A nonparametric method for gradual change problems with statistical guarantees

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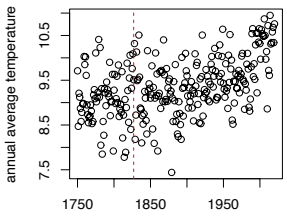
Abstract

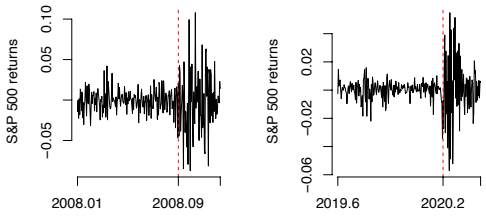
We consider the detection and localization of *gradual* changes in the distribution of a sequence of time-ordered observations. Existing literature focuses mostly on the simpler *abrupt* setting which assumes a discontinuity jump in distribution, and is unrealistic for some applied settings. We propose a general method for detecting and localizing gradual changes that does not require any specific data generating model, any particular data type, or any prior knowledge about which features of the distribution are subject to change. Despite relaxed assumptions, the proposed method possesses proven theoretical guarantees for both detection and localization.

1 Introduction

In a sequence of time-ordered observations $\{Y_{t,T} : t = 1, 2, \cdots, T\}$, the aim of change point detection (CPD) is to (a) detect: answer the question of *whether* the distribution of $Y_{t,T}$ changes, and (b) localize: if it changes, answer the question of *when*. The classic formulation of CPD usually assumes that the possible change point is *abrupt*, i.e., there is a discontinuity jump in the distribution of $Y_{t,T}$, leading to a simpler problem. However, in many real-life situations, the changes in a sequence happen *smoothly* or *gradually*, rather than abruptly. Let us consider some examples.



(a) Annual average temperature in central England.



(b) S&P 500 stock index daily returns.

Figure 1: Examples of gradual changes. The vertical red dashed lines indicate the gradual change points estimated by the method proposed in this paper.

The first example concerns climatology, and investigates the temperature patterns over years. Figure 1a depicts the annual average temperature in central England from 1750 to 2020, where we observe a smooth increase starting around 1850. The second example comes from finance. The S&P 500 stock index is an important indicator of the overall market. As shown in Figure 1b, its volatility level usually remains constant in a stable market, and then gradually increases with the development of some event such as the financial crisis in 2008 or the COVID-19 pandemic in 2020.

Despite the wide variety of applications, inference for gradual changes is under-researched, and most existing methods require domain knowledge. Early research assumed that the gradual change

Submitted to 35th Conference on Neural Information Processing Systems (NeurIPS 2021). Do not distribute.

follows a particular parametric model. For example, Lombard (1987) considers a setting where some unknown parameter changes linearly, while others (Hušková, 1999; Hušková and Steinebach, 2002; Aue and Steinebach, 2002) consider models with polynomial changes.

Recent methods also consider nonparametric settings. However, most of them still require specific assumptions on the data model. For example, Muller (1992); Raimondo (1998); Goldenshluger et al. (2006) consider the location model where first order moment of observations changes. Mallik et al. (2011, 2013) investigate a stronger assumption: the mean change is monotonic. Mercurio et al. (2004) consider the volatility model where second order moment of observations fluctuates. Quessy (2019) assumes that the sequence follows two stationary distributions at the beginning and the end, and the changing phase in-between is a mixture of them with weights changing linearly with time.

As far as we know, Vogt et al. (2015) is the only nonparametric method that applies generally to any model and any data type. Despite its generality, the method proposed in Vogt et al. (2015) requires prior knowledge about which stochastic feature(s) might change. Moreover, their method requires specification of a threshold determined through expensive simulations. Also, Vogt et al. (2015) considers only the localization problem, while ignoring the detection step which is shown to be important for false positive control in real-data applications (Van den Burg and Williams, 2020).

We propose a nonparametric method for detecting and localizing gradual changes. The proposed method requires no prior domain knowledge, and we offer theoretical guarantees on both detection (false positive rate, power) and localization (consistency).

2 Problem Statement

Suppose we observe a time-ordered independent sequence $\{Y_{t,T} : t = 1, 2, \cdots, T\}$ taking values in a general metric space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$. $Y_{t,T}$ is observed at time $u = t/T \in [0, 1]$. We are concerned with:

1. (Detection) Deciding whether the distribution of observation changes with time u. This is formulated as a hypothesis testing problem with null H_0 and alternative H_A hypotheses, where

$$H_0 : P_u = P_0, \quad \forall u \in [0, 1]$$
$$H_A : \exists\ \rho^* \in (0, 1), \varepsilon \in (0, 1 - \rho^*) \text{ s.t. } P_u = P_0,\ \forall u \in [0, \rho^*], \text{ and } P_u \neq P_0,\ \forall u \in (\rho^*, \rho^* + \varepsilon],$$

where P_u is a probability measure on $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ and $Y_{t,T} \sim P_u$ for $u = t/T$. We require that all changes in P_u are gradual (or smooth) in the sense that

$$\forall u, v \in [0, 1],\ P_v \text{ weakly converges to } P_u,\ \text{as } v \to u. \tag{1}$$

2. (Localization) If rejecting H_0 in step 1, obtain an estimator $\hat{\rho}$ of the gradual change point ρ^* where the distribution *starts* to change.

Notice that we do not put any assumptions on the data type or distribution of $Y_{t,T}$ and thus, our formulation allows a large number of special models such as

$$\text{location model:} \quad Y_{t,T} = \mu(t/T) + \varepsilon_t, \tag{2}$$
$$\text{volatility model:} \quad Y_{t,T} = \sigma(t/T)\varepsilon_t, \tag{3}$$

where $\mu(\cdot), \sigma(\cdot)$ can be any continuous function, and ε_t's are zero mean i.i.d errors.

Notations. We use $[x]$ to denote the integer part of x, $\mathbf{1}_d = (1, \cdots, 1)^\top \in \mathbb{R}^d$, $\boldsymbol{I}_d$ the identity matrix in $\mathbb{R}^{d\times d}$. We use $\mathbb{I}$ to denote indicator function, $\xrightarrow{w}$ weak convergence, $\mathbb{Z}_+$ the set of positive integers. For a set of constants a_T, b_T and random variables X_T, we write $a_T = \Theta(b_T)$ if there exist constants $C_1, C_2 > 0, t_0 \in \mathbb{Z}_+$ such that $C_1 a_T \leq b_T \leq C_2 a_T$ for all $T \geq t_0$. We write $X_T = O_p(a_T)$ if X_T/a_T is stochastically bounded, and $X_T = o_p(a_T)$ if X_T/a_T converges to zero in probability.

