Symmetry-induced Disentanglement on Graphs

Giangiacomo Mercatali*
University of Manchester

André Freitas
Idiap Research Institute
University of Manchester

Vikas Garg
YaiYai Ltd and Aalto University
vgarg@csail.mit.edu

Abstract

Learning disentangled representations is important for unraveling the underlying complex interactions between latent generative factors. Disentanglement has been formalized using a symmetry-centric notion for unstructured spaces, however, graphs have eluded a similarly rigorous treatment. We fill this gap with a new notion of *conditional symmetry* for disentanglement, and leverage tools from Lie algebras to encode graph properties into subgroups using suitable adaptations of generative models such as Variational Autoencoders. Unlike existing works on disentanglement, the proposed models segregate the latent space into uncoupled and entangled parts. Experiments on synthetic and real datasets suggest that these models can learn effective disengaged representations, and improve performance on downstream tasks such as few-shot classification and molecular generation.

1 Introduction

Disentanglement represents a fundamental desideratum in learning with limited supervision because it captures information about the salient (or explanatory) factors of variation in the data, and isolates information about each specific factor in only a few dimensions, thus unraveling the interactions underlying complex data [2, 60]. Disentangled representations have often been deemed responsible for providing neural models with the ability to improve performance on real-world downstream tasks [2, 20, 36, 59, 60]. Empirically, they have been shown to enable, or facilitate, critical properties such as sample efficiency [21] and generalization [32, 42, 70].

An interesting perspective on disentanglement has been recently proposed by Higgins et al. [20]. Specifically, they advocate viewing disentanglement as decomposition of the latent space of embeddings into subspaces, each of which can be transformed independently under the action of a single subgroup specific to the subspace, without affecting others. These subgroups arise, in turn, from the decomposition of a symmetry group. This formalism bestows several benefits, e.g., it (a) aligns with the idea that auxiliary information can be exploited for imposing a structure on the latent space [25, 40], and (b) leads to a rigorous theoretical framework for analyzing disentangled models [65, 71], e.g., those based on Variational Autoencoders (VAEs) [27, 50].

While symmetry groups provide a suitable tool both for formalizing [20] and learning [58, 65, 71] disentangled representations in unstructured domains, these techniques have not been investigated in the context of more general and complex data such as graphs. Furthermore, most works on disentanglement for graphs [1, 16, 31, 35, 38, 66] are not designed for generative settings, and the existing deep generative models (DGMs) for graph disentanglement such as [10, 17, 57] do not consider group symmetries. Note that differently from other definitions of disentanglement [2, 9, 11], leveraging group symmetries for disentanglement may unravel fundamental connections across several approaches based on graph neural networks (GNNs), i.e., the state-of-the-art models for embedding graphs [3, 5, 8, 13, 15, 39, 64].

*first-name.last-name@postgrad.manchester.ac.uk

36th Conference on Neural Information Processing Systems (NeurIPS 2022).

We, therefore, pursue two main goals in this work: (a) providing a rigorous symmetry-based formalism for disentanglement on graphs, and 2) designing novel neural architectures that are able to leverage symmetry groups to learn disengaged representations for graphs. However, accomplishing these objectives requires overcoming some challenges. In particular, graphs often abstract complex interactions, so the underlying latent space may not factorize completely into disjoint subspaces. Thus, the symmetry-based formalism introduced in [20] does not suffice for graphs. Therefore, we introduce a new, more flexible notion of *conditional disentanglement*, advocating segregating the latent space into uncoupled *and* entangled parts. Translating this idea into an efficient algorithm requires further work, and we appeal to a Lie algebra based parameterization to encode the graph properties into subgroups.

Contributions. Our contributions can be summarized as follows:

- a novel notion of symmetry-induced conditional disentanglement (Section 3.1) that generalizes the previous definition from [20] (which we call unconditional disentanglement);
- a parameterization centered on Lie algebra based on the intuition that each separable generative factor can correspond to a Lie algebra coordinate element, and control a single graph property (Section 3.2);
- two algorithms for **S**ymmetry-**I**nduced **D**isentanglement under unconditional ($\mathrm{SID_U}$) and conditional ($\mathrm{SID_C}$) settings, both using a two-level variational auto-encoding mechanism, namely, one level for the data and other for symmetry group representation (Section 4); and
- a systematic evaluation on several disentanglement metrics [7, 11, 19, 26, 44]. The results demonstrate that the proposed models can outperform contemporary GNN baselines [38, 66]; successfully learn to decouple the entangled part of latent space from the disentangled parts (Section 5.1); and improve performance on downstream tasks such as few-shot classification, compression (Section 5.2) and molecular graph generation (Section 5.3).

We begin with a review of the relevant literature, and then proceed to the proposed framework.

2 Related work

Symmetry-based disentanglement (SBD). Initial works on SBD, such as Caselles-Dupré et al. [6], Quessard et al. [48], propose to use reinforcement learning to achieve irreducible representations of group elements through observation of action transitions. Painter et al. [44] introduce a VAE method that does not require labelled action-transition pairs, and demonstrate its validity within the environment from Caselles-Dupré et al. [6].

More recently, some algebraic approaches have been introduced. Zhu et al. [71] propose to decompose a Lie group into multiple subgroups, and enforce commutativity and independence between subgroups. Yang et al. [65] impose a cyclic group structure, encouraging the group to be isomorphic to the ground truth. Bouchacourt et al. [4] invoke group representation theory for defining distributed latent operators that are able to learn disentangled representations. Tonnaer et al. [58] provide a new metric for evaluation, and a VAE-based model for learning symmetry-based disentangled representations.

Some works leverage data manifolds for SBD. For example, Fumero et al. [12] view disentanglement as a product of low-dimensional sub-manifolds underlying the data space, such that each sub-manifold encodes an explainable data factor. Pfau et al. [46] propose a non-parametric algorithm that aims to discover a decomposition of the data manifold by investigating its holonomy group.

Other works explore SBD as well. For instance, Higgins et al. [22] demonstrate the importance of symmetry transformations and disentangled representation learning in the context of neuroscience. Wang et al. [62] design a contrastive self-supervised learning method that iteratively updates a partition, modeled as a symmetry group, until the considered factor is invariant.

Disentangled GNNs. To our knowledge, Ma et al. [38] provide the first graph disentanglement approach, which involves a routing mechanism aimed at separating the underlying factors. More recently, Yang et al. [66] exploit an attention mechanism for GNNs, and provide two evaluation metrics for disentanglement on graphs. Liu et al. [35] achieve disentanglement by encouraging

independence in the latent space, while Bae and Jeon [1] propose a method for disentangling multi-relational graphs motivated by an application on pedestrian trajectory prediction. Li et al. [31] provides a contrastive framework for disentangling the latent factors in GNNs.

Differently from these methods, we provide a formal definition of disentanglement for graphs, and leverage symmetry groups in a generative model setting.

Graph DGMs. Wu et al. [63], Yu et al. [69] propose VAE-based *Information Bottleneck* methods for improving the performance of GNNs on classification tasks, by learning minimal sufficient representations. The work by Simonovsky and Komodakis [56] aims to generate molecular graphs using graph VAEs. Liu et al. [34] adapt VAEs for conditioning molecule generation on specific properties. You et al. [68] propose a deep autoregressive model that learns to generate graphs by training on a representative set of graphs and decomposes the graph generation process into a sequence of node and edge formations. Du et al. [10] propose a DGM for spatio-temporal graphs, and Guo et al. [18] investigate DGMs with spatial networks. Finally, Guo et al. [17] and Stoehr et al. [57] investigate disentangled representations with graph VAEs.

3 Proposed framework

In Section 3.1 we review the unconditional symmetry-induced disentanglement, and propose a new definition of conditional disentanglement. In Section 3.2 we extend these notions to graphs. Finally, in Section 3.3, we describe additional loss terms to encourage disentanglement.

3.1 Symmetry-based disentanglement

Definition 1 (Unconditional symmetry-based disentanglement [20])**.** Let us consider a generative process $b = W \to O$ mapping *world states* W into observations (i.e. data) O, and an inference process $h = O \to Z$ mapping the data into a latent vector space. By composing b and h, we obtain $f = h \circ b$ such that $f : W \to Z$. Given group actions on $W \cdot : G \times W \to W$ and on $Z \cdot : G \times Z \to Z$, the function f is defined equivariant between the actions on W and Z if the action commutes with f, which can be expressed as an equivariant map: $g \cdot f(w) = f(g \cdot w), \forall g \in G, \forall w \in W$. Assuming that the symmetry group G can be decomposed into subgroups $G_1 \times \ldots \times G_n$, a vector representation Z is disentangled with respect to the decomposition of G if: (1) there is an action defined on the representation set $\cdot : G \times Z \to Z$, (2) the map $f : W \to Z$ is equivariant between actions on W and Z, (3) there is a decomposition $Z = Z_1 \times Z_2 \ldots \times Z_n$ such that each Z_i is affected only by G_i and fixed for $G_j, j \neq i$.

Following the notation of Definition 1, we propose the notion of conditional disentanglement.

Definition 2 (Conditional symmetry-based disentanglement)**.** We consider: a generative process $b = W \times E \to O$ mapping a set of disentangle-able world states W and non-disentangled states E into a set of observations (i.e. data) O, and an inference process $h = O \to Z$ mapping the data into the vector space. By composing b and h, we obtain $f = b \circ h$, which maps $W \times E$ to Z. Given group actions on $W \cdot : G \times W \to W$ and on $Z \cdot : G \times Z \to Z$, the function f is defined equivariant between the actions on W and Z if the action commutes with f according to the following equivariant map: $g \cdot f(w, e) = f(g \cdot w, e), \forall g \in G, \forall w \in W, \forall e \in E$.

