ON A RELATION BETWEEN THE RATE-DISTORTION FUNCTION AND OPTIMAL TRANSPORT

Anonymous authors
Paper under double-blind review

ABSTRACT

We discuss a relationship between rate-distortion and optimal transport (OT) theory, even though they seem to be unrelated at first glance. In particular, we show that a function defined via an extremal entropic OT distance is equivalent to the rate-distortion function. We numerically verify this result as well as previous results that connect the Monge and Kantorovich problems to optimal scalar quantization. Thus, we unify solving scalar quantization and rate-distortion functions in an alternative fashion by using their respective optimal transport solvers.

Rate-Distortion. Let $X \sim P_X$ be the source supported on $\mathcal{X}$. Let $\mathcal{Y}$ be the reproduction space, and $\rho : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}$ be a distortion measure. The asymptotic limit on the minimum number of bits required to represent X with average distortion at most D is given by the rate-distortion function Cover & Thomas (2006), defined as

$$R(D) := \inf_{P_{Y|X}:\mathbb{E}_{P_{X,Y}}[\rho(X,Y)]\leq D} I(X;Y). \tag{1}$$

Any rate-distortion pair (R, D) satisfying $R > R(D)$ is achievable by some lossy source code, and no code can achieve a rate-distortion less than $R(D)$.

$R(D)$ has the following alternate form (Cover & Thomas, 2006, Ch. 10),

$$R(D) = \inf_{Q_Y} \inf_{P_{Y|X}:\mathbb{E}_{P_{X,Y}}[\rho(X,Y)]\leq D} D_{\mathsf{KL}}(P_{X,Y}||P_X \otimes Q_Y). \tag{2}$$

Due to the convex and strictly decreasing properties Cover & Thomas (2006) of $R(D)$, it suffices to fix $\lambda > 0$ and solve

$$\inf_{Q_Y} \inf_{P_{Y|X}} D_{\mathsf{KL}}(P_{X,Y}||P_X \otimes Q_Y) + \lambda \mathop{\mathbb{E}}_{P_{X,Y}}[\rho(X,Y)]. \tag{3}$$

A solution to (3) corresponds to a point on $R(D)$ corresponding to λ. The Blahut-Arimoto (BA) algorithm Blahut (1972); Arimoto (1972) solves (2) by alternating steps on $P_{Y|X}$ and Q_Y until convergence. Sweeping over λ gives the entire rate-distortion curve.

Optimal Transport. We consider optimal transport (OT) under the Kantorovich formulation, which finds the minimum distortion coupling π between measures μ and ν[1],

$$W(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \mathbb{E}_{X,Y\sim\pi}[\rho(X,Y)]. \tag{4}$$

Under certain conditions, the optimal coupling is induced by a fixed mapping, known as the Monge map. The Kantorovich problem is often regularized with an entropy term,

$$S_\epsilon(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \mathbb{E}_\pi[\rho(X,Y)] + \epsilon D_{\mathsf{KL}}(\pi||\mu \otimes \nu), \tag{5}$$

which is known as entropy-regularized optimal transport, with $\epsilon > 0$. For discrete measures μ, ν, (5) can be solved efficiently using the Sinkhorn algorithm Knopp & Sinkhorn (1967); Sinkhorn (1964).

Related Work. A connection between source coding and optimal transport was made in a talk given by Gray (2013), who discusses how scalar quantizers can be found through an extremal

[1] A joint distribution that marginalizes to μ and ν.

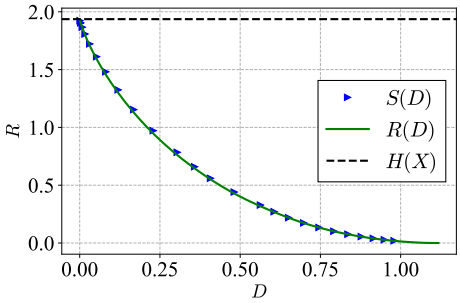

Figure 1: Equivalence of $S(D)$ and $R(D)$ on a 5-atom discrete source with $\rho(x, y) = (x - y)^2$.

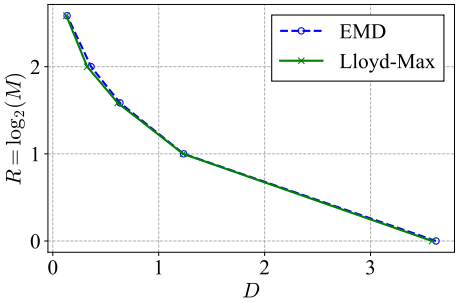

Figure 2: Equivalence of extremal EMD and Lloyd-Max for M-level scalar quantization.