3 Methodology

Existing statistic. We consider first univariate $Y_{t,T}$'s. Suppose the change is in $\mathbb{E}Y_{t,T}$; traditional CUSUM statistic (Page, 1954) solves CPD problem by defining

$$\widehat{C}_T(u, v) = 1/T \textstyle\sum_{t=1}^{[vT]} Y_{t,T} - v/(uT) \sum_{t=1}^{[uT]} Y_{t,T}, \quad \text{for any } 0 \leq v < u \leq 1.$$

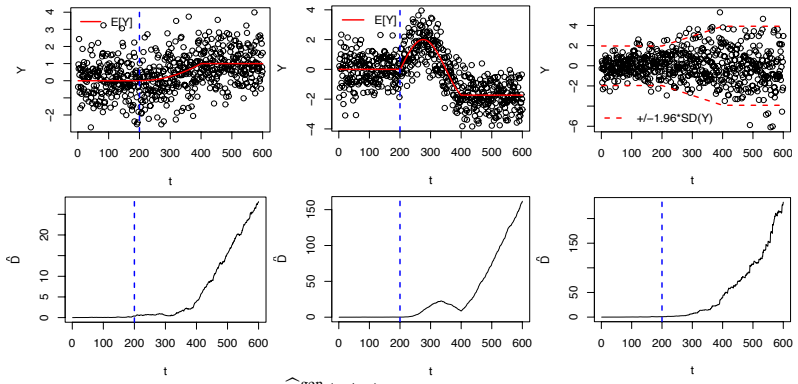


Figure 2: Plots of $Y_{t,T}$ (top row) and their $\widehat{\mathcal{D}}_T^{\text{gen}}(t/T)$ (bottom row) against t. The blue vertical line denotes true change point. Data in column 1, 2 follow location model (2) with $\varepsilon_t \sim N(0,1)$, and $\mu_1(u) = \mathbb{I}(1/3 \le u \le 2/3)(3u-1)^{1.5} + \mathbb{I}(u > 2/3)$, $\mu_2(u) = 2\sin(4\pi(u-1/3))\mathbb{I}(1/3 \le u \le 2/3) + 2\sin(4\pi/3)\mathbb{I}(u \ge 2/3)$, respectively. Data in column 3 follows volatility model (3) with $\varepsilon_t \sim N(0,1)$ and $\sigma(\cdot) = \mu_1(\cdot) + 1$. Column 1, 2 set $\mathcal{F} = \{f : x \mapsto x\}$, and column 3 $\mathcal{F} = \{f : x \mapsto x^2\}$.

which compares cumulative sums of $Y_{t,T}$ over different time spans $[0, v]$ and $[0, u]$. Then

$$\widehat{\mathcal{D}}_T^{\text{uni}}(u) = \max_{v\in[0,u]} |\widehat{C}_T(u, v)|, \quad \text{for any } 0 \le u \le 1.$$

can be used to detect changes in feature $\mathbb{E}Y_{t,T}$ over time span $[0, u]$. Intuitively, if there are no changes over $[0, u]$, $\widehat{\mathcal{D}}_T^{\text{uni}}(u)$ should be small. For example, in Figure 2, the first and second column depicts a sequence with change in $\mathbb{E}Y_{t,T}$ (shown in top row), and $\widehat{\mathcal{D}}_T^{\text{uni}}$ (shown in bottom row) take small values before $\tau^* = 200$ where $\tau^* = [T\rho^*]$, and then grow substantially. Thus, $\widehat{\mathcal{D}}_T^{\text{uni}}(u)$ essentially measures the variation over $[0, u]$ in these univariate settings.

For multivariate/non-Euclidean $Y_{t,T}$ or for changes in more general features of the form $\mathbb{E}f(Y_{t,T})$ where $f : \mathcal{Y} \to \mathbb{R}$ is a measurable function, Vogt et al. (2015) replaces $\widehat{\mathcal{D}}_T^{\text{uni}}$ with

$$\begin{aligned} &\widehat{\mathcal{D}}_T^{\text{gen}}(u) = \sup_{f\in\mathcal{F}} \max_{v\in[0,u]} |\widehat{C}_T(u, v, f)|, \quad \text{where} \\ &\widehat{C}_T(u, v, f) = 1/T \textstyle\sum_{t=1}^{[vT]} f(Y_{t,T}) - v/(uT) \sum_{t=1}^{[uT]} f(Y_{t,T}). \end{aligned} \tag{4}$$

$\widehat{\mathcal{D}}_T^{\text{gen}}$ takes supremum over a pre-specified set of functions $\mathcal{F}$ to ensure that changes in $\mathbb{E}f(Y_{t,T})$ for all $f \in \mathcal{F}$ are considered. This leads to scaling issues in the definition of $\widehat{\mathcal{D}}_T^{\text{gen}}$. Note that $\widehat{\mathcal{D}}_T^{\text{uni}}$ is a special case of $\widehat{\mathcal{D}}_T^{\text{gen}}$ with $\mathcal{F} = \{f : x \mapsto x\}$, and column 3 of Figure 2 sets $\mathcal{F} = \{f : x \mapsto x^2\}$.

There are three main issues with $\widehat{\mathcal{D}}_T^{\text{gen}}$. First, it relies heavily on the pre-specified function class $\mathcal{F}$. Also, to calculate $\widehat{\mathcal{D}}_T^{\text{gen}}$, $\mathcal{F}$ can only contain a finite (usually small) number of functions (e.g., $f : x \mapsto x$ or $f : x \mapsto x^2$), the choice of which relies heavily on prior knowledge about which features might change. When $\mathcal{F}$ is misspecified, $\widehat{\mathcal{D}}_T^{\text{gen}}$ can be non-informative and fail subsequent tasks. Second, $\widehat{\mathcal{D}}_T^{\text{gen}}$ does not consider the scale of $\widehat{C}_T(\cdot, \cdot, f)$ which could be incomparable for different f's. Third, the limiting process to which $\widehat{\mathcal{D}}_T^{\text{gen}}(\cdot)$ converges is unknown, leading to computational challenges in subsequent analyses.

Proposed statistic. We introduce a new statistic that is applicable to any data type and any generating process, and free of the issues discussed above. It is motivated by the recent success of applying kernel approaches to abrupt CPD problems (e.g., Harchaoui et al. (2008); Li et al. (2015); see Section 7 for more details). These kernel approaches assume access to a positive semidefinite kernel $k : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ that measures pairwise similarity among observations. Compared with features, kernels are more flexible and easier to specify, especially for non-Euclidean data, showing great potential for solving gradual CPD problem. Inference starts with measuring data variation in time span $[0, u]$; for each possible change point $v < u$, v divides the observations into two groups: those coming before $[Tv]$ and those after $[Tv]$. Note that the average similarity among observations within the same group is:

$$\widehat{S}_T^{\text{within}}(u, v) = 0.5(l)^{-2} \textstyle\sum_{s,t=1}^{l} k(Y_{s,T}, Y_{t,T}) + 0.5(r-l)^{-2} \sum_{s,t=l+1}^{r} k(Y_{s,T}, Y_{t,T}),$$

where $l = [vT]$, $r = [uT]$, and the average similarity among observations between different groups is

$$\widehat{S}_T^{\text{between}}(u,v) = [l(r-l)]^{-1}\sum_{s=1}^{l}\sum_{t=l+1}^{r} k(Y_{s,T}, Y_{t,T}).$$

Intuitively, if v is the true change point, we expect $\widehat{S}_T^{\text{within}}(u,v)$ to be large compared with $\widehat{S}_T^{\text{between}}(u,v)$. This intuition underlies the following statistic,

$$\widehat{\mathcal{D}}_T(u) = \max_{v\in[0,u]}\widehat{\mathcal{K}}_T(u,v) \quad \text{where} \tag{5}$$

$$\widehat{\mathcal{K}}_T(u,v) = 2v^2(u-v)^2/u^2[\widehat{S}_T^{\text{within}}(u,v) - \widehat{S}_T^{\text{between}}(u,v)]. \tag{6}$$

$\widehat{\mathcal{D}}_T$ takes the maximum over $v \in [0,u]$ using a similar idea as $\widehat{\mathcal{D}}_T^{\text{uni}}$ and $\widehat{\mathcal{D}}_T^{\text{gen}}$. The scaling factor $v^2(u-v)^2/u^2$ ensures that $\widehat{\mathcal{D}}_T$ is asymptotically well-defined for all $u \in [0,1]$ (see more details in Section 4). $\widehat{\mathcal{D}}_T$ plays the same role as $\widehat{\mathcal{D}}_T^{\text{gen}}$ and measures data variation among $[0,u]$.

