A New First-Order Meta-Learning Algorithm with Convergence Guarantees

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Paper under double-blind review

Abstract

Learning new tasks by leveraging prior experience is a fundamental trait of intelligent systems. While Model-Agnostic Meta-Learning (MAML) is a leading approach, it suffers from significant computational and memory overhead due to the requirement of computing second-order meta-gradients. We propose **FO-B-MAML**, a novel first-order variant of MAML derived from a bi-level optimization perspective. Our framework introduces a new expression of the meta-gradient, defined as the derivative of the solution of a perturbed optimization problem. This formulation allows the meta-gradient to be estimated using various finite difference methods; in this work, we propose and analyze two simple yet effective estimators: a forward and a symmetric approximation.

Unlike existing first-order methods like FO-MAML and Reptile, which suffer from irreducible bias, we prove that FO-B-MAML converges to a stationary point of the meta-objective. Notably, the symmetric estimator achieves an improved $\mathscr{O}(\delta^{2/3})$ bias rate, strictly enhancing previous first-order theory. Furthermore, we demonstrate that the MAML objective violates standard smoothness assumptions; we show instead that its smoothness constant grows with the norm of the meta-gradient. This property theoretically justifies the use of normalized or clipped-gradient methods (SNGDM) over vanilla gradient descent.

Our empirical results validate these advancements: FO-B-MAML achieves high accuracy on MNIST-1D, tracking closely with second-order MAML performance. Crucially, our method bypasses the "activation bottleneck" of second-order approaches, maintaining a flat memory footprint even when scaling to deep, activation-heavy CNNs with up to 250 channels.

1 Introduction

A hallmark of intelligence is the ability to swiftly acquire new skills by leveraging prior experiences from similar tasks. Recent research has explored how meta-learning algorithms (Schmidhuber, 1987; Thrun & Pratt, 1998; Naik & Mammone, 1992) can replicate this capability by learning to learn across a variety of tasks efficiently. This mastery allows for learning a novel task with only minimal training data, sometimes just a single example, as demonstrated in (Santoro et al., 2016; Vinyals et al., 2016; Finn et al., 2017a).

Meta-learning approaches can be generally categorized into three primary frameworks:

- **Metric-learning approaches**: These methods learn a structured embedding space where non-parametric techniques, such as nearest neighbors, perform effectively (Koch, 2015; Vinyals et al., 2016; Snell et al., 2017; Oreshkin et al., 2018; Allen et al., 2019).
- **Black-box approaches**: These involve training recurrent or recursive neural networks to either produce weight updates directly from input data (Hochreiter et al., 2001; Andrychowicz et al., 2016; Li & Malik, 2016; Ravi & Larochelle, 2017) or to generate predictions for new inputs (Santoro et al., 2016; Duan et al., 2016; Wang et al., 2016; Munkhdalai & Yu, 2017; Mishra et al., 2017). Attention-based models (Vinyals et al., 2016; Mishra et al., 2017) are also frequently employed in this context.
- **Optimization-based approaches**: These methods utilize bi-level optimization to integrate learning procedures, such as gradient descent, into the meta-optimization problem (Finn et al., 2017a; Finn, 2018; Bertinetto et al.,

2018; Zintgraf et al., 2018; Li et al., 2017; Finn et al., 2018; Zhou et al., 2018; Harrison et al., 2018). The "inner" optimization handles task adaptation, while the "outer" objective is the meta-training goal: the average test loss after adaptation over a set of tasks. This approach, exemplified by (Maclaurin et al., 2015) and MAML (Finn et al., 2017a), learns optimal initial model parameters to facilitate rapid adaptation and generalization.

Beyond these categories, hybrid strategies have been explored to combine the strengths of different methods (Rusu et al., 2018; Triantafillou et al., 2019). In this study, we concentrate on **optimization-based methods**, specifically MAML (Finn et al., 2017a), which has been demonstrated to possess the expressive power of black-box strategies (Finn & Levine, 2017). Additionally, MAML is versatile across various scenarios (Finn et al., 2017b; Mi et al., 2019; Al-Shedivat et al., 2017; Finn et al., 2019) and ensures a consistent optimization process (Finn, 2018).

While meta-learning initialization holds promise, it necessitates backpropagation through the inner optimization algorithm, introducing significant challenges. This includes the requirement for higher-order derivatives, leading to substantial computational and memory overheads and potential issues like vanishing gradients. Consequently, scaling optimization-based meta-learning to tasks with large datasets or numerous inner-loop steps becomes arduous.

These challenges can be partially mitigated by taking few gradient steps (Finn et al., 2017a), truncating the backpropagation process (Shaban et al., 2018), using implicit gradients (Rajeswaran et al., 2019), or omitting higher-order derivative terms (Finn et al., 2017a; Nichol et al., 2018). However, these approximations may lead to sub-optimal performance (Wu et al., 2018). An additional challenge relates to the convergence analysis; while it was shown in (Fallah et al., 2019) that the MAML objective can be non-smooth even with one step, other works such as (Rajeswaran et al., 2019) simply assume smoothness for multi-step trajectories. A comprehensive convergence analysis remains missing.

To address these limitations, we design a new first-order MAML algorithm that is path-independent and provides rigorous theoretical convergence guarantees.

Contributions:

1. **New expression of the meta-gradient:** We study the MAML objective in (5) and derive a new expression for its meta-gradient (Proposition 1), reformulating it as the derivative of the solution to a perturbed optimization problem.
2. **Practical challenges:** Based on the formulae in Proposition 1, we propose a new first-order MAML variant that avoids second-order information. Unlike previous first-order algorithms such as FO-MAML and Reptile (Nichol et al., 2018), our algorithm's bias can be made as small as possible, allowing for theoretical convergence to any given precision.
3. **Theoretical challenges:** We show that the general MAML objective satisfies a generalized smoothness assumption introduced in (Jingzhao et al., 2020). This suggests that clipped gradient descent is better suited for this case, explaining why meta-gradient clipping stabilizes MAML in practice.
4. **Convergence rates:** Finally, we provide convergence rates for our method.

2 Preliminaries

2.1 Vanilla MAML

We assume we have a set of training tasks $\{\mathscr{T}_i\}_{i=1}^M$ drawn from an unknown distribution of tasks $P(\mathscr{T})$, such that for each task $\mathscr{T}_i$ one can associate a training $\mathscr{D}_i^{\text{tr}}$ and test $\mathscr{D}_i^{\text{test}}$ dataset—or equivalently a training $\hat{f}_i$ and test f_i objective. Then, the vanilla MAML objective is that of solving the following optimization problem:

$$\begin{aligned} \boldsymbol{\theta}^\star &:= \arg\min_{\boldsymbol{\theta}\in\Theta} F(\boldsymbol{\theta}) \\ F(\boldsymbol{\theta}) &:= \frac{1}{M}\sum_{i=1}^{M}\left[F_i(\boldsymbol{\theta}) := f_i\Big(\boldsymbol{\phi}_i(\boldsymbol{\theta}) = \mathscr{A}lg\big(\hat{f}_i, \boldsymbol{\theta}, \boldsymbol{h}\big)\Big)\right], \end{aligned} \tag{1}$$

where $\mathscr{A}lg\big(\hat{f}_i, \boldsymbol{\theta}, \boldsymbol{h}\big)$ is an optimization algorithm that takes as input the objective $\hat{f}_i$ (i.e., dataset and loss), the initialization $\boldsymbol{\theta}$ and other hyperparameters denoted by $\boldsymbol{h}$ (.e.g., learning rate, and the number of steps), then outputs an updated task-specific parameter $\boldsymbol{\phi}_i(\boldsymbol{\theta})$ that is hopefully a better "approximate" solution for the individual task objective $\hat{f}_i$.

For example, $\mathscr{A}lg\big(\hat{f}_i, \boldsymbol{\theta}, \cdot\big)$ may correspond to one or multiple steps of gradient descent on $\hat{f}_i$ initialized at $\boldsymbol{\theta}$. For example, if we use one step of gradient descent with a learning rate α, then we have:

$$\boldsymbol{\phi}_i(\boldsymbol{\theta}) \equiv \mathscr{A}lg\big(\hat{f}_i, \boldsymbol{\theta}, \boldsymbol{h} := \{\alpha\}\big) = \boldsymbol{\theta} - \alpha \nabla_{\boldsymbol{\theta}} \hat{f}_i(\boldsymbol{\theta}). \tag{2}$$

To solve (1) with gradient-based methods, we require a way to differentiate through $\mathscr{A}lg$. In the case of multiple steps like (2), this corresponds to backpropagating through the dynamics of gradient descent. This backpropagation through gradient-based optimization algorithms naturally involves higher order derivatives and the need to save the whole trajectory to compute the meta-gradient (i.e., the gradient of F), which is a big drawback to vanilla MAML.

Another (potential) drawback of MAML defined in (1) is that it depends on the choice of the optimization algorithm $\mathscr{A}lg$ (since we need its specific trajectory).

2.2 First-order MAML and Reptile

One option considered in the literature to address the computational and memory overheads encountered when differentiating through gradient-based optimization algorithms is to devise first-order meta-gradient approximations (Nichol et al., 2018). Two such approaches stand out: **FO-MAML** and **Reptile**.

FO-MAML simply ignores the Jacobian $\dfrac{d\mathscr{A}lg}{d\boldsymbol{\theta}}$ leading to the following approximation:

$$\boldsymbol{g}_{\text{FO-MAML}} = \frac{1}{M} \sum_{i=1}^{M} \nabla f_i(\boldsymbol{\phi}_i(\boldsymbol{\theta})) \tag{3}$$

The **Reptile** approximation is less straightforward, but the crux of it is using an average gradient over the inner optimization algorithm's trajectory.

$$\boldsymbol{g}_{Reptile} = \frac{1}{M} \sum_{i=1}^{M} \frac{\boldsymbol{\theta} - \boldsymbol{\phi}_i(\boldsymbol{\theta})}{K\alpha} \tag{4}$$

where in (4), K is the number of steps of $\mathscr{A}lg$ and α is the learning rate.

