\mathbb{X}$. We use $\mathrm{d}(\mathbb{X}, \mathbb{Y}) \coloneqq \inf_{x \in \mathbb{X}, y \in \mathbb{Y}} \|x - y\|_2$ to denote the distance between $\mathbb{X}$ and $\mathbb{Y}$. The one-sided Hausdorff distance *from* $\mathbb{X}$ *to* $\mathbb{Y}$ is defined as $\mathrm{d_H}(\mathbb{X}, \mathbb{Y}) \coloneqq \sup_{x \in \mathbb{X}} \inf_{y \in \mathbb{Y}} \|x - y\|_2$. Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended real-valued function with a non-empty effective domain $\mathrm{dom}(h) = \mathbb{X} \coloneqq \{x \in \mathbb{R}^n : h(x) < \infty\}$. The range of h is denoted by $\mathrm{Rng}(h) \coloneqq \max_{x \in \mathbb{X}} h(x) - \min_{x \in \mathbb{X}} h(x)$, and the subdifferential of h at a point $x \in \mathbb{X}$ is defined as $\partial h(x) \coloneqq \left\{ y \in \mathbb{R}^n : h(\tilde{x}) \geq h(x) + \langle y, \tilde{x} - x \rangle, \forall \tilde{x} \in \mathbb{X} \right\}$. We define $\mathbb{L}(h) \coloneqq \Pi_{i=1}^n \left[\mathrm{L}_i^-(h), \mathrm{L}_i^+(h) \right]$, where $\mathrm{L}_i^+(h)$ (resp. $\mathrm{L}_i^-(h)$) is the maximum (resp. minimum) slope of the function h along the i-th dimension. Note that $\partial h(x) \subseteq \mathbb{L}(h)$ for all $x \in \mathbb{X}$. We report the complexities using the standard big O notations $\mathcal{O}$ and $\widetilde{\mathcal{O}}$, where the latter hides the logarithmic factors. In this study, we are mainly concerned with the dependence of the computational complexities on *the size of the finite sets* involved (discretization of the primal and dual domains). In particular, we ignore the possible dependence of the computational complexities on the dimension of the variables, unless they appear in the power of the size of those discrete sets.

We note that the extended version of this article, including the technical proofs, is available in the supplementary material.

2 VI in primal domain

We are concerned with the infinite-horizon, discounted cost, optimal control problems of the form

$$J_\star(x) = \min\ \mathbb{E}_{w_t}\left[\sum_{t=0}^{\infty}\gamma^t C(x_t,u_t)\middle| x_0 = x\right]$$
$$\text{s.t.}\quad x_{t+1} = g(x_t,u_t,w_t),\ x_t\in\mathbb{X},\ u_t\in\mathbb{U},\ w_t\sim\mathbb{P}(\mathbb{W}),\quad \forall t\in\{0,1,\ldots\},$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, and $w_t \in \mathbb{R}^l$ are the state, input and disturbance variables at time t, respectively; $\gamma \in (0,1)$ is the discount factor; $C : \mathbb{X}\times\mathbb{U}\to\mathbb{R}$ is the stage cost; $g : \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^n$ describes the dynamics; $\mathbb{X}\subset\mathbb{R}^n$ and $\mathbb{U}\subset\mathbb{R}^m$ describe the state and input constraints, respectively; and, $\mathbb{P}(\cdot)$ is the distribution of the disturbance over the support $\mathbb{W}\subset\mathbb{R}^l$. Assuming the stage cost C is bounded, the optimal value function solves the Bellman equation $J_\star = \mathcal{T}J_\star$, where $\mathcal{T}$ is the DP operator (C and J are extended to infinity outside their effective domains) [6, Prop. 1.2.2]

$$\mathcal{T}J(x) \coloneqq \min_u \left\{C(x,u) + \gamma\cdot\mathbb{E}_w J\big(g(x,u,w)\big)\right\},\quad \forall x\in\mathbb{X}. \tag{1}$$

Indeed, $\mathcal{T}$ is γ-contractive in the infinity-norm [6, Prop. 1.2.4]. This property then gives rise to the VI algorithm $J_{k+1} = \mathcal{T}J_k$, which converges to $J_\star$ as $k\to\infty$, for arbitrary initialization J_0. Moreover, assuming that the composition $J\circ g$ (for each w) and the cost C are jointly convex in the state and input variables, $\mathcal{T}$ also preserves convexity [7, Prop. 3.3.1].

For numerical implementation of VI, we need to address three issues. First, we need to compute the expectation in (1). In order to simplify the exposition and include the computational cost of this operation explicitly, we consider disturbances with finite support in this study:

Assumption 2.1 (Disturbance with finite support)**.** *The disturbance w has a finite support $\mathbb{W}^{\mathrm{d}}\subset\mathbb{R}^l$ with a given probability mass function (p.m.f.) $p : \mathbb{W}^{\mathrm{d}}\to[0,1]$.*

Under the preceding assumption, we have $\mathbb{E}_w J\big(g(x,u,w)\big) = \sum_{w\in\mathbb{W}^{\mathrm{d}}} p(w)\cdot J\big(g(x,u,w)\big)$. The second and more important issue is that the optimization problem (1) is infinite-dimensional for the continuous state space $\mathbb{X}$. This renders the exact implementation of VI impossible, except for a few cases with available closed-form solutions. A common solution to this problem is to deploy a sample-based approach, accompanied by a function approximation scheme. To be precise, for a finite subset $\mathbb{X}^{\mathrm{d}}$ of $\mathbb{X}$, at each iteration $k\geq 0$, we take the discrete function $J_k^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}}\to\mathbb{R}$ as the input, and compute the discrete function $J_{k+1}^{\mathrm{d}} = \big[\mathcal{T}\widetilde{J_k^{\mathrm{d}}}\big]^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}}\to\mathbb{R}$, where $\widetilde{J_k^{\mathrm{d}}} : \mathbb{X}\to\mathbb{R}$ is an extension of J_k^{d}. Finally, for each $x\in\mathbb{X}^{\mathrm{d}}$, we have to solve the minimization problem in (1) over the control input. Here, again, a common approximation is enumeration over a discretization $\mathbb{U}^{\mathrm{d}}\subset\mathbb{U}$.