Note that $\widehat{\mathcal{D}}_T$ has also a CUSUM-style representation, which is crucial for understanding its theoretical properties. Define a centered kernel $k_0(y,y') = k(y,y') - 2\mathbb{E}_{Y\sim F_0}k(y,Y) + \mathbb{E}_{Y,Y'\sim F_0}k(Y,Y')$, which can be decomposed in terms of eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ with respect to F_0 as:

$$k_0(y,y') = \textstyle\sum_{j=1}^{\infty}\lambda_j\psi_j(y)\psi_j(y') \quad \text{with} \tag{7}$$
$$\textstyle\int k_0(y,y')\psi_j(y)dF_0(y) = \lambda_j\psi_j(y'),\ \int\psi_j(y)\psi_{j'}(y)dF_0(y) = \delta_{j,j'},$$

and $\delta_{j,j'}$ is the Kronecker delta function. We denote the feature map ϕ associated with k_0 as

$$\phi(y) = (\lambda_1^{1/2}\psi_1(y), \lambda_2^{1/2}\psi_2(y), \cdots)^\top \in \mathcal{H},\ \langle\phi(y),\phi(y')\rangle_{\mathcal{H}} := \textstyle\sum_{l=1}^{\infty}\phi_l(y)\phi_l(y') = k_0(y,y').$$

Using properties of $\langle\cdot,\cdot\rangle_{\mathcal{H}}^{1/2}$ and denoting $\|\cdot\|_{\mathcal{H}} = \langle\cdot,\cdot\rangle_{\mathcal{H}}^{1/2}$, we have

$$\widehat{\mathcal{K}}_T(u,v) = \|1/T\textstyle\sum_{t=1}^{[vT]}\phi(Y_{t,T}) - v/(uT)\sum_{t=1}^{[uT]}\phi(Y_{t,T})\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty}|\widehat{C}_T(u,v,\phi_j)|^2. \tag{8}$$

Equation (8) helps the comparison of $\widehat{\mathcal{D}}_T$ against $\widehat{\mathcal{D}}_T^{\text{gen}}$. In general, $\widehat{\mathcal{D}}_T$ has three advantages. First, recall that $\widehat{\mathcal{D}}_T^{\text{gen}}$ strongly depends on the specification of the function class $\mathcal{F}$; we allow implicitly a much larger $\mathcal{F}$ with infinite functions. For example, by using universal kernels such as $k(y,y') = \exp\{-\|y-y'\|^2/2\}$, we consider any change in $\mathbb{E}f(Y_{t,T})$ where $f \in \mathcal{F} = \{f : x \mapsto x^n, n = 1,2,\cdots\}$. Under mild assumptions, there exists $f \in \mathcal{F}$ such that $\mathbb{E}f(X) \neq \mathbb{E}f(X')$ when random variables X, X' follow different distributions. Second, the asymptotic distribution of $\widehat{\mathcal{D}}_T^{\text{gen}}$ is intractable, caused by its dependence structure on $\widehat{C}_T$. There are two key facts, under H_0,

$$\widehat{C}_T(u,v,\phi_j) \text{ is asymptotically Gaussian, and } \mathbb{E}[\widehat{C}_T(u,v,\phi_j)\widehat{C}_T(u,v,\phi_{j'})] \to 0,\ \forall j,j' \in \mathbb{Z}_+.$$

It implies $\widehat{C}_T(u,v,\phi_j)$ are asymptotically independent Gaussian random variables (r.v.). Since the sum of independent Gaussian r.v. follows a known distribution (chi-square), in view of (8), the asymptotic distribution of our statistic is much simpler than that of $\widehat{\mathcal{D}}_T^{\text{gen}}$. Third, using kernels to define $\widehat{\mathcal{K}}_T$ does not lead to technical/implementation issues. In contrast, if we define $\widehat{\mathcal{K}}_T$ directly using (8) with the function class $\mathcal{F} = \{\phi_j, j = 1,2,\cdots\}$ replaced by an arbitrary function class of infinite cardinality, the infinite series will not necessarily converge, and even when it converges, it cannot not be calculated exactly.

Remark 3.1. Some useful kernels for the gradual CPD problem: (i) For $\mathcal{Y} = \mathbb{R}^d$, we recommend using the dot-product kernel $k(y,y') = \langle y,y'\rangle_{\mathbb{R}^d}$ if location model (2) holds. Here $\phi_j : x = (x_1,\cdots,x_d)^\top \mapsto x_j - a_j, \forall j = 1,\cdots,d$. When $d = 1$, $\widehat{\mathcal{D}}_T$ with this kernel equals $\widehat{\mathcal{D}}_T^{\text{gen}}$ with $\mathcal{F} = \{f : x \mapsto x - a_1\}$ and $\widehat{\mathcal{D}}_T^{\text{uni}}$. (ii) For $\mathcal{Y} = \mathbb{R}$, we recommend using $k(y,y') = y^2(y')^2$ if volatility model (3) holds. Here $\phi_j : x \mapsto x^2 - a$ where $a = \mathbb{E}_{x\sim F_0}x^2$. $\widehat{\mathcal{D}}_T$ with this kernel equals $\widehat{\mathcal{D}}_T^{\text{gen}}$ with $\mathcal{F} = \{f : x \mapsto x^2 - a\}$. (iii) For any general $\mathcal{Y}$, $k(y,y') = \exp\{-\|y-y'\|_{\mathcal{Y}}^2/(2h^2)\}$ is the RBF kernel with bandwidth $h > 0$. This can be set as the default kernel without any prior knowledge about data model.

Now we will utilize $\widehat{\mathcal{D}}_T$ for the detection and localization of gradual change points.

