Policy Gradient Algorithms Implicitly Optimize by Continuation

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Abstract

Direct policy optimization in reinforcement learning is usually solved with policy-gradient algorithms, which optimize policy parameters via stochastic gradient ascent. This paper provides a new theoretical interpretation and justification of these algorithms. First, we formulate direct policy optimization in the optimization by continuation framework. The latter is a framework for optimizing nonconvex functions where a sequence of surrogate objective functions, called continuations, are locally optimized. Second, we show that optimizing affine Gaussian policies and performing entropy regularization can be interpreted as implicitly optimizing deterministic policies by continuation. Based on these theoretical results, we argue that exploration in policy-gradient algorithms consists in computing a continuation of the return of the policy at hand, and that the variance of policies should be history-dependent functions adapted to avoid local extrema rather than to maximize the return of the policy.

1 Introduction

Applications where one has to control an environment are numerous and solving these control problems efficiently is the preoccupation of many researchers and engineers. Reinforcement learning (RL) has emerged as a solution when the environments at hand have complex and stochastic dynamics (Sutton & Barto, 2018). Direct policy optimization and more particularly (on-policy) policy gradients are methods that have been successful in recent years. These methods, reviewed by Duan et al. (2016) and Andrychowicz et al. (2020), all consist in parameterizing a policy (most often with a neural network) and adapting the parameters with a local-search algorithm in order to maximize the expected sum of rewards received when the policy is executed, called the return of the policy. We distinguish two basic elements that determine the performance of these methods. As first element, we have the formalization of the optimization problem. It is defined through two main choices: the (functional) parametrization of the policy and the learning objective function, which mostly relies on adding an entropy regularization term to the return. As second element, there is the choice of the local-search algorithm to solve the optimization problem – we focus on stochastic gradient ascent methods in this study.

The policy parameterization is the first formalization choice. In theory, there exists an optimal (parametric) deterministic policy (Sutton & Barto, 2018), which can be optimized by deterministic policy gradient (Silver et al., 2014) with a guarantee of converging towards a stationary solution (Xiong et al., 2022). However, this approach gives poor results in practice as it is subject to convergence towards local optima (Silver et al., 2014). It is therefore usual to optimize stochastic policies where this problem is mitigated in practice (Duan et al., 2016; Andrychowicz et al., 2020). For discrete state and action spaces, theoretical guarantees of global convergence hold for softmax or direct policy parameterization (Bhandari & Russo, 2019; Zhang et al., 2021; Agarwal et al., 2020). In the general case of continuous spaces, these results no longer hold and only convergence towards stationarity can be ensured under strong hypotheses (Bhatt et al., 2019; Zhang et al., 2020b; Bedi et al., 2021). Recently, convergence under milder assumptions was established assuming that the policy follows a heavy-tailed distribution, which guarantees a sufficiently spread distribution of actions (Bedi et al., 2022). Nevertheless, most of the empirical works have focused on (light-tailed) Gaussian policies (Duan et al., 2016; Andrychowicz et al., 2020) for which convergence is thus not ensured in the general case

(Bedi et al., 2022). The importance of a sufficiently spread distribution in policy gradient had already been observed in early works and was loosely interpreted as exploration (Lillicrap et al., 2015; Mnih et al., 2016). This concept originally introduced in bandit theory and value-based RL, where it consists in selecting a suboptimal action to execute in order to refine a statistical estimate (Simon, 1955; Sutton & Barto, 2018), is to our knowledge not well defined for direct policy optimization. Other empirical works also showed that relying on Beta distributions when the set of actions is bounded within an interval outperformed Gaussian policies (Chou et al., 2017; Fujita & Maeda, 2018). As a side note, another notable advantage of stochastic policies is the possibility to rely on information geometry and use efficient trust-region methods to speed up the local-search algorithms (Shani et al., 2020; Cen et al., 2022). In summary, no consensus has yet been reached on the exact policy parameterization that should be used in practice.

The second formalization choice is the learning objective and more particularly the choice of entropy regularization. Typically, a bonus enforcing the uniformity of the action distribution is added to the rewards in the objective function (Williams & Peng, 1991; Haarnoja et al., 2019). Intuitively, it avoids converging too fast towards policies with small spread, which are subject to being locally optimal. More general entropy regularizations were applied for encouraging high-variance policies while keeping the distribution sparse (Nachum et al., 2016) or enforcing the uniformity of the state-visitation distribution in addition to the action distribution (Islam et al., 2019). Again, no consensus is reached about the best regularization to use in practice.

The importance of introducing sufficient stochasticity and regularizing entropy is commonly accepted in the community. Some preliminary research has been conducted to develop a theoretical foundation for this observation. Ahmed et al. (2019) proposed an empirical analysis of the impact of the entropy regularization term. They concluded that adding this term yields a smoothed objective function. A local-search algorithm will therefore be less prone to convergence to local optima. This problem was also studied by Husain et al. (2021). They proved that optimizing a policy by regularizing the entropy is equivalent to performing a robust optimization against changes in the reward function. This result was recently reinterpreted by Brekelmans et al. (2022) who deduced that the optimization is equivalent to a game where one player adapts the policy while an adversary adapts the reward. The research papers that have been reviewed concentrate solely on learning objectives in the context of entropy regularization, leaving unanswered the question of the relationship between a policy's return and the distribution of actions. This question is of paramount importance for understanding how the formalization of the direct policy optimization problem impacts the resulting control strategy.

In this work, we propose a new theoretical interpretation of the effects of the action distribution on the objective function. Our analysis is based on the theory of optimization by continuation (Allgower & Georg, 1980), which consists in locally optimizing a sequence of surrogate objective functions. The latter are called continuations and are often constructed by filtering the optimization variables in order to remove local optima. Our main contributions are twofold. First, we define a continuation for the return of policies and formulate direct policy optimization in the optimization by continuation framework. Second, based on this framework, we study different formulations, i.e., policy parameterization and entropy regularization, of direct policy optimization. Several conclusions are drawn from the analysis. First, we show that the continuation of the return of a deterministic policy is equal to the return of a Gaussian policy. Second, we show that the continuation of the return of a Gaussian policy equals the return of another Gaussian policy with scaled variance. We then derive from the previous results that optimizing Gaussian policies using policy-gradient algorithms and performing regularization can be interpreted as optimizing deterministic policies by continuation. In this regard, exploration as it is usually understood in policy gradients, consists in computing the continuation of the return of the policy at hand. Finally, we show that for a more general continuation, the continuation of the return of a deterministic policy equals the return of a Gaussian policy where the variance is a function of the observed history of states and actions. These results provide a new interpretation for the variance of a policy: it can be seen as a parameter of the policy-gradient algorithm instead of an element of the policy parameterization. Moreover, to fully exploit the power of continuations, the variance of a policy should be a history-dependent function iteratively adapted to avoid the local extrema of the return.

Although there is no theoretical guarantee that optimization by continuation converges towards a global optimum, it has been successfully applied to several machine learning applications including image alignment

(Mobahi et al., 2012), greedy layerwise training of neural networks (Bengio, 2009), and neural network training by iteratively increasing the non-linearity of the activation functions (Pathak & Paffenroth, 2019). To our knowledge, it has never yet been applied for direct policy optimization. However, optimizing a distribution over the policy parameters rather than directly optimizing the policy is an RL technique that has been used to perform direct policy optimization (Sehnke et al., 2010; Salimans et al., 2017; Zhang et al., 2020a). Among other things, it decreases the variance of the gradient estimates in some cases. It is furthermore equivalent to optimizing the policy by Gaussian continuation (Mobahi et al., 2012; Hazan et al., 2016; 2019). Here the continuation is the convolution of the return by a Gaussian. Another method, called RL with logistic reward-weighted regression (Wierstra et al., 2008; Peters & Schaal, 2007), consists in optimizing a surrogate objective of the return. The surrogate is the expected utility of the sum of rewards. Originally justified relying on the field of decision theory (Chernoff & Moses, 2012), it can equivalently be seen as an optimization by continuation method.

The paper is organized as follows. In Section 2, the background of direct policy optimization is reminded. The framework for optimizing policies by continuation is developed in Section 3 and theoretical results relating the return of policies to their continuations are presented in Section 4. In Section 5, these results are used for elaborating on the formulations of direct policy optimization. Finally, the results are summarized and further works discussed in Section 6.

2 Theoretical Background

In this section, we remind the background of RL in Markov decision processes and discuss the direct policy optimization problem.

2.1 Markov Decision Processes

We study problems in which an agent makes sequential decisions in a stochastic environment in order to maximize an expected sum of rewards (Sutton & Barto, 2018). The environment is modeled with an infinite-time Markov Decision Process (MDP) composed of a state space $\mathcal{S}$, an action space $\mathcal{A}$, an initial state distribution with density p_0, a transition distribution (dynamic) with conditional density p, a bounded reward function ρ, and a discount factor $\gamma \in [0, 1[$. When an agent interacts with the MDP $(\mathcal{S}, \mathcal{A}, p_0, p, \rho, \gamma)$, first, an initial state $s_0 \sim p_0(\cdot)$ is sampled, then, the agent provides at each time step t an action $a_t \in \mathcal{A}$ leading to a new state $s_{t+1} \sim p(\cdot|s_t, a_t)$. Such a sequence of states and actions $h_t = (s_0, a_0, \ldots, s_{t-1}, a_{t-1}, s_t) \in H$ is called a history and H is the set of all histories of any arbitrary length. In addition, at each time step t, a reward $r_t = \rho(s_t, a_t) \in \mathbb{R}$ is observed.