Monge/Kantorovich problem, and alludes to a similar connection for Shannon's rate-distortion function. Here, we concretely provide $R(D)$'s connection with entropic OT and discuss how their respective computational methods (Blahut-Arimoto and Sinkhorn-Knopp) can compute $R(D)$. In a similar vein, we empirically verify the Gray (2013)'s results and show that Lloyd-Max and Earth Mover's distance can both compute optimal scalar quantizers.

Main Result. We first show that entropic OT can be used to upper bound $R(D)$. First, observe that the inner minimization problem in (3) looks similar to the entropic OT problem. Let us define

$$S(D) := \inf_{Q_Y} \inf_{\substack{\pi \in \Pi(P_X, Q_Y): \\ \mathbb{E}_\pi[\rho(X,Y)] \leq D}} D_{\mathsf{KL}}(\pi || P_X \otimes Q_Y), \tag{6}$$

which we call the *Sinkhorn-distortion function*, and is an extremal entropic OT distance w.r.t. P_X. Similar to $R(D)$, we can trace out $S(D)$ by sweeping over $\lambda > 0$, and solving the inner minimization (5), and then optimizing over all Q_Y, which is a convex problem in Q_Y Feydy et al. (2019). It is clear that $R(D) \leq S(D)$ by comparing (6) and (2). Next, we show that without further assumptions, $R(D)$ and $S(D)$ are equivalent.

Theorem 1. *For any source P_X and distortion function $\rho : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}$, it holds that*

$$R(D) = S(D). \tag{7}$$

See Sec. A.1 for the proof. We numerically verify the equivalence in Fig. 1 on a discrete source with 5 atoms under squared-error distortion. For $R(D)$, we use Blahut-Arimoto, and for $S(D)$, we solve the convex problem using SQP solvers Kraft (1988) with $Q \mapsto S_\epsilon(P_X, Q)$ as the objective function, showing that the two different objectives result in the same function.

Discussion. Observe that the joint $P_{X,Y} = P_X P_{Y|X}$ defined in (2) marginalizes to P_X but not necessarily Q_Y, whereas the coupling π in (6) marginalizes to both. This result says that the additional Q_Y marginalization constraint in $S(D)$ plays no role when both objectives are infimized over Q_Y. In computing $R(D)$, this provides an alternative to Blahut-Arimoto: solve (6) directly over Q_Y, using Sinkhorn iterations as a subroutine when evaluating the objective function (or its gradient). A symmetrized variant of the Sinkhorn-distortion function is often used to solve generative modeling tasks with Sinkhorn divergences Genevay et al. (2018); Salimans et al. (2018); Shen et al. (2020), where one wishes to find some $Q_Y \approx P_X$ by solving $\min_{Q_Y} S_\epsilon(P_X, Q_Y)$. However, if one leaves the objective un-symmetrized, the optimal Q_Y^* and coupling π^* are actually $R(D)$-achieving distributions with $\lambda = 1/\epsilon$.

We also verify that in discrete settings, the extremal non-entropic OT function $\min_{Q_Y : |Q_Y| \leq M} W(P_X, Q_Y)$, where $|Q_Y|$ is the size of Q_Y's alphabet, is equivalent to optimal scalar quantization of P_X as shown in Gray (2013). In Fig. 2, we solve the $\min_{Q_Y : |Q_Y| \leq M} W(P_X, Q_Y)$ on a 10-atom source using a linear program to compute the Earth Mover's distance (EMD) $W(\cdot, \cdot)$ and pass the function to a SQP solver as before. The achieved rate-distortion is equivalent to that of Lloyd-Max (M-means).

URM STATEMENT

The authors acknowledge that at least one key author of this work meets the URM criteria of ICLR 2023 Tiny Papers Track.

REFERENCES

S. Arimoto. An algorithm for computing the capacity of arbitrary discrete memoryless channels. *IEEE Transactions on Information Theory*, 18(1):14–20, 1972. doi: 10.1109/TIT.1972.1054753.

R. Blahut. Computation of channel capacity and rate-distortion functions. *IEEE Transactions on Information Theory*, 18(4):460–473, 1972. doi: 10.1109/TIT.1972.1054855.

Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory (Wiley Series in Telecommunications and Signal Processing)*. Wiley-Interscience, USA, 2006. ISBN 0471241954.

