Sharp Analysis of Stochastic Optimization under Global Kurdyka-Łojasiewicz Inequality

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Abstract

We study the complexity of finding the global solution to stochastic nonconvex optimization when the objective function satisfies global Kurdyka-Łojasiewicz (KL) inequality and the queries from stochastic gradient oracles satisfy mild expected smoothness assumption. We first introduce a general framework to analyze Stochastic Gradient Descent (SGD) and its associated nonlinear dynamics under the setting. As a byproduct of our analysis, we obtain a sample complexity of $\mathcal{O}(\epsilon^{-(4-\alpha)/\alpha})$ for SGD when the objective satisfies the so called α-PŁ condition, where α is the degree of gradient domination. We also demonstrate that the complexity bound of SGD is tight under the specific choice of stepsizes and cannot be improved with mini-batching. Furthermore, we show that a modified SGD with variance reduction and restarting (PAGER) achieves an improved sample complexity of $\mathcal{O}(\epsilon^{-2/\alpha})$ when the objective satisfies the average smoothness assumption. This leads to the first optimal algorithm for the important case of $\alpha = 1$ which appears in applications such as policy optimization in reinforcement learning.

1 Introduction

Non-convex optimization problems are ubiquitous in machine learning domains such as in training deep neural networks [16] or policy optimization in reinforcement learning [42]. Stochastic Gradient Descent (SGD) and its variants are driving the practical success of machine learning approaches. Naturally, understanding the limits of performance of SGD in the non-convex setting has become an important avenue of research in recent years [15, 3, 23, 36, 17, 10, 50].

We are interested in solving the unconstrained *stochastic*, *nonconvex* optimization problem of the form

$$\min_{x\in\mathbb{R}^d} f(x) := \mathbb{E}_{\xi\sim\mathcal{D}}\left[f_\xi(x)\right], \tag{1}$$

where $f(\cdot)$ is smooth and possibly nonconvex, and ξ is a random vector drawn from a distribution $\mathcal{D}$. Moreover, we are interested in an important special case of (1), when the expectation can be written as the average of n smooth functions:

$$\min_{x\in\mathbb{R}^d} \Big[f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x)\Big]. \tag{2}$$

As for a general nonconvex differentiable objective $f : \mathbb{R}^d \to \mathbb{R}$, finding a global minimum of f is in general intractable [35, 44], there are two common strategies to analyze optimization methods for nonconvex functions. The first one is to scale down the requirements on the solution of interest from global optimality to some relaxed version, e.g., first-order stationary point $\mathbb{E}\left[\|\nabla f(\hat{x})\|\right] \le \epsilon$. However, such solutions do not exclude the possibility of approaching a suboptimal local minima

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or a saddle point. Another approach is to study nonconvex problems with additional structural assumption in the hope of being able to establish convergence to global solutions. In this direction, several relaxations of convexity have been proposed and analyzed, for instance, star convexity, quasar-convexity, error bounds condition, restricted secant inequality, and quadratic growth [20, 18, 17]. Many of the aforementioned relaxations have limited application in real-world problems.

Recently, there has been a surge of interest in the analysis of functions satisfying the so-called Kurdyka-Łojasiewicz (KŁ) inequality [6, 7, 7]. Of particular interest is the family of functions that satisfy global KŁ inequality. Specifically, we say that $f(\cdot)$ satisfies *global KŁ inequality* if there exists some continuous function $\phi(\cdot)$ such that $||\nabla f(x)|| \geq \phi\left(f(x) - \inf_x f(x)\right)$ for all $x \in \mathbb{R}^d$. If this inequality is satisfied for $\phi(t) = \sqrt{2\mu}\, t^{1/\alpha}$, then we say that α-PŁ condition is satisfied for $f(\cdot)$. The special case of global KŁ inequality, 2-PŁ, often referred as Polyak-Łojasiewicz or PL condition, was originally discovered independently in the seminal works of B. Polyak, T. Ležanski and S. Lojasiewicz [39, 26, 31, 32]. Notably, this class of problems have found many interesting emerging applications in machine learning, for instance, policy gradient (PG) methods in reinforcement learning [34, 1, 46], generalized linear models [33], over-parameterized neural networks [2, 48], linear quadratic regulator in optimal control [9, 13], and low-rank matrix recovery [5].

Despite increased popularity of KŁ and α-PŁ assumptions, the analysis of stochastic optimization under it remains limited and the majority of works focus on deterministic gradient methods. Indeed until recently, only the special case of α-PŁ with $\alpha = 2$ was mainly addressed in the literature [20, 17, 23, 43, 40]. In this paper, we study the sample complexities of stochastic optimization for the broader class of nonconvex functions with global KŁ property.