A (stochastic) history-dependent policy $\eta \in \mathcal{E} = H \to \mathcal{P}(\mathcal{A})$ is a mapping from the set of histories H to the set of probability measures on the action space $\mathcal{P}(\mathcal{A})$, where $\eta(a|h)$ is the associated conditional probability density of action a given the history h. A (stochastic) Markov policy $\pi \in \Pi = \mathcal{S} \to \mathcal{P}(\mathcal{A})$ is a mapping from the state space $\mathcal{S}$ to the set of probability measures on the action space $\mathcal{P}(\mathcal{A})$, where $\pi(a|s)$ is the associated conditional probability density of action a in state s. Finally, deterministic policies $\mu \in M = \mathcal{S} \to \mathcal{A}$ are functions mapping an action $a = \mu(s) \in \mathcal{A}$ to each state $s \in \mathcal{S}$. We note that for each deterministic policy μ there exists an equivalent Markov policy, where the probability measure is a Dirac measure on the action $a = \mu(s)$ in each state s. In addition, for each Markov policy, there exists an equivalent history-dependent policy only accounting for the last state in the history. We therefore write by abuse of notation that $M \subsetneq \Pi \subsetneq \mathcal{E}$.

The function $J : \mathcal{E} \to \mathbb{R}$ is defined as the function mapping to any policy η the expected discounted cumulative sum of rewards gathered by an agent interacting in the MDP by sampling actions from the policy η. The value $J(\eta)$ is called the return of the policy η and is computed as follows:

$$J(\eta) = \mathop{\mathbb{E}}_{\substack{s_0 \sim p_0(\cdot) \\ a_t \sim \eta(\cdot|h_t) \\ s_{t+1} \sim p(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t \rho(s_t, a_t) \right] . \tag{1}$$

An optimal agent follows an optimal policy η^* maximizing the expected discounted sum of rewards J.

2.2 Direct Policy Optimization

Problem statement. Let $(\mathcal{S}, \mathcal{A}, p_0, p, \rho, \gamma)$ be an MDP and let $\eta_\theta \in \mathcal{E}$ be a policy parameterized by the real vector $\theta \in \mathbb{R}^{d_\Theta}$. The objective of the optimization problem is to find the optimal parameter $\theta^* \in \mathbb{R}^{d_\Theta}$ such that the return of the policy is maximized:

$$\theta^* = \underset{\theta \in \mathbb{R}^{d_\Theta}}{\operatorname{argmax}} \; J(\eta_\theta) \; . \tag{2}$$

In this work, we consider on-policy policy-gradient algorithms (Andrychowicz et al., 2020). These algorithms optimize differentiable policies with local-search methods using the derivatives of the policies. They iteratively repeat two operations. First, they approximate an ascent direction relying on histories sampled from the policy, with the current parameters, in the MDP. Second, they update these parameters in the ascent direction.

Deterministic Policies. In an MDP, there always exists an optimal deterministic policy μ^* performing at least as well as any other Markov and history-dependent policy (Sutton & Barto, 2018). It is therefore theoretically possible to find an optimal policy by solving the optimization problem described in equation equation 2 where the parameterized policy is a (universal) function approximator $\mu_\theta \in M$. In practice, optimizing deterministic policies with policy-gradient methods usually results in locally optimal policies with poor performance (Silver et al., 2014).

Gaussian Policies. In direct policy optimization, most of the works focus on learning Markov policies where the actions follow a Gaussian distribution whose mean and covariance matrix are parameterized function approximators (Duan et al., 2016; Andrychowicz et al., 2020). More precisely, a parametrized Gaussian policy $\pi_\theta^{GP} \in \Pi$ is a policy where the actions follow a Gaussian distribution of mean $\mu_\theta(s)$ and covariance matrix $\Sigma_\theta(s)$ for each state s and parameter θ, it thus has the following density:

$$\pi_\theta^{GP}(a|s) = \mathcal{N}(a|\mu_\theta(s), \Sigma_\theta(s)) \; . \tag{3}$$

Affine Policies. A parameterized policy (deterministic or stochastic) is said to be affine, if the function approximators used to construct the functional form of the policy are affine functions of the parameter θ. Formally, each function approximator f_θ of a history-dependent policy has the following form $\forall h \in H$:

$$f_\theta(h) = a(h)^T \theta + b(h) \; , \tag{4}$$

where a and b are functions building features from the histories. Such policies are often considered in theoretical studies (Busoniu et al., 2017) and perform well on complex tasks in practice (Rajeswaran et al., 2017).

3 Optimizing Policies by Continuation

In this section, we introduce the optimization by continuation methods and formulate direct policy optimization in this framework.

3.1 Optimization by Continuation

Optimization by continuation (Allgower & Georg, 1980) is a technique used to optimize nonconvex functions with the objective of avoiding local extrema. A sequence of optimization problems is solved iteratively using the optimum of the previous iteration. Each problem consists in optimizing a deformation of the original function and is typically solved by local search. Through the iterations, the function is less and less deformed. Such procedure is also sometimes referred to as graduated optimization (Blake & Zisserman, 1987) or optimization by homotopy (Watson & Haftka, 1989), as the homotopy of a function refers to its deformation in topology.

Formally, let $f : \mathcal{X} \to \mathbb{R}$ be the real-valued function to optimize. Let $g : \mathcal{Y} \to \mathbb{R}$ be another real-valued function used for building the deformation of f. Finally, let the conditional distribution function $p : \mathcal{X} \to$

$\mathcal{P}(\mathcal{Y})$ be the mapping from an optimization variable $x \in \mathcal{X}$ to the set of probability measures $\mathcal{P}(\mathcal{Y})$, such that $p(y|x)$ is the associated density function for any random event $y \in \mathcal{Y}$ given $x \in \mathcal{X}$. The continuation of the function f under the distribution p and deformation function g is defined as the function $f^p : \mathcal{X} \to \mathbb{R}$ such that $\forall x \in \mathcal{X}$:

$$f^p(x) = \mathop{\mathbb{E}}_{y \sim p(\cdot|x)} [g(y)] \ . \tag{5}$$

For the optimization by continuation described hereafter, there must exist a conditional distribution p^* for which f^p equals f in the limit as p approaches p^*. A typical example is to choose the function g equal to f, and to use a Gaussian distribution with a constant diagonal covariance matrix for the distribution p. We then have so-called Gaussian continuations Mobahi & Fisher III (2015).

Finally, optimizing a function f by continuation involves iteratively locally optimizing its continuation for a sequence of conditional distributions approaching p^* with decreasing spread. Formally, let $p_0 \succ p_1 \succ \cdots \succ p_{I-1}$ be a sequence of conditional distributions (monotonically) approaching p^* with strictly decreasing covariance matrices[1]. Then, optimizing f by continuation consists in locally optimizing its continuation f^{p_i} with a local-search algorithm initialized at x_i^* for each iteration i. This general procedure is summarized in Algorithm 1. Particular instances of this algorithm are described by Hazan et al. (2016) and Shao et al. (2019) for Gaussian continuations. algorithm algorithmic

Algorithm 1 Optimization by Continuation

1: Provide a sequence of I functions $p_0 \succ p_1 \succ \cdots \succ p_{I-1}$
2: Provide an initial variable value $x_0^* \in \mathcal{X}$ for the local search
3: **for** $i = 0, 1, \ldots, I-1$ **do**
4: $\quad x_{i+1}^* \leftarrow$ Optimize the continuation f^{p_i} by local search initialized at x_i^*
5: **end for**
6: **return** x_I^*

3.2 Continuation of the Return of a Policy

The direct policy optimization problem usually consists in maximizing a nonconvex function. Optimization by continuation is thus a good candidate for computing a solution. In this section, we introduce a novel continuation adapted to the return of policies.

The return of a policy depends on the probability of a sequence of actions through the product of the density $\eta_\theta(a_t|s_t)$ of each action a_t for a given parameter θ, see equation equation 1. We define the continuation of interest as the expectation of the return where each factor in the product of densities depends on a different parameter vector. This expectation is taken according to a distribution that disturbs these parameter vectors at each time step with a variance depending on the history. Formally, using the notations from Section 3.1, we optimize the function f that for all $x = \theta$ equals the return, $f(\theta) = J(\pi_\theta)$, over the set $\mathcal{X} = \mathbb{R}^{d_\Theta}$. Let the covariance function $\Sigma : H \to \mathbb{R}^{d_\Theta \times d_\Theta}$ be a function mapping a history $h_t \in H$ to a covariance matrix $\Sigma(h_t)$. Let the continuation distribution q be a distribution such that $q(\theta_t|\theta, \Sigma(h_t))$ is the density of θ_t distributed with mean θ and covariance matrix $\Sigma(h_t)$. Then, let $\mathcal{Y} = \left(\mathcal{S} \times \mathcal{A} \times \mathbb{R}^{d_\Theta}\right)^{\mathbb{N}}$ be the set of (infinite) sequences of states, actions and parameters and let p and g, the two functions defining the continuation, be as follows:

$$p(y|x) = p(s_0) \prod_{t=0}^{\infty} \eta_{\theta_t}(a_t|h_t) p_\theta(\theta_t|h_t) p(s_{t+1}|s_t, a_t) \tag{6}$$

$$g(y) = \sum_{t=0}^{\infty} \gamma^t \rho(s_t, a_t) \ , \tag{7}$$

[1]In this work, we consider the L^2-norm of functions and the Loewner order over the set of covariance matrices (Siotani, 1967).

where $p_\theta(\theta_t|h_t) = q(\theta_t|\theta, \Sigma(h_t))$ such that the spread of p_θ depends on the function Σ. Taken together, the continuation $f^q_\Sigma = f^p$ of the return of the policy $\eta_\theta \in \mathcal{E}$ corresponding to the distribution q and covariance function Σ, is defined $\forall \theta \in \mathbb{R}^{d_\Theta}$ as:

$$f^q_\Sigma(\theta) = \mathop{\mathbb{E}}_{\substack{s_0 \sim p_0(\cdot) \\ \theta_t \sim q(\cdot|\theta, \Sigma(h_t)) \\ a_t \sim \eta_{\theta_t}(\cdot|h_t) \\ s_{t+1} \sim p(\cdot|s_t, a_t)}} \left[\sum_{t=0}^{\infty} \gamma^t \rho(s_t, a_t) \right] . \tag{8}$$

Finally, the continuation equation equation 8 converges towards the return of η_θ in the limit as the covariance function Σ approaches zero, as required in Section 3.1.

