How Powerful is Implicit Denoising in Graph Neural Networks

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Abstract

Graph Neural Networks (GNNs), which aggregate features from neighbors, are widely used for processing graph-structured data due to their powerful representation learning capabilities. It is generally believed that GNNs can implicitly remove feature noises. However, existing works have not rigorously analyzed the implicit denoising effect in graph neural networks. In this work, we conduct a comprehensive theoretical study and analyze when and why implicit denoising happens in GNNs. Our theoretical analysis suggests that the implicit denoising largely depends on the connectivity and size of the graph, as well as the GNN architectures. Moreover, extensive empirical evaluations verify our theoretical analyses and the effectiveness of GNNs in eliminating noise in the feature matrix compared with Multi-Layer Perceptron (MLP). Moreover, motivated by adversarial machine learning in improving the robustness of neural networks and the correlation of node features and graph structure in GSD, we propose the adversarial graph signal denoising (AGSD) problem. By solving such a problem, we derive a robust graph convolution, where the smoothness of the node representations and the implicit denoising effect can be enhanced.

1 Introduction

Graph Neural Networks (GNNs) [13, 24, 8] have been widely used in graph learning and achieved remarkable performance on graph-based tasks, such as traffic prediction [7], drug discovery [5], and recommendation system [31]. A general principle behind GNNs is to perform a message passing operation that aggregates node features over neighborhoods, such that the smoothness of learned node representations on the graph is enhanced.

By promoting graph smoothness, the message passing and aggregation mechanism naturally leads to GNN models whose predictions are not only dependent on the feature of one specific node, but also the features from a set of neighboring nodes. Therefore, this mechanism can, to a certain extent, protect GNN models from noises: real-world graphs are usually noisy, e.g., Gaussian white noise exists on node features [33], however, the influence of feature noises on the model's output could be counteracted by the feature aggregation operation in GNNs. We term this effect as *implicit denoising*.

The inherent property of implicit denoising in message-passing mechanism offers new opportunities for designing GNNs with better generalizability and robustness [28, 14, 34, 29]. While many works have been conducted in the empirical exploration of GNNs, relatively fewer advances have been achieved in theoretically studying this intriguing property. Early GNN models, such as the vanilla GCN [13], GAT [24] and GraphSAGE [8], propose different designs of aggregation functions, but

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the denoising effect is not discussed in these works. Some recent attempts are made to mathematically establish the connection between a variety of GNNs and the graph signal denoising problem (GSD) [16]. By regarding the feature aggregation process in GNNs as solving a GSD problem, more advanced GNNs are proposed, such as GLP [14], S^2GC [34], and IRLS [29]. Despite these prior attempts, little efforts have been made to rigorously study the denoising effect of message passing and aggregation operation. This urges us to think about a fundamental but not clearly answered question:

Why and when implicit denoising happens in GNNs?

In this work, we conduct a comprehensive study to answer this question from both the empirical and theoretical perspectives. In order to facilitate our theoretical analysis, we first start from graph signal denoising problem and obtain a simple yet general framework based on Neumann series expansions, which we called Neumann Graph Convolution (NGC) operator. Note that NGC is not a new GNN architecture. There are also some similar GNN architectures, such as GLP [14], S^2GC [34], and GaussianMRF [12]. We just perform the theoretical analysis based on the Neumann Graph Convolution operator. Specifically, to study the denoising effect of graph convolutions, we introduce an insightful measurement on the convolution operator, termed *high-order graph connectivity factor*, which reveals how uniform the nodes are distributed in the neighborhood and reflects the strength of information diluted on a single neighboring node during the feature aggregation step. In our theoretical analysis, we observe that the denoising effect of graph convolution is largely determined by the connectivity and size of the graph structure. Besides, the GNN architecture also has an influence on the denoising effect. Extensive experimental results on homophily, heterogeneous, and large-scale graphs verify our theoretical analyses. Specifically, the denoising performance of GNNs on various types of graphs outperforms that of Multi-Layer Perceptron (MLP). In order to further improve the node representation's smoothness and denoising performance, we propose the adversarial graph signal denoising problem (AGSD). Inspired by the traditional adversarial machine learning [18] in increasing robustness, we derive a robust graph convolution model due to the correlation of node feature and graph structure, where the denoising performance can be enhanced.