3.2 Disentanglement on graphs

We aim to align the notions of symmetry groups from Definition 1 and 2 for models that encode graph-structured data. Inspired from previous work on lie groups and algebras for symmetry-induced disentangled VAEs by Zhu et al. [71], we introduce a novel formalization of disentanglement on graphs induced by Lie groups of symmetries.

Lie groups and Lie algebras A Lie group G is a group of symmetries where the symmetries are continuous [52]. A Lie group is associated with a Lie algebra $\mathfrak{g}$, which is the tangent space to the identity element of G. A Lie algebra is parameterized with a basis $\{\mathbb{A}_i\}_{i=1}^k$, where every element in $\mathfrak{g}$ can be written as $\mathbb{A} = \mathbb{A}_1 t_1 + \ldots + \mathbb{A}_k t_k$, where t_i are the coordinates. Elements of the Lie algebra can be mapped into the Lie group with an exponential function: $\exp : \mathfrak{g} \to G$. In the next paragraph,

we show how we can connect the notion of a Lie group G with the definition of disentanglement in a graph network, by associating each Lie algebra coordinate t_i element with a generative factor.

Intuitive definition A latent representation $\hat{Z}$, obtained from a graph encoder, is disentangled with respect to a Lie group G, if a change in the coordinate t_i is associated with a change in $\hat{z}_i$. In other words, disentanglement means that the semantics $(t_1, t_2, \ldots, t_k)$ are equivariant with respect to the properties that are being predicted $(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_k)$. Next, we provide the formal definitions for unconditional and conditional graph disentanglement, where the former expresses $\hat{Z}$ in function of T without accounting for Z, while the latter expresses $\hat{Z}$ based on T, after fixing an encoding Z.

Definition 3 (Symmetry-induced unconditional graph disentanglement)**.** The graph embedding $\hat{Z}$ obtained by $f(\hat{Z}|T)$ is unconditionally disentangled with respect to the Lie group coordinates $T = \{t_j\}_{j=1}^k$ if: (1) there is a group action on $\hat{Z}$: $\cdot : G \times \hat{Z} \to \hat{Z}$, (2) the map $f = \exp \mathbb{A}(T) : T \to \hat{Z}$ is equivariant between actions on T and $\hat{Z}$, (3) there is a decomposition $\hat{Z} = \hat{z}_1 \times \hat{z}_2 \ldots \times \hat{z}_k$, where each coordinate t_i affects only the corresponding component $\hat{z}_i$.

Definition 4 (Symmetry-induced conditional graph disentanglement)**.** The graph embedding $\hat{Z}$ obtained by $f(\hat{Z}|T, Z)$ is conditionally disentangled with respect to the Lie group coordinates $T = \{t_j\}_{j=1}^k$ conditioned on Z if (1) there is a group action on $\hat{Z}$: $\cdot : G \times \hat{Z} \to \hat{Z}$, (2) the map $f : \exp \mathbb{A}(T) \times Z \to \hat{Z}$ is equivariant between actions on T and $\hat{Z}$ for any fixed Z, and (3) $\exp \mathbb{A}(T)$ factorizes into the product $\prod_{i=1}^k \exp(t_i \mathbb{A}_i) \times Z$ such that each component $\exp(t_i \mathbb{A}_i) \times Z$ is affected only by the corresponding coordinate t_i for $i \in \{1, 2, \ldots, k\}$.

3.3 Disentanglement constraints

As mentioned in previous work [71, 65], a symmetry group parameterization is not sufficient for a group to be decomposed into subgroups that are parameterized independently by a single coordinate. As a result, we follow Zhu et al. [71] and provide two additional constraints for enforcing disentanglement, which can be incorporated in a loss function as regularization terms.

Commutative penalty First, disentanglement can be achieved by enforcing a coordinate t_i to be identified by single group $\exp(t_i \mathbb{A}_i)$, representing a single property variation in the data $\hat{z}_i$. Such property is expressed by the equivalence between the exponential map of the sum, and the product over exponential maps, which [71] proved to be achieved when there is commutativity between the Lie algebra basis. Namely, if $\mathbb{A}_i \mathbb{A}_j = \mathbb{A}_j \mathbb{A}_i$ then $\exp \sum_{i=1}^k t_i \mathbb{A}_i = \prod_{i=1}^k \exp t_i \mathbb{A}_i$.

Hessian penalty Furthermore, disentanglement can be strengthened using the hessian penalty [45], which states that the Hessian matrix with respect to a disentangled representation is always zero, since there should be independence between different dimensions. Zhu et al. [71] adapt this penalty in a Lie algebra parameterization setup, as $H_{ij} = \frac{\delta^2 g(T)}{\delta t_i \delta t_j}$ where $g(T) = \exp \sum_{i=1}^k t_i \mathbb{A}_i$, and also show that if $\mathbb{A}_i \mathbb{A}_j = 0, \forall i, j$ then $H_{ij} = 0$.

4 Method

We propose Symmetry-Induced Disentanglement (SID), a VAE framework for graphs [28, 56] which encourages disentangled representations using a Lie group parameterization and two regularization losses (commutative and hessian penalties). The architecture is depicted in Figure 1.

4.1 Probabilistic formulation

The naming conventions for the model variables are as follows: latent graph embeddings Z, graph features X, symmetry Lie group structure T, and the graph adjacency matrix $\mathcal{A}$.

Proposition 1 (Unconditional ELBO)**.** Given two latent variables Z and T, we instantiate the unconditional disentanglement from Definition 3 with a probabilistic model that maximize the log-likelihood of the graph data $\mathcal{G} = (X, \mathcal{A})$, as follows:

$$\mathcal{L}_{\text{u}} = \mathbb{E}_{q(Z|\mathcal{G})q(T|Z)} \log p(Z|T)p(\mathcal{G}|Z) - \mathbb{E}_{q(Z|\mathcal{G})}\text{KL}(q(T|Z)||p(T)) - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \quad (1)$$

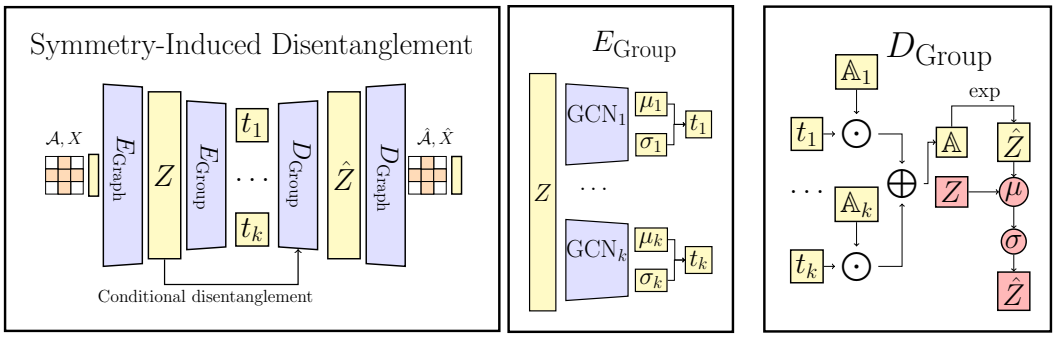

Figure 1: Left (architecture): SID is composed of 4 blocks: 2 layers of auto-encoding modules for graph data and group structure. Middle (E_{group}): the Lie algebra coordinates t_i is obtained by applying k GCNs to the graph embedding Z. Right (D_{group}): the coordinates t_i are combined with basis elements $\mathbb{A}$ to reconstruct the graph latent code $\hat{Z}$. In the conditional case (red nodes) we further condition on Z by taking the mean (μ) between Z and $\hat{Z}$ and passing it through an activation function σ (ReLu).

Proposition 2 (Conditional ELBO)**.** We extend Proposition 1 with a conditional version, corresponding to the conditional disentanglement from Definition 4.

$$\mathcal{L}_{\text{c}} = \mathbb{E}_{q(Z|\mathcal{G})q(T|Z)} \log p(\hat{Z}|T)p(\mathcal{G}|\hat{Z}, Z) - \mathbb{E}_{q(Z|\mathcal{G})}\text{KL}(q(T|Z)||p(T)) - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \tag{2}$$

Architecture Propositions 1 and (2) are proved respectively in Appendix B and C, and their correspondent losses (1 and 2) are implemented with four neural network modules as follows. Two stochastic encoders (E_{graph} and E_{group}) embed respectively 1) the graph data $\mathcal{G}$ into a latent variable Z via $q(Z|\mathcal{G})$, and 2) the graph embedding Z into the group structure T via $q(T|Z)$. Two deterministic decoders (D_{group} and D_{graph}) reconstruct respectively 1) the graph embedding $\hat{Z}$ from the lie group coordinates T via $p(\hat{Z}|T)$, and 2) the original graph data $\mathcal{G}$ from the graph embedding $\hat{Z}$ via $p(\mathcal{G}|\hat{Z})$. The decoder network D_{group} is responsible for the group structure injection, because it imposes $p(\hat{Z}|T)$ to be a Lie algebra parameterization. The D_{group} network is also responsible for the implementation of the conditional ELBO, via $p(\hat{Z}|Z, T)$, which reconstructs the latent code $\hat{Z}$ conditioning on a fixed latent variable Z and the group structure T.