This continuation is expected to be well-suited for removing local extrema of the return for three main reasons. First, marginalizing the variables of a function as in our continuation is expected to smooth this function and therefore remove local extrema – the particular case of Gaussian blurring has been widely studied in the literature (Mobahi & Fisher, 2015; Nesterov & Spokoiny, 2017). Second, we underline the interest of considering a continuation in which the disturbance of the policy parameters may vary based on the time step. Indeed, changing the parameter vector of the policy at different time steps (and changing the action distributions) may modify the objective function in significantly different ways. Third, we justify the factorization of the conditional distribution p_θ equation equation 6 by the causal effect of actions in the MDP. As the actions only influence the rewards to come, the past history is expected to provide a sufficient statistic for disturbing the parameters in order to remove local optima. We therefore chose parameter probabilities conditionally independent given the past history. This history-dependency is encoded through the covariance function Σ in equation equation 8.

Maximizing f^q_Σ to solve the optimization problem from Algorithm 1 is a complicated task. A common local-search algorithm used in machine learning is stochastic gradient ascent, which is known for performing well on several complex functions depending on many variables (Bottou, 2010). The gradient of f^q_Σ can be computed by Monte-Carlo sampling applying the reparameterization trick (Goodfellow et al., 2016) for simple continuation distributions or relying on the REINFORCE trick (Williams, 1992) in the more general case. Due to the complex time dependencies of the random events, these vanilla gradient estimates have practical limitations: the estimates may have large variance, the infinite horizon shall be truncated, and the direction provided is computed in the Euclidean space of parameters rather than in a space of distributions (Peters & Schaal, 2008). Finally, the evaluation of the continuation and its derivatives require one to sample parameter vectors, which may be computationally expensive for complex high-dimensional distributions. The study of different continuation distributions and the application of the optimization procedure from Algorithm 1 to practical problems is left for further works. In this work, we rather rely on the continuation to study existing direct policy optimization algorithms. To this end, we show in the next section that maximizing the continuation defined by equation equation 8 is equivalent to solving a direct policy optimization problem for another policy, called a mirror policy.

4 Mirror Policies and Continuations

This section is dedicated to the interpretation of the continuation of the return of a policy. We show it equals the return of another policy, called a mirror policy. The existence and closed form of mirror policies is also discussed.

4.1 Optimizing by Continuation with Mirror Policies

Definition 1. *Let $(\mathcal{S}, \mathcal{A}, p_0, p, \rho, \gamma)$ be an MDP and let $\eta_\theta \in \mathcal{E}$ be a history-dependent policy parameterized with the vector $\theta \in \mathbb{R}^{d_\Theta}$. In addition, let f^q_Σ be the continuation of the return of the policy η_θ corresponding to a continuation distribution q and covariance function Σ as defined in equation equation 8. We call a mirror policy of the original policy η_θ, under the continuation distribution q and covariance function Σ, any history-dependent policy $\eta'_\theta \in \mathcal{E}$ such that $\forall \theta \in \mathbb{R}^{d_\Theta}$:*

$$f^q_\Sigma(\theta) = J(\eta'_\theta) . \tag{9}$$

Let us assume we are provided with the continuation f_Σ^q of the return of an original policy η_θ depending on the parameter θ that shall be optimized. In addition, let us assume we can compute a mirror policy η'_θ for the original policy η_θ. By Definition 1, the continuation of the original policy equals the return of the mirror policy for all θ. In addition, under smoothness assumptions, all their derivatives are equal too. Therefore, maximizing the continuation of an original policy by stochastic gradient ascent can be performed by maximizing the return of its mirror policy by policy gradient. Applying state-of-the-art policy-gradient algorithms on the mirror policies for optimizing the continuations in Algorithm 1 may alleviate several of the shortcomings of the optimization procedure described earlier.

4.2 Existence and Closed Form of Mirror Policies

In this section, we first show that there always exists a mirror policy. In addition, several closed forms are provided depending on the original policy, the continuation distribution, and the covariance function.

Theorem 1. *For any original history-dependent policy $\eta_\theta \in \mathcal{E}$ parameterized with the vector $\theta \in \mathbb{R}^{d_\Theta}$ and for any continuation distribution q and covariance function Σ, there exists a mirror history-dependent policy $\eta'_\theta \in \mathcal{E}$ of the original policy η_θ that writes as:*

$$\eta'_\theta(a|h) = \mathop{\mathbb{E}}_{\theta' \sim q(\cdot|\theta, \Sigma(h))} \left[\eta_{\theta'}(a|h)\right] \ . \tag{10}$$

Theorem 1 guarantees the existence of mirror policies. Such a mirror policy is a function depending on the same parameters as its original policy but that has a different functional form and may therefore provide actions following a different distribution compared to the original policy.

Theorem 1 leads to two important corollary results. First, as demonstrated in Appendix A, let η'' be a mirror policy of η' and let η' be a mirror policy of the original policy η of the form of equation equation 10. Then, there exists a continuation for which η'' is a mirror policy of the original policy η. It follows that the return of the mirror policy of another mirror policy is itself equal to a continuation of the original policy. Second, Theorem 1 also reveals that for a given original policy and continuation distribution, the variance of the mirror policy is defined through the continuation covariance function Σ. Furthermore, we remind that the variance of the continuation is an hyperparameter that shall be selected for each iteration of the optimization by continuation, see Section 3. This choice of hyperparameter is thus reflected as the choice of the variance of a mirror policy. The expert making this choice sees the effect of the disturbed parameters on the environment through the variance of the mirror policy. From a practical perspective, it is probably easier to quantify the effect on the local extrema depending on the variance of the mirror policy rather than depending on the variance of the continuation.

Property 1. *Let the original policy $\pi_\theta \in \Pi$ be a Markov policy and let the covariance function depend solely on the last state in the history. Then, there exists a mirror Markov policy $\pi'_\theta \in \Pi$.*

Property 1 is an intermediate result providing sufficient assumptions on the continuation for having mirror Markov policies. Note that for this type of continuation, the parameters of the policy are disturbed independently on the history followed by the agent.

Property 2. *Let the original policy $\pi_\theta^{GP} \in \Pi$ be a Gaussian policy as defined in equation equation 3 with affine function approximators. Let the covariance function depend solely on the last state in the history and let the distribution q be a Gaussian distribution. Then, there exists a mirror Markov policy $\pi'_\theta \in \Pi$ such that for all states $s \in \mathcal{S}$, it converges towards a Gaussian policy in the limit as the affine coefficients of the covariance matrix $\Sigma_\theta(s)$ approaches zero ($\|\nabla_\theta \Sigma_\theta(s)\| \to 0$):*

$$\pi'_\theta(a|s) \to \mathcal{N}(a|\mu_\theta(s), \Sigma'_\theta(s)) \ , \tag{11}$$

where $\Sigma'_\theta(s) = C_\theta(s) + \Sigma_\theta(s)$ and $C_\theta(s) = \nabla_\theta \mu_\theta(s)^T \, \Sigma(s) \, \nabla_\theta \mu_\theta(s)$.

Under the assumptions of Property 2, a mirror policy can be approached by a policy that only differs from the original one by having a variance which is increased by the term $C_\theta(s)$ proportional to the variance of

the continuation. In particular, when the variance of the original policy π_θ^{GP} is solely dependent on the state, then $\|\nabla_\theta \Sigma_\theta(s)\| = 0$ and $\pi'_\theta(a|s) = \mathcal{N}(a|\mu_\theta(s), \Sigma'_\theta(s))$. In this case, for any θ, the covariance matrix of this mirror policy is additionally bounded from below such that $\Sigma'_\theta(s) \succeq C_\theta(s)$.

Property 3. *Let the original policy $\mu_\theta \in M$ be an affine deterministic policy. Let the covariance function depend solely on the last state in the history and let the distribution q be a Gaussian distribution. Then, the Markov policy $\pi_\theta^{GP'} \in \Pi$ is a mirror policy:*

$$\pi_\theta^{GP'}(a|s) = \mathcal{N}(a|\mu_\theta(s), \Sigma'_\theta(s)) \,, \tag{12}$$

where $\Sigma'_\theta(s) = \nabla_\theta \mu_\theta(s)^T \, \Sigma(s) \, \nabla_\theta \mu_\theta(s)$.

Therefore, under some assumptions, disturbing a deterministic policy and optimizing it afterwards can be interpreted as optimizing the continuation of the return of this policy.