1.1 Related Works and Open Questions

SGD: A plethora of existing works has studied the sample complexity of SGD and its variants for finding an ϵ-stationary point of general nonconvex function f, that is, a point $x \in \mathbb{R}^d$ for which $\mathbb{E}[||\nabla f(\hat{x})||] \leq \epsilon$. For instance, [15] showed that for a smooth objective (one with Lipschitz gradients) under bounded variance (BV) assumption, SGD with properly chosen stepsizes reaches an ϵ-stationary point with the sample complexity of $\mathcal{O}(\epsilon^{-4})$. Recently, [23, 46] further extended the result under a much milder *expected smoothness* (ES) assumption on stochastic gradient. While this sample complexity is known to be optimal for general noncovex functions, a naive application of this result to a function value using α-PŁ condition would lead to a suboptimal $\mathcal{O}(\epsilon_f^{-4/\alpha})$ sample complexity for finding an ϵ_f-optimal solution. Recently, [14] studied SGD and established convergence rates for α-PŁ functions assuming variances are bounded. Their sample complexity rates equal $\mathcal{O}(\epsilon_f^{-(4-\alpha)/\alpha})$ in our notation. Later [28] considered SGD scheme with random reshuffling under local and global KŁ and provided convergence in the iterates for $\alpha \in (1, 2]$. We note that our proof techniques are different from [14] and [28] and are not limited merely to the case of BV assumption. In this work, we will answer the following open question: *What is the exact performance limit of SGD under global KŁ condition and a more practical model of stochastic oracle?*

Variance reduction: There has been extensive research on development of algorithms which improve the dependence on n and/or ϵ for both problems (1) and (2). One important family of techniques is *variance reduction*, which has emerged from the seminal work of Le Roux et. al [41]. The main idea of *variance reduction* is to make use of the stochastic gradients computed at previous iterations to construct a better gradient estimator at a relatively small computational cost. Various modifications, generalizations, and improvements of the variance reduction technique appeared in subsequent work, for instance, [41, 22, 11] to name a few.

Finite-sum case: A number of recently proposed algorithms such as SNVRG, SARAH, STORM, SPIDER, and PAGE [49, 37, 10, 12] achieve the optimal sample complexity $\mathcal{O}\big(n + \frac{\sqrt{n}}{\epsilon^2}\big)$ to obtain $\mathbb{E}\left[||\nabla f(\hat{x})||\right] \leq \epsilon$ for minimizing a general nonconvex function [29]. Under 2-PŁ condition, these methods further improve to $\mathcal{O}\big(\,(n + \kappa\sqrt{n})\log(\frac{1}{\epsilon_f})\big)$ sample complexity[1] for finding a point x such that $\mathbb{E}\left[f(x) - f^\star\right] \leq \epsilon_f$. [21] studies SARAH in finite sum case under local KŁ assumption and proves convergence in the iterates. The analysis in [21] is only asymptotic analysis and the dependence on the parameters κ and n, which are important in practice for quantifying the improvement over GD and SGD are ignored. [27] proposes an SVRG-based algorithm for KŁ functions and [45, 37, 38]

[1] $\kappa = \mathcal{L}/\mu$ is the analogue of condition number and defined in Section 2.

study other variance reduction techniques, but they only analyze the special case $\alpha = 2$. In our work, we will answer the following open question: *What is the extent of improvement any variance reduction scheme can provide under global KŁ condition for finite-sum objectives of the form (2)?*

Online/streaming case: While variance reduction methods have been initially designed for problems of the form (2), later it was shown that they also improve over SGD in solving (1) [25, 30]. For example, the methods in [49, 37, 12, 29] achieve $\mathcal{O}(\epsilon^{-3})$ complexity under a mild additional assumptions such as the *average L-smoothness* (Assumption 7) improving over $\mathcal{O}(\epsilon^{-4})$ complexity of SGD for finding an ϵ-stationary point. Note that variance reduction for α-PŁ functions has only been addressed for the special case $\alpha = 2$ and it does not yield an improvement over SGD in terms of ϵ since SGD is already optimal. For other cases, taking $\alpha = 1$ as an example, to obtain the global convergence of SGD, we could translate the result $\mathcal{O}(\epsilon^{-3})$ for finding a stationary point to a function value by using α-PŁ condition, and obtain $\mathcal{O}\big(\epsilon_f^{-3/\alpha}\big)$ sample complexity. There are two issues with this approach. First, this complexity does not provide any improvement over SGD in the most interesting practical case $\alpha = 1$. Second, the guaranties for general nonconvex optimization hold on average, in the sense that the point $\hat{x}$ is sampled uniformly from all the iterates of the algorithm. It would be more desirable to instead derive last iterate convergence guaranties under the KL condition. In this work, we will answer the following open question: *Is it possible to accelerate the $\mathcal{O}\big(\epsilon_f^{-(4-\alpha)/\alpha}\big)$ sample complexity of SGD under global KŁ condition for stochastic objectives of the form (1)?*

1.2 Contributions.

In this work, we provide an extensive analysis of stochastic optimization under global KŁ condition and answer all the above questions in the affirmative. More precisely, our contributions are as follows

- We provide a new framework for the analysis of the dynamics of SGD under global KŁ condition (see Section 3). It is based on analysing the dynamic of SGD which is govern by a recursive inequality (see Equation (7)). As a result of this analysis, we introduce a set of conditions (see Theorem 1) for designing proper setpsizes to guarantee convergence.
- Using this framework, we provide sharp analysis of convergence rate of SGD under a general ES assumption (see Assumption 4) and demonstrate that the sample complexity $\mathcal{O}\big(\epsilon_f^{-(4-\alpha)/\alpha}\big)$ is tight for the dynamical system describing SGD. More precisely, if the recursive inequality describing the behaviour of SGD (see Equation (7)) becomes an equality, then its convergence rate cannot be better rate than $\mathcal{O}\big(\epsilon_f^{-(4-\alpha)/\alpha}\big)$ unless additional assumptions are applied.
- Next, we propose a variance reduction scheme with parameter restart PAGER. Carefully chosen sequence of parameters of PAGER allows the algorithm to adapt to the nonconvex KŁ geometry of the problem and establish state-of-the-art convergence rates for minimizing α-PŁ functions. In online setting (1), we establish $\mathcal{O}\left(\epsilon_f^{-2/\alpha}\right)$ sample complexity of PAGER, which beats $\mathcal{O}\left(\epsilon_f^{-(4-\alpha)/\alpha}\right)$ complexity of SGD for the whole spectrum of parameters $\alpha \in [1, 2)$. In particular, for the important special case of 1-PŁ, this leads to the first optimal algorithm with $O(\epsilon^{-2})$ sample complexity, which already matches with the lower bound known for stochastic convex optimization [35].
- Furthermore, we obtain faster rates with PAGER in finite sum case (2), providing the first acceleration over GD and SGD under α-PŁ condition.