Detection. As shown in Figure 2, under a good choice of k, $\widehat{\mathcal{D}}_T(u)$ summarizes the degree of variation over time span $[0, u]$ and satisfies

$$\widehat{\mathcal{D}}_T(u) \text{ is } \begin{cases} \text{small}, & \text{when} \quad u \le \rho^*, \\ \text{large}, & \text{when} \quad u > \rho^*. \end{cases} \tag{9}$$

The case of no change point is equivalent to $\rho^* = 1$. The existence of a change point can be tested using $\widehat{\mathcal{D}}_T(1)$. The p-value depends on the asymptotic null distribution of $\widehat{\mathcal{D}}_T(1)$, the rigorous establishment of which requires many technical details and is deferred to the next section (Theorem 4.4). Practitioners only need the following formula to calculate p-values:

$$\mathbb{P}(T\widehat{\mathcal{D}}_T(1) > x) \approx 2^{(\hat{q}+3)/2}[\Gamma(\hat{q}/2)]^{-1}\sqrt{\pi}(x/\hat{\lambda}_1)^{(\hat{q}-1)/2}e^{-2x/\hat{\lambda}_1}\textstyle\prod_{l=q+1}^{T}(1-\hat{\lambda}_l/\hat{\lambda}_1)^{-1/2}, \tag{10}$$

where $\hat{\lambda}_1 \ge \hat{\lambda}_2 \ge \cdots \ge \hat{\lambda}_T$ are eigenvalues of the matrix $(1/T)K_0$ where

$$K_0 = HKH \in \mathbb{R}^{T\times T},\ K = [k(Y_{i,t}, Y_{j,T})]_{i,j=1}^{T} \in \mathbb{R}^{T\times T} \text{ and } H = I_T - (1/T)\mathbf{1}_T\mathbf{1}_T^\top, \tag{11}$$

and $\hat{q}$ is the estimated multiplicity of the leading eigenvalue.

Localization. Once a significant change point is detected, the next step is to localize it. Observing property (9) with $\widehat{\mathcal{D}}_T$ replaced by $\widehat{\mathcal{D}}_T^{\text{gen}}$, Vogt et al. (2015) propose an estimator for ρ^* as:

$$\hat{\rho}^{\text{gen}} = T^{-1}\textstyle\sum_{t=1}^{T}\mathbb{I}(T^{1/2}\widehat{\mathcal{D}}_T^{\text{gen}}(t/T) \le b_T),$$

where the scaling factor $T^{1/2}$ ensures that $T^{1/2}\widehat{\mathcal{D}}_T^{\text{gen}}$ follows a non-degenerate distribution asymptotically, and b_T is set to the $(1-\alpha)$-quantile of the limiting distribution of $\sup_{v\in[0,\rho^*]}\widehat{\mathcal{D}}_T^{\text{gen}}(v)$. In practice, both ρ^* and limiting distribution of $\widehat{\mathcal{D}}_T^{\text{gen}}(\cdot)$ are unknown, thus b_T is approximated by a two-step procedure with expensive simulations. For our statistic, we find that under the null, $\widehat{\mathcal{D}}_T(u)$ and $u\widehat{\mathcal{D}}_T(1)$ follow the same limiting distribution for any u. It implies that we can estimate ρ^* by

$$\hat{\rho} = T^{-1}\textstyle\sum_{t=1}^{T}\mathbb{I}(T\widehat{\mathcal{D}}_T(t/T) \le c_T(t/T)), \quad \text{where} \quad c_T(u) = ub_T, \tag{12}$$

and the scaling factor T ensures that $T\widehat{\mathcal{D}}_T$ has a non-degenerate limiting distribution. Here, $\hat{\rho}$ is affected by c_T: a larger c_T will lead to a larger $\hat{\rho}$ and vice versa. Ideally, the optimal choice of c_T should minimize some measure of error, and we propose using $l_1(\hat{\rho}) = \mathbb{E}\,|\hat{\rho} - \rho^*|$. It depends on the finite sample distribution of $\widehat{\mathcal{D}}_T$ and could be hard to control in nonparametric settings, but we know the asymptotic distribution of $\widehat{\mathcal{D}}_T(\cdot)$ (Theorem 4.4), which is denoted here as $L(\cdot)$. Thus, we choose c_T which minimizes the l_1 error of the population version ρ^∞ of $\hat{\rho}$:

$$l_1(\rho^\infty) = \mathbb{E}|\rho^\infty - \rho^*| \quad \text{with} \quad \rho^\infty = \textstyle\int_0^{\rho^*}\mathbb{I}(L(u) \le c_T(u))du + \int_{\rho^*}^{1}\mathbb{I}(T^{1/2}L(u) \le c_T(u))du,$$

where we divide between $u \in [0, \rho^*], u \in (\rho^*, 1]$ and add scaling factor $T^{1/2}$ for $u \in (\rho^*, 1]$ to make sure that $L(u)$ is well-defined. Under some assumptions, minimizing $l_1(\rho^\infty)$ leads to

$$b_T = \hat{\lambda}_1/(2\kappa)\log T, \tag{13}$$

where $\kappa \ge 2$ is determined by the smoothness of gradual change and the smoother the change is, the larger κ is. The derivation of Equation (13) is included in the next section. The theoretical value of κ is defined in Assumption 4 in the next section, and depends on the alternative distribution of $Y_{t,T}$ and the kernel k. For practitioners, we only need to know that for abrupt changes and any choice of kernel, $\kappa = 2$ (notice that our method is also applicable for abrupt changes). For RBF kernel, if the change in $\mathbb{E}\exp\{Y_{t,T}\}$ can be approximated by $(u-\rho^*)^\beta$ in time span $u \in [\rho^*, \rho^* + \varepsilon)$ for some small $\varepsilon > 0$, we have $\kappa = 2\beta + 2$. We emphasize that choice of κ does not affect the consistency of $\hat{\rho}$. In experiments, using $\kappa = 4$ works well. An alternative that is less sensitive to κ, the max-gap estimator, is introduced next.

Max-gap estimator. Despite its good theoretical properties, $\hat{\rho}$ has often a large positive bias. This arises from the nature of gradual changes, and is common to previous gradual CPD methods as discussed in Vogt et al. (2015). Intuitively, we need to wait for enough signal strength in order to identify the gradual change point. To design a less biased estimator, recall that in Figure 2, we plotted $\widehat{\mathcal{D}}_T(\cdot)$ against time and easily visually identified the change point as the time when $\widehat{\mathcal{D}}_T(\cdot)$ starts to grow. For example, for data in the first column, a zoomed-in region is shown in Figure 3, where the black line is $T\widehat{\mathcal{D}}_T(\cdot)$ and red line $c_T(\cdot)$. In Figure 3, the growth starts around the point 285 (shown in

brown vertical line). However, using $\hat{\rho}$ gives $\hat{\tau} = 342$ (shown in green vertical line). We want an algorithm capable of identifying this elbow point. Note that from Theorem 4.4, we have

$$\mathbb{E}[c_T(u) - T\widehat{\mathcal{D}}_T(u)] \begin{cases} \text{increases with } u, & \text{if } u \le \rho^* \\ \text{decreases with } u, & \text{if } u > \rho^*. \end{cases}$$

Thus, ρ^* should be the u where $c_T(u) - T\widehat{\mathcal{D}}_T(u)$ is maximized (in Figure 3, this is where the gap between the red line and black line is maximized). It suggests setting

$$\check{\rho} = \mathrm{m}\arg\max_{u \in (0,\hat{\rho}]} [c_T(u) - T\widehat{\mathcal{D}}_T(u)], \qquad (14)$$

where $\mathrm{m}\arg\max$ takes the largest value in the set formed by $\arg\max$. In Figure 3, $\check{\rho}$ is shown by the brown line.