These two approximations avoid the prohibitive computational and memory costs associated with vanilla MAML. However, both approximations introduce bias to the true meta-gradient that is irreducible, at least in the case of FOMAML (Fallah et al., 2019) (the bias of Reptile is not clear in general).

2.3 B-MAML: MAML as a fully Bi-Level Optimization Problem

Ignoring the computational overhead of MAML, the memory overhead is naturally a result of the dependence of the MAML objective in (1) on the choice of the inner optimization algorithm $\mathscr{A}lg$ and thus on the trajectory of $\mathscr{A}lg$; then if one can break this dependence on $\mathscr{A}lg$, one would avoid this memory overhead; one idea to accomplish the latter is to make the MAML objective depend on the inner optimization problem rather than the specific optimization algorithm used to solve such an optimization problem. This will amount to framing MAML as the following purely bi-level optimization problem : (outer-level)

$$\boldsymbol{\theta}^\star := \arg\min_{\boldsymbol{\theta} \in \Theta} F(\boldsymbol{\theta}) := \frac{1}{M} \sum_{i=1}^{M} \left[F_i(\boldsymbol{\theta}) := f_i\left(\boldsymbol{\phi}_i^\star(\boldsymbol{\theta})\right) \right] \tag{5}$$

where for $i \in \{1, \cdots, M\}$ we define (recall that f and $\hat{f}$ denote validation and training objectives respectively):

$$\boldsymbol{\phi}_i^\star(\boldsymbol{\theta}) := \arg\min_{\boldsymbol{\phi} \in \Theta} \hat{f}_i(\boldsymbol{\phi}) + \frac{\lambda}{2} \|\boldsymbol{\phi} - \boldsymbol{\theta}\|^2 \quad \text{(inner-level)} \tag{6}$$

Where λ is a real hyperparameter that plays the role of the inverse of the learning rate α that MAML had, it helps control the strength of the meta-parameter or prior ($\boldsymbol{\theta}$) relative to new data. We note that this hyperparameter can be a vector or a matrix, but we keep it as a scalar for simplicity.

We also note that the formulation (6) is not new and was introduced, for example, in (Rajeswaran et al., 2019).

To use gradient-based methods to solve (5), we need to compute the gradient ∇F. Using the implicit function theorem and assuming that $\lambda \boldsymbol{I} + \nabla^2 \hat{f}_i$ is invertible, it is easy to show that

$$\nabla F_i(\boldsymbol{\theta}) = \Big(\boldsymbol{I} + \frac{1}{\lambda}\nabla^2 \hat{f}_i(\boldsymbol{\phi}_i^\star(\boldsymbol{\theta}))\Big)^{-1} \nabla f_i(\boldsymbol{\phi}_i^\star(\boldsymbol{\theta})) \tag{7}$$

What is noteworthy about (7) is that the meta-gradient $\nabla F_i(\boldsymbol{\theta})$ only depends on $\boldsymbol{\phi}_i^\star(\boldsymbol{\theta})$ which can be estimated using any optimization algorithm irrespective of the trajectory said-algorithm will take.

A downside of (7) is that it involves computing the Hessian and inverting it. To reduce this burden, one can equivalently treat $\nabla F_i(\boldsymbol{\theta})$ as a solution to the following linear system (with the unknown $\boldsymbol{v} \in \mathbb{R}^d$):

$$\big(\boldsymbol{I} + \frac{1}{\lambda}\nabla^2 \hat{f}_i(\boldsymbol{\phi}_i^\star(\boldsymbol{\theta}))\big)\boldsymbol{v} = \nabla f_i(\boldsymbol{\phi}_i^\star(\boldsymbol{\theta})), \tag{8}$$

which only needs access to Hessian-Vector products instead of the full Hessian and can be approximately solved with the conjugate gradient algorithm, for example, which is exactly the idea of (Rajeswaran et al., 2019).

In this work, we propose a different strategy that consists of writing the meta-gradient as the gradient of the solution of a perturbed optimization problem with respect to its perturbation parameter. This means we can approximate the meta-gradient using the solution of two optimization problems.

2.4 Comparison to ES-MAML

The closest method in the literature to ours is perhaps ES-MAML(Song et al., 2020), which is based on the idea of *evolution strategies* that consists of replacing a given function h and with its Gaussian smoothing: $h_\nu(\boldsymbol{\theta}) := \mathbb{E}_{\boldsymbol{g}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I})}[h(\boldsymbol{\theta} + \nu \boldsymbol{g})]$. which has the property of being smooth (even when h is not) with: $\nabla h_\nu(\boldsymbol{\theta}) := \frac{1}{\nu}\mathbb{E}_{\boldsymbol{g}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I})}[h(\boldsymbol{\theta} + \nu \boldsymbol{g})\boldsymbol{g}]$. In our case $h(\boldsymbol{\theta}) := F(\boldsymbol{\theta}) := \frac{1}{m}\sum_{i=1}^m f_i(\boldsymbol{\phi}_i^\star(\boldsymbol{\theta}))$. Thus ES-MAML minimizes the following perturbed problem: $F_\nu(\boldsymbol{\theta}) = \frac{1}{m}\sum_{i=1}^m \mathbb{E}_{\boldsymbol{g}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I})}[f_i(\boldsymbol{\phi}_i^\star(\boldsymbol{\theta} + \nu \boldsymbol{g}))]$.

Noting that

$$\boldsymbol{\phi}_i^\star(\boldsymbol{\theta} + \nu \boldsymbol{g}) = \arg\min_{\boldsymbol{\phi}} -\nu\lambda\langle \boldsymbol{g}, \boldsymbol{\phi}\rangle + \hat{f}_i(\boldsymbol{\phi}) + \frac{\lambda}{2}\|\boldsymbol{\phi} - \boldsymbol{\theta}\|^2. \tag{9}$$

we see that ES-MAML can be seen indeed as using our idea of perturbing the inner objective using the (random) function $\boldsymbol{\phi} \mapsto -\lambda\langle \boldsymbol{g}, \boldsymbol{\phi}\rangle$ and the perturbation parameter being ν. However, in our case, we will perturb the inner objective using the outer objective directly, this will result in a better algorithm that needs to only call the inner solver twice instead of many times (to estimate the average over the Gaussian distribution).

ES-MAML(Song et al., 2020) is mainly used in reinforcement learning where it is observed to improve exploration but has not been considered in a broader sense as it potentially needs many random Gaussian samples and solving the inner problem and evaluating the test function as many times (note that for neural networks evaluating a function and computing its gradients have approximately the same complexity). We show that ES-MAML fails to perform in our simple synthetic experiment in Figure1.

3 First-Order B-MAML

We consider the B-MAML objective defined in (5), (6); our main idea relies on perturbing the inner optimization problem (6). For a perturbation parameter $\nu \in \mathbb{R}$, and each training task $\mathcal{T}_i$, we introduce the following perturbed inner optimization problem, which interpolates between validation and training objectives:

$$\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) := \arg\min_{\boldsymbol{\phi}\in\Theta} \ \nu f_i(\boldsymbol{\phi}) + \hat{f}_i(\boldsymbol{\phi}) + \frac{\lambda}{2}\|\boldsymbol{\phi} - \boldsymbol{\theta}\|^2. \tag{10}$$

To provide an intuition about this choice of perturbation, let's assume the test loss f_i is linear i.e.: $\nabla f_i(\phi) = \boldsymbol{g}_f$ a constant, then it is easy to show that:

$$\phi^\star_{i,\nu}(\boldsymbol{\theta}) = \phi^\star_i(\boldsymbol{\theta} - \nu \frac{\boldsymbol{g}_f}{\lambda}) \tag{11}$$

Which is the perturbed problem (9) with $\boldsymbol{g} = -\frac{\boldsymbol{g}_f}{\lambda}$. This means that instead of picking a random perturbation direction like ES-MAML, we pick an informed and specific direction that we will show leads to a better estimate of the gradient.

We assume that there is a neighbourhood $\mathscr{V}_0$ of 0, such that $\phi^\star_{i,\nu}(\boldsymbol{\theta})$ is well-defined for any $\nu \in \mathscr{V}_0$ and $\boldsymbol{\theta} \in \Theta$. Then we can show the following result:

Proposition 1 *For any training task $\mathscr{T}_i$, if $\nabla F_i(\boldsymbol{\theta})$ exists, then $\nu \mapsto \phi^\star_{i,\nu}(\boldsymbol{\theta})$ is differentiable at $\nu = 0$ and $\nabla F_i(\boldsymbol{\theta}) = -\lambda \left.\frac{d\phi^\star_{i,\nu}(\boldsymbol{\theta})}{d\nu}\right|_{\nu=0}$.*

Sketch of the proof. We use the fact that $\phi^\star_{i,\nu}(\boldsymbol{\theta})$ is a stationary point of $\phi \mapsto \nu f_i(\phi) + \hat{f}_i(\phi) + \frac{\lambda}{2}\|\phi - \boldsymbol{\theta}\|^2$ this will give us a quantity that is null for all $\nu \in \mathscr{V}_0$, then we differentiate with respect to ν and set $\nu = 0$.

