Efficient Mirror Descent Ascent Methods for Nonsmooth Minimax Problems

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Abstract

In the paper, we propose a class of efficient mirror descent ascent methods to solve the nonsmooth nonconvex-strongly-concave minimax problems by using dynamic mirror functions, and introduce a convergence analysis framework to conduct rigorous theoretical analysis for our mirror descent ascent methods. For our stochastic algorithms, we first prove that the mini-batch stochastic mirror descent ascent (SMDA) method obtains a sample complexity of $O(\kappa^3\epsilon^{-4})$ for finding an ϵ-stationary point, where κ denotes the condition number. Further, we propose an accelerated stochastic mirror descent ascent (VR-SMDA) method based on the variance reduced technique. We prove that our VR-SMDA method achieves a lower sample complexity G $O(\kappa^3\epsilon^{-3})$. For our deterministic algorithm, we prove that our deterministic mirror descent ascent (MDA) achieves a lower sample complexity of $O(\kappa\epsilon^{-2})$ under mild conditions, which improves the best known complexity by a factor of $O(\sqrt{\kappa})$. We conduct the experiments on fair classifier and robust neural network training tasks to demonstrate the efficiency of our new algorithms.

1 Introduction

Minimax optimization recently has attracted increased interest largely due to advance in many machine learning applications such as generative adversarial networks (GANs) [12, 35], robust neural networks training [29], fair learning [28], federated learning [9], and policy evaluation [37]. In the paper, we study the following nonsmooth nonconvex-strongly-concave minimax problem:

$$\min_{x\in\mathcal{X}}\max_{y\in\mathcal{Y}} F(x,y) = \big\{f(x,y) + g(x) - h(y)\big\}, \tag{1}$$

where the function $f(x,y) : \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}$ is smooth and possibly nonconvex in $x \in \mathbb{R}^d$ and μ-strongly concave in $y \in \mathbb{R}^p$, and the functions $g(x)$ and $h(y)$ are convex and possibly nonsmooth. Here the constraint sets $\mathcal{X} \subseteq \mathbb{R}^d$ and $\mathcal{Y} \subseteq \mathbb{R}^p$ are compact and convex. In many machine learning problems, $f(x,y)$ generally represents loss function and is a stochastic form, *i.e.*, $f(x,y) = \mathbb{E}_\xi[f(x,y;\xi)]$, where the random variable ξ follows an unknown distribution. Here both $g(x)$ and $h(y)$ frequently denote the nonsmooth regularization such as $g(x) = \lambda\|x\|_1$. In fact, the above problem (1) comes from many machine learning problems, such as fair classifier, robust training, nonlinear temporal-difference learning in reinforcement learning [37] and robust federated learning [9].

When $g(x) = 0$ and $h(y) = 0$ in the problem (1), the classic gradient descent ascent (GDA) methods [34, 25] can effectively solve this problem, which alternatively conducts a gradient descent update on the variable x and a gradient ascent update on the variable y in each iteration. At the same time, some stochastic GDA methods [34, 25, 27, 16, 40, 15] have been proposed to solve the stochastic minimax problem (1), where $f(x,y) = \mathbb{E}_\xi[f(x,y;\xi)]$. More recently, some works [2, 4, 8] focus on more general minimax problem (1), where both $g(x)$ and $h(y)$ are possibly nonsmooth. Meanwhile,

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Table 1: **Sample complexity** comparison of the representative proximal gradient descent ascent (PGDA) methods for obtaining an ϵ-stationary point of the **nonsmooth nonconvex** minimax problem (1). Note that these comparison methods don't rely on specific assumptions on the problem (1). Since [8] studies the deterministic PGDA relied on the Kurdyka-Lojasiewicz (KL) geometry assumption, it be excluded. Here κ denotes condition number of objective function $f(x, y)$.

Type	Algorithm	Reference	Loop(s)	Sample Complexity
Deterministic	MAPGDA	[2]	Double	$O(\kappa^{3/2}\epsilon^{-2})$
	PAGDA	[4]	Single	$O(\kappa^{2}\epsilon^{-2})$
	MDA	Ours	Single	$O(\kappa\epsilon^{-2})$
Stochastic	PASGDA	[4]	Single	$O(\kappa^{3}\epsilon^{-4})$
	SMDA	Ours	Single	$O(\kappa^{3}\epsilon^{-4})$
	VR-SMDA	Ours	Double	$O(\kappa^{3}\epsilon^{-3})$

some (stochastic) proximal gradient descent ascent (PGDA) methods [2, 4, 8] have been presented to solve the problem (1). However, they still suffer from the large sample complexities for finding an stationary point of the minimax problem (1) without specific assumptions such as KL geometry (Please see Table 1).

In this paper, thus, we propose a class of efficient mirror descent ascent methods by using dynamic mirror function (i.e., Bregman function). Specifically, our methods perform an adaptive mirror descent update on the variable x and an adaptive mirror ascent update on the variable y alternatively in each iteration. Our new algorithmic framework is general and can generate many popular methods and their variants by adopting different mirror functions. For example, by adopting the mirror functions $\psi(x) = \frac{1}{2}\|x\|^2$ and $\phi(y) = \frac{1}{2}\|y\|^2$, our methods will include the classic (proximal) gradient descent ascent algorithms. Our main contributions are summarized as follows:

1) We propose a class of novel mirror descent ascent methods to solve the minimax problem (1) by using dynamic mirror functions. Moreover, we prove that the **stochastic** mirror descent ascent (SMDA) method has a sample complexity of $O(\kappa^3\epsilon^{-4})$ for finding an ϵ-stationary point.
2) We further introduce an accelerated **stochastic** mirror descent ascent (VR-SMDA) method by using the variance reduced technique of SARAH/SNVRG/SPIDER [31, 45, 11, 38]. Moreover, we prove that the VR-SMDA reaches a lower sample complexity of $O(\kappa^3\epsilon^{-3})$.
3) We provide a convergence analysis framework for these mirror descent ascent methods. In particular, we prove that our **deterministic** adaptive mirror descent ascent (MDA) reaches a lower sample complexity of $O(\kappa\epsilon^{-2})$ than the existing nonsmooth nonconvex minimax methods.

In fact, our methods can solve the minimax problems with nonsmooth regularization but not rely on any nonsmooth regularization. Our methods can also solve the minimax problems without nonsmooth regularization. Thus, our theoretical results can also apply these minimax problems without nonsmooth regularization studied in [25, 26].