Property 4. *Let the original policy $\mu_\theta \in M$ be an affine deterministic policy. Let the distribution q be a Gaussian distribution. Then, the policy $\eta'_\theta \in \mathcal{E}$ is a mirror policy:*

$$\eta'_\theta(a|h) = \mathcal{N}(a|\mu_\theta(s), \Sigma'_\theta(h)) \,, \tag{13}$$

where $\Sigma'_\theta(h) = \nabla_\theta \mu_\theta(s)^T \, \Sigma(h) \, \nabla_\theta \mu_\theta(s)$.

Property 4 extends Property 3 to more general continuation distributions. This extension is used later to justify the interest of optimizing history-dependent policies in order to optimize an underlying deterministic policy by continuation. The theorem and properties are shown in Appendix B.

5 Implicit Optimization by Continuation

In this section, two formulations, i.e., a parameterized policy and a learning objective each, used by several policy-gradient algorithms are analyzed relying on original and mirror policies. In Section 5.1, we show that optimizing each formulation by local search corresponds to optimizing a continuation. The optimized policy is thus the mirror policy of an unknown original policy. We show the existence of the corresponding continuation and original policy and discuss their closed form. This analysis provides a novel interpretation of the state-of-the-art algorithms for direct policy optimization. We discuss the role of stochastic policies in light of this interpretation in Section 5.2.

5.1 Gaussian Policies and Regularization

The policy-gradient literature has mainly focused on optimizing two problem formulations by local search – typically with stochastic gradient ascent and (approximate) trust-region methods. First, the vast majority of works focuses on optimizing the return of Gaussian policies (Duan et al., 2016; Andrychowicz et al., 2020). Second, in many formulations this objective function is extended by adding a bonus to the entropy of the optimized policy (Williams & Peng, 1991; Haarnoja et al., 2019). We show that when optimizing a policy according to these formulations, there exists an (unknown) deterministic original policy and a continuation such that the optimized policy is a mirror policy. Provided with the local-search algorithm from the policy-gradient method, we conclude that optimizing both formulations is equivalent to implicitly optimizing a deterministic policy by continuation.

Gaussian Policies. First, we remind that under Property 3, for any affine deterministic policy μ_θ there exists an affine Gaussian mirror policy $\pi_\theta^{GP'}$ as defined by equation equation 12. For a decreasing sequence of continuation covariances, optimizing the policy μ_θ by continuation thus consists in optimizing the return of mirror policies for each covariance matrix by local search. In this paragraph, we study the inverse of the latter statement: *under which conditions does optimizing a Gaussian policy is equivalent to (implicitly) optimizing an (unknown) deterministic policy by continuation.* Two requirements are needed for this equivalence. First, at each iteration of the policy-gradient algorithm, we must optimize a Gaussian policy that is guaranteed to

be a mirror policy of an underlying deterministic original policy. Second, the continuation covariance related to each mirror policy must decrease trough the iterations. Let us consider the affine Gaussian policy $\pi_\theta^{GP'}$ with mean function μ_θ and covariance function $\Sigma'_\theta = \Sigma'$ constant with respect to the parameters of the policy (i.e., is a function depending solely on the state). In order to satisfy the first requirement for the mirror policy $\pi_\theta^{GP'}$, we have to ensure there exists a (unknown) continuation such that $\pi_\theta^{GP'}$ is a mirror policy of μ_θ. The latter requires the transformation between covariance functions in Property 3 to be surjective, which is guaranteed if $d_\mathcal{A} \leq d_\Theta$ and $\nabla_\theta \mu_\theta(s)$ is full rank. The first assumption is always met in practice and the second is met when no action is a deterministic function of the others. The second requirement is guaranteed if the covariance of the policy $\pi_\theta^{GP'}$ is discounted through the optimization. Indeed, Property 3 ensures that the covariance of the mirror policy is proportional to the one of the continuation. In conclusion, policy-gradient algorithms that optimize the affine Gaussian policy $\pi_\theta^{GP'}$ by local search while discounting its covariance equivalently optimize its mean function μ_θ (i.e., a deterministic policy) by continuation.

Entropy Regularization. Entropy regularization consists in adding the entropy of the policy to the return in the learning objective in order to ensure that the variance of the policy remains sufficiently large during the optimization process. Note that similar objective functions are optimized with maximum entropy reinforcement learning (Haarnoja et al., 2019) or with (approximate) trust-region methods where the trust-region constraint is dualized (Schulman et al., 2015; 2017). In the following, we answer to the question: *under which conditions does optimizing a Gaussian policy by local search while ensuring the variance remains sufficiently large is equivalent to implicitly optimizing another (unknown and Gaussian) policy by continuation.* Doing so we underline an equivalence between optimization by continuation and entropy regularization. Formally, let us consider an affine Gaussian original policy π_θ^{GP} with constant covariance $\Sigma_\theta = \Sigma$. Then, under Property 2, there exists another affine Gaussian policy $\pi_\theta^{GP'}$ that is a mirror policy of π_θ^{GP}. This mirror policy has the same mean function and a covariance function bounded from below by $C_\theta = C$. Inversely, let us consider an affine Gaussian policy $\pi_\theta^{GP'}$ with a bounded covariance $\Sigma'_\theta \succeq C$, for any C, we want to ensure there exists a (unknown) continuation such that $\pi_\theta^{GP'}$ is a mirror policy of an original policy. Similar to the previous paragraph, this is guaranteed when $d_\mathcal{A} \leq d_\Theta$ and $\nabla_\theta \mu_\theta(s)$ is full rank. Therefore, policy-gradient algorithms that optimize the return of an affine Gaussian policy by local search with a lower bound on the covariance implicitly optimize the continuation of an original policy (namely another affine Gaussian policy, but with unbounded covariance). In other words, optimizing an affine Gaussian policy with entropy regularization is equivalent to optimizing the continuation of an original policy.[2] In addition, decreasing the weight of the regularization is equivalent to decreasing C, which is proportional to the (unknown) continuation covariance, see Property 2. Optimizing affine Gaussian policies with constant covariance and with a decreasing regularization term by local search can thus be interpreted as optimizing this policy by continuation. In addition, as stated previously and shown in Appendix A, optimizing the return of the mirror policy of another mirror policy is equivalent to optimizing a continuation of the original policy. As explained in the previous paragraph, the return of an affine Gaussian policy is itself the continuation of a deterministic policy. Therefore, policy-gradient algorithms that optimize affine Gaussian policies with discounted covariance and with a decreasing regularization term by local search can also be interpreted as algorithms optimizing the mean function (i.e., a deterministic policy) of this policy by continuation.

We now illustrate how policy-gradient algorithms implicitly optimize by continuation. We take as example an environment in which a car moves in a valley and must reach its lowest point (positioned in x_{target}) to maximize the expected sum of rewards gathered by the agent, see Appendix C. We assume we want to find the best K-controller, i.e., a deterministic policy $\mu_\theta(x) = \theta \times (x - x_{target})$, where x is the position of the car. Directly optimizing such a policy is in practice subject to converging to a local extremum, as explain hereafter. We thus consider the Gaussian policy $\pi_\theta^{GP}(a|x) = \mathcal{N}(a|\mu_\theta(x), \sigma')$, where $\mu_\theta(x)$ and σ' are the mean and variance of the policy, respectively. This policy is a mirror policy of the deterministic policy μ_θ under a continuation of variance $\sigma = \sigma'/(x - x_{target})^2$, see Property 3. As can be seen in Figure 1, for each value of σ', the return of the mirror policy equals the smoothed return of the original deterministic

[2]Formally, the Loewner ordering between two covariance matrices A and B implies an order between their determinant, i.e. $A \succeq B \Rightarrow |A| \geq |B|$ (Siotani, 1967). As the entropy of a Gaussain policy is a concave function of the determinant of the covariance matrix, a bounded covariance matrix implies a bounded entropy. The entropy-regularization learning objective in direct policy optimization can therefore be interpreted as the Lagrangian relaxation of the latter entropy-bounded optimization problem.

policy μ_θ. Consequently, optimizing by policy gradient the Gaussian policy is equivalent to optimizing the deterministic policy by continuation. For a well-chosen sequence of σ', with a fixed scheduling or with adequate entropy regularization, the successive solutions found by local search will escape the basin of attraction of the suboptimal parameter for any initial parameter of the local search – whereas optimizing the deterministic policy directly would provide suboptimal solutions.

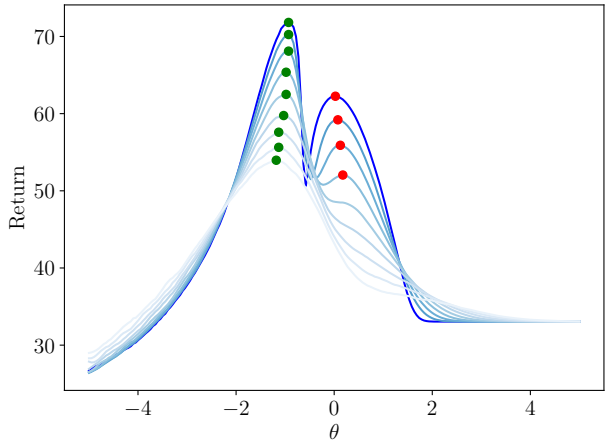

Figure 1: Illustration of the return of the policies $\mathcal{N}(a|\mu_\theta(x), \sigma')$, where $\mu_\theta(x) = \theta \times (x - x_{target})$, for different σ' values. The darker the curve, the smaller σ', and the darkest one is the return of the deterministic policy μ_θ. The green dots represent the global maxima and the red dots the local maxima. For some sufficiently large value for σ', the return of the policy has a single extremum. For a well chosen schedule of decreasing σ', a local-search algorithm will track the sequence of global extrema and converge towards the optimal deterministic policy.