In Table 1, we summarize the sample complexity results for stochastic optimization under α-PŁ and BV assumptions. We also establish sharp convergence results for convergence in the iterates to the set $X^\star$ of optimal points and provide a summary Table 2 in the Appendix.

2 Assumptions and discussion

In this section, we introduce the assumptions we make throughout the paper.

Assumption 1. *We assume that the gradient of f is Lipschitz continuous, that is, for all x and y,*

$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||, \tag{3}$$

where $L > 0$ is called the Lipschitz constant.

Table 1: Summary of sample complexity results for α-PŁ functions (Assumption 3) under average $\mathcal{L}$-smoothness (Assumptions 7) and bounded variance (Assumptions 5). Quantities: α = PL power; μ = PL constant; $\kappa = \mathcal{L}/\mu$; σ^2 = variance. The entries of the table show the expected number of stochastic gradient calls to achieve $\mathbb{E}\left[f(x_k) - f^\star\right] \leq \epsilon_f$ in $\widetilde{\mathcal{O}}\left(\cdot\right)$ notation.

Method	Finite sum case	Online case
GD	$n\kappa \left(\frac{1}{\epsilon_f}\right)^{\frac{2-\alpha}{\alpha}}$	N/A
SGD	$\frac{\kappa\sigma^2}{\mu} \left(\frac{1}{\epsilon_f}\right)^{\frac{4-\alpha}{\alpha}}$	$\frac{\kappa\sigma^2}{\mu} \left(\frac{1}{\epsilon_f}\right)^{\frac{4-\alpha}{\alpha}}$
PAGER	$n + \sqrt{n}\kappa \left(\frac{1}{\epsilon_f}\right)^{\frac{2-\alpha}{\alpha}}$ (new)	$\frac{\kappa\sigma^2}{\mu} \left(\frac{1}{\epsilon_f}\right)^{\frac{2}{\alpha}}$ (new)

Furthermore, we assume that the objective function f is lower bounded, i.e., $f^* := \inf_x f(x) > -\infty$, and it satisfies the following inequality

Assumption 2 (global KL or global Kurdyka-Łojasiewicz)**.** *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $\phi(0) = 0$ and $\phi^2(\cdot)$ is convex. The function $f(\cdot)$ is said to satisfy global Kurdyka-Łojasiewicz inequality if*

$$||\nabla f(x)|| \geq \phi\left(f(x) - f^*\right) \quad \textit{for all } x \in \mathbb{R}^d. \tag{4}$$

Assumption 3 (α-PŁ or Polyak-Łojasiewicz)**.** *There exists $\alpha \in [1, 2]$ and $\mu > 0$ such that*

$$\|\nabla f(x)\|^\alpha \geq (2\mu)^{\alpha/2}\left(f(x) - f^*\right) \quad \textit{for all } x \in \mathbb{R}^d. \tag{5}$$

We refer to α as the PL power and μ as the PL constant.

It is straightforward to see that the α-PŁ is a special case of KL with $\phi(t) = \sqrt{2\mu}\, t^{1/\alpha}$.

Connections with other assumptions. An alternative commonly adopted way to define the global KL property is: $\rho'\left(f(x) - f^\star\right) \cdot \|\nabla f(x)\| \geq 1$ for all $x \in \mathbb{R}^d$, where $\rho(t)$ is called a disingularizing function and $\rho'(\cdot)$ denotes its derivative. Moreover, disingularizing function satisfies the following conditions, it is continuous and concave, $\rho(0) = 0$, and $\rho'(t) > 0$ [28].

If the above assumption holds for $\rho(t) := \frac{1}{\theta}t^\theta$ and $\theta > 0$, then Assumption 3 holds with power $\alpha = \frac{1}{1-\theta}$. However, α-PŁ is more general since it allows to consider the case $\alpha = 1$. For example, consider the function of one variable $f(x) = (e^x + e^{-x})/2 - 1$, then $|f'(x)| \geq f(x)$ for all x. Thus $f(x)$ satisfies Assumption 3 with $\alpha = 1$ and $\mu = 1/2$. Moreover, this function is convex, but it does not satisfy inequality $\rho'\left(f(x) - f^\star\right) \cdot \|\nabla f(x)\| \geq 1$ for any choice of $\rho(t)$. We also provide several non-convex examples for which α-PŁ holds with $\alpha = 1$ in the Appendix.

The intuition behind the special case $\alpha = 1$ is that the function is allowed to be flat near the set of optimal points $X^\star = \arg\min_x f(x)$.