Notations

For two vectors x and y, $\langle x, y\rangle$ denotes their inner product. $\|\cdot\|$ denotes the ℓ_2 norm for vectors and spectral norm for matrices, respectively. $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ denote the partial derivatives *w.r.t.* variables x and y respectively, and let $\nabla f(x, y) = \big(\nabla_x f(x, y), \nabla_y f(x, y)\big)$. $\partial g(x)$ denotes the subgradient set of function $g(x)$. Given a convex closed set $\mathcal{X}$, we define a projection operation $\text{Proj}_{\mathcal{X}}(\alpha) = \min_{x\in\mathcal{X}} \|x - \alpha\|^2$. Given the mini-batch samples $\mathcal{B} = \{\xi^i\}_{i=1}^b$, we let $\nabla f_{\mathcal{B}}(x) = \frac{1}{b}\sum_{i=1}^b \nabla f(x; \xi^i)$.

2 Related Works

In this section, we review some existing typical minimax optimization methods and stochastic mirror descent methods, respectively.

2.1 Minimax Optimization Methods

Minimax optimization recently has been widely studied in machine learning community. The convergence properties of (strongly) convex-(strongly) concave minimax optimization have been

studied in [36, 24, 30, 43]. Due to the popularity of nonconvex models in machine learning, many recent studies focused on the nonconvex minimax problems such as robustly deep neural networks training and GANs. For example, some effective gradient descent ascent methods [34, 32, 25, 26, 40, 41, 15, 16, 17, 14] have been proposed for the nonconvex-(strongly) concave minimax optimization. At the same time, Yang et al. [42] and Diakonikolas et al. [10] have studied a class of specific nonconvex-nonconcave minimax problems satisfying a so-called two-sided Polyak-Lojasiewicz inequality and stampacchia variational inequality, respectively. More recently, some research works [2, 4, 8] began to study the nonsmooth nonconvex minimax problem (1). In addition, [44, 22] studied the lower bound of sample complexity for nonconvex-strongly-concave minimax optimization. [39] studied the decentralized nonconvex-strongly-concave minimax optimization.

2.2 Mirror Descent Methods

Mirror descent (a.k.a., Bregman gradient) methods [7, 3] are a class of powerful optimization methods because it can use the geometry of optimization problems by choosing proper Bregman functions [5, 6]. The mirror descent methods for convex optimization have been studied in [7, 3]. Subsequently, Lei et al. [21] integrated the variance reduced technique to the mirror descent algorithm for stochastic convex optimization. More recently, the variance-reduced adaptive stochastic mirror descent algorithm [23] was proposed to solve the nonsmooth nonconvex finite-sum mini optimization. Moreover, Babanezhad et al. [1] presented a mirror-type algorithm for convex minimax optimization. At the same time, two typical mirror descent methods [34] were proposed for weakly convex minimax optimization. In fact, few work focuses on explicitly using the mirror-decent-type method to solve the nonsmooth nonconvex minimax problems.

3 Preliminaries

In the section, we first recap some basic information about mirror descent method.

Given a ρ-strongly convex and continuously-differentiable function $\psi(x)$, *i.e.*, $\langle x_1 - x_2, \nabla\psi(x_1) - \nabla\psi(x_2)\rangle \geq \rho\|x_1 - x_2\|^2$, we define a Bregman divergence (i.e., Bregman distance) for any $x, z \in \mathcal{X}$:

$$D_\psi(z, x) = \psi(z) - \psi(x) - \langle \nabla\psi(x), z - x\rangle. \tag{2}$$

To solve the problem $\min_{x\in\mathcal{X}} f(x)$, the mirror descent method uses the following form to update the variable x, for all $t \geq 1$

$$x_{t+1} = \arg\min_{x\in\mathcal{X}} \left\{ f(x_t) + \langle \nabla f(x_t), x - x_t\rangle + \frac{1}{\alpha} D_\psi(x, x_t) \right\}, \tag{3}$$

where $\alpha > 0$ is stepsize. In the above subproblem (3), the first two terms of its objective function is a linear function approximated the function $f(x)$, and the last term is a Bregman distance between x and x_t. Note that the terms $f(x_t)$ and $\langle \nabla f(x_t), x_t\rangle$ can be omitted in the above subproblem (3). When choosing $\psi = \frac{1}{2}\|x\|^2$, we have $D_\psi(x, x_t) = \frac{1}{2}\|x - x_t\|^2$. Then the mirror descent method will reduce to the standard projected gradient descent method. Next, we introduce some mild assumptions for the problem (1).

Assumption 1. *(Smoothness) For the deterministic and mini-batch stochastic algorithms (MDA and SMDA), we assume that the function $f(x, y)$ has an L_f-Lipschitz gradient, i.e., for all $x_1, x_2 \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{Y}$, we have*

$$\|\nabla f(x_1, y_1) - \nabla f(x_2, y_2)\| \leq L_f \|(x_1, y_1) - (x_2, y_2)\|. \tag{4}$$

For our variance-reduced stochastic algorithm (VR-SMDA), we assume that each component function $f(x, y; \xi)$ has an L_f-Lipschitz gradient, i.e., for all $x_1, x_2 \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{Y}$, we have

$$\|\nabla f(x_1, y_1; \xi) - \nabla f(x_2, y_2; \xi)\| \leq L_f \|(x_1, y_1) - (x_2, y_2)\|, \ \forall \xi. \tag{5}$$

In Assumption 1, the inequality (4) is commonly used in the minimax optimization [25, 2, 4, 8]. While the inequality (5) is frequently used in the variance-reduced stochastic optimization [27, 16].

Assumption 2. *Each component function $f(x, y; \xi)$ has an unbiased stochastic gradient with bounded variance σ^2, i.e.,*

$$\mathbb{E}[\nabla f(x, y; \xi)] = \nabla f(x, y), \quad \mathbb{E}\|\nabla f(x, y; \xi) - \nabla f(x, y)\|^2 \leq \sigma^2. \tag{6}$$