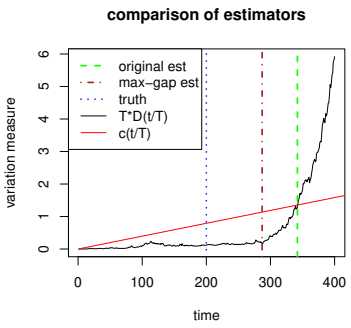


Figure 3: Comparison of max-gap estimator and original estimator in simulated data.

Compared with the original estimator $\hat{\rho}$, empirical studies show that the max-gap estimator $\check{\rho}$ has two advantages: it is more accurate, and is much less sensitive to choice of κ. Some intuition for insensitivity to κ: in Figure 3, κ changes the slope of the red line and a slight change in slope does not influence the time where its gap between the black line is maximized. The higher accuracy of $\check{\rho}$ also has a theoretical explanation, which is included in the Appendix due to space limit. In short, the l_1 error of $\check{\rho}$ consists of two parts: the overestimation error $\mathbb{E}[\check{\rho} - \rho^*]_+$ and the underestimation error $\mathbb{E}[\rho^* - \check{\rho}]_+$ with $[x]_+$ denotes the positive part of x. There is always a trade-off between overestimation and underestimation. Asymptotically, $\check{\rho}$ focuses more on controlling the overestimation error (delay) while guaranteeing consistency of the estimator, since delay is the main concern in small samples. In contrast, $\hat{\rho}$ controls the asymptotic over/under-estimation error equally, which has the optimal loss in theory but is less accurate in small samples.

4 Theory

This section establishes all theoretical results mentioned previously.

Asymptotic distribution of $\widehat{\mathcal{D}}_T$. In order to utilize $\widehat{\mathcal{D}}_T$ for downstream tasks, we need to know its asymptotic distribution. To establish that, we will first introduce some technical assumptions.

Assumption 1. *$\exists M \in (0, +\infty)$, $\forall t \in \{1, 2, \cdots, T\}$, $k(Y_{t,T}, Y_{t,T}) \le M^2$ almost surely (a.s.).*

Remark 4.1. Assumption 1 requires that the kernel is a.s. bounded for all $Y_{t,T}$. It is a weak assumption which is satisfied when $k(\cdot, \cdot)$ is continuous and $\mathcal{Y}$ is closed and bounded, or when k is RBF kernel.

Assumption 1 suffices for the asymptotic null distribution of $\widehat{\mathcal{D}}_T$. Under H_A, however, we will need to restrict the behavior of $Y_{t,T}$. One useful concept is the locally stationary process introduced in Vogt et al. (2012).

Assumption 2 (Locally Stationary Process)**.** *The array $\{Y_{t,T} : t = 1, 2, \cdots, T\}_{T=1}^{\infty}$ is a locally stationary process, i.e., $\forall u \in [0, 1]$, there exists a strictly stationary process $\{Y_t(u) : t \in \mathbb{Z}\}$ s.t.*

$$\|Y_{t,T} - Y_t(u)\|_{\mathcal{Y}} \le (|t/T - u| + 1/T)\, U_{t,T}(u) \quad \text{a.s.}$$

where $\{U_{t,T}(u) : t = 1, 2, \cdots, T\}_{T=1}^{\infty}$ is an array of positive random variables which satisfies $\mathbb{E}[U_{t,T}^{\gamma}(u)] \le c_0$ for some constant $c_0 \in (0, +\infty)$, $\gamma > 0$.

Remark 4.2. Assumption 2 is a more rigorous version of Equation (1), which ensures the change is *gradual* and has been used in Vogt et al. (2015). The intuition behind it is that $\{Y_{t,T}\}$ should be approximately stationary over short time periods. This is turned into rigorous mathematics by ensuring that locally around each $u = t/T$, $\{Y_{t,T}\}$ can be approximated by a stationary process $\{Y_t(u)\}$.

Define

$$\mathcal{D}(u) = \max_{v \in [0,u]} \mathcal{K}(u, v) \quad \text{with} \quad \mathcal{K}(u, v) = \|\textstyle\int_0^v \mu(w)dw - v/u \int_0^u \mu(w)dw\|_{\mathcal{H}}^2, \qquad (15)$$

where $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \cdots)^\top$, $\mu_j(\cdot) = \mathbb{E}\phi_j(Y_t(\cdot))$. Comparing Equations (15) and (8), we find that $\widehat{\mathcal{K}}_T(u, v)$ is in fact an estimator for $\mathcal{K}(u, v)$ and thus, $\widehat{\mathcal{D}}_T(u)$ is an estimator for $\mathcal{D}(u)$. Using the decomposition $\widehat{\mathcal{D}}_T(u) = \mathcal{D}(u) + [\widehat{\mathcal{D}}_T(u) - \mathcal{D}(u)]$, in order to study asymptotics of $\widehat{\mathcal{D}}_T$, we only need to study the approximation error $\widehat{\mathcal{D}}_T - \mathcal{D}$. We will need the following assumptions:

Assumption 3. *The feature map ϕ and stochastic processes $\{\mu_j(u) : u \in [0,1]\},\ \forall\, j \in \mathbb{Z}_+$ satisfy*

(i) $\|\phi(y) - \phi(y')\|_{\mathcal{H}}^2 \le C_1 \|y - y'\|_{\mathcal{Y}}^{\min(\gamma,1)}$ *for all $y, y' \in \mathcal{Y}$, with γ defined in Assumption 2.*

(ii) $\sum_{j=1}^{\infty} \max_{u\in(0,1)} d\mu_j(u)/du < +\infty$.

Remark 4.3. Condition (i) requires sufficient smoothness for ϕ which is always satisfied for sufficiently smooth kernels k. The better $\{Y_{t,T}\}$ is approximated by $\{Y_t(u)\}$, the larger γ is, and the smoother k should be. Condition (ii) roughly says that μ_j has a well-defined Riemann integral over $[0,1]$ so that the integral in $\mathcal{D}$ can be approximated by the Riemann sum in $\widehat{\mathcal{D}}_T$.

Now we are ready to present our main result, where $\rho^* = 1$ corresponds to no change point.

Theorem 4.4. *Suppose Assumption 1 holds.*

(1) For any $u \in (0, \rho^*]$,

$$T[\widehat{\mathcal{D}}_T(u) - \mathcal{D}(u)] \xrightarrow{w} \max_{v\in[0,u]} \sum_{l=1}^{\infty} \lambda_l [W_l(v) - \tfrac{v}{u} W_l(u)]^2, \tag{16}$$

where λ_l's are defined in (7), and $W_l(\cdot), l = 1, \cdots$ are independent standard Wiener processes.

(2) If, in addition, Assumptions 2 and 3 hold, for any $u \in (\rho^*, 1]$, we have

$$\sqrt{T}[\widehat{\mathcal{D}}_T(u) - \mathcal{D}(u)] \xrightarrow{w} \max_{v\in[0,u]} G(v,u), \tag{17}$$

where for any u, $G(\cdot, u)$ is a sample continuous Gaussian process.