Incorporating these approximations, we end up with the approximate VI algorithm $J_{k+1}^{\mathrm{d}} = \mathcal{T}^{\mathrm{d}}J_k^{\mathrm{d}}$, characterized by the *discrete* DP (d-DP) operator

$$\mathcal{T}^{\mathrm{d}}J^{\mathrm{d}}(x) \coloneqq \min_{u\in\mathbb{U}^{\mathrm{d}}} \left\{C(x,u) + \gamma\cdot\textstyle\sum_{w\in\mathbb{W}^{\mathrm{d}}} p(w)\cdot\widetilde{J^{\mathrm{d}}}\big(g(x,u,w)\big)\right\},\quad \forall x\in\mathbb{X}^{\mathrm{d}}. \tag{2}$$

The convergence of approximate VI described above depends on the properties of the extension operation $\widetilde{[\cdot]}$. In particular, if the extension operation is non-expansive (in the infinity-norm), then $\mathcal{T}^{\mathrm{d}}$ is also γ-contractive. The error of this approximation also depends on the extension operation and its representative power. We refer the interested reader to [6, 9, 22] for detailed discussions on the convergence and error of different approximation schemes for VI.

The d-DP operator and the corresponding VI algorithm will be our benchmark for evaluating the performance of the alternative algorithm developed in this study. To this end, we finish this section with some remarks on the time complexity of the d-DP operation. Let the time complexity of a single evaluation of the extension operator $\widetilde{[\cdot]}$ in (2) be of $\mathcal{O}(E)$. Then, the time complexity of the d-DP operation (2) is of $\mathcal{O}\left(XUWE\right)$. In this regard, note that the scheme described above essentially involves approximating a continuous-state/action MDP with a finite-state/action MDP, and then applying the VI algorithm. This, in turn, implies the lower bound $\Omega(XU)$ for the time complexity

(corresponding to enumeration over $u \in \mathbb{U}^{\mathrm{d}}$ for each $x \in \mathbb{X}^{\mathrm{d}}$). This lower bound is also compatible with the best existing time complexities in the literature for VI for finite MDPs; see, e.g., [3, 23]. However, as we will see in the next section, for a particular class of problems, it is possible to exploit the structure of the underlying continuous system in order to achieve a better time complexity in the corresponding discretized problem.

3 Reducing complexity via conjugate duality

In this section, we present the class of problems that allows us to employ conjugate duality and propose an alternative path for solving the corresponding DP operator. We also present the numerical scheme for implementing the proposed alternative path, and analyze its convergence, complexity, and error. We note that the proposed algorithm and its analysis are based on the d-CDP algorithm presented in [18, Sec. 5] for finite-horizon, optimal control of deterministic systems. Here, we extend those results for infinite-horizon, discounted cost, optimal control of stochastic systems. Moreover, unlike [18], our analysis includes the case where the conjugate of input cost is not analytically available and has to be computed numerically; see [18, Assump. 5.1] for more details.

3.1 VI in conjugate domain

Throughout this section, we assume that the problem data satisfy the following conditions.

Assumption 3.1 (Problem class)**.** *The problem data has the following properties:* ***(i)*** *The* disturbance *is additive; that is, $g(x,u,w) = f(x,u) + w$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ describes the deterministic dynamics.* ***(ii)*** *The* deterministic dynamics *is of the form $f(x,u) = f_{\mathrm{s}}(x) + Bu$, where $f_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz continuous, possibly nonlinear map describing the "state" dynamics and $B \in \mathbb{R}^{n\times m}$.* ***(iii)*** *The* constraint *sets $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ are compact and convex. Moreover, for each $x \in \mathbb{X}$, the set of admissible inputs $\mathbb{U}(x) \coloneqq \{u \in \mathbb{U} : g(x,u,w) \in \mathbb{X},\ \forall w \in \mathbb{W}^{\mathrm{d}}\}$ is nonempty.* ***(iv)*** *The* stage cost *C is separable in state and input; that is, $C(x,u) = C_{\mathrm{s}}(x) + C_{\mathrm{i}}(u)$, where the state cost $C_{\mathrm{s}} : \mathbb{X} \to \mathbb{R}$ and the input cost $C_{\mathrm{i}} : \mathbb{U} \to \mathbb{R}$ are Lipschitz continuous and convex.*

Some remarks are in order regarding the preceding assumptions. We first note that the setting of Assumption 3.1 goes beyond the classical LQR. In particular, it includes nonlinear dynamics, state and input constraints, and non-quadratic stage costs. Second, the properties laid out in Assumption 3.1 imply that the set of admissible inputs $\mathbb{U}(x)$ is a compact set for each $x \in \mathbb{X}$. This, in turn, implies that the optimal value in (1) is achieved if $J : \mathbb{X} \to \mathbb{R}$ is also assumed to be lower semi-continuous.

For the problem class of Assumption (3.1), we can use duality theory to present an alternative path for computing the output of the DP operator. This path forms the basis for the proposed algorithm. To this end, let us fix $x \in \mathbb{X}$ and consider the following reformulation of the optimization problem (1)

$$\mathcal{T}J(x) = C_{\mathrm{s}}(x) + \min_{u,z} \left\{ C_{\mathrm{i}}(u) + \gamma \cdot \mathbb{E}_w J(z+w) : z = f(x,u) \right\},$$

where we used additivity of disturbance and separability of stage cost in Assumptions 3.1-(i,iv). The corresponding dual problem then reads as

$$\widehat{\mathcal{T}}J(x) \coloneqq C_{\mathrm{s}}(x) + \max_{y} \min_{u,z} \left\{ C_{\mathrm{i}}(u) + \gamma \cdot \mathbb{E}_w J(z+w) + \langle y, f(x,u) - z \rangle \right\}, \tag{3}$$

where $y \in \mathbb{R}^n$ is the dual variable corresponding to the equality constraint. For the dynamics of Assumption 3.1-(ii), we can obtain the following representation for the dual problem.

Proposition 3.2 (CDP operator)**.** *The dual problem* (3) *equivalently reads as*

$$\epsilon(x) \coloneqq \gamma \cdot \mathbb{E}_w J(x+w), \quad x \in \mathbb{X}, \tag{4a}$$

$$\phi(y) \coloneqq C_{\mathrm{i}}^*(-B^\top y) + \epsilon^*(y), \quad y \in \mathbb{R}^n, \tag{4b}$$

$$\widehat{\mathcal{T}}J(x) = C_{\mathrm{s}}(x) + \phi^*\big(f_{\mathrm{s}}(x)\big), \quad x \in \mathbb{X}, \tag{4c}$$

where $h^*(y) = \sup_x \{\langle y, x\rangle - h(x)\}$ is the (convex) conjugate of h.

Following [18], we call the operator $\widehat{\mathcal{T}}$ in (4) the *conjugate* DP (CDP) operator. We next provide an alternative representation of the CDP operator that captures the essence of this operation.