4.2 Encoders

Graph encoder We first encode the graph features X and the adjacency matrix $\mathcal{A}$ in the latent space Z, using an inference network E_{graph} $q(Z|X, \mathcal{A})$. Similarly to Kipf and Welling [28], we use a single 2-layer network, where the first layer leverages a GCN to reduce the dimension of the graph features to the symmetry group size: $\overline{X} = \text{GCN}(X, \mathcal{A})$, while the second layer computes the mean and variance vectors ($\mu = \text{GCN}_\mu(\overline{X}, \mathcal{A})$, $\log \sigma = \text{GCN}_\sigma(\overline{X}, \mathcal{A})$), which are then used to sample the latent variable Z from a Gaussian distribution (reparameterization trick): $q(Z|X, \mathcal{A}) = \mathcal{N}(Z|\mu, \text{diag}(\sigma^2))$.

Group encoder We use the network E_{group} to map the latent vector Z into a vector T, which represents the k Lie algebra coordinates, where k is the subspace size (e.g. the number of disentangled elements). In our settings, each lie algebra coordinate $\{t_i\}_{i=1}^k$ is the output of a graph convolutional encoder (GCN) over Z, and each GCN consists of two layers. The first one computes a graph embedding $\overline{Z} = \text{GCN}(Z, \mathcal{A})$, and the second one produces the vectors $\mu = \text{GCN}_\mu(\overline{Z}, \mathcal{A})$, $\log \sigma = \text{GCN}_\sigma(\overline{Z}, \mathcal{A})$, which are combined into the k vectors $\{t_i\}_{i=1}^k$ with the reparameterization trick. Finally, the vectors $\{t_i\}_{i=1}^k$ are combined into a single tensor T with an MLP, as follows.

$$q(T|Z, \mathcal{A}) = \text{MLP}(q(t_i|Z, \mathcal{A})) \quad \text{with} \quad q(t_i|Z, \mathcal{A}) = \mathcal{N}(t_i|\mu_i, \text{diag}(\sigma_i^2)) \text{ for } i = 1 \ldots, k \tag{3}$$

4.3 Decoders

Group decoder The module D_{group} reconstructs the latent variable $\hat{Z}$ using a Lie group G and a Lie algebra $\mathfrak{g}$, with a Lie algebra parameterization. More specifically, we first learn a Lie algebra basis element $\{\mathbb{A}_i\}_{i=1}^k \in \mathfrak{g}$ for each coordinate $t_i \in T$, and then we aggregate the coordinates t_i and basis elements $\mathbb{A}_i$ with an exponential map, to achieve a group representation, as follows.

$$p(\hat{Z}|T) = g(T) = \exp \mathbb{A}(T) \quad \text{where } \mathbb{A}(T) = \sum\nolimits_{i=1}^{k} t_i \mathbb{A}_i \quad \text{for } g \in G,\ \mathbb{A} \in \mathfrak{g} \tag{4}$$

The conditional version of D_{group} is obtained from a modification of Eq 4, where $\hat{Z}$ is computed by feeding the mean between Z (e.g. the graph embeddings obtained from E_{graph}), and $\exp \mathbb{A}(T)$ into a ReLU non-linear activation function σ, as follows.

$$p(\hat{Z}|T, Z) = \sigma(\text{mean}[\exp \mathbb{A}(T), Z]), \quad \text{where} \quad Z = \{z_j\}_{j=1}^k \tag{5}$$

Graph decoder The network D_{graph} reconstructs the graph data $\hat{\mathcal{G}}$ from the latent $\hat{Z}$. This is achieved first passing the latent feature to a GCN layer that maps Z into the original feature size and subsequently into a sigmoid activation function.

$$p(\mathcal{A}, X|\hat{Z}) = p(\mathcal{A}, X|\sigma(\text{GCN}(\hat{Z}))) \tag{6}$$

4.4 Training

The proposed unconditional and conditional models are trained by optimizing the loss given by $\beta\mathcal{L} + \lambda h + \gamma c$, where $\mathcal{L}$ is the ELBO, from Equation (1) (unconditional) or (2) (conditional), while h and c are respectively the hessian and commutative penalties, obtained as regularization terms, following the implementation from Zhu et al. [71]. The parameters are found empirically, and we provide further insights on their effect on the model and its performance by providing an ablation study in the experiments in Section 5.1. In terms of neural architecture we set each encoder and decoder to be made of 3 of the described layers, where each layer takes as input the graph features of the previous layer.

5 Experiments

Section 5.1 evaluates the disentanglement capabilities of the models, Section 5.2 provides a compression and few-shots classification experiment, Section 5.3 assesses the generation capabilities on molecular datasets.

5.1 Disentanglement evaluation

We evaluate disentanglement with 1) an ablation study, for assessing our models components, 2) qualitative investigations and 3) quantitative metrics. In terms of metrics, we follow the evaluation protocols from Locatello et al. [36], which involve being able to randomly combine a number of generative factors to sample a new data. In our case, we provide the necessary generative factors to generate two types of synthetic random graphs (e.g. Erdos-Renyi and Watts-Strogatz). More details about our dataset construction procedure are provided in Appendix A. We compute 5 metrics, including βVAE [19] (Beta), FactorVAE (FVM) [26], Mutual Information Gap (MIG) [7], DCI Disentanglement [11], and Factor Leakage (FL) [44].

Ablation study We start by evaluating the impact of our model components on the disentanglement metrics, by maintaining all parameters fixed, except from one that varies. We probe the effect from different group sizes, hessian penalties, commutative penalties, and subspace sizes. In Figure 2 we report the mean over 10 runs for our unconditional model on the synthetic Watts-Strogatz dataset. We observe that when the group size is increased, there is an improvement in disentanglement across all metrics, and this can be explained by the idea that with a larger group size, the subgroups, represented by the Lie algebra coordinate t_i can control different data variations. The hessian penalty results to be effectively enhancing disentangled representations, while the commutative penalty is less effective, and we can explain this result because the former requires the Lie algebra basis elements to have mutual products of zeros while subgroup decomposition only requires their commutators to be zeros, which also confirms the results obtained for image datasets [71].

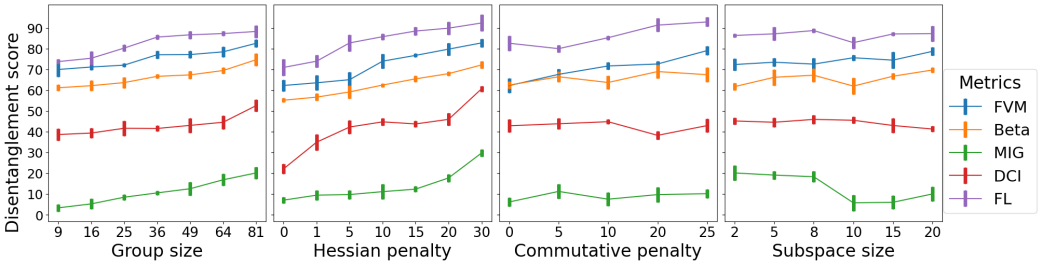

Figure 2: Impact of single model components on disentanglement metrics.

Correlation analysis Following previous work [66, 38], we report the correlation matrix of the hidden graph features, the goal of which is to show whether the features capture mutually exclusive information, which is achieved when the there is a block-wise correlation pattern. In our models we use a group-size of 32 (to match with hidden features size of the other methods), a hessian penalty of 40 and commutative penalty of 5. In Figure 3 we note that both our SID_U and SID_C models achieve a block-wise correlation pattern, however the conditional version has also some other highlighted regions, because in this model involves fixing a latent vector Z to condition the reconstruction. As a result, the conditioning may be responsible for the visualized correlation on the hidden features.

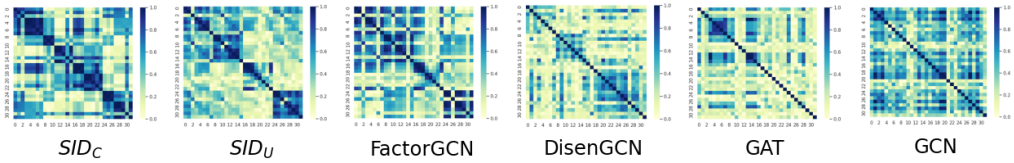

Figure 3: Correlation analysis

Learned graph variations In this experiment we train the SID_C model on Watts-Strogatz (WS) random graphs, which have 3 generative factors, n for number of nodes, p for connectivity (the probability of rewiring each edge), and k for neighborhood (k nearest neighbors in a ring topology). We adapt our model to the visualization implementations provided by Stoehr et al. [57], which report adjacency matrices (black) and graph (white) information, where variations in node attribute values are indicated in blue. In Figure 4 we report the results for the SID_C model by projecting pairs of latent variable traversals to see how the factors vary, and we observe the following. 1) The variable z_0 controls the node attribute value. For example, in $\texttt{adj}_{01}$ we see the color changing from blue to white along the horizontal axis. 2) The variable z_1 controls connectivity, for example in $\texttt{graph}_{01}$, in the z_1 axis (top to bottom) we see that the number of edges decrease, while in $\texttt{graph}_{12}$, along the z_1 axis (left to right), the number of edges increase. 3) The variable z_2 controls the number of nodes, and we see that in $\texttt{graph}_{12}$ and $\texttt{graph}_{02}$ the number of nodes decreases along the z_2 axis (bottom to top).

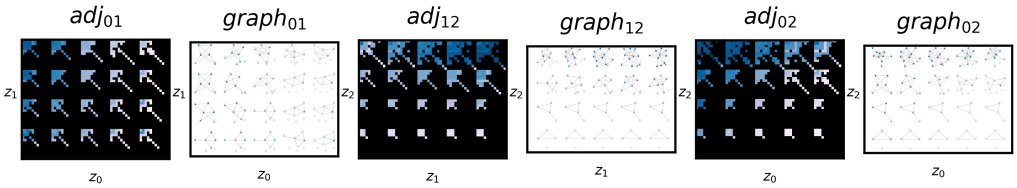

Figure 4: Graph traversals (attributes and node values) for the combinations of 3 latent variables.