Algorithm 1 (Stochastic) Mirror Descent Ascent Algorithm

1: **Input:** T, stepsizes $\{\gamma_t > 0, \lambda_t > 0, \eta_t \in (0,1]\}_{t=1}^T$, mini-batch size b ;
2: **initialize:** $x_1 \in \mathcal{X}$ and $y_1 \in \mathcal{Y}$;
3: **for** $t = 1, 2, \ldots, T$ **do**
4: **MDA:** Compute partial derivatives $v_t = \nabla_x f(x_t, y_t)$ and $w_t = \nabla_y f(x_t, y_t)$;
5: **SMDA:** Generate randomly mini-batch samples $\mathcal{B}_t = \{\xi_t^i\}_{i=1}^b$ with $|\mathcal{B}_t| = b$, and compute stochastic partial derivatives $v_t = \nabla_x f_{\mathcal{B}_t}(x_t, y_t)$ and $w_t = \nabla_y f_{\mathcal{B}_t}(x_t, y_t)$;
6: Given the mirror functions ψ_t and ϕ_t;
7: $x_{t+1} = \arg\min_{x\in\mathcal{X}} \big\{\langle v_t, x\rangle + \frac{1}{\gamma_t} D_{\psi_t}(x, x_t) + g(x)\big\}$;
8: $y_{t+1} = y_t + \eta_t(\tilde{y}_{t+1} - y_t)$ where $\tilde{y}_{t+1} = \arg\max_{y\in\mathcal{Y}} \big\{\langle w_t, y\rangle - \frac{1}{\lambda_t} D_{\phi_t}(y, y_t) - h(y)\big\}$;
9: **end for**
10: **Output:** x_ζ and y_ζ chosen uniformly random from $\{x_t, y_t\}_{t=1}^T$.
11: **Output:** (for theoretical) x_ζ and y_ζ chosen uniformly random from $\{x_t, y_t\}_{t=1}^T$.
12: **Output:** (for practical) x_T and y_T.

Assumption 3. *The function $f(x, y)$ is μ-strongly concave w.r.t y, i.e., for all $x \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{Y}$, we have $\|\nabla_y f(x, y_1) - \nabla_y f(x, y_2)\| \geq \mu\|y_1 - y_2\|$. Then the following inequality holds*

$$f(x, y_1) \leq f(x, y_2) + \langle \nabla_y f(x, y_2), y_1 - y_2\rangle - \frac{\mu}{2}\|y_1 - y_2\|^2. \tag{7}$$

Assumption 4. *The functions $g(x)$ and $h(y)$ are convex but possibly nonsmooth.*

Assumption 3 shows that the function $f(x, y)$ is μ-strongly concave w.r.t y. Assumption 4 shows that the function $h(y)$ is convex. Thus, the function $\{f(x, y) - h(y)\}$ is strongly concave in $y \in \mathcal{Y}$, there exists a unique solution to the problem $\max_{y\in\mathcal{Y}}\{f(x, y) - h(y)\}$ for any x. Let $y^*(x) = \arg\max_{y\in\mathcal{Y}}\{f(x, y) - h(y)\}$, and $\Phi(x) = f(x, y^*(x)) - h(y^*(x)) = \max_{y\in\mathcal{Y}}\{f(x, y) - h(y)\}$.

Assumption 5. *For any $\alpha \in \mathbb{R}$, the sub-level set $\{x : \Phi(x) + g(x) \leq \alpha\}$ is compact. The function $\Phi(x) + g(x)$ is bounded below in $\mathcal{X}$,* i.e., $F^* = \inf_{x\in\mathcal{X}}\{\Phi(x) + g(x)\} > -\infty$.

Assumption 5 is frequently used in nonsmooth minimax optimization [8]. In fact, when $h(y) = c$ where c is a constant, we can only assume the function $\Phi(x) + g(x)$ is bounded below in $\mathcal{X}$ instead of Assumption 5.

4 Mirror Descent Ascent Methods

In the section, we propose a class of novel mirror descent ascent methods to solve the problem (1). Specifically, we first propose a deterministic mirror descent ascent (MDA) method, and stochastic mirror descent ascent (SMDA) method. Then we further introduce an accelerated stochastic mirror descent ascent (VR-SMDA) using variance reduced technique of SPIDER [11, 38].

4.1 MDA and SMDA algorithms

When $f(x, y)$ is a deterministic function, we propose a deterministic mirror descent ascent (MDA) method to solve the deterministic problem (1). When $f(x, y) = \mathbb{E}[f(x, y; \xi)]$ is a stochastic function, we propose a stochastic mirror descent ascent (SMDA) to solve the stochastic problem (1). Specifically, Algorithm 1 shows the algorithmic framework of the MDA and SMDA algorithms.

In Algorithm 1, we use (stochastic) mirror decent to update the parameter x, and simultaneously use (stochastic) mirror ascent to update the parameter y. Specifically, at step 7 of Algorithm 1, we use the mirror descent to update x,

$$x_{t+1} = \arg\min_{x\in\mathcal{X}} \big\{\langle v_t, x\rangle + \frac{1}{\gamma_t} D_{\psi_t}(x, x_t) + g(x)\big\} \tag{8}$$

$$= \arg\min_{x\in\mathcal{X}} \big\{f(x_t, y_t) + \langle v_t, x - x_t\rangle + \frac{1}{\gamma_t} D_{\psi_t}(x, x_t) + g(x)\big\}. \tag{9}$$

In fact, we omit the constant terms $f(x_t, y_t)$ and $\langle v_t, x_t\rangle$ in the above subproblem (8). In the above subproblem (9), the first two terms of its objective function is a linear function approximated the

function $f(x,y)$ based on (stochastic) derivative estimator v_t, and the third term is a Bregman distance between x and x_t based on the Bregman function ψ_t. Since the function $g(x)$ is possibly nonsmooth, we keep it in the above subproblem (9) as the standard proximal descent algorithm [33]. Similarly, at step 8 of Algorithm 1, we use the mirror ascent to update y,

$$\tilde{y}_{t+1} = \arg\max_{y\in\mathcal{Y}} \big\{\langle w_t, y\rangle - \frac{1}{\lambda_t} D_{\phi_t}(y, y_t) - h(y)\big\} \tag{10}$$

$$= \arg\max_{y\in\mathcal{Y}} \big\{f(x_t, y_t) + \langle w_t, y - y_t\rangle - \frac{1}{\lambda_t} D_{\phi_t}(y, y_t) - h(y)\big\}. \tag{11}$$

In the above subproblem (11), the first two terms of its objective function is a linear function approximated the function $f(x,y)$ based on (stochastic) derivative estimator w_t, and the third term is a Bregman distance between y and y_t based on the Bregman function ϕ_t. Moreover, at the step 8 of Algorithm 1, we further use a momentum iteration to update y.