Proposition 3.3 (CDP reformulation). *The CDP operator $\widehat{\mathcal{T}}$ equivalently reads as*

$$\widehat{\mathcal{T}}J(x) = C_{\mathrm{s}}(x) + \min_{u} \left\{ C_{\mathrm{i}}(u) + \gamma \cdot [\mathbb{E}_w J(\cdot + w)]^{**}\big(f(x,u)\big) \right\}, \tag{5}$$

where $[\cdot]^{**} = [[\cdot]^*]^*$ denotes the biconjugate operation.

The preceding result implies that the indirect path through the conjugate domain essentially involves substituting the (expectation of the) value function by its biconjugate. In particular, it points to a sufficient condition for zero duality gap.

Corollary 3.4 (Equivalence of $\mathcal{T}$ and $\widehat{\mathcal{T}}$). *If $J : \mathbb{X} \to \mathbb{R}$ is convex, then $\widehat{\mathcal{T}}J = \mathcal{T}J$.*

Hence, $\widehat{\mathcal{T}}$ has the same properties as $\mathcal{T}$ if J is convex. In particular, if $\widehat{\mathcal{T}}$ preserves convexity, then the *conjugate* VI (CVI) algorithm $J_{k+1} = \widehat{\mathcal{T}}J_k$ converges to the optimal value function $J_\star$, with arbitrary convex initialization J_0. For $\widehat{\mathcal{T}}$ to preserve convexity, a sufficient condition is the convexity of $J \circ f$ (jointly in x and u), given that J is convex (e.g., for linear dynamics $f(x,u) = Ax + Bu$). We note that, if $\widehat{\mathcal{T}}$ does not preserve convexity, then the alternative path suffers from duality gap.

3.2 CVI algorithm

The approximate CVI algorithm involves consecutive applications of an approximate implementation of the CDP operator (4) until some termination condition is satisfied. Algorithm 1 provides the pseudo-code of this procedure. In particular, we consider solving (4) for a finite set $\mathbb{X}^{\mathrm{d}} \subset \mathbb{X}$, and terminate the iterations when the difference between two consecutive discrete value functions (in the infinity-norm) is less than a given constant $e_{\mathrm{t}} > 0$. Since we are we are working with a finite subset of the state space, we can restrict the feasibility condition of Assumption 3.1-(iii) to all $x \in \mathbb{X}^{\mathrm{d}}$:

Assumption 3.5 (Feasibile discretization). *We have $\mathbb{U}(x) \neq \emptyset$ for all $x \in \mathbb{X}^{\mathrm{d}}$.*

In what follows, we describe the main steps within the initialization and iterations of Algorithm 1. In particular, the conjugate operations in (4) are handled numerically via the linear-time Legendre transform (LLT) algorithm [19]. LLT is an efficient algorithm for computing the *discrete* conjugate function over a finite *grid-like* dual domain. Precisely, to compute the conjugate of the function $h : \mathbb{X} \to \mathbb{R}$, LLT takes its discretization $h^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$ as an input, and outputs $h^{\mathrm{d}*\mathrm{d}} : \mathbb{Y}^{\mathrm{g}} \to \mathbb{R}$, for the grid-like dual domain $\mathbb{Y}^{\mathrm{g}}$. We refer the reader to [19] for a detailed description of LLT.

The main steps of the proposed approximate implementation of the CDP operator (6) are as follows: **(i)** For the expectation operation in (4a), by Assumption 2.1, we again have $\mathbb{E}_w J(\cdot + w) = \sum_{w \in \mathbb{W}^{\mathrm{d}}} p(w) \cdot J(\cdot + w)$. Hence, we need to pass the value function $J^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$ through the "scaled expection filter" to obtain $\varepsilon^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \overline{\mathbb{R}}$ in (6a) as an approximation of ϵ in (4a). Notice that here we are using an extension $\widetilde{J^{\mathrm{d}}} : \mathbb{X} \to \mathbb{R}$ of J^{d} (recall that we only have access to the discrete value function J^{d}). **(ii)** In order to compute ϕ in (4b), we need access to two conjugate functions. First, for ϵ^*, we use the approximation $\varepsilon^{\mathrm{d}*\mathrm{d}} : \mathbb{Y}^{\mathrm{g}} \to \mathbb{R}$ in (6b), by applying LLT to the data points $\varepsilon^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \overline{\mathbb{R}}$ for a properly chosen state dual grid $\mathbb{Y}^{\mathrm{g}} \subset \mathbb{R}^n$. We also need the conjugate C_{i}^* of the input cost. If this function is not analytically available, we approximate it as follows: For a properly chosen input dual grid $\mathbb{V}^{\mathrm{g}} \subset \mathbb{R}^m$, we employ LLT to compute $C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}} : \mathbb{V}^{\mathrm{g}} \to \mathbb{R}$ in (6c), using the data points $C_{\mathrm{i}}^{\mathrm{d}} : \mathbb{U}^{\mathrm{d}} \to \mathbb{R}$, where $\mathbb{U}^{\mathrm{d}}$ is a finite subset of $\mathbb{U}$. With these conjugate functions at hand, we can now compute $\varphi^{\mathrm{d}} : \mathbb{Y}^{\mathrm{g}} \to \mathbb{R}$ in (6d), as an approximation of ϕ in (4b). In particular, notice that we use the LERP extension $\overline{C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}}}$ of $C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}}$ to approximate $C_{\mathrm{i}}^{\mathrm{d}*}$ at the required point $-B^\top y$ for each $y \in \mathbb{Y}^{\mathrm{g}}$. **(i)** To be able to compute the output according to (4c), we need to perform another conjugate transform. In particular, we need the value of ϕ^* at $f_{\mathrm{s}}(x)$ for $x \in \mathbb{X}^{\mathrm{d}}$. Here, we use the approximation $\varphi^{\mathrm{d}*\mathrm{d}} : \mathbb{Z}^{\mathrm{g}} \to \mathbb{R}$ in (6e), by applying LLT to the data points $\varphi^{\mathrm{d}} : \mathbb{Y}^{\mathrm{g}} \to \mathbb{R}$ for a properly chosen grid $\mathbb{Z}^{\mathrm{g}} \subset \mathbb{R}^n$. Finally, we use the LERP extension $\overline{\varphi^{\mathrm{d}*\mathrm{d}}}$ of $\varphi^{\mathrm{d}*\mathrm{d}}$ to approximate $\varphi^{\mathrm{d}*}$ at the required point $f_{\mathrm{s}}(x)$ for each $x \in \mathbb{X}^{\mathrm{d}}$, and compute $\widehat{\mathcal{T}}^{\mathrm{d}} J^{\mathrm{d}}$ in (6f) as an approximation of $\widehat{\mathcal{T}}J$ in (4c). With these

Algorithm 1 CVI: Approximate VI in conjugate domain

Input: dynamics $f_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$; finite state space $\mathbb{X}^{\mathrm{d}} \subset \mathbb{X}$; finite input space $\mathbb{U}^{\mathrm{d}} \subset \mathbb{U}$; state cost function $C_{\mathrm{s}}^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$; input cost function $C_{\mathrm{i}}^{\mathrm{d}} : \mathbb{U}^{\mathrm{d}} \to \mathbb{R}$; finite disturbance space $\mathbb{W}^{\mathrm{d}}$ and its p.m.f. $p : \mathbb{W}^{\mathrm{d}} \to [0,1]$; discount factor γ; termination bound e_{t}.