Remark 4.5. Both $\sum_{l=1}^{\infty} \lambda_l [W_l(\cdot) - \frac{\cdot}{u} W_l(u)]^2$ and $G(\cdot, u)$ are sample continuous and thus, the right hand size of (16) (17) are well-defined. λ_l's are determined by F_0, k and (16) states that the higher the noise level of F_0 is, the more dispersed the asymptotic null of $\widehat{\mathcal{D}}_T$ will be. Note that the asymptotic behavior of $\widehat{\mathcal{D}}_T$ is completely different before and after the change point: before change point, $\widehat{\mathcal{D}}_T = O_p(T^{-1})$ and after re-scaling, $\widehat{\mathcal{D}}_T$ is maximum of a chi-square process, while after change point, $\widehat{\mathcal{D}}_T = \Theta(T^{1/2}) + O_p(T^{-1/2})$ and after re-centering and re-scaling, $\widehat{\mathcal{D}}_T$ is maximum of a Gaussian process. This distinct property of $\widehat{\mathcal{D}}_T$ is critical for the success of both detection and localization.

Detection. To calculate p-values, Theorem 2.1 of Liu and Ji (2014) says that for $\forall n \in \mathbb{Z}_+$ and $\lambda_1 = \cdots = \lambda_q > \lambda_{q+1} \ge \lambda_{q+2} \ge \cdots \ge \lambda_n > 0$, as $x \to \infty$,

$$\begin{aligned}&\mathbb{P}(\max_{v\in[0,1]} \textstyle\sum_{l=1}^{n} \lambda_l \left[W_l(v) - v W_l(1)\right]^2 > x) \\ &= 2^{(q+3)/2} [\Gamma(q/2)]^{-1} \sqrt{\pi}\, (x/\lambda_1)^{(q-1)/2} \exp\{-2x/\lambda_1\} \textstyle\prod_{l=q+1}^{n} (1 - \lambda_l/\lambda_1)^{-1/2} (1 + o(1)).\end{aligned}$$

Combined with Theorem 4.4, it implies (10). Also, we have the following:

Corollary 4.1 (Power Consistency). Suppose Assumption 1, 2, 3 hold. If $\sqrt{T}\mathcal{D}(1) \to \infty$,

$$\forall x > 0, \quad \mathbb{P}(T\widehat{\mathcal{D}}_T(1) > x) \to 1, \quad T \to \infty.$$

Remark 4.6. Corollary 4.1 shows that power of the proposed test is affected by the magnitude of change measured in $\mathcal{D}(1)$. As long as $\mathcal{D}(1)$ goes to zero at a rate slower than $T^{-1/2}$, the change will be detected if it exists; it ensures correctness of the detection step.

Localization. Recall we need to optimize c_T. This requires regulating the local behavior of $\mathcal{D}$ at ρ^*:

Assumption 4. *There is a cusp of order κ at ρ^* for $\mathcal{D}(\cdot)$, i.e., $\frac{\mathcal{D}(u)}{(u-\rho^*)^\kappa} \to m > 0, \quad u \to \rho^* + .$*

Remark 4.7. Assumption 4 says $\mathcal{D}$ can be locally approximated by a Taylor-type expansion around ρ^*, which is a common assumption in gradual CPD literature (Mallik et al., 2013; Vogt et al., 2015).

Theorem 4.8. *Suppose Assumptions 1, 2, 3, 4 hold, and $c_T(u) = u b_T$. The c_T minimizing $l_1(\rho^\infty)$ satisfies*

$$c_T(u) = (u \lambda_1 r \log T)/2, \ r \ge 1/\kappa. \tag{18}$$

Remark 4.9. In Equation (18), the larger the noise level λ_1 is, the larger c_T is. The smoother the gradual change is (the larger κ is), the smaller c_T is. And r can be viewed as a tuning parameter s.t. if we are less tolerant to delays in $\hat{\rho}$, we could set r to be small, and vice versa. In practice, $\hat{\rho}$ is often overestimated. Thus, we suggest choosing $r = 1/\kappa$, which ultimately leads to (13).

Theorem 4.10. *Under Assumptions 1, 2, 3, 4 and Equation (18), $\hat{\rho} - \rho^* = o_p(1)$, $\check{\rho} - \rho^* = o_p(1)$.*

Remark 4.11. Theorem 4.10 shows that the original estimator and the max-gap estimator are both consistent, and establishes theoretical guarantees for the localization step.

Table 1: Comparison of average l_1 localization error over 20 simulations. Numbers after $\pm$ are the standard error of the average. Methods marked with '-' means not applicable to that model.

MODEL	LOCATION							VOLATILITY		NETWORK
DIM	1	1	1	1	10	20	50	1	1	10^2
CHANGE	LINEAR	QUADRATIC	ONE-SIDED	COMPLEX	LINEAR	LINEAR	LINEAR	LINEAR	COMPLEX	LINEAR
$\check{\rho}$	0.09±0.01	0.15±0.01	0.03±0.00	**0.03±0.01**	**0.07±0.01**	**0.06±0.01**	**0.05±0.01**	0.15±0.01	**0.05±0.00**	**0.10±0.02**
$\hat{\rho}$	0.10±0.01	0.24±0.01	0.08±0.00	0.05±0.01	0.08±0.01	0.10±0.01	0.09±0.01	0.26±0.01	0.12±0.00	0.11±0.02
$\hat{\rho}^{\text{POLY}}$	**0.05±0.01**	**0.09±0.02**	0.10±0.01	0.23±0.00	-	-	-	-	-	-
$\hat{\rho}^{\text{ONE-SIDE}}$	0.07±0.01	**0.09±0.02**	**0.02±0.00**	0.62±0.00	-	-	-	-	-	-
$\hat{\rho}^{\text{MIX}}$	**0.05±0.01**	**0.09±0.01**	0.14±0.00	0.12±0.00	0.08±0.00	0.18±0.02	0.43±0.00	**0.08±0.01**	0.14±0.00	-
$\hat{\rho}^{\text{GEN}}$	0.17±0.01	0.24±0.01	0.07±0.00	0.05±0.00	0.13±0.01	0.15±0.01	0.14±0.00	0.26±0.01	0.12±0.00	0.27±0.00
Q	0.18±0.01	0.23±0.01	0.05±0.00	0.27±0.00	0.16±0.01	0.17±0.00	0.16±0.00	0.21±0.01	0.06±0.00	0.16±0.01
KCPA	0.18±0.01	0.23±0.01	0.05±0.00	0.27±0.00	0.16±0.01	0.16±0.00	0.16±0.00	0.21±0.01	0.06±0.00	0.16±0.01
Z_w	0.24±0.04	0.29±0.04	0.09±0.02	0.29±0.01	0.16±0.01	0.17±0.01	0.18±0.01	0.18±0.03	0.16±0.03	0.16±0.02

5 Simulations

To better understand finite sample properties of the proposed method, we evaluate its performance in simulations and against baselines.