When choosing the mirror functions $\psi_t(x) = \frac{1}{2}\|x\|^2$ and $\phi_t(y) = \frac{1}{2}\|y\|^2$ for all $t \geq 1$, we have $D_{\psi_t}(x, x_t) = \frac{1}{2}\|x - x_t\|^2$ and $D_{\phi_t}(y, y_t) = \frac{1}{2}\|y - y_t\|^2$. Under this case, Algorithm 1 will reduce the standard (stochastic) proximal gradient descent ascent algorithm. When choose the mirror functions $\psi_t(x) = \frac{1}{2}x^T H_t x$ and $\phi_t(y) = \frac{1}{2}y^T G_t y$ for all $t \geq 1$, we have $D_{\psi_t}(x, x_t) = \frac{1}{2}(x - x_t)^T H_t (x - x_t)$ and $D_{\phi_t}(y, y_t) = \frac{1}{2}(y - y_t)^T G_t (y - y_t)$, where $H_t \succeq \rho I_d$ and $G_t \succeq \rho I_p$. For example, given $\alpha \in (0,1)$ and $\rho > 0$, we can generate the matrices H_t and G_t like as in Adam-type algorithms [19, 18], defined as

$$\tilde{v}_0 = 0,\ \tilde{v}_t = \alpha\tilde{v}_{t-1} + (1-\alpha)\nabla_x f(x_t, y_t; \xi_t)^2, \quad H_t = \text{diag}(\sqrt{\tilde{v}_t} + \rho),\ t \geq 1 \tag{12}$$

$$\tilde{w}_0 = 0,\ \tilde{w}_t = \alpha\tilde{w}_{t-1} + (1-\alpha)\nabla_y f(x_t, y_t; \xi_t)^2, \quad G_t = \text{diag}(\sqrt{\tilde{w}_t} + \rho),\ t \geq 1 \tag{13}$$

Under this case, our SMDA algorithm will reduce a novel adaptive gradient descent ascent algorithm.

In the problem (1), the functions $g(x)$ and $h(y)$ are generally nonsmooth, e.g., $g(x) = \nu_1\|x\|_1$ and $h(y) = \nu_2\|y\|_1$ with $\nu_1, \nu_2 > 0$. When choosing the mirror functions $\psi_t(x) = \frac{1}{2}x^T H_t x$ and $\phi_t(y) = \frac{1}{2}y^T G_t y$ for all $t \geq 1$, and the matrices H_t and G_t are diagonal, e.g., generated from the above (12) and (13), we can use the soft thresholding operator $S(a,\lambda) = \text{sign}(a)\max(|a| - \lambda, 0) = \arg\min_z\{\frac{1}{2}(z-a)^2 + \lambda|z|\}$ to obtain the closed-form solutions of the following subproblems:

$$\min_{x\in\mathcal{X}} \big\{\langle v_t, x\rangle + \frac{1}{2\gamma_t}(x - x_t)^T H_t (x - x_t) + \nu_1\|x\|_1\big\}, \tag{14}$$

$$\max_{y\in\mathcal{Y}} \big\{\langle w_t, y\rangle - \frac{1}{2\lambda_t}(y - y_t)^T G_t (y - y_t) - \nu_2\|y\|_1\big\}, \tag{15}$$

where $H_t = \text{diag}(h_{1,t}, \cdots, h_{d,t})$ with $h_{i,t} > 0$ for $i \in [d]$, and $G_t = \text{diag}(g_{1,t}, \cdots, g_{p,t})$ with $g_{j,t} > 0$ for $j \in [p]$. Without loss of generality, assume the constraint sets $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \mathbb{R}^p$, we have

$$S\big(x_{i,t} - \frac{\gamma_t}{h_{i,t}} v_{i,t}, \frac{\gamma_t \nu_1}{h_{i,t}}\big) = \arg\min_{x_i\in\mathbb{R}} \big\{\langle v_{i,t}, x_i\rangle + \frac{h_{i,t}}{2\gamma_t}(x_i - x_{i,t})^2 + \nu_1|x_i|\big\},\ i \in [d] \tag{16}$$

$$S\big(y_{j,t} + \frac{\lambda_t}{g_{j,t}} w_{j,t}, \frac{\lambda_t \nu_2}{g_{j,t}}\big) = \arg\max_{y_j\in\mathbb{R}} \big\{\langle w_{j,t}, y_j\rangle - \frac{g_{j,t}}{2\lambda_t}(y_j - y_{j,t})^2 - \nu_2|y_j|\big\},\ j \in [p]. \tag{17}$$

4.2 VR-SMDA Algorithm

In this subsection, we propose an accelerated stochastic mirror descent ascent (VR-SMDA) algorithm to solve the stochastic problem (1). Algorithm 2 describes the detailed algorithmic framework of the VR-SMDA method.

In Algorithm 1, we only draw a mini-batch samples $\mathcal{B}_t = \{\xi_t^i\}_{i=1}^b$ at each iteration. Clearly, the mini-batch samples will take large variances in our SMDA algorithm. Thus, we use the variance reduced technique of SPIDER in our VR-SMDA algorithm to accelerate it. Specifically, when $\bmod\ (t,q) = 0$, we draw a relative large batch samples $\mathcal{B}_t = \{\xi_t^i\}_{i=1}^b$ to estimate our stochastic partial derivatives v_t and w_t; when $\bmod\ (t,q) \neq 0$, we only draw a mini-batch samples $\mathcal{I}_t =$

Algorithm 2 Accelerated Stochastic Mirror Descent Ascent (VR-SMDA) Algorithm

Input: T, q, stepsizes $\{\gamma_t > 0, \lambda_t > 0, \eta_t \in (0,1]\}_{t=1}^T$, mini-batch sizes b and b_1;
initialize: $x_1 \in \mathcal{X}$ and $y_1 \in \mathcal{Y}$;
for $t = 1, 2, \ldots, T$ **do**
  if $\mod (t, q) = 0$ **then**
    Randomly generate mini-batch samples $\mathcal{B}_t = \{\xi_t^i\}_{i=1}^b$ with $|\mathcal{B}_t| = b$;
    Compute stochastic partial derivatives $v_t = \nabla_x f_{\mathcal{B}_t}(x_t, y_t)$ and $w_t = \nabla_y f_{\mathcal{B}_t}(x_t, y_t)$;
  else
    Randomly generate mini-batch samples $\mathcal{I}_t = \{\xi_t^i\}_{i=1}^{b_1}$ with $|\mathcal{I}_t| = b_1$;
    Compute stochastic partial derivatives