Output: discrete value function $\widehat{J}^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$.

initialization:

construct the grids $\mathbb{V}^{\mathrm{g}}$, $\mathbb{Z}^{\mathrm{g}}$, and $\mathbb{Y}^{\mathrm{g}}$;

use LLT to compute $C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}} : \mathbb{V}^{\mathrm{g}} \to \mathbb{R}$ from $C_{\mathrm{i}}^{\mathrm{d}} : \mathbb{U}^{\mathrm{d}} \to \mathbb{R}$;

$J^{\mathrm{d}}(x) \leftarrow 0$ and $J_+^{\mathrm{d}}(x) \leftarrow C_{\mathrm{s}}^{\mathrm{d}}(x) - \min C_{\mathrm{i}}^{\mathrm{d}}$ for $x \in \mathbb{X}^{\mathrm{d}}$;

iteration:

while $\left\| J_+^{\mathrm{d}} - J^{\mathrm{d}} \right\|_\infty \geq e_{\mathrm{t}}$ **do**

  $J^{\mathrm{d}} \leftarrow J_+^{\mathrm{d}}$;

  d-CDP operation:

  $\varepsilon^{\mathrm{d}}(x) \leftarrow \gamma \cdot \sum_{w\in\mathbb{W}^{\mathrm{d}}} p(w) \cdot \widetilde{J^{\mathrm{d}}}(x+w)$ for $x \in \mathbb{X}^{\mathrm{d}}$; use LLT to compute $\varepsilon^{\mathrm{d}*\mathrm{d}} : \mathbb{Y}^{\mathrm{g}} \to \overline{\mathbb{R}}$ from $\varepsilon^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$;

  for each $y \in \mathbb{Y}^{\mathrm{g}}$ **do**

    use LERP to compute $\overline{C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}}}(-B^\top y)$ from $C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}} : \mathbb{V}^{\mathrm{g}} \to \mathbb{R}$; $\varphi^{\mathrm{d}}(y) \leftarrow \overline{C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}}}(-B^\top y) + \varepsilon^{\mathrm{d}*\mathrm{d}}(y)$;

  end for

  use LLT to compute $\varphi^{\mathrm{d}*\mathrm{d}} : \mathbb{Z}^{\mathrm{g}} \to \mathbb{R}$ from $\varphi^{\mathrm{d}} : \mathbb{Y}^{\mathrm{g}} \to \mathbb{R}$;

  for each $x \in \mathbb{X}^{\mathrm{d}}$ **do**

    use LERP to compute $\overline{\varphi^{\mathrm{d}*\mathrm{d}}}\big(f_{\mathrm{s}}(x)\big)$ from $\varphi^{\mathrm{d}*\mathrm{d}} : \mathbb{Z}^{\mathrm{g}} \to \mathbb{R}$; $J_+^{\mathrm{d}}(x) \leftarrow C_{\mathrm{s}}(x) + \overline{\varphi^{\mathrm{d}*\mathrm{d}}}\big(f_{\mathrm{s}}(x)\big)$;

  end for

end while

$\widehat{J}^{\mathrm{d}} \leftarrow J_+^{\mathrm{d}}$.

approximations, we can introduce the *discrete* CDP (d-CDP) operator as follows

$$\varepsilon^{\mathrm{d}}(x) \coloneqq \gamma \cdot \sum_{w\in\mathbb{W}^{\mathrm{d}}} p(w) \cdot \widetilde{J^{\mathrm{d}}}(x+w), \quad x \in \mathbb{X}^{\mathrm{d}}, \tag{6a}$$

$$\varepsilon^{\mathrm{d}*\mathrm{d}}(y) = \max_{x\in\mathbb{X}^{\mathrm{d}}} \left\{ \langle x, y\rangle - \varepsilon^{\mathrm{d}}(x) \right\}, \quad y \in \mathbb{Y}^{\mathrm{g}}, \tag{6b}$$

$$C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}}(v) = \max_{u\in\mathbb{U}^{\mathrm{d}}} \left\{ \langle u, v\rangle - C_{\mathrm{i}}^{\mathrm{d}}(u) \right\}, \quad v \in \mathbb{V}^{\mathrm{g}}, \tag{6c}$$

$$\varphi^{\mathrm{d}}(y) \coloneqq \overline{C_{\mathrm{i}}^{\mathrm{d}*\mathrm{d}}}(-B^\top y) + \varepsilon^{\mathrm{d}*\mathrm{d}}(y), \quad y \in \mathbb{Y}^{\mathrm{g}}, \tag{6d}$$

$$\varphi^{\mathrm{d}*\mathrm{d}}(z) = \max_{y\in\mathbb{Y}^{\mathrm{g}}} \left\{ \langle y, z\rangle - \varphi^{\mathrm{d}}(y) \right\}, \quad z \in \mathbb{Z}^{\mathrm{g}}, \tag{6e}$$

$$\widehat{\mathcal{T}}^{\mathrm{d}} J^{\mathrm{d}}(x) \coloneqq C_{\mathrm{s}}(x) + \overline{\varphi^{\mathrm{d}*\mathrm{d}}}\big(f_{\mathrm{s}}(x)\big), \quad x \in \mathbb{X}^{\mathrm{d}}. \tag{6f}$$

We will discuss the proper construction of the grids $\mathbb{Y}^{\mathrm{g}}$, $\mathbb{V}^{\mathrm{g}}$, and $\mathbb{Z}^{\mathrm{g}}$ in Section 3.4.

3.3 Analysis of CVI algorithm

We now provide our main theoretical results concerning the convergence, complexity, and error of the proposed algorithm. Let us begin with presenting the assumptions to be called in this subsection.