2 Graph Signal Denoising and Implicit Denoising in GNNs

Notations. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represent a undirected graph, where $\mathcal{V}$ is the set of vertices $\{v_1, \cdots, v_n\}$ with $|\mathcal{V}| = n$ and $\mathcal{E}$ is the set of edges. The adjacency matrix is defined as $\mathbf{A} \in \{0,1\}^{n\times n}$, and $\mathbf{A}_{i,j} = 1$ if and only if $(v_i, v_j) \in \mathcal{E}$. Let $\mathcal{N}_i = \{v_j | \mathbf{A}_{i,j} = 1\}$ denote the neighborhood of node v_i and $\mathbf{D}$ denote the diagonal degree matrix, where $\mathbf{D}_{i,i} = \sum_{j=1}^{n} \mathbf{A}_{i,j}$. The feature matrix is denoted as $\mathbf{X} \in \mathbb{R}^{n\times d}$ where each node v_i is associated with a d-dimensional feature vector $\mathbf{X}_i$. $\mathbf{Y} \in \{0,1\}^{n\times c}$ denotes the matrix, where $\mathbf{Y}_i \in \{0,1\}^c$ is a one-hot vector and $\sum_{j=1}^{c} \mathbf{Y}_{i,j} = 1$ for any $v_i \in V$.

Graph Signal Denoising. Graph Signal Denoising (GSD) [4] is a fundamental problem in signal processing. The goal of graph signal denoising is to recover the the clean graph signal from the noisy observations. In this study, we view the vertex feature as the graph signal and thus we can define the graph denoising problem from the signal processing perspective [22].

Problem 1. *Given an observed noisy graph feature matrix $\mathbf{X} = \mathbf{X}^* + \boldsymbol{\eta}$, where $\boldsymbol{\eta} \in \mathbb{R}^{n\times d}$ is the noise matrix and we wish to obtain a filtered feature matrix $\mathbf{F}$ that can recover the unobserved clean feature matrix $\mathbf{X}^* \in \mathbb{R}^{n\times d}$ by solving the following optimization problem:*

$$q(\mathbf{F}) = \min_{\mathbf{F}} \|\mathbf{F}-\mathbf{X}\|_F^2 + \lambda \operatorname{tr}\left(\mathbf{F}^\top \widetilde{\mathbf{L}} \mathbf{F}\right) = \min_{\mathbf{F}} \|\mathbf{F}-\mathbf{X}\|_F^2 + \lambda \sum_{(i,j)\in\mathcal{E}} \mathbf{A}_{ij} \left\| \frac{\mathbf{F}_i}{\sqrt{d_i}+1} - \frac{\mathbf{F}_j}{\sqrt{d_j}+1} \right\|_2^2, \tag{1}$$

where $\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-\frac{1}{2}} \widetilde{\mathbf{A}} \widetilde{\mathbf{D}}^{-\frac{1}{2}}$ or $\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-1} \widetilde{\mathbf{A}}$ denotes the graph Laplacian with respect to the adjacency matrix with self-loop $\widetilde{\mathbf{A}} = \mathbf{A} + \mathbf{I}$.

Note that the first term in Eq. (1) enforces the solution to be close to the observed data $\mathbf{X}$ while the second term encourages the smoothness of the filtered feature matrix $\mathbf{F}$ over the graph, i.e., nearby vertices should have similar vertex features.

3 Related Work

Implicit Denoising in GNNs. Existing graph denoising works are mainly based on the graph smoothing technique [4, 33]. It is well known that GNNs can increase the smoothness of node features through aggregating information from neighbors, thus the influence from noisy features can be counteracted in GNN's output. Some recent GNN models are derived from the perspective of signal denoising, such as S^2GC [34], GLP [14], and IRLS [29]. Moreover, Ma et al. [16] builds the connection between signal denoising and existing popular GNNs by regarding the message passing scheme as a process of solving GSD problem. The relationship between GSD and GCN can be briefly illustrated as follows [16]. The gradient of $q(\mathbf{F})$ with respect to $\mathbf{F}$ at $\mathbf{X}$ is $\frac{\partial q(\mathbf{F})}{\partial \mathbf{F}}\Big|_{\mathbf{F}=\mathbf{X}} = 2\lambda\widetilde{\mathbf{L}}\mathbf{X}$, thus one-step gradient descent for GSD is $\mathbf{F} \leftarrow \mathbf{X} - b\frac{\partial q(\mathbf{F})}{\partial \mathbf{F}}\Big|_{\mathbf{F}=\mathbf{X}} = (1-2b\lambda)\mathbf{X} + 2b\lambda\widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}\mathbf{X}$. When setting $b = \frac{1}{2\lambda}$, we have $\mathbf{F} \leftarrow \widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}\mathbf{X}$, which is exactly the aggregation scheme of GCN. This suggests a possibility for us to understand the behavior of GNNs through the lens of signal denoising. To our best knowledge, we are the first to offer the theoretical analysis to understand the denoising effect of GNNs. Besides, there is a recent work [32] to conduct the empirical study of the denoising effect in GNNs. In this work, we perform an extensive analysis to understand the denoising effect of GNNs from both theoretical and experimental perspectives.