$$v_t = \nabla_x f_{\mathcal{I}_t}(x_t, y_t) - \nabla_x f_{\mathcal{I}_t}(x_{t-1}, y_{t-1}) + v_{t-1}, \tag{18}$$
$$w_t = \nabla_y f_{\mathcal{I}_t}(x_t, y_t) - \nabla_y f_{\mathcal{I}_t}(x_{t-1}, y_{t-1}) + w_{t-1}; \tag{19}$$

  end if
  Given the mirror functions ψ_t and ϕ_t;
  $x_{t+1} = \arg\min_{x\in\mathcal{X}} \big\{\langle v_t, x\rangle + \frac{1}{\gamma_t} D_{\psi_t}(x, x_t) + g(x)\big\}$;
  $y_{t+1} = y_t + \eta_t(\tilde{y}_{t+1} - y_t)$ where $\tilde{y}_{t+1} = \arg\max_{y\in\mathcal{Y}} \big\{\langle w_t, y\rangle - \frac{1}{\lambda_t} D_{\phi_t}(y, y_t) - h(y)\big\}$;
end for
Output: (for theoretical) x_ζ and y_ζ chosen uniformly random from $\{x_t, y_t\}_{t=1}^T$.
Output: (for practical) x_T and y_T.

$\{\xi_t^i\}_{i=1}^{b_1}$ $(b > b_1)$ to estimate v_t and w_t in (18) and (19). Since samples $\mathcal{I}_t$ are independent to variables $\{x_t, x_{t-1}, y_t, y_{t-1}, v_{t-1}\}$, by using Assumption 2, we have

$$\mathbb{E}_{\mathcal{I}_t}[v_t] = \nabla_x f(x_t, y_t) - \nabla_x f(x_{t-1}, y_{t-1}) + v_{t-1} \neq \nabla_x f(x_t, y_t), \tag{20}$$
$$\mathbb{E}_{\mathcal{I}_t}[w_t] = \nabla_y f(x_t, y_t) - \nabla_y f(x_{t-1}, y_{t-1}) + w_{t-1} \neq \nabla_y f(x_t, y_t). \tag{21}$$

Thus, the partial derivative estimators v_t and w_t are biased. As in Algorithm 1, we also use the mirror descent iteration to update x, and use both the mirror ascent and momentum iterations to update y in Algorithm 2.

5 Convergence Analysis

In this section, we study the convergence properties of our algorithms (*i.e.*, MDA, SMDA and VR-SMDA) under some mild conditions. All related proofs are provided in supplementary materials. We first introduce a useful convergence metric $\|\mathcal{G}_t\|$ or $\mathbb{E}\|\mathcal{G}_t\|$ to measure convergence properties of our algorithms as in [23]. Given the generated parameters x_t at t-th iteration by our algorithms, we define a gradient mapping as

$$\mathcal{G}_t = \frac{1}{\gamma_t}(x_t - x_{t+1}^+), \tag{22}$$
$$x_{t+1}^+ = \arg\min_{x\in\mathcal{X}} \big\{\langle \nabla\Phi(x_t), x\rangle + \frac{1}{\gamma_t} D_{\psi_t}(x, x_t) + g(x)\big\}, \tag{23}$$

where $\Phi(x) = f(x, y^*(x)) - h(y^*(x)) = \max_{y\in\mathcal{Y}}\{f(x,y) - h(y)\}$. When $\mathcal{X} = \mathbb{R}^d$ and $g(x)$ is a constant, and $\psi_t(x) = \frac{1}{2}\|x\|^2$, we have $\mathcal{G}_t = \nabla\Phi(x) = \nabla_x f(x, y^*(x))$. Under this case, our convergence metric $\mathbb{E}\|\mathcal{G}_t\| = \mathbb{E}\|\nabla_x f(x, y^*(x))\|$ is a common convergence metric used in [25].

Since the objective function $f(x,y)$ is μ-strongly concave in y, the standard (stochastic) proximal gradient ascent can easily obtain the global solution of the subproblem $\max_{y\in\mathcal{Y}}\{f(x,y) - h(y)\}$. Without loss of generalization, in our theoretical analysis, we give the mirror functions $\phi_t(y) = \frac{1}{2}\|y\|^2$ for all $t \geq 1$, and the mirror functions $\psi_t(x)$ for all $t \geq 1$ are ρ-strong convex. Here, the constant ρ can be seen as a lower bound of the strong convexity of all functions $\{\psi_t(x)\}_{t=1}^T$ as in [23], i.e., all functions $\{\psi_t(x)\}_{t=1}^T$ are ρ-strongly concave.

5.1 Convergence Analysis of the SMDA and MDA Algorithms

In the subsection, we provide the convergence properties of our SMDA and MDA algorithms.

Theorem 1. *Suppose the sequence $\{x_t, y_t\}_{t=1}^T$ be generated from Algorithm 1 using stochastic partial derivatives (i.e., SMDA algorithm). Let $0 < \eta = \eta_t \leq 1$, $0 < \gamma = \gamma_t \leq \min(\frac{3\rho}{4L}, \frac{9\eta\rho\mu\lambda}{800\kappa^2}, \frac{2\eta\mu\rho\lambda}{25L_f^2})$ and $0 < \lambda = \lambda_t \leq \frac{1}{6L_f}$, we have*

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\|\mathcal{G}_t\| \leq \frac{4\sqrt{2(\tilde{F}(x_1) - F^*)}}{\sqrt{3T\gamma\rho}} + \frac{4\sqrt{2}\Delta_1}{\sqrt{3T\gamma\rho}} + \frac{10\sigma}{\sqrt{3b\rho}} + \frac{20\sigma\sqrt{\eta\lambda}}{3\sqrt{\gamma\rho\mu b}}, \tag{24}$$

where $\kappa = L_f/\mu$, $L = L_f(1+\kappa)$, $\tilde{F}(x) = \Phi(x) + g(x)$ and $\Delta_1 = \|y_1 - y^*(x_1)\|$.