Disentanglement metrics We perform 10 random seed runs, by training for 200 epochs on the proposed synthetic datasets and measure the quantitative metrics on 1000 data points. In terms of baselines, we consider graph VAEs with GAT [61] and GCN [29] encoders, as well as disentangled

Table 1: Disentanglement results.

	Watts-Strogatz					Erdos-Renyi				
Model	FVM	Beta	MIG	DCI	FL	FVM	Beta	MIG	DCI	FL
GCN	$69.6_{\pm 0.8}$	$66.5_{\pm 0.7}$	$23.2_{\pm 2.5}$	$43.8_{\pm 1.6}$	$84.1_{\pm 0.5}$	$57.0_{\pm 4.5}$	$61.0_{\pm 1.4}$	$22.5_{\pm 3.3}$	$54.1_{\pm 2.1}$	$71.7_{\pm 2.6}$
GAT	$72.6_{\pm 1.3}$	$66.2_{\pm 3.4}$	$17.4_{\pm 1.0}$	$53.2_{\pm 2.7}$	$79.0_{\pm 1.1}$	$56.0_{\pm 3.2}$	$66.3_{\pm 2.5}$	$31.4_{\pm 1.9}$	$54.0_{\pm 2.9}$	$70.2_{\pm 2.1}$
FAC	$74.2_{\pm 1.3}$	$65.7_{\pm 2.6}$	$26.4_{\pm 2.0}$	$53.4_{\pm 4.0}$	$80.3_{\pm 0.9}$	$67.2_{\pm 1.7}$	$66.3_{\pm 3.3}$	$35.1_{\pm 2.4}$	$53.8_{\pm 1.5}$	$78.1_{\pm 3.0}$
DIS	$73.0_{\pm 3.5}$	$69.3_{\pm 2.1}$	$27.1_{\pm 1.7}$	$52.2_{\pm 3.9}$	$85.2_{\pm 2.0}$	$68.1_{\pm 1.8}$	$64.1_{\pm 0.5}$	$41.3_{\pm 1.4}$	$53.5_{\pm 2.3}$	$72.4_{\pm 2.3}$
SID_U	$80.5_{\pm 1.5}$	$71.6_{\pm 4.5}$	$28.5_{\pm 2.8}$	$55.4_{\pm 2.0}$	$91.0_{\pm 2.3}$	$83.0_{\pm 2.9}$	$71.8_{\pm 3.2}$	$\mathbf{53.2}_{\pm 0.8}$	$58.6_{\pm 1.9}$	$87.1_{\pm 4.0}$
SID_C	$\mathbf{81.9}_{\pm 1.2}$	$\mathbf{74.5}_{\pm 3.2}$	$\mathbf{49.1}_{\pm 3.1}$	$\mathbf{59.7}_{\pm 1.2}$	$\mathbf{92.3}_{\pm 0.9}$	$\mathbf{83.3}_{\pm 0.7}$	$\mathbf{74.1}_{\pm 1.2}$	$52.4_{\pm 3.1}$	$\mathbf{60.5}_{\pm 1.2}$	$\mathbf{89.8}_{\pm 2.3}$

Table 2: Compression and few shot classification results.

	FRANKENSTEIN			COLORS-3			Mutagenicity			NCI1		
Model	MSE	Size	Acc.	MSE	Size	Acc.	MSE	Size	Acc.	MSE	Size	Acc.
MIA	$\mathbf{1.6}_{\pm 0.5}$	4.9M	$65_{\pm 1}$	$\mathbf{1.8}_{\pm 0.2}$	729K	$33_{\pm 2}$	$4.1_{\pm 0.8}$	753K	$78_{\pm 2}$	$\mathbf{3.2}_{\pm 0.5}$	777K	$61_{\pm 1}$
GCN	$12.6_{\pm 2.5}$	4.2M	$39_{\pm 3}$	$11.2_{\pm 1.1}$	657K	$33_{\pm 2}$	$16.5_{\pm 2.1}$	673K	$72_{\pm 1}$	$7.4_{\pm 1.2}$	697K	$53_{\pm 4}$
GAT	$9.0_{\pm 1.2}$	4.2M	$26_{\pm 2}$	$10.1_{\pm 3.1}$	665K	$30_{\pm 1}$	$11.3_{\pm 2.8}$	681K	$79_{\pm 4}$	$8.1_{\pm 0.3}$	705K	$50_{\pm 2}$
SID_U	$4.3_{\pm 1.1}$	**4.1M**	$68_{\pm 2}$	$5.1_{\pm 1.5}$	**625K**	$\mathbf{39}_{\pm 1}$	$4.2_{\pm 0.3}$	**633K**	$81_{\pm 3}$	$3.5_{\pm 0.3}$	**657K**	$63_{\pm 2}$
SID_C	$2.8_{\pm 0.5}$	**4.1M**	$\mathbf{69}_{\pm 2}$	$1.9_{\pm 0.4}$	**625K**	$38_{\pm 3}$	$\mathbf{3.2}_{\pm 1.1}$	**633K**	$\mathbf{84}_{\pm 2}$	$3.9_{\pm 0.9}$	**657K**	$\mathbf{66}_{\pm 2}$

encoders such as FAC [66] and DIS [38]. We set our models to have a group size of 81, a hessian penalty of 40, and a commutative penalty of 5.

In Table 1 we observe that our models are able to improve on previous baselines on all metrics on the evaluated datasets, and the conditional version achieves the best performance in most cases. We notice that the FL metric shows a significant performance increase, justified by its ability to capture the disentanglement on each generative factor, which, in our models is controlled by the group size, while in previous approaches is not considered. In terms of the other previously proposed disentangled approaches, DIS and FAC demonstrate higher disentanglement in most cases, when compared to GCN and GAT, however they are outperformed by our methods.

5.2 Compression and few-shots classification

Following the setup from Ge et al. [14], we consider the task of learning a compressed model via multiple pooling operations, and test on few-shots classification capabilities on datasets with multiple classes, including FRANKENSTEIN [43] (2 classes), COLORS-3 [30] (11 classes), Mutagenicity [51] (2 classes), NCI1 [54] (2 c). The baselines include MIA [14], which propose an attention-based pooling for compression, as well as GCN [29], and GAT [61] based autoencoders, equipped with the same pooling layer. All models are set to have a pooling rate of 0.8, and a depth of 3 layers. Our models are set with a group size of 81, hessian penalty of 20 and commutative penalty of 5. The task consists in training first without labels to learn the representation, and then using the compressed model as input for training a GCN classifier a labelled subset of the dataset, which involves 100 samples for training and 100 for testing.

In Table 2 we report the mean and variance across 10 runs for MSE, the classification accuracy, and the size of the trained model. The results show that SID_C and SID_U outperform the considered baselines in terms of few-shots classification accuracy on all datasets, while achieving a smaller compressed size. We hypothesize that the superior accuracy performance of our models is due to a more efficient learning provided by disentangled representations, which have previously demonstrated the ability to enhance the accuracy of predictions in the context of generalization tasks [41, 70, 32]. In terms of MSE, our models show an improvement compared to GCN- and GAT-based autoencoders, but they are outperformed by MIA in most cases.

5.3 Molecular graph generation

Molecular generation involves three tasks, random generation, property optimization and constrained optimization, and a suitable setup can be obtained following the approaches implemented via Torch-

Table 3: Random graph generation.

	ZINC				QM9				MOSES			
Model	Val	WR	Uni	Nov	Val	WR	Uni	Nov	Val	WR	Uni	Nov
JT-VAE	**100**	n/a	**100**	**100**	n/a	n/a	n/a	n/a	**100**	n/a	**99.96**	91.43
GCPN	**100**	20	99.97	**100**	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
GraphAF	**100**	68	99.1	**100**	**100**	67	94.15	88.83	**100**	71	99.99	**100**
GraphDF	**100**	**89.03**	99.16	**100**	**100**	82.67	**97.62**	**98.1**	**100**	**87.58**	99.55	**100**
SID_U	**100**	78.21	99.12	**100**	**100**	75.43	95.22	90.33	**100**	81.62	99.62	**100**
SID_C	**100**	86.02	99.14	**100**	**100**	**82.92**	96.43	92.28	**100**	84.29	99.79	**100**

Table 4: Property optimization performance.

	Penalized logP			QED		
	1st	2nd	3rd	1st	2nd	3rd
ZINC	4.52	4.3	4.23	0.948	0.948	0.948
JT-VAE	5.3	4.93	4.49	0.925	0.911	0.910
GCPN	7.98	7.85	7.80	**0.948**	0.947	0.946
GraphAF	12.23	11.29	11.05	**0.948**	**0.948**	0.947
GraphDF	**13.7**	**13.18**	**13.17**	**0.948**	**0.948**	**0.948**
SID_U	12.56	12.46	12.03	0.947	**0.948**	0.947
SID_C	12.89	12.82	12.27	**0.948**	**0.948**	**0.948**

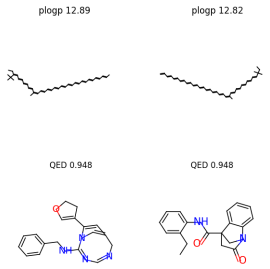

Figure 5: Molecules from SID_C

Drug[2] and DIG [33]. The generation process involves 1) learning the distribution of the molecular data and 2) fine-tuning the pre-trained model on one of the tasks. We employ our models during the training phase, using a group size of 81, a hessian penalty of 40 and commutative penalty of 5, and then we follow the reinforcement learning fine-tuning method from GCPN [67]. Similar setups are used from previous molecule generation methods, which we include as baselines: JT-VAE [24], GraphAF [55], GraphDF [37], GCPN [67].