Assumption 4 (k-ES, Expected Smoothness of order k)**.** *The stochastic gradient estimator $g_k(x, \xi)$ is an unbiased estimate of the gradient $\nabla f(x)$ at any given point x and its second moment satisfies*

$$\mathbb{E}\left[\|g_k(x,\xi)\|^2\right] \leq 2A \cdot h\big(f(x) - f^*\big) + B \cdot ||\nabla f(x)||^2 + \frac{C}{b_k}, \quad \textit{for all } x \in \mathbb{R}^d, \tag{6}$$

where A, B, C are non-negative constants. $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave continuously differentiable function with $h'(t) \geq 0$, $h(0) = 0$, and b_k is a non-decreasing sequence of k. The expectation is taken over random variable $\xi \sim \mathcal{D}$. We call b_k the cost of such estimator.

This assumption encompasses previous assumptions in the literature. For instance, it is straightforward to see that an estimator satisfies *bounded variance assumption* [15] when $h(t) = 0$, $B = 1$, and $\tau = 0$ in (6). Gradient estimators with *relaxed growth assumption* [4, 8] are also special cases of (6) for $h(t) = 0$ and $b_k = 1$. A closely related assumption to the relaxed growth was introduced in [43] which holds when $h(t) = 0$ and $b_k = \infty$ in (6). *Expected smoothness assumption* [23, 18, 47] is the closest assumption to ours and it holds when $h(t) = t$ and $b_k = 1$. Notably, ES assumption is satisfied in practical scenarios such as mini-batching, importance sampling and compressed communication [23]. More recently, it has been shown that PG method with softmax policies and log barrier regularization can be modeled using ES assumption [46].

We highlight two special cases for the sequence b_k: $b_k = \Theta(k^\tau)$ with $\tau \geq 0$ and $b_k = \Theta(q^k)$ with $q > 1$. For example, Monte Carlo sampling and mini-batching allow us to design estimators with such b_ks. When a gradient estimator satisfies Assumption 4, unless b_k is bounded for all k, it essentially means that we have a mechanism to reduce its variance. More precisely, the variance decreases according to the sequence $1/b_k$. As we show in Section 4, if such estimator exists, it results in a better convergence rate compared to vanilla SGD and the improvement is captured by sequence b_k.

Note that due to the first term in (6), i.e., $2A \cdot h(f(x) - f^*)$, the second moment can be larger when the objective gap at x is large as well. Such property does not hold for estimators with bounded variance assumption. An advantage of existing this term is discussed by [19]. Namely, [19] shows that when $h(t) = t$ and $A > 0$, then this model allows functions with up to quadratic growth. Hence, it allows us to consider many classic problems which fundamentally have quadratic growth (e.g., any quadratically regularized problem such as training a support vector machine). Here we further extend this model to cover functions beyond the quadratic growth regime, by allowing $h(t)$ to be a larger class of non-decreasing concave functions.

Assumption 5 (k-BV, bounded variance). *Let Assumption 4 hold with $A = 0$, $B = 1$ and $C = \sigma^2$. Then k-BV assumption holds, i.e.,* $\mathbb{E}\left[\|g_k(x,\xi) - \nabla f(x)\|^2\right] \leq \frac{\sigma^2}{b_k}$.

3 Stochastic Gradient Method

Algorithm 1 summarizes the steps of a slightly modified SGD which we analyze in this work. We call this algorithm a SGD with restarts. This algorithm updates the point x for T number of iterations within an inner-loop. Note that, in the inner-loop, the step-size remains unchanged and the iterates are updated via $x_{t+1} = x_t - \eta g_k(x_t, \xi_t)$, where η is the step-size and ξ_ts are independent random variables. The cost and the variance of the gradient estimator $g_k(x, \xi)$ remain the same within the inner loop of Algorithm 1.

Algorithm 1: SGD with restarts

```
Initialization: x, T, K, {η_k : k = 0, ..., K − 1}
for k = 0, ..., K − 1 do
    η ← η_k
    for t = 0, ..., T − 1 do
        x ← x − η g_k(x, ξ_t)
return x
```

3.1 Dynamics of SGD

Let $\{x_t\}_{t\geq 0}$ be the sequence of points generated by the inner loop of Algorithm 1. Given Assumptions 1, 2, 4, and the update rule in Algorithm 1, the dynamics of SGD in the inner-loop are as follows.

Lemma 1. *Under Assumptions 1, 2, and 4 with constant cost, i.e., $b := b_k$, we obtain*

$$\delta_{t+1} \leq \delta_t + a\eta^2 \cdot h(\delta_t) - \frac{\eta}{2}\phi^2(\delta_t) + \frac{d\eta^2}{b}, \tag{7}$$

where $\delta_t := \mathbb{E}\left[f(x_t) - f^\star\right]$, $a := LA$, $d := \frac{LC}{2}$, $\eta := \eta_k$.

Understanding the dynamics of this recursion, allows us to establish the global convergence of SGD. Our approach consists of two main steps: i) Finding the stationary[2] point of (7) when the inequality is replaced by equality and for a fixed step-size, $\eta_k = \eta$, which we denote by $r(\eta)$. ii) Selecting the step-sizes $\{\eta_k\}$ and sequence $\{b_k\}$, such that the corresponding stationary points $\{r(\eta_k)\}$ (defined below) converge to zero as k increases.