Remark 1. *Without loss of generality, let $L_f \geq \frac{1}{\mu}$. Given $0 < \eta \leq 1$, $\lambda = O(\frac{1}{L_f})$, $\gamma = \min(\frac{3\rho}{4L}, \frac{9\eta\rho\mu\lambda}{800\kappa^2}, \frac{2\eta\mu\rho\lambda}{25L_f^2})$ and $\rho = O(L_f^{\nu})$, $(\nu \geq 0)$, we have $\gamma = O(\kappa^{\nu-3})$ and $\gamma\rho = O(\kappa^{2\nu-3})$. Thus, our SMDA algorithm has a convergence rate of $O(\sqrt{\frac{\kappa^{3-2\nu}}{T}} + \sqrt{\frac{\kappa^{-2\nu}}{b}} + \sqrt{\frac{\kappa^{3-2\nu}}{b}})$. When let $\nu = \frac{1}{2}$ (i.e., $\rho = O(\sqrt{L_f})$), $b = T/\kappa$ and $\sqrt{\frac{\kappa^2}{T}} = \epsilon/3$, we have $T = O(\kappa^2\epsilon^{-2})$ and $b = O(\kappa\epsilon^{-2})$. Since our SMDA algorithm requires b samples to estimate the stochastic partial directives v_t and w_t at each iteration, and needs T iterations, it has a sample complexity of $bT = O(\kappa^3\epsilon^{-4})$ for finding an ϵ-stationary point, the same complexity as in [4]. When let $\nu = 5/6$ (i.e., $\rho = L_f^{4/3}$), $b = T/\kappa^{1/3}$ and $\sqrt{\frac{\kappa^{1/3}}{T}} = \epsilon/3$, we have $T = O(\kappa^{1/3}\epsilon^{-2})$ and $b = O(\epsilon^{-2})$. Thus, our SMDA algorithm has a near optimal sample complexity of $bT = O(\kappa^{1/3}\epsilon^{-4})$, the same complexity as in [22] in solving the problem* (1) *without the nonsmooth regularization.*

Theorem 2. *Suppose the sequence $\{x_t, y_t\}_{t=1}^T$ be generated from Algorithm 1 using the deterministic partial derivatives (i.e., MDA algorithm). Let $0 < \eta = \eta_t \leq 1$, $0 < \gamma = \gamma_t \leq \min(\frac{3\rho}{4L}, \frac{9\eta\rho\mu\lambda}{800\kappa^2}, \frac{2\eta\mu\rho\lambda}{25L_f^2})$ and $0 < \lambda = \lambda_t \leq \frac{1}{6L_f}$, we have*

$$\frac{1}{T}\sum_{t=1}^{T}\|\mathcal{G}_t\| \leq \frac{4\sqrt{2(\tilde{F}(x_1) - F^*)}}{\sqrt{3T\gamma\rho}} + \frac{4\sqrt{2}\Delta_1}{\sqrt{3T\gamma\rho}}, \tag{25}$$

where $\kappa = L_f/\mu$, $L = L_f(1+\kappa)$, $\tilde{F}(x) = \Phi(x) + g(x)$ and $\Delta_1 = \|y_1 - y^*(x_1)\|$.

Remark 2. *Without loss of generality, let $L_f \geq \frac{1}{\mu}$. Given $0 < \eta \leq 1$, $\lambda = O(\frac{1}{L_f})$, $\gamma = \min(\frac{3\rho}{4L}, \frac{9\eta\rho\mu\lambda}{800\kappa^2}, \frac{2\eta\mu\rho\lambda}{25L_f^2})$ and $\rho = O(L_f^{(\frac{1}{2}+\nu)})$, $(\nu \geq 0)$, we have $\frac{1}{\gamma\rho} = O(\kappa^{(2-2\nu)})$. Under this case, our MDA algorithm has a sample complexity of $T = O(\kappa^{(2-2\nu)}\epsilon^{-2})$, $\nu \geq 0$ for finding an ϵ-stationary point. When $\nu = 0$, our MDA algorithm has a sample complexity of $T = O(\kappa^2\epsilon^{-2})$, the same complexity as in [4]. When $\nu = 1/2$, our MDA algorithm has a lower sample complexity of $T = O(\kappa\epsilon^{-2})$ than the sample complexity of [4, 2]. In solving the problem* (1) *without the nonsmooth regularization, when $\nu = 3/4$, our MDA algorithm has a near optimal sample complexity of $T = O(\sqrt{\kappa}\epsilon^{-2})$, the same complexity as in [26].*

5.2 Convergence Analysis of the VR-SMDA Algorithm

In the subsection, we provide the convergence properties of the VR-SMDA algorithm.

Theorem 3. *Suppose the sequence $\{x_t, y_t\}_{t=1}^T$ be generated from Algorithm 2. Let $b_1 = q$, $0 < \eta = \eta_t \leq 1$, $0 < \gamma = \gamma_t \leq \min(\frac{3\rho}{4L}, \frac{\eta\mu\lambda\rho}{38L_f^2}, \frac{3\rho}{19L_f^2\eta}, \frac{\rho\eta}{8}, \frac{9\rho\eta\mu\lambda}{400\kappa^2})$ and $0 < \lambda = \lambda_t \leq \min(\frac{1}{6L_f}, \frac{9\mu}{100\eta^2L_f^2})$, we have*

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\|\mathcal{G}_t\| \leq \frac{4\sqrt{2(\tilde{F}(x_1) - F^*)}}{\sqrt{3T\gamma\rho}} + \frac{4\sqrt{2}\Delta_1}{\sqrt{3T\gamma\rho}} + \frac{2\sqrt{2}\sigma}{\sqrt{\gamma\rho\eta b}L_f}, \tag{26}$$

where $\kappa = L_f/\mu$, $L = L_f(1+\kappa)$, $\tilde{F}(x) = \Phi(x) + g(x)$ and $\Delta_1 = \|y_1 - y^*(x_1)\|$.

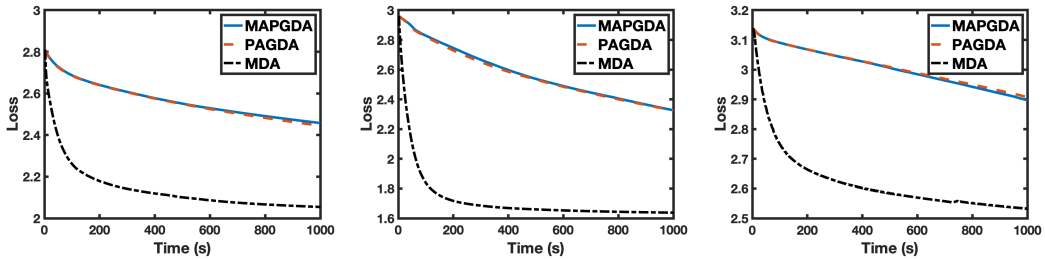
Remark 3. *Without loss of generality, let $L_f \geq \frac{1}{\mu}$. Given $0 < \eta \leq 1$, $\lambda = O(\frac{1}{\kappa L_f})$, $\gamma = \min(\frac{3\rho}{4L}, \frac{\eta\mu\lambda\rho}{38L_f^2}, \frac{3\rho}{19L_f^2\eta}, \frac{\rho\eta}{8}, \frac{9\rho\eta\mu\lambda}{400\kappa^2})$ and $\rho = O(L_f^{1+\nu})$ $(\nu \geq 0)$, we have $\frac{1}{\gamma\rho} = O(\kappa^{2-2\nu})$. Thus, our*