Random generation This task evaluates the quality of randomly generated samples on 4 standard metrics in percentage, including: valid molecules with resampling (Val), valid molecules without resampling (WR), unique molecules (Uni), novel molecules that not appear in the training data (Nov). We train our models for 10 epochs, with a batch size of 32 and the learning rate of 0.001, and compute the metrics over 10K generated molecules. In Table 3 we report results on ZINC [23], QM9 [49], and MOSES [47]. We observe that both SID_U and SID_C are able to improve the performance of GraphAF and GCPN on most metrics, while achieving comparable performance with GraphDF, furthermore the SID_C model achieves the highest score of validity without resampling on the QM9 dataset. Since our model is not optimized for molecule generation, the positive results of our models indicate that disentangled representations may represent a flexible tool for generating more realistic graphs, and that encouraging the separation of semantic factors into different latent codes, may be beneficial for improving the quality of the samples.

Property optimization The goal of the task is to generate novel molecules with high property scores for the penalized logP property, and the quantitative estimation of drug-likeness property (QED). The setup involves pretraining our models for 300 epochs on ZINC random generation, applying the RL fine-tuning procedure from GCPN, and measuring the scores from the top 3 generated molecules. In Table 4 we observe that our models are able to outperform GCPN and GraphAF for both penalized logP and QED, and achieve comparable performance to the more advanced GraphDF model. In Figure 5 we show the molecules generated using SID_C. While our model does not outperform GraphDF, our results are relevant because they demonstrate that the disentangled representations learned from our models can be leveraged to improve the performance of the downstream task of property optimization.

[2]https://torchdrug.ai/ (Apache license)

Table 5: Constrained optimization performance on 800 molecules used in GraphAF.

δ	GraphAF			GraphDF			SID_C		
	Imp	Sim	Suc	Imp	Sim	Suc	Imp	Sim	Suc
0.0	$13.13_{\pm 6.89}$	$0.29_{\pm 0.15}$	100	$\mathbf{14.15}_{\pm 6.86}$	$0.29_{\pm 0.13}$	100	$13.26_{\pm 3.24}$	$0.28_{\pm 0.21}$	100
0.2	$11.90_{\pm 6.86}$	$0.33_{\pm 0.12}$	100	$\mathbf{12.77}_{\pm 6.59}$	$0.32_{\pm 0.11}$	100	$12.35_{\pm 5.62}$	$0.32_{\pm 0.13}$	100
0.4	$8.21_{\pm 6.51}$	$0.49_{\pm 0.09}$	99.8	$\mathbf{9.19}_{\pm 6.43}$	$0.48_{\pm 0.08}$	99.6	$8.25_{\pm 4.93}$	$0.50_{\pm 0.18}$	98.4
0.6	$\mathbf{4.98}_{\pm 6.49}$	$0.66_{\pm 0.05}$	96.8	$4.51_{\pm 5.80}$	$0.65_{\pm 0.05}$	92.1	$4.67_{\pm 5.39}$	$0.66_{\pm 0.03}$	93.2

Constrained Optimization This task aims to modify the input molecular graph for improving its penalized logP score while keeping the similarity between the input and modified molecules higher than the threshold δ. We follow the setup of GraphAF and GraphDF, which select 800 molecules from ZINC with low penalized logP scores as the input molecules to be optimized. We pretrain our model on the random generation task for 300 epochs, and then fine-tune with the RL procedure from GraphAF. We report the mean and standard deviation of the largest property improvement (Imp), and similarities (Sim) between them and their corresponding input molecules, as well as the success rate (Suc). In Table 5 we compare the results for our SID_C model with GraphAF and GraphDF. We observe that SID_C is able to outperform GraphAF 3 out of 4 times for improvement values, and achieves comparable results with GraphDF, which we motivate by the fact that GraphDF provides further fine-tuning in order to improve the baseline. In terms of similarity and success rate, our model has comparable results with the baselines for all δ setups.

6 Discussion

Limitations In terms of quantitative evaluation, we rely on synthetic dataset, because current metrics are based on the ability to sample each factor and combine them into an observation. Since this is an open problem in unsupervised representation learning, as mentioned by Kim and Mnih [26], future work should investigate how to provide metrics that can evaluate disentanglement on real-world datasets.

A recent line of work from Khemakhem et al. [25] shows that disentanglement is achieved with model identifiability, which means that a representation is disentangled when a set of learned parameters is uniquely identified with the parameters in the true distribution. Our approach has not considered or provided proofs for, identifiability, and thus, this more principled notion of disentanglement can be explored in future work.

Conclusion We formalize the notion of conditional disentanglement on graphs and propose a novel framework for graph disentanglement by leveraging tools from Lie algebras. Based on the new definition of disentanglement, we design a graph VAE based on a Lie group parameterization, and provide a novel ELBO criteria for optimizing conditional disentanglement. Our method achieves superior performance on quantitative disentanglement benchmarks when compared to contemporary disentangled GNNs and other convolutional layers. Finally, we demonstrate strong capabilities on few-shots classification and molecular generation experiments.

7 Acknowledgements

We thank the anonymous reviewers for their helpful comments. GM research was partially funded by EPSRC and the BBC under iCASE.

References

[1] Inhwan Bae and Hae-Gon Jeon. Disentangled multi-relational graph convolutional network for pedestrian trajectory prediction. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pages 911–919, 2021.

[2] Yoshua Bengio, Aaron Courville, and Pascal Vincent. Representation learning: A review and new perspectives. *IEEE transactions on pattern analysis and machine intelligence*, 35(8): 1798–1828, 2013.

[3] Beatrice Bevilacqua, Fabrizio Frasca, Derek Lim, Balasubramaniam Srinivasan, Chen Cai, Gopinath Balamurugan, Michael M. Bronstein, and Haggai Maron. Equivariant subgraph aggregation networks. In *International Conference on Learning Representations*, 2022.

[4] Diane Bouchacourt, Mark Ibrahim, and Stéphane Deny. Addressing the topological defects of disentanglement via distributed operators. *arXiv preprint arXiv:2102.05623*, 2021.

[5] Michael M Bronstein, Joan Bruna, Yann LeCun, Arthur Szlam, and Pierre Vandergheynst. Geometric deep learning: going beyond euclidean data. *IEEE Signal Processing Magazine*, 34 (4):18–42, 2017.

[6] Hugo Caselles-Dupré, Michael Garcia Ortiz, and David Filliat. Symmetry-based disentangled representation learning requires interaction with environments. *Advances in Neural Information Processing Systems*, 32, 2019.

[7] Ricky TQ Chen, Xuechen Li, Roger Grosse, and David Duvenaud. Isolating sources of disentanglement in vaes. In *Proceedings of the 32nd International Conference on Neural Information Processing Systems*, pages 2615–2625, 2018.

[8] Zhengdao Chen, Soledad Villar, Lei Chen, and Joan Bruna. On the equivalence between graph isomorphism testing and function approximation with gnns. In *NeurIPS*, 2019.

[9] Kien Do and Truyen Tran. Theory and evaluation metrics for learning disentangled representations. In *International Conference on Learning Representations*, 2019.

[10] Yuanqi Du, Xiaojie Guo, Hengning Cao, Yanfang Ye, and Liang Zhao. Disentangled spatiotemporal graph generative models. In *Proceedings of the AAAI Conference on Artificial Intelligence*, 2022.

[11] Cian Eastwood and Christopher KI Williams. A framework for the quantitative evaluation of disentangled representations. In *International Conference on Learning Representations*, 2018.

[12] Marco Fumero, Luca Cosmo, Simone Melzi, and Emanuele Rodolà. Learning disentangled representations via product manifold projection. In *International Conference on Machine Learning*, pages 3530–3540. PMLR, 2021.

[13] V. Garg, S. Jegelka, and T. Jaakkola. Generalization and representational limits of graph neural networks. In *International Conference on Machine Learning (ICML)*, 2020.

[14] Yunhao Ge, Yunkui Pang, Linwei Li, and Laurent Itti. Graph autoencoder for graph compression and representation learning. In *Neural Compression: From Information Theory to Applications–Workshop@ ICLR 2021*, 2021.

[15] M. Gori, G. Monfardini, and F. Scarselli. A new model for learning in graph domains. In *IEEE International Joint Conference on Neural Networks (IJCNN)*, 2005.

[16] Jingwei Guo, Kaizhu Huang, Xinping Yi, and Rui Zhang. Lgd-gcn: Local and global disentangled graph convolutional networks. *arXiv preprint arXiv:2104.11893*, 2021.

[17] Xiaojie Guo, Liang Zhao, Zhao Qin, Lingfei Wu, Amarda Shehu, and Yanfang Ye. Interpretable deep graph generation with node-edge co-disentanglement. In *Proceedings of the 26th ACM SIGKDD international conference on knowledge discovery & data mining*, pages 1697–1707, 2020.

[18] Xiaojie Guo, Yuanqi Du, and Liang Zhao. Deep generative models for spatial networks. In *Proceedings of the 27th ACM SIGKDD Conference on Knowledge Discovery & Data Mining*, pages 505–515, 2021.

[19] Irina Higgins, Loic Matthey, Arka Pal, Christopher Burgess, Xavier Glorot, Matthew Botvinick, Shakir Mohamed, and Alexander Lerchner. beta-vae: Learning basic visual concepts with a constrained variational framework. In *International Conference on Learning Representations*, 2017.