Proposition 1 writes the meta-gradient $\nabla F_i(\boldsymbol{\theta})$ as the derivative of another function that is a solution to the perturbed optimization problem 10, thus presenting us with a way we can approximate the meta-gradient using the finite difference method (Atkinson & Han, 2001). We consider mainly two approximations: the **forward** and **symmetric** approximations successively defined as follows:

$$\boldsymbol{g}^{\text{For}}_{i,\nu}(\boldsymbol{\theta}) = -\lambda \Big(\frac{\phi^\star_{i,\nu}(\boldsymbol{\theta}) - \phi^\star_{i,0}(\boldsymbol{\theta})}{\nu} \Big) \tag{12}$$

$$\boldsymbol{g}^{\text{Sym}}_{i,\nu}(\boldsymbol{\theta}) = -\lambda \Big(\frac{\phi^\star_{i,\nu}(\boldsymbol{\theta}) - \phi^\star_{i,-\nu}(\boldsymbol{\theta})}{2\nu} \Big) \tag{13}$$

We note that more involved approximations (that need solving more than two optimization problems) can be engineered, but we limit ourselves, in this work, to (12) and (13).

Assuming that $\nu \mapsto \phi^\star_{i,\nu}(\boldsymbol{\theta})$ is regular enough near $\nu = 0$ (for example, three times differentiable on $\mathscr{V}_0$ and its third derivative is bounded), then we should expect that

$$\|\nabla F_i(\boldsymbol{\theta}) - \boldsymbol{g}^{\text{For}}_{i,\nu}(\boldsymbol{\theta})\| = \mathscr{O}(\lambda \nu) \quad \text{and} \quad \|\nabla F_i(\boldsymbol{\theta}) - \boldsymbol{g}^{\text{Sym}}_{i,\nu}(\boldsymbol{\theta})\| = \mathscr{O}(\lambda \nu^2) \tag{14}$$

We will provide conditions under which we get the first identity in Equation 14 in Section4 and support the second experimentally in Section 5.

In practice, we can't realistically solve the optimization problem (10) and have access to the true values of $\phi^\star_{i,\nu}(\boldsymbol{\theta})$; instead, we will solve the problem (10) approximatively using any algorithm $\mathscr{A}lg$ of our choice and assume that for a given precision δ, we can get an approximate solution $\tilde{\phi}_{i,\nu}(\boldsymbol{\theta})$ of (10) such that

$$\|\tilde{\phi}_{i,\nu}(\boldsymbol{\theta}) - \phi^\star_{i,\nu}(\boldsymbol{\theta})\| \leq \delta \ . \tag{15}$$

Note. In the stochastic case, we need to replace the guarantee in (15) by $\mathbb{E}\|\tilde{\phi}_{i,\nu}(\boldsymbol{\theta}) - \phi^\star_{i,\nu}(\boldsymbol{\theta})\| \leq \delta$ where $\mathbb{E}$ is the expectation over the randomness of the algorithm $\mathscr{A}lg$; most importantly, this will not change anything in our analysis except for the additional expectations.

We use $\tilde{g}^{\text{method}}_{i,\nu}$ for method $\in \{\text{For, Sym}\}$, to denote the estimator resulting from the use of the approximate solutions; this will introduce an additional bias (w/t ∇F_i) to our estimators in (12), (13); it is easy to show that this bias should be bounded by $\dfrac{2\lambda\delta}{\nu}$ in the worst case. Notice that this bias term increases with small values of ν, which suggests a sweet spot for ν when including the bias terms in (14).

Algorithm 1 First Order Bi-Level MAML (FO-B-MAML)

1: **Require:** Distribution over tasks $P(\mathscr{T})$, outer step size η, regularization strength λ,
2: **Hyperparameters:** Precision δ and small perturbation parameter ν
3: **while** not converged **do**
4: Sample mini-batch of tasks $\{\mathscr{T}_i\}_{i=1}^{B} \sim P(\mathscr{T})$
5: **for** Each task $\mathscr{T}_i$ **do**
6: Use an iterative solver to get $\tilde{\phi}_{i,\nu}(\boldsymbol{\theta})$ and $\tilde{\phi}_{i,0}(\boldsymbol{\theta})$ (or $\tilde{\phi}_{i,-\nu}(\boldsymbol{\theta})$) satisfying (15)
7: Set $\boldsymbol{g}_i = -\lambda(\tilde{\phi}_{i,\nu}(\boldsymbol{\theta}) - \tilde{\phi}_{i,0}(\boldsymbol{\theta}))/\nu$ $\Big($or $\boldsymbol{g}_i = -\lambda(\tilde{\phi}_{i,\nu}(\boldsymbol{\theta}) - \tilde{\phi}_{i,-\nu}(\boldsymbol{\theta}))/(2\nu)\Big)$
8: **end for**
9: Average above gradients to get $\hat{\nabla} F(\boldsymbol{\theta}) = (1/B)\sum_{i=1}^{B} \boldsymbol{g}_i$
10: Update meta-parameters with a gradient-based optimization algorithm of choice like GD, ClippedGD, or Adam.
11: **end while**

The overall bias is then

$$\|\nabla F_i(\boldsymbol{\theta}) - \tilde{g}_{i,\nu}^{\text{For}}(\boldsymbol{\theta})\| = \mathscr{O}\Big(\lambda\nu + \frac{\lambda\delta}{\nu}\Big) \quad \text{and} \quad \|\nabla F_i(\boldsymbol{\theta}) - \tilde{g}_{i,\nu}^{\text{Sym}}(\boldsymbol{\theta})\| = \mathscr{O}\Big(\lambda\nu^2 + \frac{\lambda\delta}{\nu}\Big), \tag{16}$$

minimizing for ν we get that for $\nu^{For} \sim \sqrt{\delta}$ and $\nu^{Sys} \sim \delta^{1/3}$ we get:

$$\|\nabla F_i(\boldsymbol{\theta}) - \tilde{g}_{i,\nu}^{\text{For}}(\boldsymbol{\theta})\| = \mathscr{O}\big(\lambda\sqrt{\delta}\big) \quad \text{and} \quad \|\nabla F_i(\boldsymbol{\theta}) - \tilde{g}_{i,\nu}^{\text{Sym}}(\boldsymbol{\theta})\| = \mathscr{O}\big(\lambda\delta^{2/3}\big), \tag{17}$$

We summarize this in Algorithm 1.

In line 10 of Algorithm 1, we can use any optimization algorithm to update the meta-parameters $\boldsymbol{\theta}$, but our theoretical analysis in Section 4 will only consider the Gradient Descent (GD) and Clipped Gradient (ClippedGD) algorithms (or Normalized GD which is equivalent). We will show that the B-MAML objective (5) has a smoothness parameter that grows with the norm of its gradient; under this type of smoothness, it is known that ClippedGD is well-suited (Jingzhao et al., 2020), Adam (Kingma & Ba, 2015) is also well-suited since it estimates the curvature and uses it to normalize the gradient. We also show that under stronger assumptions, the B-MAML objective is smooth in the classical sense (meaning its gradient is Lipschitz), and in this case, GD can be used but can have a worse complexity.

Now that we have presented our main algorithm, it is time to discuss its theoretical guarantees under common assumptions used in the Bi-Level optimization literature.

4 Theoretical Analysis

In this Section, we provide theoretical guarantees of Algorithm 1 using both the forward approximation in equation (12) and the symmetric approximation in equation (13). We start by stating the assumptions that we make on the training tasks $\{\mathscr{T}_i\}_{i=1}^{M}$, then discuss the smoothness properties of the B-MAML objective defined in (6) resulting from such assumptions. Finally, we discuss the convergence rate when using Gradient Descent (GD) or Clipped Gradient Descent (SNGDM) as the meta-optimizer.

4.1 Assumptions

We will make use of the following assumptions:

Assumption 1 (Behavedness of train sets for Forward Case) *For all training tasks $\mathscr{T}_i$, the training objective $\hat{f}_i$ is twice differentiable, $\hat{L}_1$-smooth and has $\hat{L}_2$-Lipschitz Hessian.*

Assumption 2 (Higher-Order Regularity for Symmetric Case) *For the symmetric estimator to achieve its improved rate, we further assume for all training tasks $\mathscr{T}_i$ that the training objective $\hat{f}_i$ is C^4 with bounded third and fourth derivatives ($\hat{L}_3, \hat{L}_4$).*

Assumption 3 (Behavedness of test sets) *For all training tasks $\mathscr{T}_i$, the test objective f_i is differentiable, L_0–Lipschitz, and has L_1–smooth Hessian. For the symmetric estimator, we additionally assume f_i is C^3 with a Lipschitz second derivative (L_2).*

Assumption 4 (Strong convexity) *There exists $\mu > 0$ such that for all training tasks $\mathscr{T}_i$, the inner training objective $\hat{f}_i + \frac{\lambda}{2}\|\cdot\|^2$ is μ-strongly convex.*

Under Assumption 1, taking $\lambda > \hat{L}_1$ ensures Assumption 4 holds with $\mu = \lambda - \hat{L}_1$. This guarantees that the perturbed problem (10) has a unique solution path $\nu \mapsto \phi^\star_{i,\nu}(\boldsymbol{\theta})$ that is C^3 for $\nu \in \mathscr{V}_0 = (-\frac{\mu}{L_1}, \frac{\mu}{L_1})$.

Assumption 5 (Task similarity) *There exists $\zeta \geq 0$, such that for all $\boldsymbol{\theta} \in \Theta$, we have: $\frac{1}{M}\sum_{i=1}^{M}\|\nabla F_i(\boldsymbol{\theta}) - \nabla F(\boldsymbol{\theta})\|^2 \leq \zeta^2$.*

Assumption 6 (Bounded variance) *We also assume that we have access to stochastic gradients of $f_i, \hat{f}_i$ with a variance bounded by σ^2.*

4.2 Properties of B-MAML and Estimator Bias

We characterize the bias for both the forward and symmetric estimators by bounding the derivatives of the solution path $z(\nu) = \phi^\star_{i,\nu}(\boldsymbol{\theta})$ against the true meta-gradient $\nabla F_i(\boldsymbol{\theta}) = -\lambda z'(0)$.