VR-SMDA algorithm has a convergence rate of $O(\sqrt{\frac{\kappa^{(2-2\nu)}}{T}} + \sqrt{\frac{\kappa^{(1-2\nu)}}{b}})$. *When let* $\nu = 0$ $(i.e., \rho = O(L_f))$, $b = T/\kappa$ *and* $\sqrt{\frac{\kappa^2}{T}} = \epsilon/2$, *we have* $T = O(\kappa^2\epsilon^{-2})$. *Further let* $b_1 = q = O(\kappa\epsilon^{-1})$ *and* $b = O(\kappa\epsilon^{-2})$. *Since Algorithm 2 requires* b *samples to estimate the stochastic directives* v_t *and* w_t *at each iteration when* $\mod(t, q) = 0$, *otherwise needs* b_1 *samples, and need* T *iterations, it has a lower sample complexity of* $b_1T + bT/q = O(\kappa^3\epsilon^{-3})$ *for finding an* ϵ*-stationary point of the problem* (1) *than the sample complexity of [4], which is the same complexity of [27] in solving the problem* (1) *without the nonsmooth regularization.*

Remark 4. *The above sample complexities are obtained when given* $\rho = O(L_f^{\nu})$ $(\nu > 0)$, *where* L_f *is the smooth parameter of objective function* $f(x, y)$. *Although in the objective function* $f(x, y)$, L_f *may be large, we can easily change the original objective function* $f(x, y)$ *to a new function* $\hat{f}(x, y) = \tau f(x, y)$, $0 < \tau < 1$. *Since* $\nabla\hat{f}(x, y) = \tau\nabla f(x, y)$, *the gradient of function* $\hat{f}(x, y)$ *is* $\hat{L}$*-Lipschitz continuous* ($\hat{L} = \tau L_f$). *Thus, we can choose a suitable hyper-parameter* τ *to let this new objective function* $\hat{f}(x, y)$ *satisfy the condition* $\rho = O(\hat{L})$.

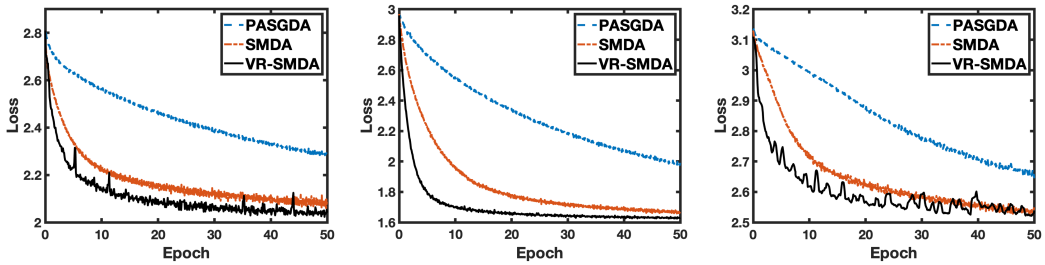
6 Numerical Experiments

In this section, we perform two tasks (i.e., fair classifier and robust neural network training) to validate efficiency of our algorithms. Specifically, we conduct these tasks on the Fashion-MNIST dataset as in [32] as well MNIST dataset and CIFAR-10 dataset. Fashion-MNIST dataset and MNIST dataset consist of 28×28 arrays of grayscale pixel images classified into 10 categories, and includes $60,000$ training images and $10,000$ testing images. CIFAR-10 dataset includes $60,000$ 32×32 colour images ($50,000$ training images and $10,000$ testing images). In the experiment, we compare our algorithms (MDA, SMDA and VR-SMDA) with the existing proximal gradient descent ascent algorithms (MAPGDA [2], PAGDA [4] and PASGDA[4]) for solving these nonsmooth nonconvex minimax problems. Note that the Proximal-GDA algorithm in [8] only is a non-accelerated version of MAPGDA algorithm [2], so we omit it in the comparison methods. The experiments are run on CPU machines with 2.3 GHz Intel Core i9 as well as NVIDIA Tesla P40 GPU.



(a) Fashion-MNIST (b) MNIST (c) CIFAR-10

Figure 1: Results of different deterministic methods on the fair classifier task.



(a) Fashion-MNIST (b) MNIST (c) CIFAR-10

Figure 2: Results of different stochastic methods on the fair classifier task.

6.1 Fair Classifier

The first task is to train a fair classifier to minimize the maximum loss over categories. Here, we use a nonconvex Convolutional Neural Network (CNN) model as classifier. Similar to [32], we limit our experiment to the three categories. To be precise, the Fashion-MNIST dataset is limited to T-shirt/top,

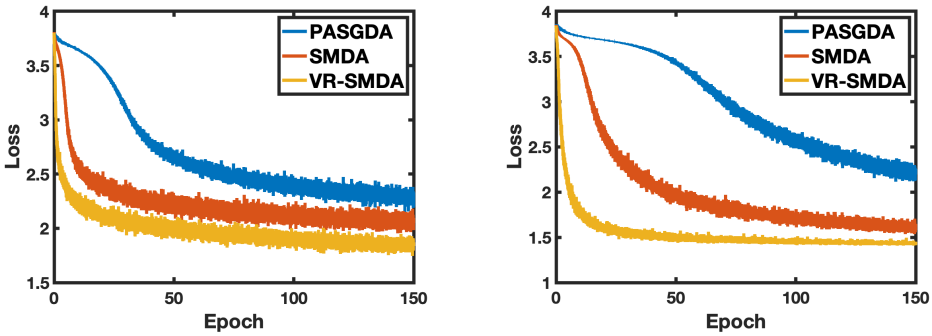
Coat and Shirt categories. The MNIST dataset is limited to digital numbers $\{0, 2, 3\}$, and CIFAR10 dataset is limited to airplane, automobile and bird. Specifically, we will solve the minimax problem:

$$\min_{w} \max_{y \in \mathcal{Y}} \left\{ \sum_{i=1}^{3} y_i \mathcal{L}_i(w) + g(w) - h(y) \right\}, \quad \text{s.t. } \mathcal{Y} = \left\{ y \in \mathbb{R}^3 \mid y_i \geq 0, \sum_{i=1}^{3} y_i = 1 \right\}, \tag{27}$$

where w denotes the parameters in CNN model, and $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are the loss functions corresponding to the samples in three different categories. Here we let $g(w) = \nu_1 \|w\|_1$ and $h(y) = \nu_2 \|y\|_2^2$, where $\nu_1 > 0$ and $\nu_2 > 0$. Clearly the inner maximization problem is strongly concave, and the outer minimization problem is nonconvex nonsmooth. Thus our theory can be applied.