Assumption 3.6 (Grids)**.** *Consider the following properties for the grids in Algorithm 1 (consult the Notations in Section 1): **(i)** The grid $\mathbb{V}^{\mathrm{g}}$ is constructed such that $\mathrm{co}(\mathbb{V}_{\mathrm{sub}}^{\mathrm{g}}) \supseteq \mathbb{L}(C_{\mathrm{i}}^{\mathrm{d}})$. **(ii)** The grid $\mathbb{Z}^{\mathrm{g}}$ is constructed such that $\mathrm{co}(\mathbb{Z}^{\mathrm{g}}) \supseteq f_{\mathrm{s}}\big(\mathbb{X}^{\mathrm{d}}\big)$. **(iii)** The construction of $\mathbb{Y}^{\mathrm{g}}$, $\mathbb{V}^{\mathrm{g}}$, and $\mathbb{Z}^{\mathrm{g}}$ requires at most $\mathcal{O}(X+U)$ operations. The cardinality of the grids $\mathbb{Y}^{\mathrm{g}}$ and $\mathbb{Z}^{\mathrm{g}}$ (resp. $\mathbb{V}^{\mathrm{g}}$) in each dimension is the same as that of $\mathbb{X}^{\mathrm{d}}$ (resp. $\mathbb{U}^{\mathrm{d}}$) in that dimension so that $Y, Z = X$ and $V = U$.*

Assumption 3.7 (Extension operator)**.** *Consider the following properties for the operator $\widetilde{[\cdot]}$ in (6a): **(i)** The the extension operator is non-expansive w.r.t. the infinity norm; **(i)** Given a function $J : \mathbb{X} \to \mathbb{R}$ and its discretization $J^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$, we have $\|J - \widetilde{J^{\mathrm{d}}}\|_\infty \leq e_{\mathrm{e}}$ for some constant $e_{\mathrm{e}} \geq 0$.*

Assumption 3.8 (Dynamics)**.** *Given a convex function $J : \mathbb{X} \to R$, the composition $J \circ f$ is jointly convex in the state and input variables, where f describes the deterministic dynamics.*

Our first result concerns the contractiveness of the d-CDP operator.

Theorem 3.9 (Convergence)**.** *Let Assumptions 3.6-(ii) and 3.7-(i) hold. Then, the d-CDP operator* (6) *is γ-contractive w.r.t. the infinity-norm.*

The preceding theorem implies that the approximate CVI Algorithm 1 is indeed convergent given that the required conditions are satisfied. In particular, for deterministic dynamics, $\mathrm{co}(\mathbb{Z}^{\mathrm{g}}) \supseteq f_{\mathrm{s}}(\mathbb{X}^{\mathrm{d}})$ is sufficient for Algorithm 1 to be convergent. We next consider the time complexity of our algorithm.

Theorem 3.10 (Complexity)**.** *Let Assumption 3.6-(iii) hold. Also assume that each evaluation of the extension operator $\widetilde{[\cdot]}$ in* (6a) *requires $\mathcal{O}(E)$ operations. Then, the time complexity of initialization and each iteration in Algorithm 1 are of $\mathcal{O}(X+U)$ and $\widetilde{\mathcal{O}}(XWE)$, receptively.*

The requirements of Assumption 3.6-(iii) will be discussed in Section 3.4. Recall that each iteration of VI (in primal domain) has a complexity of $\mathcal{O}(XUWE)$, where E denotes the complexity of the extension operation used in (2). This observation points to a basic characteristic of the proposed approach: CVI reduces the quadratic complexity of VI to a linear one by replacing the minimization operation in the primal domain with a simple addition in the conjugate domain. Hence, for problem class of Assumption 3.1, CVI is expected to lead to a reduction in the computational cost. We note that CVI, like VI and other approximation schemes that utilize discretization/abstraction of the continuous state and input spaces, still suffers from the so-called "curse of dimensionality." This is because the sizes X and U of the discretizations increase exponentially with the dimensions n and m of the corresponding space. However, for CVI, this exponential increase is of rate $\max\{m,n\}$, compared to the rate $m+n$ for VI.

Theorem 3.11 (Error)**.** *Let Assumptions 3.6-(i,ii), 3.7-(i), and 3.8 hold. Consider the true optimal value function $J_\star = \mathcal{T}J_\star : \mathbb{X} \to \mathbb{R}$ and its discretization $J_\star^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$, and let Assumption 3.7-(ii) hold for $J_\star$. Also, let $\widehat{J}^{\mathrm{d}} : \mathbb{X}^{\mathrm{d}} \to \mathbb{R}$ be the output of Algorithm 1 with grids $\mathbb{Y}^{\mathrm{g}}$, $\mathbb{V}^{\mathrm{g}}$, and $\mathbb{Z}^{\mathrm{g}}$. Then,*

$$\|\widehat{J}^{\mathrm{d}} - J_\star^{\mathrm{d}}\|_\infty \leq \frac{\gamma(e_{\mathrm{e}} + e_{\mathrm{t}}) + e_{\mathrm{d}}}{1-\gamma}, \tag{7}$$

where $e_{\mathrm{d}} = e_{\mathrm{u}} + e_{\mathrm{v}} + e_{\mathrm{x}} + e_{\mathrm{y}} + e_{\mathrm{z}}$, and

$$\begin{aligned} &e_{\mathrm{u}} = c_{\mathrm{u}} \cdot \mathrm{d_H}(\mathbb{U}, \mathbb{U}^{\mathrm{d}}),\ e_{\mathrm{v}} = c_{\mathrm{v}} \cdot \mathrm{d_H}\left(\mathrm{co}(\mathbb{V}^{\mathrm{g}}), \mathbb{V}^{\mathrm{g}}\right),\ e_{\mathrm{x}} = c_{\mathrm{x}} \cdot \mathrm{d_H}\left(\mathbb{X}, \mathbb{X}^{\mathrm{d}}\right), \\ &e_{\mathrm{y}} = c_{\mathrm{y}} \cdot \max_{x \in \mathbb{X}^{\mathrm{d}}} \mathrm{d}\left(\partial(V_\star - C_{\mathrm{s}})(x), \mathbb{Y}^{\mathrm{g}}\right),\ e_{\mathrm{z}} = c_{\mathrm{z}} \cdot \mathrm{d_H}\left(f_{\mathrm{s}}(\mathbb{X}^{\mathrm{d}}), \mathbb{Z}^{\mathrm{g}}\right), \end{aligned} \tag{8}$$

with constants $c_{\mathrm{u}}, c_{\mathrm{v}}, c_{\mathrm{x}}, c_{\mathrm{y}}, c_{\mathrm{z}} > 0$ depending on the problem data.