In this section, we have established an equivalence between the optimization of some policies by policy gradient and the optimization of an underlying policy by continuation. It opens up new questions about the hypothesis space of the (mirror) policy to consider in practice in order to exploit the properties of continuations at best. These considerations are made in the next section. We finally recall that a central assumption in the previous results is the affinity of policies. Seemingly restrictive, such policies allow to optimally control complex environments in practice (Rajeswaran et al., 2017) and give first-order results for non-affine policies.

5.2 Continuations for Interpreting Stochastic Policies

In practice, we know that optimizing stochastic policies tends to converge to a final policy with low variance and better performance than if we had directly optimized a deterministic policy. Practitioners often justify this observation by the need to explore through a stochastic policy. Nevertheless, to our knowledge, this concept inherited from bandit theory is not well defined for direct policy optimization. The previous analysis establishes an equivalence between optimizing stochastic policies with policy-gradient algorithms and optimizing deterministic policies by continuation. Furthermore, as explained in Section 3.2, the continuation equation equation 8 consists in smoothing the return of this deterministic policy through the continuation distribution. Local optima tend to be removed when the variance of the continuation is sufficiently large. Optimizing stochastic policies and regularizing the entropy, as in most state-of-the-art policy-gradient algorithms, is therefore expected to avoid local extrema before converging towards policies with small variance. We thus provide a theoretical motivation for the performance reached by algorithms applying exploration as understood in direct policy optimization.

The relationships between optimization by continuation and policy gradient in Section 5.1 have been established relying on Property 2 and Property 3. They assume continuations where the covariance matrix depends only on the current state and not on the whole observed history. In the general case, Property 4

allows one to extend these results by performing an analysis similar to Section 5.1. To be more specific, let us assume an affine Gaussian policy $\pi_\theta^{GP'}$, where the mean μ_θ is a function of the state and where the covariance $\Sigma_\theta = \Sigma$ is a function of the history and is constant with respect to θ. Under this assumption, if $d_\mathcal{A} \leq d_\Theta$ and $\nabla_\theta \mu_\theta(s)$ is full rank, the return of the policy $\pi_\theta^{GP'}$ is equal to a (unknown) continuation of the mean function μ_θ (i.e., a deterministic policy). Furthermore, optimizing the Gaussian policy by policy gradient while discounting the covariance can be interpreted as optimizing the deterministic policy μ_θ by continuation. In practice, this result suggests to optimize history-dependent policies by policy gradient to take advantage of the most general regularization of the objective function through implicit continuation. A similar observation was recently made by Mutti et al. (2022) who argued that history-dependent policies are required when more complex regularizations are involved.

Finally, a last point has been left open in the previous discussions, namely the update of the covariance matrix of the mirror policies. The latter is defined through the covariance of the continuation. Therefore, the covariance must decrease through the optimization and must be chosen to avoid local optima. One direction to investigate in order to select a variance that removes local extrema is to update the parameters of the policy by following a combination of two directions: the functional gradient of the optimized policy's return with respect to the policy mean and the functional gradient of another measure (to be defined) with respect to the policy variance. An example of heuristic measure for smoothness might be the entropy of the actions and/or states encountered in histories. This strategy obviously does not follow the classical approach when optimizing stochastic policies where the covariance is adapted by the policy-gradient algorithm to locally maximize the return and the exact procedure for updating the variance will require future studies. However the empirical inefficiency of this classical approach has already been highlighted in previous works that improved the performance of policy-gradient algorithms by exploring alternative learning objective functions (Houthooft et al., 2018; Papini et al., 2020).

6 Conclusion

In this work, we have studied the problem formulation, i.e., policy parameterization and reward-shaping strategy, when solving direct policy optimization problems. More particularly, we established connections between formulations of state-of-the-art policy-gradient algorithms and the optimization by continuation framework (Allgower & Georg, 1980). We have shown that algorithms optimizing stochastic policies and regularizing the entropy inherit the properties of optimization by continuation and are thus less subject to converging towards local optima. In addition, the role of the variance of the policies is reinterpreted in this framework: it is a parameter of the optimization procedure to adapt in order to avoid local extrema. Additionally, to inherit the properties of generic continuations, it may be beneficial to consider variances that are functions of the history of states and actions observed at each time step.

Our study leaves several questions open. Firstly, our results rely on several assumptions that may not hold in practice. Specifically, it is unclear how our findings can be generalized to non-affine policies and alternative to Gaussian policies. Nonetheless, our results can be extended in cases where we can obtain an analytic expression for the mirror policy outlined in Theorem 1. While finding such an expression may be challenging in general, we can easily extend our conclusions to non-affine policies by considering the first-order approximation. Additionally, our study is focused on Gaussian policies, which are commonly used in continuous state-action spaces. However, for discrete spaces, a natural choice of policy is a Bernoulli distribution on actions (or a categorical distribution for more than one action) for each discrete state. In the case of a Beta continuation distribution, a mirror policy can be derived where actions follow a Beta-binomial distribution, a result known in Bayesian inference as the Beta distribution is a conjugate distribution of the binomial distribution (Bishop & Nasrabadi, 2006). An analysis of this mirror policy would allow us to draw conclusions equivalent to those of the continuous case studied in this paper. Secondly, the study focused on entropy regularization of the policy only. Recent works have underlined the benefits of other regularization strategies that enforce the spread of other distributions as the state visiting frequency or the marginal state probability (Hazan et al., 2019; Guo et al., 2021; Mutti et al., 2022). Future research is also needed to better understand the effect of these regularizations on the optimization procedure.

Finally, we give a new interpretation for the variance of policies that suggests it shall be updated to avoid local extrema rather than to maximize the return locally. A first strategy for updating the variance is proposed in Section 5.2, which opens the door to further research and new algorithm development.

References

Alekh Agarwal, Sham M Kakade, Jason D Lee, and Gaurav Mahajan. Optimality and approximation with policy gradient methods in markov decision processes. In *Conference on Learning Theory*, pp. 64–66. PMLR, 2020.

Zafarali Ahmed, Nicolas Le Roux, Mohammad Norouzi, and Dale Schuurmans. Understanding the impact of entropy on policy optimization. In *International Conference on Machine Learning*, pp. 151–160. PMLR, 2019.

Eugene L Allgower and Kurt Georg. *Numerical continuation methods: an introduction*, volume 13. Springer Series in Computational Mathematics, 1980.

Marcin Andrychowicz, Anton Raichuk, Piotr Stańczyk, Manu Orsini, Sertan Girgin, Raphael Marinier, Léonard Hussenot, Matthieu Geist, Olivier Pietquin, Marcin Michalski, et al. What matters in on-policy reinforcement learning? a large-scale empirical study. *arXiv preprint arXiv:2006.05990*, 2020.

Amrit Singh Bedi, Anjaly Parayil, Junyu Zhang, Mengdi Wang, and Alec Koppel. On the sample complexity and metastability of heavy-tailed policy search in continuous control. *arXiv preprint arXiv:2106.08414*, 2021.

Amrit Singh Bedi, Souradip Chakraborty, Anjaly Parayil, Brian M Sadler, Pratap Tokekar, and Alec Koppel. On the hidden biases of policy mirror ascent in continuous action spaces. In *International Conference on Machine Learning*, pp. 1716–1731. PMLR, 2022.

Yoshua Bengio. Learning deep architectures for ai. *Foundations and Trends in Machine Learning*, 2(1): 1–127, 2009.

Jalaj Bhandari and Daniel Russo. Global optimality guarantees for policy gradient methods. *arXiv preprint arXiv:1906.01786*, 2019.

Sujay Bhatt, Alec Koppel, and Vikram Krishnamurthy. Policy gradient using weak derivatives for reinforcement learning. In *Conference on Decision and Control (CDC)*, volume 58, pp. 5531–5537. IEEE, 2019.

Christopher M Bishop and Nasser M Nasrabadi. *Pattern recognition and machine learning*, volume 4. Springer, 2006.

Andrew Blake and Andrew Zisserman. *Visual reconstruction*. MIT press, 1987.

Léon Bottou. Large-scale machine learning with stochastic gradient descent. In *Proceedings of COMPSTAT'2010*, pp. 177–186. Springer, 2010.

Rob Brekelmans, Tim Genewein, Jordi Grau-Moya, Grégoire Delétang, Markus Kunesch, Shane Legg, and Pedro Ortega. Your policy regularizer is secretly an adversary. *arXiv preprint arXiv:2203.12592*, 2022.

Lucian Busoniu, Robert Babuska, Bart De Schutter, and Damien Ernst. *Reinforcement learning and dynamic programming using function approximators*. CRC press, 2017.

Shicong Cen, Chen Cheng, Yuxin Chen, Yuting Wei, and Yuejie Chi. Fast global convergence of natural policy gradient methods with entropy regularization. *Operations Research*, 70(4):2563–2578, 2022.

Herman Chernoff and Lincoln E Moses. *Elementary decision theory*. Courier Corporation, 2012.

Po-Wei Chou, Daniel Maturana, and Sebastian Scherer. Improving stochastic policy gradients in continuous control with deep reinforcement learning using the beta distribution. In *International Conference on Machine Learning*, pp. 834–843. PMLR, 2017.