In the experiment, we let $\nu_1 = 0.001$ and $\nu_2 = 0.1$ in the above problem (27). For fair comparison, we use the same step size for all methods. Specifically, step-size for w is 0.001 and step-size for y is 0.00001. We apply xavier normal initialization to CNN layer. In our algorithms, we choose the mirror functions $\psi_t(w) = \frac{1}{2} w^T H_t w$ and $\phi_t(y) = \frac{1}{2} y^T G_t y$ for all $t \geq 1$, where H_t and G_t are generated from (12) and (13) respectively, given $\alpha = 0.1$ and $\rho = 0.00005$. We set $\eta = \eta_t = 1$ in our algorithms. We run all deterministic algorithms for 1000 seconds and all stochastic algorithms for 50 epochs. Then we record the loss value. For stochastic methods, batch sizes of PASGDA and SMDA are 3000. For our VR-SMDA, we set the large batch size $b = 60000$ and the mini-batch size $b_1 = q = 3000$.

Figure 1 shows the loss vs time of different deterministic methods. Figure 2 plots the loss vs epoch of different stochastic methods. From these results, we can find that our algorithms consistently outperform the other algorithms with a great margin. The main reason is that our algorithms use the preconditioned (adaptive) matrices H_t and G_t in updating x and y, respectively.



(a) Fashion-MNIST (b) MNIST

Figure 3: Results of different stochastic methods on the robust NN training task at Fashion-MNIST and MNIST datasets.

6.2 Robust Neural Network Training

The second task is to train robust Neural Networks (NNs). Although the NNs have been widely used in many applications such as image classification, they are vulnerable to adversarial attacks such as Fast Gradient Sign Method (FGSM) [13] and Projected Gradient Descent (PGD) attack [20]. In other word, a small perturbation in the input of NN can significantly change its output. Thus, we try to train a robust NN against these adversarial attacks, which generally reformulate this robust training into the following minimax problem:

$$\min_{w} \sum_{i=1}^{n} \max_{y_i \in \mathcal{Y}} \mathcal{L}\big(f(a_i + y_i; w), b_i\big), \quad \mathcal{Y} = \left\{ \|y_i\|_\infty \leq \varepsilon, \ i \in [n] \right\} \tag{28}$$

where (a_i, b_i) denotes the i-th data point, and w is the parameter of NN, and y_i denotes is the perturbation added to the i-th data point. Following [32], we approximate the inner maximization problem of the above minimax problem (28) with the following finite max problem

$$\min_{w} \sum_{i=1}^{n} \max \left\{ \mathcal{L}\big(f(\hat{a}_{i,0}(w); w), b_i\big), \cdots, \mathcal{L}\big(f(\hat{a}_{i,9}(w); w), b_i\big) \right\}, \tag{29}$$

where $\hat{a}_{i,j}(w)$ is the result of a targeted attack on data point a_i that is changed the output of NN to label j. Following [32], we can obtain $\hat{a}_{i,j}(w)$ by using the following procedure: In the last layer of

the NN architecture for learning classification on MNIST (Fashion-MNIST) we have 10 different neurons, each corresponding with one category of classification. For any sample (a_i, b_i) in the dataset and starting from $a_{i,j}^0 = a_i$ for any $j = 1, 2, \cdots, 9$, we run projected gradient ascent to obtain the following chain of points:

$$a_{i,j}^{k+1} \;=\; \text{Proj}_{\mathcal{Y}}\big[a_{i,j}^k + \mu \nabla_a \big(Z_j(a_{i,j}^k, w) - Z_{b_i}(a_{i,j}^k, w)\big)\big],\; k = 0, 1, \cdots, K-1 \tag{30}$$

where $\mu > 0$ is a stepsize, and Z_j is the network logit before softmax corresponding to label j. Finally, we can set $\hat{a}_{i,j}(w) = a_{i,j}^K$ in the above minimax problem (29).

Next, we can replace the above problem (29) with the following nonconvex nonsmooth problem:

$$\min_w \sum_{i=1}^n \max_{u \in U} \Big\{ \sum_{i=0}^9 u_i \, \mathcal{L}\big(f(a_{i,j}^K; w), b_i\big) + \nu_1 \|w\|_1 - \nu_2 \|u\|_2^2 \Big\}, \tag{31}$$

$$\text{s.t. } U = \big\{ u \in \mathbb{R}^{10} \mid u_i \geq 0,\; \sum_{i=0}^9 u_i = 1 \big\},$$

where $\nu_1 > 0$ and $\nu_2 > 0$. Clearly, the inner maximization problem in (31) is strongly-concave, and its outer minimization problem is nonconvex nonsmooth.

In the experiment, we set $\nu_1 = 0.0001$ and $\nu_2 = 0.1$ in the above problem (31). In the above problem (30), we set $K = 5$. For fair comparison, we use the same step size for all methods. Specifically, step-size for w is 0.0005 and step-size for u is 0.00001. We set $\eta = \eta_t = 1$ in our algorithms. For our algorithms, we choose the mirror functions $\psi_t(w) = \frac{1}{2} w^T H_t w$ and $\phi_t(u) = \frac{1}{2} u^T G_t u$ for all $t \geq 1$, where H_t and G_t are generated from (12) and (13) respectively, given $\alpha = 0.1$ and $\rho = 0.0005$. Here we only conduct experiments with stochastic methods, and batch-sizes of PASGDA and SMDA are 600. For our VR-SMDA, we set $b = 1200$ and $b_1 = q = 600$. Following [32], we set $\varepsilon = 0.4$ in the above problem (28). Figure 3 shows the loss vs epoch of different stochastic methods. From these results, we can find that our algorithms outperform the other algorithms, and the VR-SMDA consistently outperforms the SMDA.

7 Conclusions

In the paper, we proposed a class of novel adaptive mirror descent ascent methods to solve the nonconvex-strongly-concave minimax optimization problems with nonsmooth regularization. Moreover, we provided an effective convergence analysis framework for our methods. Some experimental results on fair classifier and robust neural network training tasks verify that our new algorithms consistently outperform the related algorithms.

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