Let us first note that Assumptions 3.6-(i,ii) on the grids $\mathbb{V}^{\mathrm{g}}$ and $\mathbb{Z}^{\mathrm{g}}$ are required for bounding the error of approximate discrete conjugations using LERP in (6d) and (6f); for more details see the proof of Theorem 3.11 in Section 5 of the extended manuscript in the supplementary material. Moreover, Assumption 3.8 implies that the DP and CDP operators preserve convexity, and they both have the true optimal value function $J_\star$ as their fixed point (i.e., the duality gap is zero). Otherwise, the proposed scheme can suffer from large errors due to dualization. The remaining sources of error in the proposed approximate implementation of CVI are captured by the three error terms in (7): **(i)** e_{e} is due to the approximation of the value function using the extension operator $\widetilde{[\cdot]}$; **(ii)** e_{t} corresponds to the termination of the algorithm after a finite number of iterations; **(i)** e_{d} captures the error due to the discretization of the primal and dual state and input domains.

3.4 Construction of the grids

In this subsection, we provide specific guidelines for the construction of the grids $\mathbb{Y}^{\mathrm{g}}$, $\mathbb{V}^{\mathrm{g}}$ and $\mathbb{Z}^{\mathrm{g}}$. We note that these discrete sets must be *grid-like* since they form the dual grid for the three conjugate transforms that are handled using LLT. The presented guidelines aim to minimize the error terms in (8) while taking into account the properties laid out in Assumption 3.6. In particular, the schemes described below satisfy the requirements of Assumption 3.6-(iii).

Construction of $\mathbb{V}^{\mathrm{g}}$. Assumption 3.6-(i) and the error term e_{v} in (8) suggest that we find the smallest input dual grid $\mathbb{V}^{\mathrm{g}}$ such that $\mathrm{co}(\mathbb{V}^{\mathrm{g}}_{\mathrm{sub}}) \supseteq \mathbb{L}(C_{\mathrm{i}}^{\mathrm{d}})$. This latter condition essentially means that $\mathbb{V}^{\mathrm{g}}$ must "more than cover the range of slopes" of the function $C_{\mathrm{i}}^{\mathrm{d}}$; recall that $\mathbb{L}(C_{\mathrm{i}}^{\mathrm{d}}) = \Pi_{j=1}^{m}\left[\mathrm{L}_j^-(C_{\mathrm{i}}^{\mathrm{d}}), \mathrm{L}_j^-(C_{\mathrm{i}}^{\mathrm{d}})\right]$, where $\mathrm{L}_j^-(C_{\mathrm{i}}^{\mathrm{d}})$ (resp. $\mathrm{L}_j^+(C_{\mathrm{i}}^{\mathrm{d}})$) is the minimum (resp. maximum) "slope" of $C_{\mathrm{i}}^{\mathrm{d}}$ along the j-th dimension. Hence, we need to compute/approximate $\mathrm{L}_j^{\pm}(C_{\mathrm{i}}^{\mathrm{d}})$ for

$j = 1, \dots, m$. A conservative approximation for $\mathrm{L}_j^-(C_{\mathrm{i}}^{\mathrm{d}})$ (resp. $\mathrm{L}_j^+(C_{\mathrm{i}}^{\mathrm{d}})$) is $\mathrm{L}_j^-(C_{\mathrm{i}}) = \min \partial C_{\mathrm{i}}/\partial u_j$ (resp. $\mathrm{L}_j^+(C_{\mathrm{i}}) = \max \partial C_{\mathrm{i}}/\partial u_j$), assuming C_{i} is differentiable. Alternatively, we can directly use the discrete input cost $C_{\mathrm{i}}^{\mathrm{d}}$ for computing $\mathrm{L}_j^{\pm}(C_{\mathrm{i}}^{\mathrm{d}})$. In particular, if the domain $\mathbb{U}^{\mathrm{d}} = \mathbb{U}^{\mathrm{g}} = \Pi_{j=1}^m \mathbb{U}_j^{\mathrm{g}}$ of $C_{\mathrm{i}}^{\mathrm{d}}$ is grid-like, we can use the fact that C_{i} is convex, and take $\mathrm{L}_j^-(C_{\mathrm{i}}^{\mathrm{d}})$ (resp. $\mathrm{L}_j^+(C_{\mathrm{i}}^{\mathrm{d}})$) to be the minimum first forward difference (resp. maximum last backward difference) of $C_{\mathrm{i}}^{\mathrm{d}}$ along the j-th dimension (this scheme requires $\mathcal{O}(U)$ operations). Having $\mathrm{L}_j^{\pm}(C_{\mathrm{i}}^{\mathrm{d}})$ at our disposal, we can then construct $\mathbb{V}_{\mathrm{sub}}^{\mathrm{g}} = \Pi_{j=1}^m \mathbb{V}_{\mathrm{sub}\,j}^{\mathrm{g}}$ such that, in each dimension j, $\mathbb{V}_{\mathrm{sub}\,j}^{\mathrm{g}}$ is uniform with the same cardinality as $\mathbb{U}_j^{\mathrm{g}}$, and $\mathrm{co}(\mathbb{V}_{\mathrm{sub}\,j}^{\mathrm{g}}) = \big[\mathrm{L}_j^-(C_{\mathrm{i}}^{\mathrm{d}}), \mathrm{L}_j^+(C_{\mathrm{i}}^{\mathrm{d}})\big]$. Finally, we construct $\mathbb{V}^{\mathrm{g}}$ by extending $\mathbb{V}_{\mathrm{sub}}^{\mathrm{g}}$ uniformly in each dimension (by adding a smaller and a larger element to $\mathbb{V}_{\mathrm{sub}}^{\mathrm{g}}$ in each dimension).

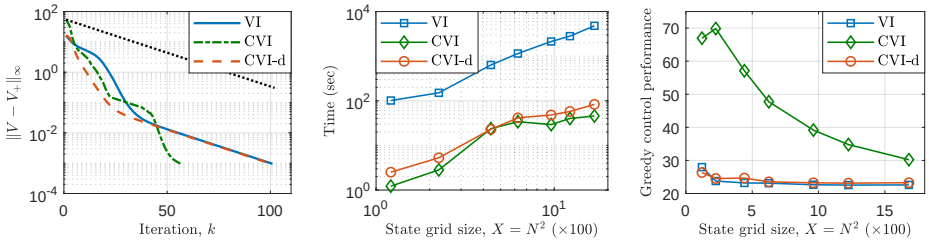
Construction of $\mathbb{Z}^{\mathrm{g}}$. According to Assumption 3.6-(ii), the grid $\mathbb{Z}^{\mathrm{g}}$ must be constructed such that $\mathrm{co}(\mathbb{Z}^{\mathrm{g}}) \supseteq f_{\mathrm{s}}\big(\mathbb{X}^{\mathrm{d}}\big)$. This can be simply done by finding the vertices of the smallest box that contains the set $f_{\mathrm{s}}\big(\mathbb{X}^{\mathrm{d}}\big)$. Those vertices give the range of $\mathbb{Z}^{\mathrm{g}}$ in each dimension. We can then, for example, take $\mathbb{Z}^{\mathrm{g}}$ to be the uniform grid with the same cardinality as $\mathbb{Y}^{\mathrm{g}}$ in each dimension (so that $Z = Y$). This way, $\mathrm{d_H}\left(f_{\mathrm{s}}(\mathbb{X}^{\mathrm{d}}), \mathbb{Z}^{\mathrm{g}}\right) \leq \mathrm{d_H}\left(\mathrm{co}(\mathbb{Z}^{\mathrm{g}}), \mathbb{Z}^{\mathrm{g}}\right)$, and hence e_{z} in (8) reduces by using finer grids $\mathbb{Z}^{\mathrm{g}}$. This construction has a time complexity of $\mathcal{O}(X)$.