[20] Irina Higgins, David Amos, David Pfau, Sebastien Racaniere, Loic Matthey, Danilo Rezende, and Alexander Lerchner. Towards a definition of disentangled representations. *arXiv preprint arXiv:1812.02230*, 2018.

[21] Irina Higgins, Nicolas Sonnerat, Loic Matthey, Arka Pal, Christopher P Burgess, Matko Bošnjak, Murray Shanahan, Matthew Botvinick, Demis Hassabis, and Alexander Lerchner. Scan: Learning hierarchical compositional visual concepts. In *International Conference on Learning Representations*, 2018.

[22] Irina Higgins, Sébastien Racanière, and Danilo Rezende. Symmetry-based representations for artificial and biological general intelligence. *Frontiers in Computational Neuroscience*, page 28, 2022.

[23] John J Irwin, Teague Sterling, Michael M Mysinger, Erin S Bolstad, and Ryan G Coleman. Zinc: a free tool to discover chemistry for biology. *Journal of chemical information and modeling*, 52 (7):1757–1768, 2012.

[24] Wengong Jin, Regina Barzilay, and Tommi Jaakkola. Junction tree variational autoencoder for molecular graph generation. In *International conference on machine learning*, pages 2323–2332. PMLR, 2018.

[25] Ilyes Khemakhem, Diederik Kingma, Ricardo Monti, and Aapo Hyvarinen. Variational autoencoders and nonlinear ica: A unifying framework. In *International Conference on Artificial Intelligence and Statistics*, pages 2207–2217. PMLR, 2020.

[26] Hyunjik Kim and Andriy Mnih. Disentangling by factorising. In *International Conference on Machine Learning*, pages 2649–2658, 2018.

[27] Diederik P Kingma and Max Welling. Auto-encoding variational bayes. In *2nd International Conference on Learning Representations, ICLR 2014, Banff, AB, Canada, April 14-16, 2014, Conference Track Proceedings*, 2014.

[28] Thomas N Kipf and Max Welling. Variational graph auto-encoders. *arXiv preprint arXiv:1611.07308*, 2016.

[29] Thomas N. Kipf and Max Welling. Semi-supervised classification with graph convolutional networks. In *International Conference on Learning Representations (ICLR)*, 2017.

[30] Boris Knyazev, Graham W Taylor, and Mohamed Amer. Understanding attention and generalization in graph neural networks. *Advances in neural information processing systems*, 32, 2019.

[31] Haoyang Li, Xin Wang, Ziwei Zhang, Zehuan Yuan, Hang Li, and Wenwu Zhu. Disentangled contrastive learning on graphs. *Advances in Neural Information Processing Systems*, 34, 2021.

[32] Xiangyu Li, Zhe Xu, Kun Wei, and Cheng Deng. Generalized zero-shot learning via disentangled representation. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pages 1966–1974, 2021.

[33] Meng Liu, Youzhi Luo, Limei Wang, Yaochen Xie, Hao Yuan, Shurui Gui, Haiyang Yu, Zhao Xu, Jingtun Zhang, Yi Liu, et al. Dig: a turnkey library for diving into graph deep learning research. *Journal of Machine Learning Research*, 22(240):1–9, 2021.

[34] Qi Liu, Miltiadis Allamanis, Marc Brockschmidt, and Alexander Gaunt. Constrained graph variational autoencoders for molecule design. *Advances in neural information processing systems*, 31, 2018.

[35] Yanbei Liu, Xiao Wang, Shu Wu, and Zhitao Xiao. Independence promoted graph disentangled networks. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 4916–4923, 2020.

[36] Francesco Locatello, Stefan Bauer, Mario Lucic, Gunnar Raetsch, Sylvain Gelly, Bernhard Schölkopf, and Olivier Bachem. Challenging common assumptions in the unsupervised learning of disentangled representations. In *international conference on machine learning*, pages 4114–4124. PMLR, 2019.

[37] Youzhi Luo, Keqiang Yan, and Shuiwang Ji. Graphdf: A discrete flow model for molecular graph generation. In *International Conference on Machine Learning*, pages 7192–7203. PMLR, 2021.

[38] Jianxin Ma, Peng Cui, Kun Kuang, Xin Wang, and Wenwu Zhu. Disentangled graph convolutional networks. In *International conference on machine learning*, pages 4212–4221. PMLR, 2019.

[39] Haggai Maron, Heli Ben-Hamu, Hadar Serviansky, and Yaron Lipman. Provably powerful graph networks. *Advances in Neural Information Processing Systems*, 32:2156–2167, 2019.

[40] Graziano Mita, Maurizio Filippone, and Pietro Michiardi. An identifiable double vae for disentangled representations. In *International Conference on Machine Learning*, pages 7769–7779. PMLR, 2021.

[41] Milton Llera Montero, Casimir JH Ludwig, Rui Ponte Costa, Gaurav Malhotra, and Jeffrey Bowers. The role of disentanglement in generalisation. In *International Conference on Learning Representations*, 2020.

[42] Milton Llera Montero, Casimir JH Ludwig, Rui Ponte Costa, Gaurav Malhotra, and Jeffrey Bowers. The role of disentanglement in generalisation. In *International Conference on Learning Representations*, 2021. URL `https://openreview.net/forum?id=qbH974jKUVy`.

[43] Francesco Orsini, Paolo Frasconi, and Luc De Raedt. Graph invariant kernels. In *Twenty-Fourth International Joint Conference on Artificial Intelligence*, 2015.

[44] Matthew Painter, Adam Prugel-Bennett, and Jonathon Hare. Linear disentangled representations and unsupervised action estimation. *Advances in Neural Information Processing Systems*, 33: 13297–13307, 2020.

[45] William Peebles, John Peebles, Jun-Yan Zhu, Alexei Efros, and Antonio Torralba. The hessian penalty: A weak prior for unsupervised disentanglement. In *European Conference on Computer Vision*, pages 581–597. Springer, 2020.

[46] David Pfau, Irina Higgins, Alex Botev, and Sébastien Racanière. Disentangling by subspace diffusion. *Advances in Neural Information Processing Systems*, 33:17403–17415, 2020.

[47] Daniil Polykovskiy, Alexander Zhebrak, Benjamin Sanchez-Lengeling, Sergey Golovanov, Oktai Tatanov, Stanislav Belyaev, Rauf Kurbanov, Aleksey Artamonov, Vladimir Aladinskiy, Mark Veselov, et al. Molecular sets (moses): a benchmarking platform for molecular generation models. *Frontiers in pharmacology*, 11:1931, 2020.

[48] Robin Quessard, Thomas Barrett, and William Clements. Learning disentangled representations and group structure of dynamical environments. *Advances in Neural Information Processing Systems*, 33:19727–19737, 2020.

[49] Raghunathan Ramakrishnan, Pavlo O Dral, Matthias Rupp, and O Anatole Von Lilienfeld. Quantum chemistry structures and properties of 134 kilo molecules. *Scientific data*, 1(1):1–7, 2014.

[50] Danilo Jimenez Rezende, Shakir Mohamed, and Daan Wierstra. Stochastic backpropagation and approximate inference in deep generative models. In *International conference on machine learning*, pages 1278–1286. PMLR, 2014.

[51] Kaspar Riesen and Horst Bunke. Iam graph database repository for graph based pattern recognition and machine learning. In *Joint IAPR International Workshops on Statistical Techniques in Pattern Recognition (SPR) and Structural and Syntactic Pattern Recognition (SSPR)*, pages 287–297. Springer, 2008.

[52] Wulf Rossmann. *Lie groups: an introduction through linear groups*, volume 5. Oxford University Press on Demand, 2006.

[53] Michael Schlichtkrull, Thomas N Kipf, Peter Bloem, Rianne van den Berg, Ivan Titov, and Max Welling. Modeling relational data with graph convolutional networks. In *European semantic web conference*, pages 593–607. Springer, 2018.

[54] Nino Shervashidze, Pascal Schweitzer, Erik Jan Van Leeuwen, Kurt Mehlhorn, and Karsten M Borgwardt. Weisfeiler-lehman graph kernels. *Journal of Machine Learning Research*, 12(9), 2011.

[55] Chence Shi, Minkai Xu, Zhaocheng Zhu, Weinan Zhang, Ming Zhang, and Jian Tang. Graphaf: a flow-based autoregressive model for molecular graph generation. In *International Conference on Learning Representations*, 2020.

[56] Martin Simonovsky and Nikos Komodakis. Graphvae: Towards generation of small graphs using variational autoencoders. In *International conference on artificial neural networks*, pages 412–422. Springer, 2018.

[57] Niklas Stoehr, Emine Yilmaz, Marc Brockschmidt, and Jan Stuehmer. Disentangling interpretable generative parameters of random and real-world graphs. *arXiv preprint arXiv:1910.05639*, 2019.

[58] Loek Tonnaer, Luis Armando Perez Rey, Vlado Menkovski, Mike Holenderski, and Jim Portegies. Quantifying and learning linear symmetry-based disentanglement. In *Proceedings of the 39th International Conference on Machine Learning*, volume 162 of *Proceedings of Machine Learning Research*, pages 21584–21608, 17–23 Jul 2022.

[59] Michael Tschannen, Olivier Bachem, and Mario Lucic. Recent advances in autoencoder-based representation learning. *arXiv preprint arXiv:1812.05069*, 2018.

[60] Sjoerd Van Steenkiste, Francesco Locatello, Jürgen Schmidhuber, and Olivier Bachem. Are disentangled representations helpful for abstract visual reasoning? *Advances in Neural Information Processing Systems*, 32, 2019.