The stationary point of (7) after replacing inequality with equality must satisfy the equation:

$$a\eta^2 h(t) + \frac{d\eta^2}{b} = \frac{\eta}{2}\phi^2(t). \tag{8}$$

[2] Stationary point of a dynamic is its convergence point.

Let us call this stationary point $r(\eta)$. To complete the first step, we approximate $r(\eta)$ by a polynomial function of η. In other words, we find $\nu \in \mathbb{R}^+$ such that $r(\eta) = \Theta(\eta^\nu)$.

For the second step of our framework, we should design the stepsizes. Next result introduces a set of conditions that allow us to design the stepsizes and obtain its corresponding convergence rate. All proofs are presented in the Appendix.

Theorem 1. *Suppose there exist $\nu, \{\omega_j\}_{j\geq 0}$, and $\zeta \geq 0$ such that $\eta_k = \Theta(k^{-\zeta})$, $r(\eta_k) = \Theta(k^{-\zeta\nu})$, $|1-\omega_k| < 1$, and*

$$1 + a\eta_k^2 h'\big(r(\eta_k)\big) - \eta_k \phi'\big(r(\eta_k)\big)\phi\big(r(\eta_k)\big) = 1 - \omega_k k^{-1}. \tag{9}$$

Then, $\delta_k = \mathcal{O}(k^{-\zeta\nu})$ and the iteration complexity of Algorithm 1 is $\mathcal{O}(\epsilon_f^{-1/(\zeta\nu)})$.

As an application of Theorem 1, we present the convergence rate of SGD for α-PŁ functions.

Corollary 1. *Consider a special case of Assumption 4 with $h(t) = t^\beta$ and $b_k = k^\tau$, where $\beta \in (0, 1]$ and $\tau \geq 0$. Suppose the objective function f satisfies Assumptions 1 and 3. Let $\gamma := \alpha\beta$. Then, for any $\epsilon_f > 0$, Algorithm 1 returns a point x with $\mathbb{E}\left[f(x) - f^\star\right] \leq \epsilon_f$ after $N := K \cdot T$ iterations.*
i) When $\gamma = 2$ ($\alpha = 2$ and $\beta = 1$), we have

$$N = \mathcal{O}(\epsilon_f^{-\frac{1}{1+\tau}}), \text{ with } \eta_k = \Theta(k^{-1}).$$

ii) When $\gamma < 2$, the convergence rate is

$$N = \mathcal{O}\big(\epsilon_f^{-\frac{4-\alpha}{\alpha(\tau+1)}}\big) \text{ with } \eta_k = \Theta(k^{-\frac{\tau+1}{2-\alpha/2}+\tau}) \text{ if } \tau \leq \frac{\gamma}{4-\alpha-\gamma}, \text{and}$$
$$N = \mathcal{O}\big(\epsilon_f^{-\frac{4-\alpha-\gamma}{\alpha}}\big) \text{ with } \eta_k = \Theta(k^{-\frac{2-\gamma}{4-\alpha-\gamma}}) \text{ if } \tau > \frac{\gamma}{4-\alpha-\gamma}.$$

To verify the above result empirically, we simulated δ_t in (7) throughout all iterations of Algorithms 1 for different sets of parameters and presented the results in Figure 1 along with their corresponding convergence rates given in Corollary 1. As it is shown in these figure, the above convergence rates correctly capture the behaviour of the dynamics in (7). As an example, in Figure 1(b), the red solid curve shows the rate of δ_k, i.e., $\log(\delta_k)$ as a function of $\log(k)$ for when $\gamma = 1.1, \alpha = 1.4$, and $\tau = 0.9$. Based on Corollary 1, the convergence rate of Algorithm 1 for this setting is $\mathcal{O}(\epsilon_f^{-0.98})$ or equivalently $\log(\delta_k) = (-\frac{\alpha(\tau+1)}{4-\alpha})\log(k) \approx -1.02\log(k)$ which is shown by red dashed line. Next, we discuss how the results in Corollary 1 generalizes the existing work in the literature.

Remark 1. *Authors in [23] studied the convergence of SGD for 2-PŁ objectives, under a stronger assumption than Assumption (6). More precisely, they assumed an estimator that satisfies Assumption 4 with $\tau = 0$ and $h(t) = t$ and obtained the convergence rate of $\mathcal{O}(\epsilon_f^{-1})$. This is consistent with our rate presented in the Corollary 1. It is noteworthy in this setting, $\mathcal{O}(\epsilon_f^{-1})$ is optimal [23].*

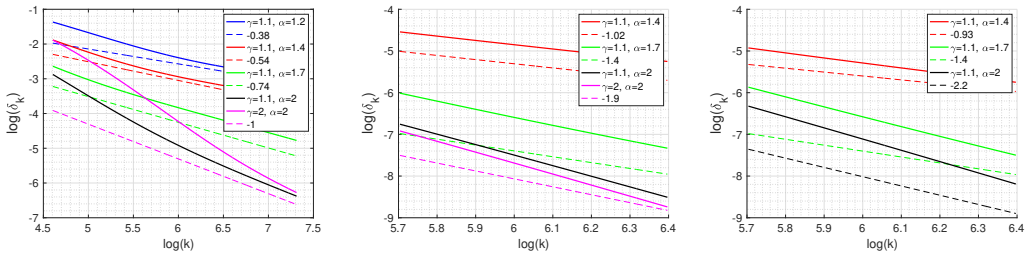
The authors in [47] studied the performance of SGD for 1-PŁ objectives. Having an estimator satisfying Assumption 4 with $\tau = 0$ and $h(t) = t$, they showed the convergence rate of $\mathcal{O}(\epsilon_f^{-3})$ which can be obtained from Corollary 1 by setting $\tau = 0, \gamma = 1$, and $\alpha = 1$.