Yan Duan, Xi Chen, Rein Houthooft, John Schulman, and Pieter Abbeel. Benchmarking deep reinforcement learning for continuous control. In *International Conference on Machine Learning*, pp. 1329–1338. PMLR, 2016.

Yasuhiro Fujita and Shin-ichi Maeda. Clipped action policy gradient. In *International Conference on Machine Learning*, pp. 1597–1606. PMLR, 2018.

Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep learning*. MIT press, 2016.

Zhaohan Daniel Guo, Mohammad Gheshlaghi Azar, Alaa Saade, Shantanu Thakoor, Bilal Piot, Bernardo Avila Pires, Michal Valko, Thomas Mesnard, Tor Lattimore, and Rémi Munos. Geometric entropic exploration. *arXiv preprint arXiv:2101.02055*, 2021.

Tuomas Haarnoja, Aurick Zhou, Kristian Hartikainen, George Tucker, Sehoon Ha, Jie Tan, Vikash Kumar, Henry Zhu, Abhishek Gupta, Pieter Abbeel, et al. Soft actor-critic algorithms and applications. *arXiv preprint arXiv:1812.05905*, 2019.

Elad Hazan, Kfir Yehuda Levy, and Shai Shalev-Shwartz. On graduated optimization for stochastic non-convex problems. In *International Conference on Machine Learning*, pp. 1833–1841. PMLR, 2016.

Elad Hazan, Sham Kakade, Karan Singh, and Abby Van Soest. Provably efficient maximum entropy exploration. In *International Conference on Machine Learning*, pp. 2681–2691. PMLR, 2019.

Rein Houthooft, Yuhua Chen, Phillip Isola, Bradly Stadie, Filip Wolski, OpenAI Jonathan Ho, and Pieter Abbeel. Evolved policy gradients. *Advances in Neural Information Processing Systems*, 31, 2018.

Hisham Husain, Kamil Ciosek, and Ryota Tomioka. Regularized policies are reward robust. In *International Conference on Artificial Intelligence and Statistics*, pp. 64–72. PMLR, 2021.

Riashat Islam, Zafarali Ahmed, and Doina Precup. Marginalized state distribution entropy regularization in policy optimization. *arXiv preprint arXiv:1912.05128*, 2019.

Timothy P Lillicrap, Jonathan J Hunt, Alexander Pritzel, Nicolas Heess, Tom Erez, Yuval Tassa, David Silver, and Daan Wierstra. Continuous control with deep reinforcement learning. *arXiv preprint arXiv:1509.02971*, 2015.

Volodymyr Mnih, Adria Puigdomenech Badia, Mehdi Mirza, Alex Graves, Timothy Lillicrap, Tim Harley, David Silver, and Koray Kavukcuoglu. Asynchronous methods for deep reinforcement learning. In *International Conference on Machine Learning*, pp. 1928–1937. PMLR, 2016.

Hossein Mobahi and John W Fisher. On the link between gaussian homotopy continuation and convex envelopes. In *International Workshop on Energy Minimization Methods in Computer Vision and Pattern Recognition*, pp. 43–56. Springer, 2015.

Hossein Mobahi and John Fisher III. A theoretical analysis of optimization by gaussian continuation. In *Conference on Artificial Intelligence*, volume 29. AAAI, 2015.

Hossein Mobahi, C Lawrence Zitnick, and Yi Ma. Seeing through the blur. In *Conference on Computer Vision and Pattern Recognition*, pp. 1736–1743. IEEE, 2012.

Mirco Mutti, Riccardo De Santi, and Marcello Restelli. The importance of non-markovianity in maximum state entropy exploration. *arXiv preprint arXiv:2202.03060*, 2022.

Ofir Nachum, Mohammad Norouzi, and Dale Schuurmans. Improving policy gradient by exploring under-appreciated rewards. *arXiv preprint arXiv:1611.09321*, 2016.

Yurii Nesterov and Vladimir Spokoiny. Random gradient-free minimization of convex functions. *Foundations of Computational Mathematics*, 17(2):527–566, 2017.

Matteo Papini, Andrea Battistello, and Marcello Restelli. Balancing learning speed and stability in policy gradient via adaptive exploration. In *International conference on artificial intelligence and statistics*, pp. 1188–1199. PMLR, 2020.

Harsh Nilesh Pathak and Randy Paffenroth. Parameter continuation methods for the optimization of deep neural networks. In *International Conference on Machine Learning And Applications (ICMLA)*, volume 18, pp. 1637–1643. IEEE, 2019.

Jan Peters and Stefan Schaal. Reinforcement learning by reward-weighted regression for operational space control. In *International conference on Machine learning*, volume 24, pp. 745–750, 2007.

Jan Peters and Stefan Schaal. Reinforcement learning of motor skills with policy gradients. *Neural networks*, 21(4):682–697, 2008.

Aravind Rajeswaran, Kendall Lowrey, Emanuel V Todorov, and Sham M Kakade. Towards generalization and simplicity in continuous control. *Advances in Neural Information Processing Systems*, 30, 2017.

Calyampudi Radhakrishna Rao. *Linear statistical inference and its applications*, volume 2. Wiley New York, 1973.

Tim Salimans, Jonathan Ho, Xi Chen, Szymon Sidor, and Ilya Sutskever. Evolution strategies as a scalable alternative to reinforcement learning. *arXiv preprint arXiv:1703.03864*, 2017.

John Schulman, Sergey Levine, Pieter Abbeel, Michael Jordan, and Philipp Moritz. Trust region policy optimization. In *International Conference on Machine Learning*, pp. 1889–1897. PMLR, 2015.

John Schulman, Filip Wolski, Prafulla Dhariwal, Alec Radford, and Oleg Klimov. Proximal policy optimization algorithms. *arXiv preprint arXiv:1707.06347*, 2017.

Frank Sehnke, Christian Osendorfer, Thomas Rückstieß, Alex Graves, Jan Peters, and Jürgen Schmidhuber. Parameter-exploring policy gradients. *Neural Networks*, 23(4):551–559, 2010.

Lior Shani, Yonathan Efroni, and Shie Mannor. Adaptive trust region policy optimization: Global convergence and faster rates for regularized mdps. In *Conference on Artificial Intelligence*, volume 34, pp. 5668–5675. AAAI, 2020.

Weijia Shao, Christian Geißler, and Fikret Sivrikaya. Graduated optimization of black-box functions. *arXiv preprint arXiv:1906.01279*, 2019.

David Silver, Guy Lever, Nicolas Heess, Thomas Degris, Daan Wierstra, and Martin Riedmiller. Deterministic policy gradient algorithms. In *International Conference on Machine Learning*, pp. 387–395. PMLR, 2014.

Herbert A Simon. A behavioral model of rational choice. *The quarterly journal of economics*, 69(1):99–118, 1955.

Minoru Siotani. Some applications of loewner's ordering on symmetric matrices. *Annals of the Institute of Statistical Mathematics*, 19:245–259, 1967.

Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

Layne T Watson and Raphael T Haftka. Modern homotopy methods in optimization. *Computer Methods in Applied Mechanics and Engineering*, 74(3):289–305, 1989.

Daan Wierstra, Tom Schaul, Jan Peters, and Juergen Schmidhuber. Episodic reinforcement learning by logistic reward-weighted regression. In *International Conference on Artificial Neural Networks*, pp. 407–416. Springer, 2008.

Ronald J Williams. Simple statistical gradient-following algorithms for connectionist reinforcement learning. *Machine learning*, 8(3-4):229–256, 1992.

Ronald J Williams and Jing Peng. Function optimization using connectionist reinforcement learning algorithms. *Connection Science*, 3(3):241–268, 1991.

Huaqing Xiong, Tengyu Xu, Lin Zhao, Yingbin Liang, and Wei Zhang. Deterministic policy gradient: Convergence analysis. In *Conference on Uncertainty in Artificial Intelligence*, volume 28, 2022.

Jiaxin Zhang, Hoang Tran, Dan Lu, and Guannan Zhang. A novel evolution strategy with directional gaussian smoothing for blackbox optimization. *arXiv preprint arXiv:2002.03001*, 2020a.

Junzi Zhang, Jongho Kim, Brendan O'Donoghue, and Stephen Boyd. Sample efficient reinforcement learning with reinforce. In *Conference on Artificial Intelligence*, volume 35, pp. 10887–10895. AAAI, 2021.

Kaiqing Zhang, Alec Koppel, Hao Zhu, and Tamer Basar. Global convergence of policy gradient methods to (almost) locally optimal policies. *SIAM Journal on Control and Optimization*, 58(6):3586–3612, 2020b.

A Composition of Mirror Policies

Theorem 2. *Let q be a continuation distribution and let Σ be a covariance function as defined in Section 3.2. In addition, let η_θ, η'_θ and η''_θ be three parameterized history-dependent policies such that:*

$$\eta'_\theta(a|h) = \int \eta_{\theta'}(a|h) q(\theta'|\theta, \Sigma(h))\, d\theta' \tag{14}$$

$$\eta''_\theta(a|h) = \int \eta'_{\theta''}(a|h) q(\theta''|\theta, \Sigma(h))\, d\theta'' \ . \tag{15}$$

Then, η'_θ is a mirror policy of the original policy η_θ and η''_θ is a mirror policy of the original policy η'_θ under continuation distribution q and covariance function Σ. In addition, there exists a continuation for which η''_θ is a mirror policy of the original policy η_θ.