[61] Petar Veličković, Guillem Cucurull, Arantxa Casanova, Adriana Romero, Pietro Liò, and Yoshua Bengio. Graph attention networks. In *International Conference on Learning Representations*, 2018.

[62] Tan Wang, Zhongqi Yue, Jianqiang Huang, Qianru Sun, and Hanwang Zhang. Self-supervised learning disentangled group representation as feature. *Advances in Neural Information Processing Systems*, 34, 2021.

[63] Tailin Wu, Hongyu Ren, Pan Li, and Jure Leskovec. Graph information bottleneck. *Advances in Neural Information Processing Systems*, 33:20437–20448, 2020.

[64] K. Xu, W. Hu, J. Leskovec, and S. Jegelka. How powerful are graph neural networks? In *International Conference on Learning Representations (ICLR)*, 2019.

[65] Tao Yang, Xuanchi Ren, Yuwang Wang, Wenjun Zeng, and Nanning Zheng. Towards building a group-based unsupervised representation disentanglement framework. In *International Conference on Learning Representations*, 2022.

[66] Yiding Yang, Zunlei Feng, Mingli Song, and Xinchao Wang. Factorizable graph convolutional networks. *Advances in Neural Information Processing Systems*, 33:20286–20296, 2020.

[67] Jiaxuan You, Bowen Liu, Zhitao Ying, Vijay Pande, and Jure Leskovec. Graph convolutional policy network for goal-directed molecular graph generation. *Advances in neural information processing systems*, 31, 2018.

[68] Jiaxuan You, Rex Ying, Xiang Ren, William Hamilton, and Jure Leskovec. Graphrnn: Generating realistic graphs with deep auto-regressive models. In *International conference on machine learning*, pages 5708–5717. PMLR, 2018.

[69] Junchi Yu, Tingyang Xu, Yu Rong, Yatao Bian, Junzhou Huang, and Ran He. Graph information bottleneck for subgraph recognition. In *International Conference on Learning Representations*, 2021. URL `https://openreview.net/forum?id=bM4Iqfg8M2k`.

[70] Hanlin Zhang, Yi-Fan Zhang, Weiyang Liu, Adrian Weller, Bernhard Schölkopf, and Eric P Xing. Towards principled disentanglement for domain generalization. *arXiv preprint arXiv:2111.13839*, 2021.

[71] Xinqi Zhu, Chang Xu, and Dacheng Tao. Commutative lie group vae for disentanglement learning. In *International Conference on Machine Learning*, pages 12924–12934. PMLR, 2021.

Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See Section 6
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes] Provided in Appendix
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] Provided at `https://mercatali.gitlab.io/sid/`
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] We specify the experimental setups in Section 5
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] See beginning of Section 5
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See beginning of Section 5
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] See Section 5.3
   (b) Did you mention the license of the assets? [Yes] See Section 5.3
   (c) Did you include any new assets either in the supplemental material or as a URL? [Yes] We include our assets at `https://mercatali.gitlab.io/sid/`
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

A Synthetic datasets

We construct two datasets with the PyTorch Geometric library[3] and the NetworkX library[4], based respectively on Watts-Strogatz, and Erdos-Renyi random graphs. The datasets consist in a list of graphs, where we control the generative factors necessary for synthesizing node feature and adjacency matrices in each graph. This allows us to create labels and latent codes for each graph, similarly to the canonical dataset for disentanglement such as Dsprites[5].

[3] `https://github.com/pyg-team/pytorch_geometric`
[4] `https://networkx.org/`
[5] `https://github.com/deepmind/dsprites-dataset`

Node feature factors The node feature matrix is computed for both datasets as a random tensor of dimension $N_{\text{nodes}} \times N_{\text{features}}$. The number of node is fixed to 20, while the features are 64. The elements of the feature matrix are sampled from a normal distribution with a given mean and variance, which are two of the generative factors that we control.

Adjacency matrix factors The adjacency matrices are computed via the NetworkX library, where we are able to control the factors responsible for generating the graph.

For Watts-Strogatz graphs we have (1) a factor k, which represents the k nearest neighbors in a ring topology, and (2) factor p, which is the probability of rewiring each edge. For Erdos-Renyi graphs, we control only one factor p, that represents the probability for edge creation.

Combination Watts-Strogatz data has 2 adjacency factors, while Erdos-Renyi has only one, thus, in order to achieve the same number of graphs in both datasets, we fix the variance to 2 on Watts-Strogatz, while we provide 20 values in Erdos-Renyi. The final Watts-Strogatz and Erdos-Renyi datasets have $40 \times 40 \times 20 = 32.000$ graphs. The combinations for the factors: mean, variance, k, and p are provided in Table 6. Note that number of nodes is not considered a factor.

Table 6: Graph datasets for disentanglement.

		Node feature		Adjacency matrix		
Dataset	Num nodes	Mean	Variance	k	p	Num graphs
Watts-Strogatz	20	40	1	20	40	32K
Erdos-Renyi	20	40	20	n/a	40	32K

B Unconditional Disentanglement

The unconditional ELBO (Proposition 1), states that given two latent variables Z and T modeling the log-likelihood of the graph data $\mathcal{G}$ is bounded by the following $\mathcal{L}_{\text{u}}$ ELBO:

$$\mathcal{L}_{\text{u}} = \mathbb{E}_{q(Z|\mathcal{G})q(T|Z)} \log p(\mathcal{G}|Z)p(Z|T) - \mathbb{E}_{q(Z|\mathcal{G})}\text{KL}(q(T|Z)||p(T)) - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \quad (7)$$

Proof. Using Jensen inequality, we have:

$$\begin{aligned}
\log p(\mathcal{G}) &= \log \int_Z \int_T p(\mathcal{G}, Z, T) \\
&= \log \int_Z \int_T p(\mathcal{G}, Z, T) \frac{q(T|\mathcal{G}, Z)q(Z|\mathcal{G})}{q(T|\mathcal{G}, Z)q(Z|\mathcal{G})} \\
&\geq \int_Z q(Z|\mathcal{G}) \log \int_T p(\mathcal{G}, Z, T) \frac{q(T|\mathcal{G}, Z)}{q(T|\mathcal{G}, Z)q(Z|\mathcal{G})} \\
&= \int_Z q(Z|\mathcal{G}) \log \int_T p(\mathcal{G}, Z, T) \frac{q(T|\mathcal{G}, Z)}{q(T|\mathcal{G}, Z)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G})
\end{aligned}$$

By applying Jensen inequality again, we obtain:

$$\begin{aligned}
\log p(\mathcal{G}) &\geq \int_Z q(Z|\mathcal{G}) \int_T q(T|\mathcal{G}, Z) \log \frac{p(\mathcal{G}, Z, T)}{q(T|\mathcal{G}, Z)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \\
&= \int_Z q(Z|\mathcal{G}) \int_T q(T|\mathcal{G}, Z) \log \frac{p(\mathcal{G}, Z|T)p(T)}{q(T|\mathcal{G}, Z)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \\
&= \mathbb{E}_{q(Z|\mathcal{G})}\mathbb{E}_{q(T|\mathcal{G},Z)} \log p(\mathcal{G}, Z|T) - \mathbb{E}_{q(Z|\mathcal{G})} \int_T q(T|\mathcal{G}, Z) \log \frac{q(T|\mathcal{G}, Z)}{p(T)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \\
&= \mathbb{E}_{q(Z|\mathcal{G})}\mathbb{E}_{q(T|\mathcal{G},Z)} \log p(\mathcal{G}, Z|T) - \mathbb{E}_{q(Z|\mathcal{G})}\text{KL}(q(T|\mathcal{G}, Z)||p(T)) - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G})
\end{aligned}$$

Assuming Markov settings, which imply $q(T|Z) = q(T|\mathcal{G}, Z)$ and $p(\mathcal{G}|Z,T) = p(\mathcal{G}|Z)$, we have:

$$\log p(\mathcal{G}) \quad = \quad \mathbb{E}_{q(Z|\mathcal{G})q(T|Z)} \log p(\mathcal{G}|Z)p(Z|T) - \mathbb{E}_{q(Z|\mathcal{G})}\text{KL}(q(T|Z)||p(T)) - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G})$$

□

C Conditional Disentanglement

Using Jensen's inequality, we note that for any fixed $\hat{Z}$

$$\begin{aligned}
\log p(\mathcal{G}, \hat{Z}) &= \log \int_Z \int_T p(\mathcal{G}, Z, T, \hat{Z}) \\
&= \log \int_Z \int_T p(\mathcal{G}, Z, T, \hat{Z}) \frac{q(Z,T|\mathcal{G})}{q(Z,T|\mathcal{G})} \\
&= \log \int_Z \int_T p(\mathcal{G}, Z, T, \hat{Z}) \frac{q(T|\mathcal{G},Z)q(Z|\mathcal{G})}{q(T|\mathcal{G},Z)q(Z|\mathcal{G})} \\
&\geq \int_Z q(Z|\mathcal{G}) \log \int_T p(\mathcal{G}, Z, T, \hat{Z}) \frac{q(T|\mathcal{G},Z)}{q(T|\mathcal{G},Z)q(Z|\mathcal{G})} \\
&= \int_Z q(Z|\mathcal{G}) \log \int_T p(\mathcal{G}, Z, T, \hat{Z}) \frac{q(T|\mathcal{G},Z)}{q(T|\mathcal{G},Z)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G})
\end{aligned}$$