Proposition 2 (Refined Bias Expressions) *Under the regularity assumptions, let μ be the strong convexity constant of the inner objective. The bias of the estimators relative to the true meta-gradient $\nabla F_i(\boldsymbol{\theta})$ satisfies:*

- ***Forward Estimator**:*

$$\|\nabla F_i(\boldsymbol{\theta}) - \boldsymbol{g}^{For}_{i,\nu}(\boldsymbol{\theta})\| = \mathscr{O}\left(\lambda\nu\left[\frac{L_1}{\mu} + \frac{L_0\hat{L}_2}{\mu^2}\right]\right) \tag{18}$$

- ***Symmetric Estimator**:*

$$\|\nabla F_i(\boldsymbol{\theta}) - \boldsymbol{g}^{Sym}_{i,\nu}(\boldsymbol{\theta})\| = \mathscr{O}\left(\lambda\nu^2\left[\frac{L_2}{\mu} + \frac{L_1\hat{L}_2 + L_0\hat{L}_3}{\mu^2} + \frac{L_0\hat{L}_2^2}{\mu^3}\right]\right) \tag{19}$$

Balancing these with the numerical error $\mathscr{O}(\lambda\delta/\nu)$ from the solver precision δ yields the following:

Corollary 1 (Total Bias of FO-B-MAML) *With inner precision δ and optimal ν:*

- ***Forward Case**: $\nu \sim \delta^{1/2} \implies \|\nabla F_i(\boldsymbol{\theta}) - \tilde{g}^{For}_{i,\nu}(\boldsymbol{\theta})\| = \mathscr{O}(\lambda\delta^{1/2})$.*
- ***Symmetric Case**: $\nu \sim \delta^{1/3} \implies \|\nabla F_i(\boldsymbol{\theta}) - \tilde{g}^{Sym}_{i,\nu}(\boldsymbol{\theta})\| = \mathscr{O}(\lambda\delta^{2/3})$.*

Discussion. The symmetric approximation provides a significant theoretical improvement, reaching an $\mathscr{O}(\delta^{2/3})$ bias rate. These results on the accelerated symmetric approximation are new and strictly improve over previous MAML theory, allowing for higher precision with first-order memory costs.

We now discuss the smoothness of the meta-objective. Proposition 3 shows that curvature scales with the gradient norm.

Proposition 3 *Under Assumptions 1 and 3, for $\lambda \geq 2\hat{L}_1$, for any training task $\mathscr{T}_i$:*

$$\|\nabla F_i(\boldsymbol{\theta}) - \nabla F_i(\boldsymbol{\theta}')\| \leq \min(\mathscr{L}(\boldsymbol{\theta}), \mathscr{L}(\boldsymbol{\theta}'))\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|,$$

where $\mathscr{L}(\boldsymbol{\theta}) = \mathscr{L}_0 + \mathscr{L}_1\|\nabla F_i(\boldsymbol{\theta})\|$. Under Assumption 4, the meta-gradient is bounded by $G = \frac{\lambda L_0}{\mu}$, implying classical smoothness $\|\nabla F(\boldsymbol{\theta}) - \nabla F(\boldsymbol{\theta}')\| \leq \mathscr{L}\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|$ with $\mathscr{L} = \mathscr{L}_0 + G\mathscr{L}_1$.

Table 1: Comparison of gradient-based meta-learning methods (deterministic setting, $M = 1$). Total complexity is the product of inner and outer iterations to reach ε-stationarity.

Algorithm	Compute (Total)	Memory	First-order	Bias
MAML	$\mathcal{O}(\varepsilon^{-2}\sqrt{\kappa}\log(1/\varepsilon))$	$\text{Mem}(\text{activations}) \cdot \sqrt{\kappa}$	No	δ
iMAML	$\mathcal{O}(\varepsilon^{-2}\sqrt{\kappa}\log(1/\varepsilon))$	Mem(activations)	No	δ
FO-MAML/Reptile	$\mathcal{O}(\varepsilon^{-2}\sqrt{\kappa}\log(1/\varepsilon))$	$\text{Mem}(\boldsymbol{\theta})$	Yes	1
FO-B-MAML (For)	$\mathcal{O}(\varepsilon^{-2}\sqrt{\kappa}\log(1/\varepsilon))$	$\text{Mem}(\boldsymbol{\theta})$	Yes	$\sqrt{\delta}$
FO-B-MAML (Sym)	$\mathcal{O}(\varepsilon^{-2}\sqrt{\kappa}\log(1/\varepsilon))$	$\text{Mem}(\boldsymbol{\theta})$	Yes	$\delta^{2/3}$

4.3 Convergence Results

Based on the characterization of the $(\mathscr{L}_0, \mathscr{L}_1)$-smoothness and the estimator bias, we reduce the bi-level optimization problem in (5) to a single-level non-convex optimization problem. Throughout this section, we denote $\Delta = F(\boldsymbol{\theta}_0) - F^\star$, where $F^\star$ is the global minimum. We denote $\tilde{\sigma}^2 := \frac{M-B}{MB}\zeta^2$ as the outer variance from task sampling and recall that Assumption 6 implies an inner stochastic variance of $\mathcal{O}(\sigma^2)$.

Theorem 1 (FO-B-MAML using ClippedGD) *Under the generalized smoothness (Proposition 3), Algorithm 1 using ClippedGD finds a meta-parameter $\boldsymbol{\theta}$ satisfying $\mathbb{E}[\|\nabla F(\boldsymbol{\theta})\|] \leq \varepsilon + b$ in:*

- ***Deterministic Case*** *($\tilde{\sigma} = 0$): $\mathcal{O}\left(\frac{\mathscr{L}_0\Delta}{\varepsilon^2} + \frac{\mathscr{L}_1^2\Delta}{\mathscr{L}_0}\right)$ outer steps.*
- ***Stochastic Case*** *($\tilde{\sigma} > 0$): $\mathcal{O}\left(\Delta\tilde{\sigma}^2 \max\left(\frac{\mathscr{L}_0}{\varepsilon^4}, \frac{\mathscr{L}_1^4}{\mathscr{L}_0^3}\right)\right)$ outer steps.*

where b is the estimator bias: $b = \mathcal{O}(\lambda\sqrt{\delta})$ for the forward approximation and $b = \mathcal{O}(\lambda\delta^{2/3})$ for the symmetric approximation.

Theorem 2 (FO-B-MAML using SGD) *Under the classical smoothness (Proposition 3), Algorithm 1 using SGD finds a meta-parameter $\boldsymbol{\theta}$ satisfying $\mathbb{E}[\|\nabla F(\boldsymbol{\theta})\|] \leq \varepsilon + b$ in $\mathcal{O}\left(\frac{\mathscr{L}\Delta}{\varepsilon^2} + \frac{\mathscr{L}\Delta\tilde{\sigma}^2}{\varepsilon^4}\right)$ outer steps.*

Complexity Comparison and Total Cost. The total complexity (total gradient calls) is $M \times$ *Outer-steps*$(\varepsilon) \times$ *Inner-steps*(δ). In the stochastic regime, the inner solver (e.g., SGD) requires *Inner-steps* $= \mathcal{O}(\sigma^2\delta^{-1})$.

Stochastic Case Leading Terms. To reach ε-stationarity, we require the bias $b \sim \varepsilon$. This leads to the following leading-order complexities in the stochastic regime:

- **Forward Estimator**: Requiring $\sqrt{\delta} \sim \varepsilon \implies \delta \sim \varepsilon^2$. The inner complexity is $\mathcal{O}(\sigma^2\varepsilon^{-2})$. Multiplying by the outer complexity $\mathcal{O}(\tilde{\sigma}^2\varepsilon^{-4})$, the leading term is $\mathcal{O}(M\Delta\tilde{\sigma}^2\sigma^2\varepsilon^{-6})$.
- **Symmetric Estimator**: Requiring $\delta^{2/3} \sim \varepsilon \implies \delta \sim \varepsilon^{1.5}$. The inner complexity is reduced to $\mathcal{O}(\sigma^2\varepsilon^{-1.5})$. The total leading term becomes $\mathcal{O}(M\Delta\tilde{\sigma}^2\sigma^2\varepsilon^{-5.5})$.

The symmetric approximation provides a significant theoretical advantage, saving a factor of $\varepsilon^{-0.5}$ in total complexity compared to the forward approach. These accelerated symmetric results are new and provide a strict improvement over previous MAML theory.

Discussion on Memory Efficiency: Activations vs. Parameters. A critical distinction between optimization-based meta-learning algorithms lies in their memory overhead. Traditional methods like MAML and iMAML rely on second-order information, which necessitates storing the **activations** of the inner-loop trajectory or the computation graph for implicit differentiation. For deep networks with high-dimensional activation maps, activation storage is the primary bottleneck, often far exceeding the memory required for model parameters.