Proof. First, η'_θ is a mirror policy of the original policy η_θ and η''_θ is a mirror policy of the original policy η'_θ under continuation distribution q and covariance function Σ, see Theorem 1. Then, let us substitute equation equation 14 in equation equation 15:

$$\eta''_\theta(a|h) = \int \eta'_{\theta''}(a|h) q(\theta''|\theta, \Sigma(h))\, d\theta'' \tag{16}$$

$$= \int \left(\int \eta_{\theta'}(a|h) q(\theta'|\theta'', \Sigma(h))\, d\theta' \right) q(\theta''|\theta, \Sigma(h))\, d\theta'' \tag{17}$$

$$= \int \eta_{\theta'}(a|h) \left(\int q(\theta'|\theta'', \Sigma(h)) q(\theta''|\theta, \Sigma(h))\, d\theta'' \right) d\theta' \ . \tag{18}$$

We thus have that:

$$\eta''_\theta(a|h) = \int \eta_{\theta'}(a|h) p_\theta(\theta'|h)\, d\theta' \tag{19}$$

$$p_\theta(\theta'|h) = \int q(\theta'|\theta'', \Sigma(h)) q(\theta''|\theta, \Sigma(h))\, d\theta'' \ . \tag{20}$$

The distribution p_θ is a continuation distribution with a spread depending on the history h through the covariance function Σ. By Theorem 1, η''_θ is a mirror policy of the original policy η_θ.

□

B Theoretical Results on Mirror Policies

Theorem 1. *For any original history-dependent policy $\eta_\theta \in \mathcal{E}$ parameterized with the vector $\theta \in \mathbb{R}^{d_\Theta}$ and for any continuation distribution q and covariance function Σ, there exists a mirror history-dependent policy $\eta'_\theta \in \mathcal{E}$ of the original policy η_θ that writes as:*

$$\eta'_\theta(a|h) = \mathop{\mathbb{E}}_{\theta' \sim q(\cdot|\theta, \Sigma(h))} \left[\eta_{\theta'}(a|h) \right] \ . \tag{21}$$

Proof. Let $h = (s_0, a_0, s_1, a_1, \dots) \in H$ be a history and let $R(h)$ be the discounted sum of rewards computed from this history. In addition, let $h_t = (s_0, a_0, \dots, s_t) \in H$ be the history composed of the t first actions and $t+1$ first states in h. By definition of the continuation f^q_Σ and given equation equation 21, we

have that:

$$f^q_\Sigma(\theta) = \mathop{\mathbb{E}}_{\substack{s_0\sim p_0(\cdot)\\ \theta_t\sim q(\cdot|\theta,\Sigma(h_t))\\ a_t\sim\eta_{\theta_t}(\cdot|h_t)\\ s_{t+1}\sim p(\cdot|s_t,a_t)}}\left[\sum_{t=0}^{\infty}\gamma^t\rho(s_t,a_t)\right] \tag{22}$$

$$= \int\left(\prod_{t=0}^{\infty}\int \eta_{\theta_t}(a_t|h_t)q(\theta_t|\theta,\Sigma(h_t))\,d\theta_t\right)\left(p_0(s_0)\prod_{t=0}^{\infty}p(s_{t+1}|s_t,a_t)\right)R(h)\,dh \tag{23}$$

$$= \int\left(\prod_{t=0}^{\infty}\eta'_\theta(a_t|h_t)\right)\left(p_0(s_0)\prod_{t=0}^{\infty}p(s_{t+1}|s_t,a_t)\right)R(h)\,dh\,. \tag{24}$$

By definition, the latter equation is equal to the return $J(\eta'_\theta)$ of the policy η'_θ for any parameter vector θ. Therefore, η'_θ is a mirror policy of the original policy η_θ under the process q and covariance matrix Σ.

□

Property 1. *Let the original policy $\pi_\theta \in \Pi$ be a Markov policy and let the covariance function depend solely on the last state in the history. Then, there exists a mirror Markov policy $\pi'_\theta \in \Pi$.*

Proof. By hypotheses, the covariance matrix only depends on the last state s_t of the history h_t, therefore:

$$q(\theta_t|\theta,\Sigma(h_t)) = q(\theta_t|\theta,\Sigma(s_t))\,. \tag{25}$$

In addition, the original policy π_θ is a Markov policy, therefore :

$$\eta_\theta(a_t|h_t) = \pi_\theta(a_t|s_t)\,. \tag{26}$$

The closed form of the mirror policy, provided by equation equation 21, can thus be simplified as:

$$\eta'_\theta(a_t|h_t) = \int \eta_{\theta_t}(a_t|h_t)q(\theta_t|\theta,\Sigma(h_t))\,d\theta_t \tag{27}$$

$$= \int \pi_{\theta_t}(a_t|s_t)q(\theta_t|\theta,\Sigma(s_t))\,d\theta_t\,. \tag{28}$$

The previous equation is independent on h_t knowing s_t, there thus exists a Markov mirror policy $\pi'_\theta \in \Pi$ respecting Theorem 1 such that:

$$\eta'_\theta(a_t|h_t) = \pi'_\theta(a_t|s_t)\,. \tag{29}$$

□

Property 2. *Let the original policy $\pi^{GP}_\theta \in \Pi$ be a Gaussian policy as defined in equation equation 3 with affine function approximators. Let the covariance function depend solely on the last state in the history and let the distribution q be a Gaussian distribution. Then, there exists a mirror Markov policy $\pi'_\theta \in \Pi$ such that for all states $s \in \mathcal{S}$, it converges towards a Gaussian policy in the limit as the affine coefficients of the covariance matrix $\Sigma_\theta(s)$ approaches zero ($\|\nabla_\theta\Sigma_\theta(s)\| \to 0$):*

$$\pi'_\theta(a|s) \to \mathcal{N}(a|\mu_\theta(s),\Sigma'_\theta(s))\,, \tag{30}$$

where $\Sigma'_\theta(s) = C_\theta(s) + \Sigma_\theta(s)$ and $C_\theta(s) = \nabla_\theta\mu_\theta(s)^T\,\Sigma(s)\,\nabla_\theta\mu_\theta(s)$.

Proof. First, the existence of a Markov mirror policy results from Property 1 and is provided by equation equation 29:

$$\pi'_\theta(a_t|s_t) = \int \pi_{\theta_t}(a_t|s_t)q(\theta_t|\theta,\Sigma(s_t))\,d\theta_t\,. \tag{31}$$

In addition, π_{θ_t} and q are Gaussian distributions by hypotheses:

$$\pi_{\theta_t}(a_t|s_t) = \mathcal{N}(a_t|\mu_{\theta_t}(s_t), \Sigma_{\theta_t}(s)) \tag{32}$$
$$q(\theta_t|\theta, \Sigma(s_t)) = \mathcal{N}(\theta_t|\theta, \Sigma(s_t)) \ , \tag{33}$$

where $\mu_{\theta_t}(s_t)$ and $\Sigma_{\theta_t}(s_t)$ are affine functions of θ_t. Therefore, these functions can be written as follows:

$$\mu_{\theta_t}(s_t) = (\nabla_{\theta_t}\mu_{\theta_t}(s_t))\,\theta_t + \mu'(s_t) \tag{34}$$
$$\Sigma_{\theta_t}(s_t) = (\nabla_{\theta_t}\Sigma_{\theta_t}(s_t))\,\theta_t + \Sigma'(s_t) \ . \tag{35}$$

For any state s_t, in the limit as affine coefficients of the covariance approaches zero, the covariance is such that:

$$\lim_{\|\nabla_{\theta_t}\Sigma_{\theta_t}(s_t)\|\to 0} \Sigma_{\theta_t}(s_t) = \Sigma'(s_t) \ . \tag{36}$$

In this limit, equation equation 31 consists in marginalizing a conditional linear Gaussian transition model with a Gaussian prior and is such that (Bishop & Nasrabadi, 2006):

$$\lim_{\|\nabla_{\theta}\Sigma_{\theta}(s_t)\|\to 0} \pi'_\theta(a_t|s_t) = \mathcal{N}\left(a_t|\,(\nabla_\theta\mu_\theta(s_t))\,\theta + \mu'(s_t), (\nabla_\theta\mu_\theta(s_t))^T\,\Sigma(s_t)\,(\nabla_\theta\mu_\theta(s_t)) + \Sigma'(s_t)\right) \tag{37}$$
$$= \mathcal{N}\left(a_t|\mu_\theta(s_t), (\nabla_\theta\mu_\theta(s_t))^T\,\Sigma(s_t)\,(\nabla_\theta\mu_\theta(s_t)) + \Sigma_\theta(s_t)\right) \ . \tag{38}$$

□

Property 3. *Let the original policy $\mu_\theta \in M$ be an affine deterministic policy. Let the covariance function depend solely on the last state in the history and let the distribution q be a Gaussian distribution. Then, the Markov policy $\pi_\theta^{GP'} \in \Pi$ is a mirror policy:*

$$\pi_\theta^{GP'}(a|s) = \mathcal{N}(a|\mu_\theta(s), \Sigma'_\theta(s)) \ , \tag{39}$$

where $\Sigma'_\theta(s) = \nabla_\theta\mu_\theta(s)^T\,\Sigma(s)\,\nabla_\theta\mu_\theta(s)$.