[14] studied SGD with α-PŁ objectives for $\alpha \in [1, 2]$ with an estimator that satisfies Assumption 4 with $A = 0, B = 1$, and $b_k = 1$ and obtained similar convergence rate to ours. More precisely, their results is a special case of Corollary 1, when $\beta = 0$ and $\tau = 0$.

3.2 Mini-batching and SGD

In what follows, we prove that mini-batching technique cannot improve the rates that SGD achieves in Corollary 1.

Consider a mini-batch setting in which the unbiased gradient estimator is given by $g_k(x_k, \xi_k) := \frac{1}{b_k}\sum_{i=1}^{b_k} \nabla f_{\xi_k^i}(x_k)$, where $\{\nabla f_{\xi^i}(x_k)\}_{i\geq 1}$ are queries provided by an oracle such that $\mathbb{E}[\nabla f_{\xi^i}(x_k)] = \nabla f(x_k)$ and $\mathbb{E}[||\nabla f_{\xi^i}(x_k) - \nabla f(x_k)||^2] \leq \sigma^2$ for all k. The variance of this estimator diminishes linearly in the size of the mini-batch b_k, i.e., when $\mathbb{E}[||g_k(x_k, \xi_k) - \nabla f(x_k)||^2] \leq \sigma^2/b_k$.



(a) $\tau = 0$. (b) $\tau = 0.9$. (c) $\tau = 2$.

Figure 1: Behavior of dynamics in (7) for $h(t) = t^{\beta}$, $\phi(t) = \sqrt{2\mu}\, t^{1/\alpha}$, $\tau \in \{0, 0.9, 2\}$, and different α, β. Each solid line shows $\log(\delta_k)$ as a function of $\log(k)$, for a given set of parameters and each dashed line shows the corresponding theoretical convergence rate of δ_k presented in Corollary 1. (a) and (b) verify the case corresponding to $\tau \leq \gamma/(4-\alpha-\gamma)$ and (c) verifies the case pertaining to $\tau > \gamma/(4-\alpha-\gamma)$. Note that the distance between the dashed and solid lines is due to constant factors.

Lemma 2. *Consider the setting of Corollary 1. The optimal adaptive mini-batch size for Algorithm 1 is* $b_k = \Omega(k^{\frac{\gamma}{4-\alpha-\gamma}})$.

Given the above result, Algorithm 1 requires total of $\sum_{k=1}^{N} k^{\frac{\gamma}{4-\alpha-\gamma}} = \Omega(N^{\frac{4-\alpha}{4-\alpha-\gamma}})$ number of queries after N iterations. This means that Algorithm 1 with optimal mini-batch sizes finds a global ϵ_f-stationary point with the total complexity of the order $\mathcal{O}\big(\epsilon_f^{-\frac{4-\alpha}{\alpha}}\big)$. On the other hand, as it is shown in Corollary 1, Algorithm 1 with an estimator that satisfies Assumption 4 with $b_k = 1$ (without mini-batching) can achieve the same sample complexity. In other words, mini-batching cannot improve the complexity of Algorithm 1.

3.3 Tightness of rates in Corollary 1

In this section, we show that when $\tau = 0$, the convergence rates presented in Corollary 1 are tight for the dynamic in (7). More precisely, if there exists a function f and a gradient estimator satisfying the assumptions in Corollary 1 such that its corresponding recursive inequality (7) is an equality, then its convergence rate, presented in Corollary 1 is optimal, i.e., it cannot be improved by any choices of stepsizes. To do so, for any given $0 < \varepsilon \leq 1$, we introduce a sequence $r_{0,\varepsilon}, r_{1,\varepsilon}, r_{2,\varepsilon}, \ldots$ such that there exists $K_0 > 0$ and for all $k \geq K_0$, $r_{k,\varepsilon} \leq \delta_k$. Moreover, we show that to ensure $r_{k,\varepsilon} \leq \epsilon$, k must be of the order $\Omega(\epsilon^{-\frac{4-\alpha-\varepsilon}{\alpha+\varepsilon}})$. Next proposition summarizes our results about the tightness of our convergence rates in Corollary 1.

Proposition 1. *Consider the following recursion*

$$\delta_{k+1} = \delta_k + a\eta_k^2 \cdot h(\delta_k) - \frac{\eta_k}{2}\phi^2(\delta_k) + \frac{d\eta_k^2}{b_k}, \quad \forall k,$$

where $a, d > 0$, $h(t) = t^{\beta}$ *with* $\beta \in (0, 1]$, $\phi(t) = \sqrt{2\mu}t^{1/\alpha}$ *with* $\alpha \in [1, 2]$, *and* $b_k = \mathcal{O}(1)$. *Then, the optimal convergence rate of* δ_k *is* $\mathcal{O}(k^{-\frac{\alpha}{4-\alpha}})$. *Moreover, this rate can be achieved by* $\eta_k = \Theta(k^{-\frac{1}{2-\alpha/2}})$.