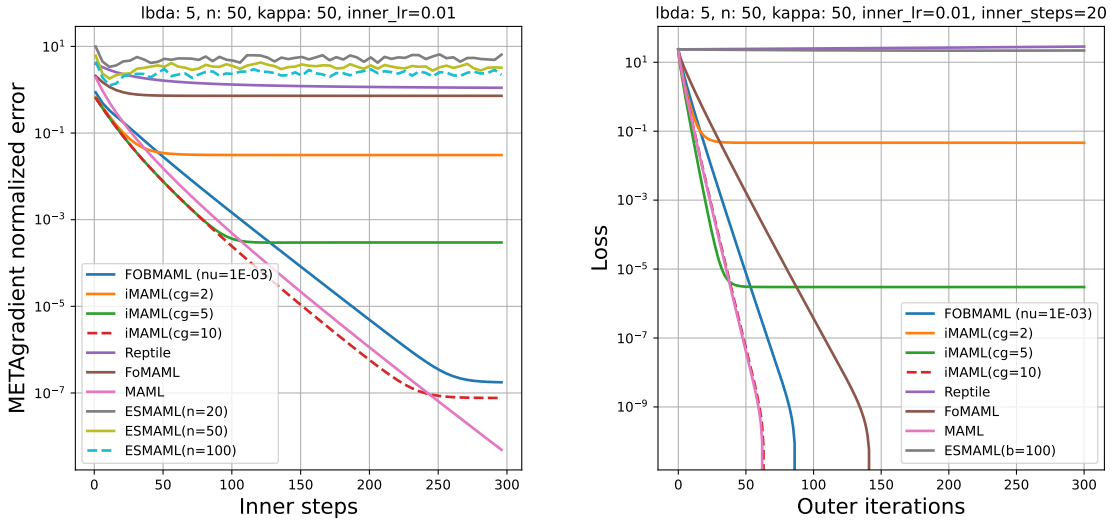

Figure 1: **(Above)** Quality of the meta-gradient estimates as a function of inner steps. **(Below)** Outer loss as a function of iterations for 20 inner steps.

In contrast, **FO-B-MAML** only requires the calculation of point estimates $\tilde{\phi}_{i,v}(\boldsymbol{\theta})$, offering two primary benefits:

- **Parameter-Only Storage**: FO-B-MAML relies on $\text{Mem}(\boldsymbol{\theta})$. While the symmetric approximation involves two subproblems ($\tilde{\phi}_{i,v}$ and $\tilde{\phi}_{i,-v}$), we avoid a $2\times$ parameter overhead by solving these subproblems **sequentially**. By computing the first estimate and then accumulating the second, the peak memory footprint remains identical to standard first-order training.

- **Scalability**: This allows FO-B-MAML to scale to activation-heavy architectures where MAML would typically trigger Out-of-Memory (OOM) errors, as evidenced by our ConvNet experiments.

5 Experiments

We evaluate Algorithm 1 across three distinct settings: a synthetic linear regression problem to measure meta-gradient quality, an MNIST-1D benchmark for classification performance, and a CNN scalability test to verify memory efficiency on deep architectures. We compare our results against iMAML, MAML, Reptile, and FO-MAML.

5.1 Meta-Gradient Quality and Convergence

We first consider a synthetic linear regression problem which allows for a closed-form meta-gradient calculation. Figure 1 illustrates that FO-B-MAML's meta-gradient approximation benefits continuously from increased inner steps (decreasing δ), a property not shared by FO-MAML or Reptile. FO-B-MAML is highly competitive with iMAML; notably, it outperforms iMAML when the latter is restricted to a small number of conjugate gradient steps ($cg \in \{2, 5\}$).

5.2 MNIST-1D Classification

To assess performance on non-linear tasks, we utilize the MNIST-1D dataset in a 3-way 5-shot meta-learning setup. As shown in Figure 2, FO-B-MAML tracks the performance of second-order MAML much more closely than other first-order methods. Specifically, FO-B-MAML reaches an accuracy above 0.85 within 100 iterations and maintains a final accuracy competitive with MAML (approx. 0.95) while using only first-order information. In contrast, FO-MAML remains below 0.55 accuracy throughout the training.

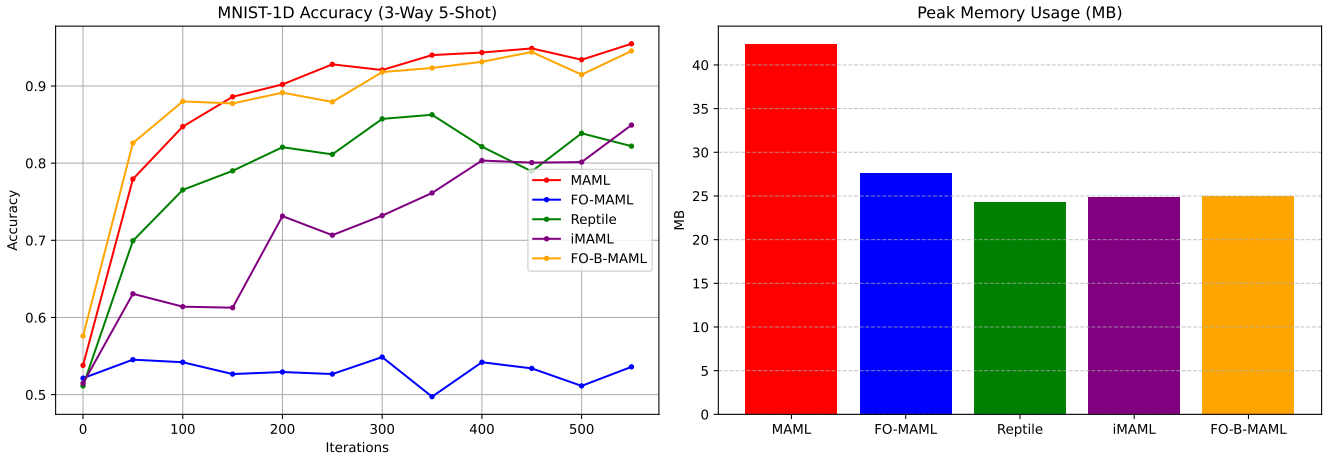

Figure 2: MNIST-1D 3-Way 5-Shot accuracy comparison across 600 iterations.

5.3 Memory Scalability on Deep CNNs

A core contribution of FO-B-MAML is its ability to scale to deep architectures where activation storage becomes a bottleneck. We measured peak memory usage on an activation-heavy ConvNet by varying the number of channels.

Activation vs. Parameter Bottleneck. Traditional methods like MAML and iMAML rely on second-order information, which necessitates storing the **activations** of the inner-loop trajectory. As shown in our memory benchmarks, MAML's memory usage grows aggressively with the model width. For the MNIST-1D task, MAML requires over 40 MB, while FO-B-MAML requires only 25 MB.

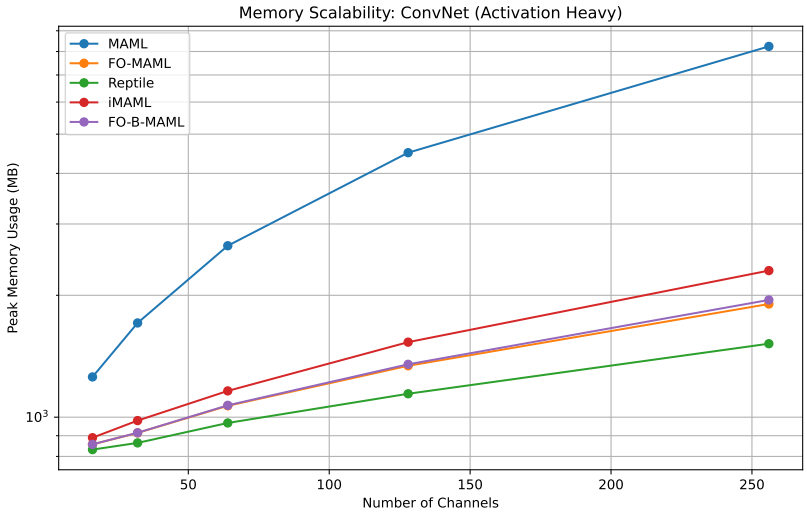

Figure 3: Peak memory usage comparison (MB) as a function of ConvNet channels.

In contrast, **FO-B-MAML** only requires the calculation of point estimates $\tilde{\phi}_{i,v}(\boldsymbol{\theta})$. As seen in Figure 3, FO-B-MAML maintains a memory footprint around 10^3 MB as channel counts increase to 250, whereas MAML's requirements scale steeply toward significantly higher values. This confirms the practical utility of our method for activation-heavy architectures.

6 General Discussion

Extension. In this work, we focused on the specific regularization form in (6) and (5). However, our approach is inherently general and can be extended to meta-learn other hyperparameters or shared components, such as common decoders. For example, consider a general inner problem:

$$\phi_i^\star\Big(\boldsymbol{h} := \{\boldsymbol{\theta}, \boldsymbol{\theta}_d, \lambda\}\Big) = \arg\min_{\boldsymbol{\phi}} \Big[\hat{f}_i(\boldsymbol{\theta}_d, \boldsymbol{\phi}) + \lambda \mathscr{R}eg(\boldsymbol{\phi}; \boldsymbol{\theta})\Big] , \tag{20}$$

where $\boldsymbol{\theta}_d$ denotes shared parameters. By adding a perturbation $\nu f(\boldsymbol{\phi})$, the meta-gradient satisfies $\nabla F_i(\boldsymbol{h}) = \frac{d\partial_{\boldsymbol{h}} g(\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{h}), \boldsymbol{h})}{d\nu}\Big|_{\nu=0}$. This allows for high-precision meta-gradients in complex architectures without second-order derivatives.

Practical Impact of Memory Efficiency. Our experimental results on deep CNNs highlight that FO-B-MAML bypasses the "activation bottleneck" inherent in second-order methods. While MAML requires memory that scales steeply with model width due to activation storage, FO-B-MAML relies only on parameter estimates.

By solving subproblems sequentially, we maintain a memory footprint nearly identical to standard first-order training, enabling meta-learning on architectures that would otherwise trigger Out-of-Memory (OOM) errors.