Construction of $\mathbb{Y}^{\mathrm{g}}$. Construction of the state dual grid $\mathbb{Y}^{\mathrm{g}}$ is more involved. According to Theorem 3.11, we need to choose a grid that minimizes e_{y} in (8). This can be done by choosing $\mathbb{Y}^{\mathrm{g}}$ such that $\mathbb{Y}^{\mathrm{g}} \cap \partial(J_\star - C_{\mathrm{s}}) \neq \emptyset$ for all $x \in \mathbb{X}^{\mathrm{d}}$ so that $e_{\mathrm{y}} = 0$. Even if we had access to the optimal value function $J_\star$, satisfying such a condition could lead to dual grids $\mathbb{Y}^{\mathrm{g}} \subset \mathbb{R}^n$ of size $\mathcal{O}(X^n)$. Such a large size violates Assumption 3.6-(iii) on the size of $\mathbb{Y}^{\mathrm{g}}$, and essentially renders the proposed algorithm impractical for dimensions $n \geq 2$. A more practical condition is $\mathrm{co}(\mathbb{Y}^{\mathrm{g}}) \cap \partial(J_\star - C_{\mathrm{s}}) \neq \emptyset$ for all $x \in \mathbb{X}^{\mathrm{d}}$ so that $\max_{x \in \mathbb{X}^{\mathrm{d}}} \mathrm{d}\left(\partial(J_\star - C_{\mathrm{s}})(x), \mathbb{Y}^{\mathrm{g}}\right) \leq \mathrm{d_H}\left(\mathrm{co}(\mathbb{Y}^{\mathrm{g}}), \mathbb{Y}^{\mathrm{g}}\right)$, and hence e_{y} reduces by using finer grids $\mathbb{Y}^{\mathrm{g}}$. The latter condition is satisfied if $\mathrm{co}(\mathbb{Y}^{\mathrm{g}}) \supseteq \mathbb{L}(J_\star - C_{\mathrm{s}})$, i.e., if $\mathbb{Y}^{\mathrm{g}}$ "covers the range of slops" of $(J_\star - C_{\mathrm{s}})$. Hence, we need to approximate the range of slopes of $(J_\star - C_{\mathrm{s}})$. To this end, we first use the fact that $J_\star$ is the fixed point of DP operator (1) to approximate $\mathrm{Rng}(J_\star - C_{\mathrm{s}})$ by $R = \frac{\mathrm{Rng}(C_{\mathrm{i}}^{\mathrm{d}}) + \gamma \cdot \mathrm{Rng}(C_{\mathrm{s}}^{\mathrm{d}})}{1-\gamma}$. We then construct the gird $\mathbb{Y}^{\mathrm{g}} = \Pi_{i=1}^n \mathbb{Y}_i^{\mathrm{g}}$ such that, for each dimension i, we have $\pm \alpha R / \Delta_{\mathbb{X}_i^{\mathrm{d}}} \in \mathrm{co}(\mathbb{Y}_i^{\mathrm{g}})$, where $\Delta_{\mathbb{X}_i^{\mathrm{d}}}$ (with some abuse of notation) denotes the diameter of the projection of $\mathbb{X}^{\mathrm{d}}$ on the i-th dimension, and $\alpha > 0$ is a scaling factor mainly depending on the dimension of the state space. This construction has a one-time computational cost of $\mathcal{O}(X + U)$.

Remark 3.12 (Dynamic construction of $\mathbb{Y}^{\mathrm{g}}$)**.** *Alternatively, we can construct* $\mathbb{Y}^{\mathrm{g}}$ dynamically *at each iteration in order to minimize the corresponding error in each application of the d-CDP operator given by* $e_{\mathrm{y}} = c_{\mathrm{y}} \cdot \max_{x \in \mathbb{X}^{\mathrm{d}}} \mathrm{d}\left(\partial(\mathcal{T}J - C_{\mathrm{s}})(x), \mathbb{Y}^{\mathrm{g}}\right)$*; see Lemma 5.6 and Proposition 5.8 of the extended manuscript in the supplementary material. This means that the construction* $\mathbb{Y}^{\mathrm{g}}$ *in Algorithm 1 is moved from line 1 to inside the iterations, after line 5. Similar to the static scheme described above, the aim here is to construct* $\mathbb{Y}^{\mathrm{g}}$ *such that* $\mathrm{co}(\mathbb{Y}^{\mathrm{g}}) \supseteq \mathbb{L}(\mathcal{T}J - C_{\mathrm{s}})$*. Since we do not have access* $\mathcal{T}J$ *at our disposal (it is the output of the current iteration), we can again use the definition of the DP operator* (2) *to approximate Rng*$(\mathcal{T}J - C_{\mathrm{s}})$ *by* $R = \mathit{Rng}(C_{\mathrm{i}}^{\mathrm{d}}) + \gamma \cdot \mathit{Rng}(V^{\mathrm{d}})$ *where* V^{d} *is the output of the previous iteration. We then construct the gird* $\mathbb{Y}^{\mathrm{g}} = \Pi_{i=1}^n \mathbb{Y}_i^{\mathrm{g}}$ *such that, for each dimension* i*, we again have* $\pm \alpha R / \Delta_{\mathbb{X}_i^{\mathrm{d}}} \in \mathrm{co}(\mathbb{Y}_i^{\mathrm{g}})$*. This construction has a one-time computational cost of* $\mathcal{O}(U)$ *for computing Rng*$(C_{\mathrm{i}}^{\mathrm{d}})$ *and a per iteration computational cost of* $\mathcal{O}(U)$ *for computing Rng*(V^{d})*. Notice, however, that under this dynamic construction, the error bound of Theorem 3.11 does not hold true. More importantly, with a dynamic grid* $\mathbb{Y}^{\mathrm{g}}$ *that varies in each iteration, there is no guarantee for CVI to converge. However, as we will see in the numerical examples, the proposed scheme leads to (possibly non-monotone) convergence of CVI.*

4 Numerical simulations

We now showcase the application of the CVI Algorithm 1 in solving the optimal control problem of a noisy inverted pendulum, and compare its performance with the VI algorithm. The details of these simulations (and more numerical examples) are provided in Section 4 of the extended manuscript in the supplementary material. The MATLAB package for the implementation of the proposed CVI algorithm is provided in the supplementary material. We also note that for the discrete conjugation (LLT), we used the MATLAB package (in particular, the `LLTd` routine) provided in [19].