Data generating process. We set $T = 600, \rho^* = 1/3$. Following Vogt et al. (2015), we consider a location model, a volatility model, and we add a network model. For the location model (2), we include univariate cases with $\varepsilon_t \sim N(0,1)$ and four different types of change ordered in increasing difficulty: (i) linear change $\mu_1(u) = \mathbb{I}(1/3 \le u \le 2/3)(3u-3) + \mathbb{I}(u \ge 2/3)$; (ii) quadratic change $\mu_2(u) = \mathbb{I}(1/3 \le u \le 2/3)(3u-1)^2 + \mathbb{I}(u \ge 2/3)$; (iii) one-sided change $\mu_3(u) = 2\sin(2.5\pi(u-1/3))\mathbb{I}(1/3 \le u \le 2/3) + \mathbb{I}(u \ge 2/3)$ in the sense that $\mu_3(u) > \mu_3(\rho^*)$ for all $u > \rho^*$; and (iv) a complex change $\mu_4(u) = 2\sin(4\pi(u-1/3))\mathbb{I}(1/3 \le u \le 2/3) + 2\sin(4\pi/3)\mathbb{I}(u \ge 2/3)$. We also consider multivariate $Y_{t,T} \in \mathbb{R}^d$ where $\mu_5 = \mu_1 \mathbf{1}_d, \varepsilon_t \sim N_d(0, I_d)$. For volatility model (3), we consider $\sigma_i(u) = \mu_i(u) + 1, \varepsilon_t \sim N(0,1), \forall i = 1, 4$. For network model, we set $Y_{t,T}$ as the Erdos-Renyi random graph with 10 nodes. At each time $u \in [0,1]$, there exists a 3-node community such that the possibility of forming an edge among them follows Binomial$(1, p(u))$ independently. Here $p(u) = 0.8\mathbb{I}(1/3 \le u \le 2/3)(3u-1) + 0.8\mathbb{I}(u \ge 2/3) + 0.1$. The probability of forming an edge between other pair of nodes follows a Binomial$(1, 0.1)$.

Baselines. We consider four gradual CPD baselines, ordered in increasing generality: $\hat{\rho}^{\text{poly}}$ (Hušková, 1999) which requires univariate location model with polynomial change, $\hat{\rho}^{\text{one-side}}$ (Mallik et al., 2013) which requires univariate location model with one-sided change, $\hat{\rho}^{\text{mix}}$ (Quessy, 2019) which requires any general model with a mixture type of change whose mixture weight changes linearly with time, and $\hat{\rho}^{\text{gen}}$ (Vogt et al., 2015) which does not have any constraints on model or type of change. We also include three nonparametric abrupt CPD methods: KCpA (Harchaoui et al., 2008), Z_w (Chu et al., 2019), and Q (Matteson and James, 2014)).

Detailed setup. For $\hat{\rho}^{\text{one-side}}$ we tune the bandwidth on 20 independently generated datasets among $\{0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 5\}$. For each dataset, for fairness we use the same kernel for $\hat{\rho}, \check{\rho}$ and KCpA, and use its corresponding distance for Q, Z_w and function class $\mathcal{F}$ for $\hat{\rho}^{\text{gen}}$. For location model, $\mathcal{F} = \{f : x \mapsto x_i, \forall i = 1, \cdots, d\}$; for network model, $\mathcal{F} = \{f : x \mapsto x_{ij}, \forall i, j = 1, \cdots, 10\}$; forvolatility model, $\mathcal{F} = \{f : x \mapsto x^2\}$. For $\hat{\rho}^{\text{poly}}$ we set the polynomial to the true degree if the polynomial model is correct, and 1 otherwise. As recommended by their authors, we use a granularity of 20 for $\hat{\rho}^{\text{mix}}$ and minimum spanning tree to construct the binary graph for Z_w. Threshold for $\hat{\rho}^{\text{gen}}$ is computed using strategy described in Section 6 of Vogt et al. (2015). More details are in the Appendix.

Metrics and Results. We report the power and l_1 error of estimated change points. For fairness, power of all methods are computed via 500 permutations under significance level $\alpha = 0.05$. Due to space limit, detailed results on power are included in the Appendix - performance of all abrupt as well as gradual CPD methods are similar. In terms of localization, however, performance varies. In Table 1, the abrupt CPD methods (KCpA, Q, Z_w) have a large error in most settings, which is not surprising because KCpA, Q can be proved as inconsistent for some gradual changes. For $\hat{\rho}^{\text{poly}}, \hat{\rho}^{\text{one-side}}$ which require assumptions on the changing form, the localization is accurate when assumptions are satisfied, but poor otherwise. $\hat{\rho}^{\text{mix}}$ performs well in low dimensions and when the change (approximately) satisfies its assumption, but poorly when either one is violated. The proposed estimators $\hat{\rho}, \check{\rho}$ are robust across different settings and $\check{\rho}$ has improved performance over $\hat{\rho}$. $\hat{\rho}^{\text{gen}}$ is also significantly outperformed by $\check{\rho}$. Finally, note that $\hat{\rho}^{\text{mix}}, \hat{\rho}^{\text{gen}}$ are much more computationally expensive than the others.

6 Real Data Applications

Different from most machine learning tasks, there are currently no benchmarking dataset with human annotations for gradual CPD. Thus, we consider the applications introduced in Section 1, and compare our result with known external events and/or other CPD estimators.

Central England Temperature. The Central England Temperature (CET) record (Parker et al., 1992) under Open Government License is the oldest temperature record worldwide and is a valuable source for studying climate change. It contains the monthly mean temperature in central England from 1750 to 2020. Since there is a cycle of 12 months for the measurements, following Horváth et al. (1999), we view the data as $n = 271$ curves with 12 measurements on each curve. We set $k(y, y') = y^\top y'$ where $y, y' \in \mathbb{R}^{12}$. Using max-gap estimator, we identify 1827 as the change point (shown in red vertical line in Figure 1a), which roughly corresponds to the beginning of mass industrialization and is close to the 1850 estimated by Berkes et al. (2009).

S&P 500 Index. The S&P 500 is a stock market index which tracks the stock of 500 large US companies and is usually used as a benchmark of the overall market. We investigate the daily return data of the S&P 500 index[1] in two periods, one from 2008/01/02 to 2008/12/31 and another from 2019/06/03 to 2020/06/01. Both time periods contains a change point where volatility level gradually increases. Following Vogt et al. (2015), the daily return $Y_{t,T}$ roughly follows the volatility model (3) and our task is to identify changes in $\sigma(\cdot)$. We define the kernel as $k(y, y') = y^2(y')^2$ where $y, y' \in \mathbb{R}$. In both periods, we detect a change under $\alpha = 0.05$. The first period has an esimated change point 2008/09/16, following Lehman Brothers Bankruptcy in September 15 which is often viewed as a turning point in the crisis. The second period has an estimated change point 2020/02/24, days in the initial phase of the community spread of COVID-19 in the United States. The estimated change points are shown in red vertical lines in Figure 1b.

7 Related Work

Difference with Vogt et al. (2015). The major improvements of this work over Vogt et al. (2015) are discussed in detail in Section 1, 3. Other differences include: Vogt et al. (2015) allow correlated observations, while we assume independence; Vogt et al. (2015) uses estimator (3), while we propose a refined max-gap estimator that performs better empirically. We note that the our method can also be adapted for the correlated case, a possible direction for future work.

Abrupt CPD. Abrupt CPD methods assume the distribution remains stationary until the change point when it jumps to another distribution, and remains stationary there. There is a rich literature on them; see Niu et al. (2016); Aminikhanghahi and Cook (2017); Truong et al. (2020) for detailed surveys.