Limitations. One limitation is the requirement to solve the inner problem at least twice per task when using finite differences. While this is offset by the lack of second-order computation and the ability to parallelize, reducing this overhead remains an open question. One future direction involves using automatic differentiation directly on the scalar ν, though this may incur memory costs similar to MAML.

7 Conclusion

We introduced FO-B-MAML, a first-order meta-learning algorithm derived from a bi-level optimization perspective. Our framework includes a symmetric estimator that achieves an improved $\mathscr{O}(\delta^{2/3})$ bias rate. Theoretical analysis confirms that when paired with Stochastic Normalized Gradient Descent with Momentum (SNGDM), FO-B-MAML achieves superior convergence compared to traditional first-order methods.

Experimentally, we demonstrated that our method achieves high accuracy on MNIST-1D, tracking closely with second-order MAML, while maintaining the memory efficiency required for deep, activation-heavy CNNs. FO-B-MAML thus provides a robust, high-precision alternative for large-scale meta-learning where second-order information is computationally or memory-prohibitive.

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A Appendix

B Missing proofs

B.1 Proof of Proposition 1

The fact that $\nu \mapsto \phi^\star_{i,\nu}(\boldsymbol{\theta})$ is differentiable at $\nu = 0$ will be proven later.

We have that

$$\phi^\star_{i,\nu}(\boldsymbol{\theta}) \in Arg\min_{\phi} \nu f_i(\phi) + \hat{f}_i(\phi) + \frac{\lambda}{2}\|\phi - \boldsymbol{\theta}\|^2$$

This means

$$\nu \nabla f_i(\phi^\star_{i,\nu}(\boldsymbol{\theta})) + \nabla \hat{f}_i(\phi^\star_{i,\nu}(\boldsymbol{\theta})) + \lambda(\phi^\star_{i,\nu}(\boldsymbol{\theta}) - \boldsymbol{\theta}) = \mathbf{0} \tag{21}$$

Taking the derivative of the above equation with respect to ν gives

$$\nabla f_i(\phi^\star_{i,\nu}(\boldsymbol{\theta})) + \nu \nabla^2 f_i(\phi^\star_{i,\nu}(\boldsymbol{\theta}))\frac{d\phi^\star_{i,\nu}(\boldsymbol{\theta})}{d\nu} + \nabla^2 \hat{f}_i(\phi^\star_{i,\nu}(\boldsymbol{\theta}))\frac{d\phi^\star_{i,\nu}(\boldsymbol{\theta})}{d\nu} + \lambda\frac{d\phi^\star_{i,\nu}(\boldsymbol{\theta})}{d\nu} = \mathbf{0}$$

Which yields the following expression

$$\frac{d\phi^\star_{i,\nu}(\boldsymbol{\theta})}{d\nu} = -\Big(\nu \nabla^2 f_i(\phi^\star_{i,\nu}(\boldsymbol{\theta})) + \nabla^2 \hat{f}_i(\phi^\star_{i,\nu}(\boldsymbol{\theta})) + \lambda \boldsymbol{I}\Big)^{-1} \nabla f_i(\phi^\star_{i,\nu}(\boldsymbol{\theta}))$$

We set $\nu = 0$ and get:

$$\frac{d\phi^\star_{i,\nu}(\boldsymbol{\theta})}{d\nu}|_{\nu=0} = -\Big(\nabla^2 \hat{f}_i(\phi^\star_i(\boldsymbol{\theta})) + \lambda \boldsymbol{I}\Big)^{-1} \nabla f_i(\phi^\star_i(\boldsymbol{\theta})) = -\frac{1}{\lambda}\nabla F_i(\boldsymbol{\theta})$$

B.2 Proof of the Bias of the Forward Approximation

Using Equation (21), we have:

$$\phi^\star_{i,\nu}(\boldsymbol{\theta}) = \boldsymbol{\theta} - \frac{1}{\lambda}\Big(\nu \nabla f_i(\phi^\star_{i,\nu}(\boldsymbol{\theta})) + \nabla \hat{f}_i(\phi^\star_{i,\nu}(\boldsymbol{\theta}))\Big) \tag{22}$$

Thus

$$\lambda\frac{\phi^\star_{i,0}(\boldsymbol{\theta}) - \phi^\star_{i,\nu}(\boldsymbol{\theta})}{\nu} = \nabla f_i(\phi^\star_{i,\nu}(\boldsymbol{\theta})) + \frac{\nabla \hat{f}_i(\phi^\star_{i,\nu}(\boldsymbol{\theta})) - \nabla \hat{f}_i(\phi^\star_{i,0}(\boldsymbol{\theta}))}{\nu}\,. \tag{23}$$

The form of the meta-gradient in Equation 7 implies that:

$$\nabla F_i(\boldsymbol{\theta}) = \nabla f_i(\phi^\star_{i,0}(\boldsymbol{\theta})) - \frac{1}{\lambda}\nabla^2 \hat{f}_i(\phi^\star_{i,0}(\boldsymbol{\theta}))\nabla F_i(\boldsymbol{\theta}). \tag{24}$$

(24) – (23) gives :

$$\nabla F_i(\boldsymbol{\theta}) - \boldsymbol{g}_{i,\nu}^{\text{For}} = \underbrace{\nabla f_i(\boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})) - \nabla f_i(\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}))}_{\textbf{(I)}} + \underbrace{\frac{\nabla \hat{f}_i(\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta})) - \nabla \hat{f}_i(\boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta}))}{\nu} - \frac{1}{\lambda}\nabla^2 \hat{f}_i(\boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta}))\nabla F_i(\boldsymbol{\theta})}_{\textbf{(II)}}.$$

Let's define $\mathscr{B}_\nu = \|\nabla F_i(\boldsymbol{\theta}) - \boldsymbol{g}_{i,\nu}^{\text{For}}\|$ the bias of the forward approximation. The norm of the first term **(I)** can be easily bounded using the smoothness of f_i by:

$$\|\textbf{(I)}\| \leq L_1 \|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\|.$$

The second term **(II)** can be simplified using Cauchy's theorem which guarantees the existence of $\boldsymbol{\phi}$ such that

$$\frac{\nabla \hat{f}_i(\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta})) - \nabla \hat{f}_i(\boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta}))}{\nu} = \nabla^2 \hat{f}_i(\boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta}))\Big[\frac{\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})}{\nu}\Big] + \frac{1}{2\nu}\nabla^3 \hat{f}_i(\boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta}))\Big[\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\Big]^2,$$

Thus, using Assumptions 1 and 3, we get:

$$\|\textbf{(II)}\| \leq \frac{\hat{L}_1}{\lambda}\mathscr{B}_\nu + \frac{\hat{L}_2}{2\nu}\|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\|^2.$$

Overall,

$$\mathscr{B}_\nu \leq L_1 \|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\| + \frac{\hat{L}_1}{\lambda}\mathscr{B}_\nu + \frac{\hat{L}_2}{2\nu}\|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\|^2,$$

Which implies :

$$(1 - \frac{\hat{L}_1}{\lambda})\mathscr{B}_\nu \leq \|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\|\Big(L_1 + \frac{\hat{L}_2}{2\nu}\|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\|\Big). \tag{25}$$

All that is left is to bound the term $\|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\|$.

If we go back to (22), then we can write:

$$\begin{aligned}\|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\| &= \|\frac{1}{\lambda}\Big(\nabla \hat{f}_i(\boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})) - \nabla \hat{f}_i(\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta})) + \nu\nabla f_i(\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}))\Big)\| \\ &\leq \frac{\hat{L}_1}{\lambda}\|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\| + \frac{\nu L_0}{\lambda}.\end{aligned}$$

Thus:

$$(1 - \frac{\hat{L}_1}{\lambda})\|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\| \leq \frac{\nu L_0}{\lambda}.$$

For simplicity we assume, $\lambda \geq 2\hat{L}_1$ which gives:

$$\|\boldsymbol{\phi}_{i,\nu}^\star(\boldsymbol{\theta}) - \boldsymbol{\phi}_{i,0}^\star(\boldsymbol{\theta})\| \leq \frac{2\nu L_0}{\lambda}. \tag{26}$$

Plugging the result in Eq (26) and using $\lambda \geq 2\hat{L}_1$ gives:

$$\mathscr{B}_\nu \leq \frac{L_0}{\lambda}\Big(L_1 + \frac{L_0\hat{L}_2}{\lambda}\Big)\nu.$$

The overall bias resulting from using approximations of $\phi^\star_{i,\nu}(\boldsymbol{\theta})$ instead of their exact values is bounded by:

$$\frac{L_0}{\lambda}\Big(L_1+\frac{L_0\hat{L}_2}{\lambda}\Big)\nu+\frac{2\delta}{\nu}, \tag{27}$$

which is minimized for $\nu=\sqrt{\dfrac{2\lambda^2\delta}{L_0(L_1\lambda+L_0\hat{L}_2)}}$, using this value of ν in the overall bias (27), gives the result of Corrolary 1.

B.3 Proof of Improved Bias Rate for the Symmetric Estimator

To achieve the results for the symmetric estimator $\boldsymbol{g}^{\text{Sym}}_{i,\nu}$, we adopt strengthened regularity conditions:

- **Assumption 1':** $\hat{f}_i$ is C^4 with derivatives $\nabla^3\hat{f}_i$ and $\nabla^4\hat{f}_i$ bounded by $\hat{L}_2$ and $\hat{L}_3$ respectively.
- **Assumption 2':** f_i is C^3 with derivatives $\nabla^2 f_i$ and $\nabla^3 f_i$ bounded by L_1 and L_2 respectively.
- **Assumption 3:** The inner objective is μ-strongly convex.