4 Faster Rates with Variance Reduction

To simplify the exposition of the results in this section, let us assume that $g_k(x_t, \xi_t)$ is constructed explicitly via mini-batching for which Assumption 5 holds. Additionally, we assume that we have access to $g'_k(x, \xi)$, which satisfies Assumption 7.

Assumption 6 (Average $\mathcal{L}$-smoothness (of order k))**.** *Let* $g'_k(x, \xi) := \frac{1}{b'_k}\sum_{i=1}^{b'_k} \nabla f_{\xi^i}(x)$ *and* $g'_k(y, \xi) := \frac{1}{b'_k}\sum_{i=1}^{b'_k} \nabla f_{\xi^i}(y)$ *be unbiased mini-batch estimators of the gradient of* $f(\cdot)$ *at points* x *and* y, *respectively for shared stochasticity* $\xi^i \sim \mathcal{D}$ *for each* $i = 1, \ldots, b_k$ *and* $\xi = (\xi_1, \ldots, \xi_{b_k})$.

Define $\widetilde{\Delta}(x,y) := g'_k(x,\xi) - g'_k(y,\xi)$. The average $\mathcal{L}$-smoothness (of order k) holds if there exists $\mathcal{L} \geq 0$ such that

$$\mathbb{E}\left[\left\|\widetilde{\Delta}(x,y) - \Delta(x,y)\right\|^2\right] \leq \frac{\mathcal{L}^2}{b'_k}\|x-y\|^2 \qquad \text{for all } x,y \in \mathbb{R}^d,$$

where $\Delta(x,y) := \nabla f(x) - \nabla f(y)$.

Remark 2. *The Assumption 7 holds in several standard settings. For instance, if each $\nabla f_{\xi^i}(x)$ is Lipschitz with constant $\bar{L}$ (almost surely or on average), then Assumption 7 holds with $\mathcal{L} \leq \bar{L}$. Another example is when $f(\cdot)$ is of the form (2) and $b_k = n$, then $\mathcal{L} = 0$.*

We remark from the analysis of Algorithm 1 in Section 3 that merely playing with choice of η_k and b_k (chosen as polynomial functions of k) is not sufficient to improve the convergence, hence, we need to construct more sophisticated gradient estimator and reduce the variance using control variate. Now, we highlight the main algorithmic ingredients of our construction. First, let us describe the variance reduced estimator named PAGE, which was originally proposed and shown to be optimal for finding first order stationary point [29]. Moreover, PAGE is easy to implement and designed via a small modification to mini-batch SGD. The update of PAGE estimator writes as

$$g_{t+1} = \begin{cases} \frac{1}{b}\sum_{i=1}^{b} \nabla f_{\xi^i_{t+1}}(x_{t+1}), & \text{w.p.} \quad p, \\ g_t + \frac{1}{b'}\sum_{i=1}^{b'}\left(\nabla f_{\xi^i_{t+1}}(x_{t+1}) - \nabla f_{\xi^i_{t+1}}(x_t)\right), & \text{w.p. } 1-p, \end{cases}$$

where p is a small probability and mini-batch sizes satisfy $b > b'$.

Now we are ready to introduce PAGER (Algorithm 2) – a Probabilistic Average Gradient Estimator with parameter Restart, which is formally described in Algorithm 2. It takes as input the sequence of parameters $\{\Lambda_k := (\eta_k, T_k, p_k, b_k, b'_k) : k = 0, ..., K-1\}$, where T_k is the length of stage k, η_k, p_k, b_k, b'_k step-size, probability, and batch-sizes at stage k. PAGER updates this sequence of parameters in the outer loop $k = 0, \ldots, K-1$ and applies PAGE estimator with a fixed set of parameters sequences in the inner loop $t = 0, \ldots, T_k - 1$. We will select $\{\Lambda_k\}_{k\geq 0}$ depending on the PŁ power α to capture the dependence on the geometry of the problem and establish fast rates for each α in settings (1) and (2).

Algorithm 2: PAGER (PAGE with restarts)

```
Initialization: x̄_0, ḡ_0, K, {(η_k, T_k, p_k, b_k, b'_k) : t = 0, ..., T − 1}
for k = 0, ..., K − 1 do
    (x_0, g_0) ← (x̄_k, ḡ_k)
    (η, p, b, b') ← (η_k, p_k, b_k, b'_k)
    for t = 0, ..., T_k − 1 do
        x_{t+1} = x_t − η g_t
        Sample χ ~ Be(p)
        if χ = 1 then
            Query g_k(x_{t+1}, ξ_{t+1}) with batch-size b_k
            Set g_{t+1} = g_k(x_{t+1}, ξ_{t+1})
        else
            Query g'_k(x_{t+1}, ξ_{t+1}) and g'_k(x_t, ξ_{t+1}) each with batch-size b'_k
            Set g_{t+1} = g_t + g'_k(x_{t+1}, ξ_{t+1}) − g'_k(x_t, ξ_{t+1})
    (x̄_{t+1}, ḡ_{t+1}) ← (x_{t+1}, g_{t+1})
Return: x̄_T
```

4.1 Online case

We present convergence guaranties for Algorithm 2 in the setting (1).