Kernel-based CPD methods. Existing kernel-based CPD methods all focus on the abrupt settings (Harchaoui et al., 2008; Arlot et al., 2012; Li et al., 2015; Garreau et al., 2018). We emphasize that their method is fundamentally different from ours, and, as far as we know, none of them produces a consistent localization estimator in settings considered in this paper.

CUSUM. The CUSUM principle was proposed by Page (1954) and has led to a rich literature. Some papers have investigated using CUSUM under gradual changes (Bissell, 1984a,b; Gan, 1992), but they considered only simple settings with a linear trend in the mean of univariate data, and their analyses are based mostly on empirical studies.

8 Discussion

We propose a general method to detect and to localize gradual changes in sequence data. Despite the relaxed assumptions, the proposed method is theoretically guaranteed, and the proposed max-gap estimator achieves good empirical performance. Note that the proposed method also works for abrupt CPD with Corollary 4.1 and Theorem 4.10 hold. In contrast, many abrupt CPD methods perform poorly in gradual change settings. The trade-off is that for abrupt changes or gradual changes with a known pattern (e.g., polynomial), our method often does not perform as good as the ones especially designed for those settings. There are no foreseeable negative social impacts of this work.

[1] S&P Dow Jones Indices LLC, S&P 500 [SP500], retrieved from https://finance.yahoo.com/quote/%5EGSPC/history/.

References

Aminikhanghahi, S. and D. J. Cook (2017). A survey of methods for time series change point detection. *Knowledge and information systems 51*(2), 339–367.

Arlot, S., A. Celisse, and Z. Harchaoui (2012). A kernel multiple change-point algorithm via model selection. *arXiv:1202.3878*.

Aue, A. and J. Steinebach (2002). A note on estimating the change-point of a gradually changing stochastic process. *Statistics & probability letters 56*(2), 177–191.

Berkes, I., R. Gabrys, L. Horváth, and P. Kokoszka (2009). Detecting changes in the mean of functional observations. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 71*(5), 927–946.

Bissell, A. (1984a). Estimation of linear trend from a cusum chart or tabulation. *Journal of the Royal Statistical Society: Series C (Applied Statistics) 33*(2), 152–157.

Bissell, A. (1984b). The performance of control charts and cusums under linear trend. *Journal of the Royal Statistical Society: Series C (Applied Statistics) 33*(2), 145–151.

Brown, B. M. et al. (1971). Martingale central limit theorems. *Annals of Mathematical Statistics 42*(1), 59–66.

Chu, L., H. Chen, et al. (2019). Asymptotic distribution-free change-point detection for multivariate and non-euclidean data. *The Annals of Statistics 47*(1), 382–414.

Dehling, H., A. Rooch, M. S. Taqqu, et al. (2017). Power of change-point tests for long-range dependent data. *Electronic Journal of Statistics 11*(1), 2168–2198.

Gan, F. (1992). Cusum control charts under linear drift. *Journal of the Royal Statistical Society: Series D (The Statistician) 41*(1), 71–84.

Garreau, D., S. Arlot, et al. (2018). Consistent change-point detection with kernels. *Electronic Journal of Statistics 12*(2), 4440–4486.

Goldenshluger, A., A. Tsybakov, A. Zeevi, et al. (2006). Optimal change-point estimation from indirect observations. *The Annals of Statistics 34*(1), 350–372.

Harchaoui, Z., E. Moulines, and F. Bach (2008). Kernel change-point analysis. *Advances in neural information processing systems 21*, 609–616.

Horváth, L., P. Kokoszka, and J. Steinebach (1999). Testing for changes in multivariate dependent observations with an application to temperature changes. *Journal of Multivariate Analysis 68*(1), 96–119.

Hušková, M. (1999). Gradual changes versus abrupt changes. *Journal of Statistical Planning and Inference 76*(1-2), 109–125.

Hušková, M. and J. Steinebach (2002). Asymptotic tests for gradual changes. *Statistics & Risk Modeling 20*(1-4), 137–152.

Inglot, T. and T. Ledwina (2006). Asymptotic optimality of new adaptive test in regression model. In *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, Volume 42, pp. 579–590. Elsevier.

Li, S., Y. Xie, H. Dai, and L. Song (2015). M-statistic for kernel change-point detection. In *Advances in Neural Information Processing Systems*, pp. 3366–3374.

Liu, P. and L. Ji (2014). Extremes of chi-square processes with trend. *arXiv preprint arXiv:1407.6501*.

Lombard, F. (1987). Rank tests for changepoint problems. *Biometrika 74*(3), 615–624.

Mallik, A., M. Banerjee, B. Sen, et al. (2013). Asymptotics for p-value based threshold estimation in regression settings. *Electronic Journal of Statistics 7*, 2477–2515.

Mallik, A., B. Sen, M. Banerjee, and G. Michailidis (2011). Threshold estimation based on ap-value framework in dose-response and regression settings. *Biometrika 98*(4), 887–900.

Matteson, D. S. and N. A. James (2014). A nonparametric approach for multiple change point analysis of multivariate data. *Journal of the American Statistical Association 109*(505), 334–345.

Mercurio, D., V. Spokoiny, et al. (2004). Statistical inference for time-inhomogeneous volatility models. *The Annals of Statistics 32*(2), 577–602.

Muller, H.-G. (1992). Change-points in nonparametric regression analysis. *The Annals of Statistics*, 737–761.

Niu, Y. S., N. Hao, and H. Zhang (2016). Multiple change-point detection: A selective overview. *Statistical Science*, 611–623.

Page, E. S. (1954). Continuous inspection schemes. *Biometrika 41*(1/2), 100–115.

Parker, D. E., T. P. Legg, and C. K. Folland (1992). A new daily central england temperature series, 1772–1991. *International Journal of Climatology 12*(4), 317–342.

Piterbarg, V. (1994). High excursions for nonstationary generalized chi-square processes. *Stochastic Processes and their Applications 53*(2), 307–337.

Quessy, J.-F. (2019). Consistent nonparametric tests for detecting gradual changes in the marginals and the copula of multivariate time series. *Statistical Papers 60*(3), 717–746.

Raimondo, M. (1998). Minimax estimation of sharp change points. *Annals of statistics*, 1379–1397.

Tewes, J. (2017). *Change-point tests and the bootstrap under long-and short-range dependence*. Ph. D. thesis, Ruhr-Universität Bochum.

Truong, C., L. Oudre, and N. Vayatis (2020). Selective review of offline change point detection methods. *Signal Processing 167*, 107299.

Van den Burg, G. J. J. and C. K. I. Williams (2020). An evaluation of change point detection algorithms. *arXiv preprint arXiv:2003.06222*.

Vogt, M. et al. (2012). Nonparametric regression for locally stationary time series. *The Annals of Statistics 40*(5), 2601–2633.

Vogt, M., H. Dette, et al. (2015). Detecting gradual changes in locally stationary processes. *The Annals of Statistics 43*(2), 713–740.

Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See Section 8.
   (c) Did you discuss any potential negative societal impacts of your work? [Yes] See Section 8.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes] See Appendix.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] All methods included in the experiment are unsupervised and choice of other hyperparameters are specified in Section 5 and the Appendix.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] See Section 6.
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   (c) Did you include any new assets either in the supplemental material or as a URL? [No]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A] Data are publicly available.
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]