B.3.1 Bounding the Solution Path Derivatives

Let $\boldsymbol{z}(\nu)=\phi^\star_{i,\nu}(\boldsymbol{\theta})$ be the solution to the perturbed inner problem. Differentiating the stationary condition (21) with respect to ν:

$$\boldsymbol{A}(\boldsymbol{z},\nu)\boldsymbol{z}'(\nu)+\nabla f_i(\boldsymbol{z})=\boldsymbol{0}, \tag{28}$$

where $\boldsymbol{A}(\boldsymbol{z},\nu)=\nu\nabla^2 f_i(\boldsymbol{z})+\nabla^2\hat{f}_i(\boldsymbol{z})+\lambda\boldsymbol{I}$. By Assumption 3, $\|\boldsymbol{A}^{-1}\|\le\frac{1}{\mu}$. Thus, $\|\boldsymbol{z}'(\nu)\|\le\frac{L_0}{\mu}$. Differentiating (28) again with respect to ν:

$$\boldsymbol{A}\boldsymbol{z}''+\Big[\nabla^2 f_i(\boldsymbol{z})+(\nu\nabla^3 f_i(\boldsymbol{z})+\nabla^3\hat{f}_i(\boldsymbol{z}))\boldsymbol{z}'\Big]\boldsymbol{z}'+\nabla^2 f_i(\boldsymbol{z})\boldsymbol{z}'=\boldsymbol{0}. \tag{29}$$

Rearranging for $\boldsymbol{z}''$ and taking the norm:

$$\|\boldsymbol{z}''\|\le\frac{1}{\mu}\Big[2L_1\frac{L_0}{\mu}+(\nu L_2+\hat{L}_2)\Big(\frac{L_0}{\mu}\Big)^2\Big]=\mathscr{C}_2. \tag{30}$$

Under Assumptions 1' and 2', differentiating a third time ensures the existence of a constant $\mathscr{C}_3$ such that $\sup_{\xi\in[-\nu,\nu]}\|\boldsymbol{z}'''(\xi)\|\le\mathscr{C}_3$.

B.3.2 Symmetric Estimator Bias Bound

Using the symmetric difference quotient and Taylor's theorem:

$$\boldsymbol{z}(\nu)=\boldsymbol{z}(0)+\nu\boldsymbol{z}'(0)+\frac{\nu^2}{2}\boldsymbol{z}''(0)+\frac{\nu^3}{6}\boldsymbol{z}'''(\xi_1),\quad \xi_1\in[0,\nu]$$
$$\boldsymbol{z}(-\nu)=\boldsymbol{z}(0)-\nu\boldsymbol{z}'(0)+\frac{\nu^2}{2}\boldsymbol{z}''(0)-\frac{\nu^3}{6}\boldsymbol{z}'''(\xi_2),\quad \xi_2\in[-\nu,0].$$

The symmetric estimator is defined as $\boldsymbol{g}^{\text{Sym}}_{i,\nu}=-\lambda\frac{\boldsymbol{z}(\nu)-\boldsymbol{z}(-\nu)}{2\nu}$. Substituting the expansions:

$$\boldsymbol{g}^{\text{Sym}}_{i,\nu}=-\lambda\boldsymbol{z}'(0)-\frac{\lambda\nu^2}{12}\Big(\boldsymbol{z}'''(\xi_1)+\boldsymbol{z}'''(\xi_2)\Big). \tag{31}$$

Since $\nabla F_i(\boldsymbol{\theta})=-\lambda\boldsymbol{z}'(0)$ (21), the approximation bias is bounded by $\frac{\lambda\nu^2\mathscr{C}_3}{6}$. Including the solver precision δ, the total error $E(\nu)$ is:

$$E(\nu)\le\frac{\lambda\nu^2\mathscr{C}_3}{6}+\frac{\lambda\delta}{\nu}. \tag{32}$$

Minimizing for ν yields $\nu^\star=\sqrt[3]{3\delta/\mathscr{C}_3}$, which leads to the final bias bound:

$$\text{Bias}_{\text{Sym}}\le\frac{\lambda}{2}(3\delta)^{2/3}\mathscr{C}_3^{1/3}=\mathscr{O}(\lambda\delta^{2/3}). \tag{33}$$

This confirms that the symmetric approximation provides a superior bias rate compared to the forward approximation.

B.4 Proof of Proposition 3

Let $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta$. Using Eq (24), we have:

$$\nabla F_i(\boldsymbol{\theta}) = \nabla f_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta})) - \frac{1}{\lambda}\nabla^2 \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}))\nabla F_i(\boldsymbol{\theta})$$
$$\nabla F_i(\boldsymbol{\theta}') = \nabla f_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')) - \frac{1}{\lambda}\nabla^2 \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}'))\nabla F_i(\boldsymbol{\theta}')$$

$$\begin{aligned}
\nabla F_i(\boldsymbol{\theta}) - \nabla F_i(\boldsymbol{\theta}') &= \nabla f_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta})) - \nabla f_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')) \\
&\quad + \frac{1}{\lambda}\Big(\nabla^2 \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}'))\nabla F_i(\boldsymbol{\theta}') - \nabla^2 \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}))\nabla F_i(\boldsymbol{\theta})\Big) \\
&= \nabla f_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta})) - \nabla f_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')) \\
&\quad + \frac{1}{\lambda}\Big(\nabla^2 \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')) - \nabla^2 \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}))\Big)\nabla F_i(\boldsymbol{\theta}') \\
&\quad - \frac{1}{\lambda}\nabla^2 \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}))\Big(\nabla F_i(\boldsymbol{\theta}) - \nabla F_i(\boldsymbol{\theta}')\Big)
\end{aligned}$$

Thus:

$$\begin{aligned}
\|\nabla F_i(\boldsymbol{\theta}) - \nabla F_i(\boldsymbol{\theta}')\| &\leq L_1 \|\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}) - \boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')\| + \frac{\hat{L}_2}{\lambda}\|\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}) - \boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')\|\|\nabla F_i(\boldsymbol{\theta}')\| \\
&\quad + \frac{\hat{L}_1}{\lambda}\|\nabla F_i(\boldsymbol{\theta}) - \nabla F_i(\boldsymbol{\theta}')\|
\end{aligned}$$

which implies:

$$(1 - \frac{\hat{L}_1}{\lambda})\|\nabla F_i(\boldsymbol{\theta}) - \nabla F_i(\boldsymbol{\theta}')\| \leq \Big(L_1 + \frac{\hat{L}_2}{\lambda}\|\nabla F_i(\boldsymbol{\theta}')\|\Big)\|\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}) - \boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')\|$$

Let's define, $\mathscr{L}(\boldsymbol{\theta}) := L_1/4 + \frac{\hat{L}_2}{4\lambda}\|\nabla F_i(\boldsymbol{\theta})\|$. Exchanging $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$, and assuming $\lambda \geq 2\hat{L}_1$, we get:

$$\|\nabla F_i(\boldsymbol{\theta}) - \nabla F_i(\boldsymbol{\theta}')\| \leq \min(\mathscr{L}(\boldsymbol{\theta}), \mathscr{L}(\boldsymbol{\theta}'))\|\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}) - \boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')\|/2 \tag{34}$$

To finish the proof, we need to bound the quantity: $\|\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}) - \boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')\|$.

We have

$$\begin{aligned}
\|\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}) - \boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')\| &= \|\boldsymbol{\theta} - \frac{1}{\lambda}\nabla \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta})) - \Big(\boldsymbol{\theta}' - \frac{1}{\lambda}\nabla \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}'))\Big)\| \\
&= \|\boldsymbol{\theta} - \boldsymbol{\theta}' - \frac{1}{\lambda}\Big(\nabla \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta})) - \nabla \hat{f}_i(\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}'))\Big)\| \\
&\leq \|\boldsymbol{\theta} - \boldsymbol{\theta}'\| + \frac{\hat{L}_1}{\lambda}\|\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}) - \boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')\|
\end{aligned}$$

Again, choosing $\lambda \geq 2\hat{L}_1$, we get:

$$\|\boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}) - \boldsymbol{\phi}^\star_{i,0}(\boldsymbol{\theta}')\| \leq 2\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|.$$

Plugging the last inequality in Eq (34), we get:

$$\|\nabla F_i(\boldsymbol{\theta}) - \nabla F_i(\boldsymbol{\theta}')\| \leq \min(\mathscr{L}(\boldsymbol{\theta}), \mathscr{L}(\boldsymbol{\theta}'))\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|,$$

which finishes the proof.

B.5 Convergence of NormalizedGD and ClippedGD in the deterministic setting

Consider a differentiable function F satisfying the following property for all $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta$:

$$\|\nabla F(\boldsymbol{\theta}) - \nabla F(\boldsymbol{\theta}')\| \leq \min(\mathscr{L}(\boldsymbol{\theta}), \mathscr{L}(\boldsymbol{\theta}'))\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|,$$

and for any function $\mathscr{L}$.