Proof. The statement results from the particularization of Property 2 to the case of deterministic policies. Let $\pi_\theta^{GP} \in \Pi$ be an affine Gaussian policy with constant covariance matrix for any state $\Sigma_\theta(s_t) = C$. In that case, we have by Property 2 that $\pi'_\theta \in \Pi$ is a mirror policy as follows:

$$\pi'_\theta(a_t|s_t) = \mathcal{N}\left(a_t|\mu_\theta(s_t), (\nabla_\theta\mu_\theta(s_t))^T\,\Sigma(s_t)\,(\nabla_\theta\mu_\theta(s_t)) + \Sigma_\theta(s_t)\right) \tag{40}$$
$$= \mathcal{N}\left(a_t|\mu_\theta(s_t), (\nabla_\theta\mu_\theta(s_t))^T\,\Sigma(s_t)\,(\nabla_\theta\mu_\theta(s_t)) + C\right) \ . \tag{41}$$

Taking the limit of π_θ^{GP} as the constant covariance matrix C approaches zero, we get that the original policy from Property 2 converges to the one of Property 3, namely the deterministic policy μ_θ. This implies that the policy $\pi_\theta^{GP'} \in \Pi$ provided by the limit of the mirror policy from Property 2, see equation equation 41, is a mirror policy of the original policy μ_θ from Property 3:

$$\pi_\theta^{GP'}(a_t|s_t) = \lim_{C\to 0} \pi'_\theta(a_t|s_t) = \mathcal{N}\left(a_t|\mu_\theta(s_t), (\nabla_\theta\mu_\theta(s_t))^T\,\Sigma(s_t)\,(\nabla_\theta\mu_\theta(s_t))\right) \ . \tag{42}$$

□

Property 4. *Let the original policy $\mu_\theta \in M$ be an affine deterministic policy. Let the distribution q be a Gaussian distribution. Then, the policy $\eta'_\theta \in \mathcal{E}$ is a mirror policy:*

$$\eta'_\theta(a|h) = \mathcal{N}(a|\mu_\theta(s), \Sigma'_\theta(h)) \ , \tag{43}$$

where $\Sigma'_\theta(h) = \nabla_\theta \mu_\theta(s)^T \, \Sigma(h) \, \nabla_\theta \mu_\theta(s)$.

Proof. The policy μ_{θ_t} is an affine function of the parameter vector θ_t and can thus be written as follows:

$$\mu_{\theta_t}(s_t) = \left(\nabla_{\theta_t} \mu_{\theta_t}(s_t)\right) \theta_t + \mu'(s_t) \; . \tag{44}$$

In addition, the samples drawn from the process q are distributed according to a Gaussian distribution:

$$q(\theta_t|\theta, \Sigma(h_t)) = \mathcal{N}(\theta_t|\theta, \Sigma(h_t)) \; . \tag{45}$$

The closed form of the density of the mirror policy, provided by equation equation 21, is thus simplified as:

$$\eta'_\theta(a_t|h_t) = \int \eta_{\theta_t}(a_t|h_t) q(\theta_t|\theta, \Sigma(h_t)) \, d\theta_t \tag{46}$$

$$= \int \eta_{\theta_t}(a_t|h_t) \mathcal{N}(\theta_t|\theta, \Sigma(h_t)) \, d\theta_t \; , \tag{47}$$

where η_{θ_t} is the policy where each action respecting equation equation 44 has a probability one. The policy is a degenerated Gaussian distribution (Rao, 1973), it provides a dirac measure to each state, and its (generalized) density function may be approached as follows:

$$\eta_{\theta_t}(a_t|h_t) = \lim_{\|\Sigma\| \to 0} \mathcal{N}(a_t| \left(\nabla_{\theta_t} \mu_{\theta_t}(s_t)\right) \theta_t + \mu'(s_t), \Sigma) \; . \tag{48}$$

By substitution, we therefore get that the mirror policy η'_θ writes as follow:

$$\eta'_\theta(a_t|h_t) = \int \eta_{\theta_t}(a_t|h_t) \mathcal{N}(\theta_t|\theta, \Sigma(h_t)) \, d\theta_t \tag{49}$$

$$= \int \lim_{\|\Sigma\| \to 0} \mathcal{N}\left(a_t| \left(\nabla_{\theta_t} \mu_{\theta_t}(s_t)\right) \theta_t + \mu'(s_t), \Sigma\right) \mathcal{N}\left(\theta_t|\theta, \Sigma(h_t)\right) \, d\theta_t \; . \tag{50}$$

The product of the Gaussian prior over parameters and the linear Gaussian transition model of the actions provides a joint Gaussian distribution of actions and parameters (Bishop & Nasrabadi, 2006), which is degenerated but has a density for the (marginal) Gaussian distribution of actions (Rao, 1973). The density of the mirror policy η'_θ can thus be computed taking the limit of the marginalization:

$$\eta'_\theta(a_t|h_t) = \lim_{\|\Sigma\| \to 0} \int \mathcal{N}\left(a_t| \left(\nabla_{\theta_t} \mu_{\theta_t}(s_t)\right) \theta_t + \mu'(s_t), \Sigma\right) \mathcal{N}\left(\theta_t|\theta, \Sigma(h_t)\right) \, d\theta_t \tag{51}$$

$$= \lim_{\|\Sigma\| \to 0} \mathcal{N}\left(a_t| \left(\nabla_\theta \mu_\theta(s_t)\right) \theta + \mu'(s_t), \left(\nabla_\theta \mu_\theta(s_t)\right)^T \Sigma(h_t) \left(\nabla_\theta \mu_\theta(s_t)\right) + \Sigma\right) \tag{52}$$

$$= \lim_{\|\Sigma\| \to 0} \mathcal{N}\left(a_t|\mu_\theta(s_t), \left(\nabla_\theta \mu_\theta(s_t)\right)^T \Sigma(h_t) \left(\nabla_\theta \mu_\theta(s_t)\right) + \Sigma\right) \tag{53}$$

$$= \mathcal{N}\left(a_t|\mu_\theta(s_t), \left(\nabla_\theta \mu_\theta(s_t)\right)^T \Sigma(h_t) \left(\nabla_\theta \mu_\theta(s_t)\right)\right) \; . \tag{54}$$

We note that this result can be obtained without working on degenerated Gaussian distributions. The policy is an affine function of the parameters, which follow a Gaussian distribution, the marginal distribution of actions is thus also a Gaussian distribution of the form of equation equation 54. This distribution is furthermore the one of a mirror policy, see Theorem 1.

□

C Description of the Car Environment

In this section, we formalize the reinforcement learning environment that models the movement of a car in a valley with two floors separated by a peak, as depicted in Figure 2. The car always starts at the topmost floor and receives rewards proportionally to its depth in the valley. An optimal agent drives the car from

the initial position to the lowest floor in the valley by passing the peak. In the following, we describe each element composing the environment.

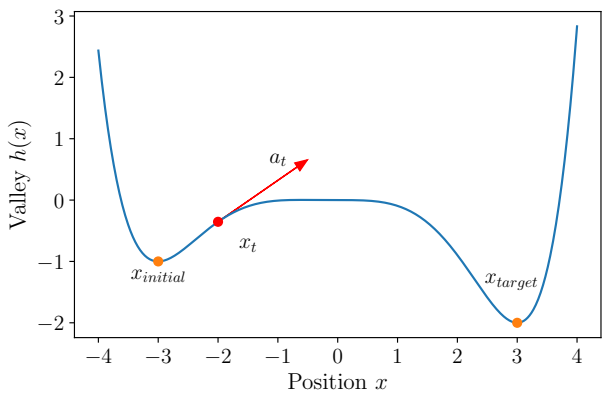

Figure 2: Valley in which the car moves.

State Space. The state $s_t \in \mathbb{R}^2$ of the environment is composed of two scalar values, namely the position $x_t \in \mathbb{R}$ of the pointmass representing the car and its tangent speed $v_t \in \mathbb{R}$.

Action Space. At each time step, the agent controls the force applied on the car through the actions $a_t \in \mathbb{R}$ it executes.

Initial Position. The car always starts at the topmost floor $x_{initial} = -3$ in the valley at rest. The initial state distribution thus provides a probability one to the state

$$x_0 = x_{initial} \tag{55}$$
$$v_0 = 0\,. \tag{56}$$

Transition Distribution. The continuous motion of the car in the valley is derived for Newton's formula. The valley's analytical description is provided by the function h, the car's mass is denoted by $m = 0.5$, gravitational acceleration by $g = 9.81$, and damping factor by $e = 0.65$. The position x and speed v of the car follow the subsequent continuous-time dynamics as a function of the force a:

$$\dot{x} = v \tag{57}$$
$$\dot{v} = \frac{a}{m(1 + h'(x)^2)} - \frac{gh'(x)}{1 + h'(x)^2} + \frac{v^2 h'(x) h''(x)}{1 + h'(x)^2} - ev^2\,. \tag{58}$$

The position and force are furthermore bounded to intervals as part of the dynamics such that

$$x \in [x_m, x_M] = [-4, 5] \tag{59}$$
$$a \in [a_m, a_M] = [-10, 10]\,. \tag{60}$$

Clamped force values are therefore used in equation equation 58. Similarly, the position is clamped in equation equation 57.

In discrete time, the state s_{t+1} is computed through Euler integration of the continuous-time dynamics, considering an initial position given by the current state s_t. The force a remains constant during a discretization time $\Delta = 0.1$ and is equal to the action a_t, with an additive noise drawn from $\mathcal{N}(\cdot|0, 1)$ and clamped before integration.

Reward Function. The rewards correspond to the depth of the valley at the current position. The reward function thus solely depends on the position

$$\rho(s_t, a_t) = -h(x_t) \ . \tag{61}$$

Discount Factor. The discount factor equals $\gamma = 0.99$ and the horizon is curtailed to $T = 100$ in each numerical computation.