(a) (b) (c)

Figure 1: VI vs. CVI: (a) Convergence rate for grid size $X = 41^2$; (b) Running time; (c) Average cost of fifty instances of the control problem with random initial states over $T = 400$ time steps. (d) Difference between outputs of VI and CVI algorithms (over the discrete state space $\mathbb{X}^{\mathrm{g}}$). The black dotted line in (a) corresponds to exponential convergence with rate $\gamma = 0.95$.

We use the setup (model and stage cost) of [18, App. C.2.2], with discount factor $\gamma = 0.95$ and termination bound $e_{\mathrm{t}} = 0.001$. In particular, the state and input costs are both quadratic ($\|\cdot\|_2^2$), and the discrete-time, nonlinear dynamics is of the form $x^+ = f_{\mathrm{s}}(x) + Bu + w$, where where $f_{\mathrm{s}}(x_1, x_2) = [x_1 + \alpha_{12} x_2,\ \alpha_{21} \sin x_1 + \alpha_{22} x_2]^\top$ and $B = [0,\ \beta]^\top$ ($\alpha_{12}, \alpha_{21}, \alpha_{22}, \beta \in \mathbb{R}$). State and input constraints are described by $\mathbb{X} = [-\frac{\pi}{3}, \frac{\pi}{3}] \times [-\pi, \pi] \subset \mathbb{R}^2$ and $\mathbb{U} = [-3, 3] \subset \mathbb{R}$. The disturbance has a uniform distribution over the finite support $\mathbb{W}^{\mathrm{g}} = \{0, \pm 0.025\frac{\pi}{3}, \pm 0.05\frac{\pi}{3}\} \times \{0, \pm 0.025\pi, \pm 0.05\pi\} \subset \mathbb{R}^2$ of size $W = 5^2$. We use uniform, grid-like discretizations $\mathbb{X}^{\mathrm{g}}$ and $\mathbb{U}^{\mathrm{g}}$ for the state and input spaces such that $\mathrm{co}(\mathbb{X}^{\mathrm{g}}) = [-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\pi, \pi] \subset \mathbb{X}$ and $\mathrm{co}(\mathbb{U}^{\mathrm{g}}) = \mathbb{U}$. This choice of discrete state spaces $\mathbb{X}^{\mathrm{g}}$ particularly satisfies the feasibility condition of Assumption 3.5 Also, we use *nearest neighbor* extension (which is non-expansive) for the extension operators in (2) for VI and in (6a) for CVI. The grids $\mathbb{V}^{\mathrm{g}} \subset \mathbb{R}$ and $\mathbb{Z}^{\mathrm{g}}, \mathbb{Y}^{\mathrm{g}} \subset \mathbb{R}^2$ are also constructed uniformly, following the guidelines of Section 3.4 (with $\alpha = 1$). In particular, we also consider the *dynamic* scheme of Remark 3.12 for the construction of $\mathbb{Y}^{\mathrm{g}}$ (hereafter, referred to as CVI-d). Moreover, in each implementation of VI and CVI, all of the involved grids are chosen to be of the same size in each dimension, i.e., $X = Y = Z = N^2$ and $U = V = N$. We are particularly interested in the performance of these algorithms, as the size of the discretizations increases.

The results of our numerical simulations are shown in Figure 1. As shown in Figures 1a, both VI and CVI algorithms are indeed convergent with a rate greater than or equal to $\gamma = 0.95$. In particular, CVI terminates in 57 iterations, compared to 101 iterations required for VI to terminate. As expected, this faster convergence, combined with the lower time complexity of CVI in each iteration, lead to a significant reduction in the running time of this algorithm compared to VI. This effect can be clearly seen in Figure 1b, where the run-time of CVI for $N = 41^2$ is an order of magnitude less than that of VI for $N = 11^2$. Since we do not have access to the true optimal value function, in order to evaluate the performance of the outputs of the VI and CVI algorithm, we consider the performance of the greedy policy w.r.t. the discrete value function J^{d} computed using these algorithms. Figure 1c reports the average cost of fifty instances of the optimal control problem with greedy control actions. As can be seen, the reduction in the running time in CVI comes with an increase in the cost of underlying optimal control problem particularly for coarse discretizations of the state space.

Let us now consider the effect of *dynamic* construction of the state dual grid $\mathbb{Y}^{\mathrm{g}}$. As can be seen in Figure 1a, using a dynamic $\mathbb{Y}^{\mathrm{g}}$ leads to an slower convergence (CVI-d converges in 100 iterations). We note that our simulations show that for the *deterministic* system (with dynamics $x^+ = f_{\mathrm{s}}(x) + Bu$), CVI-d has a similar converge rate as that of CVI. Moreover, for greater values of the scaling factor (e.g., $\alpha = 3$), we observed the non-monotone convergence of CVI-d. See the extended manuscript in the supplementary material for these results. We also note that the relative behaviour of the convergence rates in Figures 1a was also seen for other grid sizes in the discretization scheme. However, thanks to the low time complexity of CVI in each iteration, we see a small increase in the running time of CVI-d compared to CVI; see Figure 1b. More importantly, as depicted in Figure 1c, CVI-d shows almost the same performance as VI when it comes to the quality of the greedy actions generated using the outputs of these algorithms. This is because the dynamic construction of $\mathbb{Y}^{\mathrm{g}}$ in CVI-d uses the available computational power (related to size of the discretization) smartly by finding the smallest grid $\mathbb{Y}^{\mathrm{g}}$ in each iteration, in order to minimize the error of that same iteration.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] See Theorems 3.9, 3.10 and 3.11.
   (b) Did you describe the limitations of your work? [Yes] See the discussions following Theorems 3.10 and 3.11.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See, in particular, Assumptions 3.6, 3.7, and 3.8.
   (b) Did you include complete proofs of all theoretical results? [Yes] See Section 5 of the extended manuscript in the supplementary material.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] See the MATLAB codes provided in the supplementary material.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See Section 4 of the extended manuscript in the supplementary material.
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] We used the MATLAB package for LLT provided in [19] as mentioned in Section 4.
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [Yes] We are providing a MATLAB package for the implementation of the CVI algorithm introduced in this study. See the MATLAB codes in the supplementary material.
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]