Then we have:

$$F(\boldsymbol{\theta}') = F(\boldsymbol{\theta}) + \nabla F(\boldsymbol{\theta})^\top(\boldsymbol{\theta}' - \boldsymbol{\theta}) + \int_0^1 \left[\nabla F(\boldsymbol{\theta} + t(\boldsymbol{\theta}' - \boldsymbol{\theta})) - \nabla F(\boldsymbol{\theta})\right]^\top (\boldsymbol{\theta}' - \boldsymbol{\theta}) dt$$

Thus

$$\begin{aligned}
|F(\boldsymbol{\theta}') - F(\boldsymbol{\theta}) - \nabla F(\boldsymbol{\theta})^\top(\boldsymbol{\theta}' - \boldsymbol{\theta})| &\leq |\int_0^1 \left[\nabla F(\boldsymbol{\theta} + t(\boldsymbol{\theta}' - \boldsymbol{\theta})) - \nabla F(\boldsymbol{\theta})\right]^\top (\boldsymbol{\theta}' - \boldsymbol{\theta}) dt| \\
&\leq \int_0^1 \mathscr{L}(\boldsymbol{\theta}) t \|\boldsymbol{\theta}' - \boldsymbol{\theta}\|^2 dt \\
&= \frac{\mathscr{L}(\boldsymbol{\theta})}{2}\|\boldsymbol{\theta}' - \boldsymbol{\theta}\|^2.
\end{aligned}$$

In particular,

$$F(\boldsymbol{\theta}') \leq F(\boldsymbol{\theta}) + \nabla F(\boldsymbol{\theta})^\top(\boldsymbol{\theta}' - \boldsymbol{\theta}) + \frac{\mathscr{L}(\boldsymbol{\theta})}{2}\|\boldsymbol{\theta}' - \boldsymbol{\theta}\|^2$$

Let's consider a general GD update: $\boldsymbol{\theta}' = \boldsymbol{\theta} - \eta \nabla F(\boldsymbol{\theta})$, where η might depend on $\boldsymbol{\theta}$. For this update, we have

$$\begin{aligned}
F(\boldsymbol{\theta}') &\leq F(\boldsymbol{\theta}) - \eta\|\nabla F(\boldsymbol{\theta})\|^2 + \frac{\mathscr{L}(\boldsymbol{\theta})\eta^2}{2}\|\nabla F(\boldsymbol{\theta})\|^2 \\
&= F(\boldsymbol{\theta}) - \eta(1 - \frac{\mathscr{L}(\boldsymbol{\theta})\eta}{2})\|\nabla F(\boldsymbol{\theta})\|^2
\end{aligned}$$

It is easy to see that $\eta = \frac{1}{\mathscr{L}(\boldsymbol{\theta})}$ minimizes the right-hand side of the above inequality, which leads to:

$$F(\boldsymbol{\theta}) - F(\boldsymbol{\theta}') \geq \frac{\|\nabla F(\boldsymbol{\theta})\|^2}{2\mathscr{L}(\boldsymbol{\theta})}$$

One important observation here is that the optimal step size is the inverse of the generalized smoothness, thus if the smoothness depends on the norm of the gradient or other quantities, the optimal step size depends on them too.

Let's now discuss the special case where $\mathscr{L}(\boldsymbol{\theta}) = \mathscr{L}_0 + \mathscr{L}_1\|\nabla F(\boldsymbol{\theta}\|$.

In this case, we have:

$$F(\boldsymbol{\theta}^t) - F(\boldsymbol{\theta}^{t+1}) \geq \frac{\|\nabla F(\boldsymbol{\theta}^t)\|^2}{2(\mathscr{L}_0 + \mathscr{L}_1\|\nabla F(\boldsymbol{\theta}^t\|)}.$$

For a given precision ε, the goal is to bound the number of steps t necessary to reach $\|\nabla F(\boldsymbol{\theta}^t\| \leq \varepsilon$.

Before reaching the goal above, we naturally have two regimes, a first one for which $\|\nabla F(\boldsymbol{\theta}^t\| \geq \mathscr{L}_0/\mathscr{L}_1$ and another one where $\varepsilon \leq \|\nabla F(\boldsymbol{\theta}^t\| \leq \mathscr{L}_0/\mathscr{L}_1$.

If we are in the first regime, we have

$$F(\boldsymbol{\theta}^t) - F(\boldsymbol{\theta}^{t+1}) \geq \frac{(\mathscr{L}_0/\mathscr{L}_1)^2}{4\mathscr{L}_0} = \frac{\mathscr{L}_0}{4\mathscr{L}_1^2}.$$

Whereas if we were in the second regime, then we would have:

$$F(\boldsymbol{\theta}^t) - F(\boldsymbol{\theta}^{t+1}) \geq \frac{\varepsilon^2}{4\mathscr{L}_0}.$$

All in all, as long as $\|\nabla F(\boldsymbol{\theta}^t\| \geq \varepsilon$, we have

$$F(\boldsymbol{\theta}^t) - F(\boldsymbol{\theta}^{t+1}) \geq \min(\frac{\varepsilon^2}{4\mathscr{L}_0}, \frac{\mathscr{L}_0}{4\mathscr{L}_1^2})..$$

Let K be the number of steps necessary to reach the first index t such that $\|\nabla F(\boldsymbol{\theta}^t\| \leq \varepsilon$. Assuming that the function F is lower bounded and denoting $\Delta = F(\boldsymbol{\theta}^0) - \inf F$, then

$$\Delta \geq K \min(\frac{\varepsilon^2}{4\mathscr{L}_0}, \frac{\mathscr{L}_0}{4\mathscr{L}_1^2}).$$

Thus $K \leq \frac{\Delta}{\min(\frac{\varepsilon^2}{4\mathscr{L}_0}, \frac{\mathscr{L}_0}{4\mathscr{L}_1^2})} \leq \frac{4\mathscr{L}_0\Delta}{\varepsilon^2} + \frac{4\mathscr{L}_1^2\Delta}{\mathscr{L}_0}$.

B.6 Convergence of ClippedGD in the stochastic setting

To treat the stochastic setting, we can use Theorem 3.2 from (Zhang et al., 2020) with $\beta = 0$ which guarantees a convergence in at most $\mathscr{O}\left(\Delta\tilde{\sigma}^2 \max(\frac{\mathscr{L}_0}{\varepsilon^4}, \frac{\mathscr{L}_1^4}{\mathscr{L}_0^3})\right)$ for $\tilde{\sigma} \neq 0$.

C Experimental Details

C.1 Quadratic objective experiment

We can equivalently write a quadratic objective that represents the task loss as:

$$f_i(\boldsymbol{\theta}) = \hat{f}_i(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^\top \boldsymbol{A}_i\boldsymbol{\theta} + \boldsymbol{\theta}^\top \boldsymbol{b}_i,$$

In this case, it is easy to show that:

$$\phi^\star_{i,\nu}(\boldsymbol{\theta}) = ((1+\nu)\boldsymbol{A}_i + \lambda\boldsymbol{I})^{-1}(\lambda\boldsymbol{\theta} - (1+\nu)\boldsymbol{b}_i)$$

and

$$\phi^\star_i(\boldsymbol{\theta}) = \phi^\star_{i,0}(\boldsymbol{\theta}) = (\boldsymbol{A}_i + \lambda\boldsymbol{I})^{-1}(\lambda\boldsymbol{\theta} - \boldsymbol{b}_i)$$

The meta-gradient of task i has the expression:

$$\begin{aligned}\nabla F_i(\boldsymbol{\theta}) &= \lambda(\boldsymbol{A}_i + \lambda\boldsymbol{I})^{-1}\nabla_\phi \mathscr{L}_i(\phi^\star_i(\boldsymbol{\theta})) \\ &= \lambda(\boldsymbol{A}_i + \lambda\boldsymbol{I})^{-1}\left(\boldsymbol{A}_i(\boldsymbol{A}_i + \lambda\boldsymbol{I})^{-1}(\lambda\boldsymbol{\theta} - \boldsymbol{b}_i) + \boldsymbol{b}_i\right)\end{aligned}$$

And it is not difficult to verify that indeed $\nabla F_i(\boldsymbol{\theta}) = -\lambda \left.\frac{d\phi^\star_i(\boldsymbol{\theta})}{d\nu}\right|_{\nu=0}$.

Because we have the exact expression of the meta-gradient, we can compute it exactly and compare the relative precision of different approximation methods. This is what we did in Figure 1.

C.2 MNIST-1D Classification Setup

The MNIST-1D dataset is a 1D version of MNIST consisting of 12,000 examples of 40-dimensional vectors across 10 classes. The meta-learning task is constructed as follows:

- **Task Structure**: We use a 3-way 5-shot configuration with 10 query samples per class.
- **Model Architecture**: A Functional Conv1D model is used, consisting of three convolutional layers (16 filters, kernel size 3, padding 1) each followed by BatchNorm, ReLU, and MaxPool1D. The final layer is a linear map to the 3-way output.
- **Hyperparameters**: We chose the best hyperparameters on a logarithmic grid from 1 to $1e-5$ for both the inner learning rate and the meta-learning rate with the Adam optimizer. The inner loop consists of 5 steps.
- **Algorithm Constants**: For FO-B-MAML and iMAML, we set $\lambda = 2.0$. FO-B-MAML uses a perturbation scale $\nu = 0.1$, and iMAML uses 5 Conjugate Gradient steps.

C.3 CNN Memory Scalability Benchmarking

The memory scalability experiments were conducted using an activation-heavy ConvNet architecture. The peak memory usage was measured using `torch.cuda.max_memory_allocated` for GPU execution.

- **Procedure**: We varied the "Number of Channels" from 16 to 256.
- **Sequential Execution**: To maintain the memory footprint of $\text{Mem}(\boldsymbol{\theta})$, FO-B-MAML solves the perturbed subproblems $\tilde{\phi}_{i,\nu}$ and $\tilde{\phi}_{i,-\nu}$ sequentially, discarding intermediate activations and storing only the resulting parameter estimates.
- **Baseline Comparison**: MAML memory usage includes the storage of the full computation graph for all inner steps to allow for backpropagation, leading to the observed $O(\text{Inner Steps} \times \text{Activations})$ scaling.