Theorem 2. *Let $f(\cdot)$ has the form (1) and satisfy Assumptions 1, 3 (with $\alpha \in [1,2)$) and 7, let the sequences in Algorithm 2 be chosen as $b'_k = \Theta(2^{\frac{(2-\alpha)k}{\alpha}})$, $p_k = \Theta(2^{\frac{-(2-\alpha)k}{\alpha}})$, $b_k = \Theta(2^{\frac{2k}{\alpha}})$, $T_k = \Theta(2^{\frac{(2-\alpha)k}{\alpha}})$, $\eta_k = \Theta(2^{\frac{-(2-\alpha)k}{\alpha}})$. Then, for any $\epsilon_f > 0$ Algorithm 2 returns a point x*

with $\mathbb{E}\left[f(x) - f^\star\right] \leq \epsilon_f$ *after* $N := \sum_{k=0}^{K-1} T_k = \mathcal{O}\big(\epsilon_f^{-\frac{2-\alpha}{\alpha}}\big)$ *iterations. The expected total computational cost (sample complexity) is*

$$cost := \sum_{k=0}^{K-1} T_k \left(p_k b_k + 2(1-p_k) b'_k\right) = \mathcal{O}\left(\epsilon_f^{-\frac{2}{\alpha}}\right).$$

Remark 3. *For brevity, in Theorem 6 we define the input sequences up to constants hidden in* $\Theta(\cdot)$ *notation. In fact, our analysis allows to specify these constants and we present detailed derivations in Appendix.*

Theorem 2 implies that PAGER improves the convergence of SGD by the order of $\mathcal{O}\big(\epsilon_f^{-\frac{2-\alpha}{\alpha}}\big)$ under α-PŁ condition. In the interesting case of $\alpha = 1$, this leads to $\mathcal{O}\big(\epsilon_f^{-2}\big)$ sample complexity compared to the best known $\mathcal{O}\big(\epsilon_f^{-3}\big)$ for SGD, see discussion in Section 3.

Additionally, as a consequence of our analysis we obtain optimal last iterate convergence for convex stochastic optimization under the additional assumption that the iterates of PAGER remain bounded, i.e., $\|x_t - x^\star\|^2 \leq D$ for all $t \geq 0$, where $x^\star \in \arg\min_x f(x)$.[3] Consider the interesting special case of 1-PŁ. In this setting PAGER has $\mathcal{O}\left(\epsilon^{-2}\right)$ sample complexity for the last iterate of the algorithm. Since the iterates of the algorithm are bounded, convexity $\langle \nabla f(x), x - x^\star\rangle \geq f(x) - f(x^\star)$ implies 1-PŁ with $\mu = \frac{1}{2D}$. This observation implies the last iterate convergence of PAGER for convex optimization under additional average $\mathcal{L}$-smoothness assumption. The standard analysis of SGD for convex functions provides convergence for the average iterate [24]. The last iterate convergence for convex functions was recently established for SGD in [14] by following an involved analysis with a careful control of iterates via suffix-averaging scheme.

4.2 Finite sum case

Let $f(\cdot)$ have the finite sum form (2). Then we obtain the following result.

Theorem 3. *Let* $f(\cdot)$ *has the form (2) and satisfy Assumptions 1, 3 (with* $\alpha \in [1, 2)$*) and 7, let the sequences in Algorithm 2 be chosen as* $p_k = \frac{1}{n+1}$, $b'_k = 1$, $b_k = n$, $T_k = \Theta\big(2^{\frac{(2-\alpha)k}{\alpha}}\big)$, $\eta_t = \min\left\{\frac{1}{\sqrt{n}\mathcal{L}}, \Theta\big(2^{\frac{(2-\alpha)k}{\alpha}}\big)\right\}$. *Then, for any* $\epsilon_f > 0$, *Algorithm 2 returns a point* x *with* $\mathbb{E}\left[f(x) - f^\star\right] \leq \epsilon_f$ *after* $N := \sum_{k=0}^{K-1} T_k = \widetilde{\mathcal{O}}\big(n + \sqrt{n}\kappa\epsilon_f^{-\frac{2-\alpha}{\alpha}}\big)$ *iterations. The expected total computational cost (sample complexity) is*

$$cost := \sum_{k=0}^{K-1} T_k \left(p_k b_k + 2(1-p_k) b'_k\right) = \widetilde{\mathcal{O}}\big(n + \sqrt{n}\kappa\epsilon_f^{-\frac{2-\alpha}{\alpha}}\big).$$

Theorem 3 quantifies the improvement of PAGER over GD in the finite sum setting in terms of n. Recall that GD has sample complexity $\mathcal{O}\big(n\kappa\epsilon_f^{\frac{-(2-\alpha)}{\alpha}}\big)$, when n is large, we get the improvement of order $\sqrt{n}$. Notice that in the limit $\alpha \to 2$, this improvement matches the best known improvement for 2-PŁ objectives [29].

5 Conclusion

We analyzed the complexity of SGD when the objective satisfies global KL inequality and the queries from stochastic gradient oracle satisfy weak expected smoothness. We introduced a general framework for this analysis which resulted in a sample complexity of $\mathcal{O}(\epsilon^{-(4-\alpha)/\alpha})$ for SGD with objectives satisfying α-PŁ condition. We also demonstrated the tightness of this rate under the specific choice of stepsizes. Last but not least, we developed a modified SGD with variance reduction and restarting (PAGER) that achieves the optimal rate for the important case of 1-PŁ objectives.

[3] Note that this assumption is mild since it holds for the iterates of PAGER, for example, if we additionally assume that $f(\cdot)$ is coercive, i.e., $f(x) \to \infty$ for $x \to \infty$.

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1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] We accurately state the main contributions in introduction.
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
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