Using the conditional independence assumption $q(T|\mathcal{G}, Z) = q(T|Z)$, and applying Jensen's inequality again, we get

$$\begin{aligned}
\log p(\mathcal{G}, \hat{Z}) &\geq \int_Z q(Z|\mathcal{G}) \log \int_T p(\mathcal{G}, Z, T, \hat{Z}) \frac{q(T|Z)}{q(T|Z)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \\
&\geq \int_Z q(Z|\mathcal{G}) \int_T q(T|Z) \log \frac{p(\mathcal{G}, Z, T, \hat{Z})}{q(T|Z)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \\
&= \mathbb{E}_{q(Z|\mathcal{G})}\mathbb{E}_{q(T|Z)} \log \frac{p(\mathcal{G}, Z, T, \hat{Z})}{q(T|Z)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \\
&= \mathbb{E}_{q(Z|\mathcal{G})}\mathbb{E}_{q(T|Z)} \log \frac{p(T)p(\hat{Z}|T)p(Z|\hat{Z},T)p(\mathcal{G}|\hat{Z},T,Z)}{q(T|Z)} - \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \\
&= \mathbb{E}_{q(Z|\mathcal{G})}\mathbb{E}_{q(T|Z)} \log \Big(p(\hat{Z}|T)p(Z|\hat{Z},T)p(\mathcal{G}|\hat{Z},T,Z)\Big) \; - \; \mathbb{E}_{q(Z|\mathcal{G})} \text{KL}(q(T|Z)||p(T)) \\
&\quad - \; \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \; .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}_{p(\hat{Z})} \log p(\mathcal{G}, \hat{Z}) \quad \geq \quad & \mathbb{E}_{p(\hat{Z})}\mathbb{E}_{q(Z|\mathcal{G})}\mathbb{E}_{q(T|Z)} \log \Big(p(\hat{Z}|T)p(Z|\hat{Z},T)p(\mathcal{G}|\hat{Z},T,Z)\Big) \; - \; \mathbb{E}_{q(Z|\mathcal{G})} \text{KL}(q(T|Z)||p(T)) \\
& - \; \mathbb{E}_{q(Z|\mathcal{G})} \log q(Z|\mathcal{G}) \; .
\end{aligned}$$

Finally, we get our conditional objective when we assume

$$\mathbb{E}_{p(\hat{Z})}\mathbb{E}_{q(Z|\mathcal{G})}\mathbb{E}_{q(T|Z)} \log \Big(p(\hat{Z}|T)p(Z|\hat{Z},T)p(\mathcal{G}|\hat{Z},T,Z)\Big) \; \geq \; \mathbb{E}_{q(Z|\mathcal{G})}\mathbb{E}_{q(T|Z)} \log \Big(p(\hat{Z}|T)p(\mathcal{G}|\hat{Z},Z)\Big) \; .$$

D Scalability experiment

We perform a graph classification task on large scale datasets, and report their dimensions in Table 7. In Table 8 we report the results for our SID_C model on social network datasets including IMDB binary IMDB-multi, COLLAB, as well as macro molecules such as PROTEINS and MUTAG. The baselines for this experiment include FactorGCN [66] and DGCL [31]. For our model we set group-size to 81, hessian penalty to 40 and commutative penalty to 5. We follow the classification setup from Yang et al. [66], with a ten-fold cross-validation procedure, and report accuracy and standard deviation. The results show that both SID_U and SID_C are able to improve the performance of the baselines. In particular, SID_C achieves the highest accuracy on all the datasets.

Table 7: Dimensions of datasets

	IMDB-B	IMDB-M	COLLAB	PROTEINS	MUTAG
Graphs	1000	500	5000	1113	188
Classes	2	3	3	2	2
Avg. Nodes	19.77	13.00	74.49	39.06	17.93
Avg. Edges	96.53	65.94	2457.78	72.82	19.79

Table 8: Scalability

	IMDB-B	IMDB-M	COLLAB	PROTEINS	MUTAG
FactorGCN	75.3 ± 2.7	-	81.2 ± 1.4	-	89.9 ± 6.5
DGCL	75.9 ± 0.7	51.9 ± 0.4	81.2 ± 0.3	76.4 ± 0.5	92.1 ± 0.8
SID_U	76.1 ± 0.2	51.2 ± 0.9	81.9 ± 0.8	76.5 ± 1.5	91.4 ± 1.2
SID_C	**76.5 ± 0.3**	**52.5 ± 0.2**	**82.5±0.2**	**76.9 ± 0.1**	**92.5 ± 0.5**

E Molecular experiment

In order to have a fair comparison with Flow models, such as GraphAF [55] and GraphDF [37], we incorporate our parameterization into the training process of a Flow model. Both GraphAF and GraphDF are designed using relational GCN [53] (RGCN) as building block to compute an embedding of a graph using a layer R-GCN: $H_i^L = \text{R-GCN}(G_i)$, $\tilde{h}_i = \text{sum}(H_i^L)$ where sum denotes the sum-pooling operation, and $H_{i,j}^L \in \mathbb{R}^k$ denotes the embedding of the j-th node in the embeddings H_i^L.

We incorporate some parts of our models after the RGCN encoding, as follows. From the embedding H, we compute the lie algebra coordinates, with an encoder $q(T|H)$, following our E_group network from Section 4. We then apply the lie algebra exponential mapping to reconstruct the embedding H as $p(H|T)$, following the D_group network from Section 4. Furthermore, we incorporate the hessian penalty as a regularization. In all the three tasks, we follow the experimental setup from GraphAF, we set the model with group-size of 81, and a hessian penalty of 40.

We first perform the the random graph generation task, which involves quantifying 4 metrics in percentage, including: valid molecules with resampling (Val), valid molecules without resampling (Res), unique molecules (Uni), and novel molecules not appearing in the training data (Nov). In Table 9 we report results for GCPN, GraphAF, GraphDF and our models. The results indicate that with the inclusion of a lie algebra reparameterization and the hessian penalty, the performance on random generation is enhanced, and the Flow + SID_C model achieves the best performance on all metrics a part from uniqueness, where it achieves comparable results.

Table 9: Random graph generation.

	ZINC				QM9				MOSES			
Model	Val	Res	Uni	Nov	Val	Res	Uni	Nov	Val	Res	Uni	Nov
GCPN	**100**	20	**99.97**	**100**	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
GraphAF	**100**	68	99.1	**100**	**100**	67	94.15	88.83	**100**	71	**99.99**	**100**
GraphDF	**100**	89.03	99.16	**100**	**100**	82.67	**97.62**	98.1	**100**	87.58	99.55	**100**
SID_U	**100**	78.21	99.12	**100**	**100**	75.43	95.22	90.33	**100**	81.62	99.62	**100**
SID_C	**100**	86.02	99.14	**100**	**100**	82.92	96.43	92.28	**100**	84.29	99.79	**100**
Flow + SID_C	**100**	**90.12**	99.53	**100**	**100**	**83.45**	97.46	**98.62**	**100**	**88.42**	99.79	**100**

Secondly, we report the results for the property optimization task in Table 10, which show the scores from the top 3 generated molecules for two selected properties (penalized logp and QED). The baselines are the same ones as in the random generation task. We observe that our Flow + SID_C model is the top performing method on penalized logP for the top 3 scores, and it achieves the top results, together with the other methods, for QED.

Table 10: Property optimization performance.

	Penalized logP			QED		
	1st	2nd	3rd	1st	2nd	3rd
ZINC	4.52	4.3	4.23	0.948	0.948	0.948
GCPN	7.98	7.85	7.80	**0.948**	0.947	0.946
GraphAF	12.23	11.29	11.05	**0.948**	**0.948**	0.947
GraphDF	13.7	13.18	13.17	**0.948**	**0.948**	**0.948**
SID_U	12.56	12.46	12.03	0.947	**0.948**	0.947
SID_C	12.89	12.82	12.27	**0.948**	**0.948**	**0.948**
Flow + SID_C	**13.97**	**13.35**	**13.39**	**0.948**	**0.948**	**0.948**

Finally, we show in Table 11 the results for the constrained optimization task. We report only GraphDF as baseline, which is the top performing model among the baselines. The evaluation involves reporting the mean and standard deviation of metrics including: the largest property improvement (Imp), and similarities (Sim) between them and their corresponding input molecules, as well as the success rate (Suc).

We observe that by combining the disentangled principles developed in our models into a Flow-based model, we are able to further improve the results, and achieve the top performance.

Table 11: Constrained optimization performance on 800 molecules used in GraphAF.

δ	Flow + SID_C			GraphDF			SID_C		
	Imp	Sim	Suc	Imp	Sim	Suc	Imp	Sim	Suc
0.0	**14.28**$_{\pm 5.24}$	0.29$_{\pm 0.10}$	100	14.15$_{\pm 6.86}$	0.29$_{\pm 0.13}$	100	13.26$_{\pm 3.24}$	0.28$_{\pm 0.21}$	100
0.2	**12.89**$_{\pm 4.33}$	0.34$_{\pm 0.09}$	99.93	12.77$_{\pm 6.59}$	0.32$_{\pm 0.11}$	100	12.35$_{\pm 5.62}$	0.32$_{\pm 0.13}$	100
0.4	**9.32**$_{\pm 5.45}$	0.50$_{\pm 0.03}$	99.3	9.19$_{\pm 6.43}$	0.48$_{\pm 0.08}$	99.6	8.25$_{\pm 4.93}$	0.50$_{\pm 0.18}$	98.4
0.6	**4.98**$_{\pm 6.21}$	0.67$_{\pm 0.04}$	97.3	4.51$_{\pm 5.80}$	0.65$_{\pm 0.05}$	92.1	4.67$_{\pm 5.39}$	0.66$_{\pm 0.03}$	93.2