Jean Feydy, Thibault Séjourné, François-Xavier Vialard, Shun-ichi Amari, Alain Trouvé, and Gabriel Peyré. Interpolating between optimal transport and mmd using sinkhorn divergences. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pp. 2681–2690. PMLR, 2019.

Aude Genevay, Gabriel Peyre, and Marco Cuturi. Learning generative models with sinkhorn divergences. In Amos Storkey and Fernando Perez-Cruz (eds.), *Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics*, volume 84 of *Proceedings of Machine Learning Research*, pp. 1608–1617. PMLR, 09–11 Apr 2018.

Robert M. Gray. Transportation distance, shannon information, and source coding. GRETSI 2013 Symposium on Signal and Image Processing, 2013. URL `https://ee.stanford.edu/~gray/gretsi.pdf`.

Paul Knopp and Richard Sinkhorn. Concerning nonnegative matrices and doubly stochastic matrices. *Pacific Journal of Mathematics*, 21(2):343 – 348, 1967. doi: pjm/1102992505. URL `https://doi.org/`.

D. Kraft. *A Software Package for Sequential Quadratic Programming*. Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt Köln: Forschungsbericht. Wiss. Berichtswesen d. DFVLR, 1988. URL `https://books.google.com/books?id=4rKaGwAACAAJ`.

Gabriel Peyré and Marco Cuturi. Computational optimal transport: With applications to data science. *Foundations and Trends® in Machine Learning*, 11(5-6):355–607, 2019. ISSN 1935-8237. doi: 10.1561/2200000073. URL `http://dx.doi.org/10.1561/2200000073`.

Tim Salimans, Han Zhang, Alec Radford, and Dimitris Metaxas. Improving gans using optimal transport. *arXiv preprint arXiv:1803.05573*, 2018.

Zebang Shen, Zhenfu Wang, Alejandro Ribeiro, and Hamed Hassani. Sinkhorn natural gradient for generative models. *Advances in Neural Information Processing Systems*, 33:1646–1656, 2020.

Richard Sinkhorn. A Relationship Between Arbitrary Positive Matrices and Doubly Stochastic Matrices. *The Annals of Mathematical Statistics*, 35(2):876 – 879, 1964. doi: 10.1214/aoms/1177703591. URL `https://doi.org/10.1214/aoms/1177703591`.

A APPENDIX

A.1 PROOFS

Theorem 1. *For any source P_X and distortion function $\rho : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}$, it holds that*

$$R(D) = S(D). \tag{7}$$

Proof. From (Cover & Thomas, 2006, Ch. 9), the optimizers $Q_Y^*, P_{Y|X}^*$ of (3) for a fixed $\lambda > 0$ satisfy

$$\frac{dP_{Y|X=x}^*}{dQ_Y}(x,y) = \frac{e^{-\lambda\,\rho(x,y)}}{\int_{\mathcal{Y}} e^{-\lambda\,\rho(x,\tilde{y})} dQ_Y^*}, \tag{8}$$

$$Q_Y^* = \int_{\mathcal{X}} dP_{Y|X}^* dP_X, \tag{9}$$

simultaneously, which achieves a unique point on $R(D)$ corresponding to λ. To show that $S(D)$ achieves the same objective as $R(D)$ on the same P_X and distortion measure, it suffices to show that the $R(D)$-optimal Q_Y^* and $P_{Y|X}^*$ are feasible for $S(D)$, since $R(D) \leq S(D)$. From (Peyré & Cuturi, 2019, Ch. 4, Prop. 4.3), the optimal coupling π^* in entropic OT is unique and has the form

$$\frac{d\pi^*}{dP_X dQ_Y}(x,y) = u(x)e^{-\lambda\,\rho(x,y)}v(y), \tag{10}$$

where $u(x), v(y)$ are dual variables that ensure π^* is a valid coupling. The $R(D)$-optimal joint distribution $P_X P_{Y|X}^*$, which is guaranteed to be a coupling between P_X and Q_Y^* due to (9), indeed has the form

$$\frac{dP_X P_{Y|X}^*}{dP_X dQ_Y^*}(x,y) = \frac{1}{\int_{\mathcal{Y}} e^{-\lambda\,\rho(x,y')} dQ_Y^*} \cdot e^{-\lambda\,\rho(x,y)} \cdot 1, \tag{11}$$

where the first term only depends on x and the last term only depends on y. Since $R(D)$ is a lower bound of $S(D)$, we are done. □