4 A Simple Unifying Framework: Neumann Graph Convolution

A General Framework. In this section, we discuss a simple yet general framework for solving graph signal denoising problem, namely Neumann Graph Convolution (NGC). We focus on the theoretical analysis of the denoising effect in GNNs in this work. NGC can facilitate our theoretical analysis. By taking the derivative $\nabla q(\mathbf{F}) = 2\widetilde{\mathbf{L}}\mathbf{F} + 2(\mathbf{F}-\mathbf{X})$ to zero, we obtain the solution of GSD optimization problem as follows:

$$\mathbf{F} = (\mathbf{I} + \lambda\widetilde{\mathbf{L}})^{-1}\mathbf{X}. \tag{2}$$

To avoid the expensive computation of the inverse matrix, we can use Neumann series [23] expansion to approximate Eq. (2) up to up to S-th order:

$$\left(\mathbf{I} + \lambda\widetilde{\mathbf{L}}\right)^{-1} = \frac{1}{\lambda+1}\left(\mathbf{I} - \frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)^{-1} \approx \frac{1}{\lambda+1}\sum_{s=0}^{S}\left(\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)^{s}, \tag{3}$$

where $\widetilde{\boldsymbol{\mathcal{A}}}$ can take the form of $\widetilde{\boldsymbol{\mathcal{A}}} = \widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}$ or $\widetilde{\boldsymbol{\mathcal{A}}} = \widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}$, and the proof can be found in Appendix B. Based on the Neumann series expansion of the solution of GSD, we introduce a general graph convolution model – Neumann Graph Convolution defined as the following expansion:

$$\mathbf{H} = \widetilde{\boldsymbol{\mathcal{A}}}_S\mathbf{X}\mathbf{W} = \frac{1}{\lambda+1}\sum_{s=0}^{S}\left(\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)^{s}\mathbf{X}\mathbf{W}, \tag{4}$$

where $\widetilde{\boldsymbol{\mathcal{A}}}_S = \frac{1}{\lambda+1}\sum_{s=0}^{S}\left(\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)^{s}$ and $\mathbf{W}$ is the weight matrix. Our spectral convolution $\widetilde{\boldsymbol{\mathcal{A}}}_S\mathbf{X}$ on graphs is a multi-scale graph convolution [1, 15], which covers the single-scale graph convolution models such as SGC [26] since the graph convolution of SGC is $\widetilde{\boldsymbol{\mathcal{A}}}^2\mathbf{X}$ and $\widetilde{\boldsymbol{\mathcal{A}}}^2$ is the third term of $\widetilde{\boldsymbol{\mathcal{A}}}_S$. Besides, if we remove the non-linear functions in GCN [13], it also can be covered by our model. Therefore, we can draw the conclusion that our proposed NGC is a general framework.

High-order Graph Connectivity Factor. Based on NGC, we obtain the filtered graph signal via $\mathbf{F} = \widetilde{\boldsymbol{\mathcal{A}}}_S\mathbf{X}$. Intuitively, $\widetilde{\boldsymbol{\mathcal{A}}}_S$ captures not only the connectivity of the graph structure (represented by $\widetilde{\boldsymbol{\mathcal{A}}}$), but also the higher order connectivities (represented by $\widetilde{\boldsymbol{\mathcal{A}}}^2, \widetilde{\boldsymbol{\mathcal{A}}}^3, \ldots, \widetilde{\boldsymbol{\mathcal{A}}}^S$). As will be discussed in Sec. 5, such high order graph connectivity will largely influence the intriguing implicit denoising effect in GNNs. To formally quantify the high order graph connectivity, we give the following definition:

Definition 1 (High-order Graph Connectivity Factor)**.** *We define the high-order graph connectivity factor τ as*

$$\tau = \max_i \tau_i, \quad \textit{where } \tau_i = n \sum_{j=1}^{n} \left[\widetilde{\boldsymbol{\mathcal{A}}}_S\right]_{ij}^2 \Bigg/ \left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2. \tag{5}$$

Remark 1. *Here we give some intuitions about why Eq. (5) represents high-order graph connectivity. Note that each element in $\widetilde{\boldsymbol{\mathcal{A}}}_S$ is non-negative and each row sum satisfies*[1]

$$\sum_{j=1}^{n} \left[\widetilde{\boldsymbol{\mathcal{A}}}_S\right]_{ij} = 1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}. \tag{6}$$

Based on Eq. (6), the sum of squares of elements in each row satisfy:

$$\left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2 \Big/ n \leq \sum_{j=1}^{n} \left[\widetilde{\boldsymbol{\mathcal{A}}}_S\right]_{ij}^2 \leq \left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2. \tag{7}$$

When the high-order graph has a high connectivity, i.e., the elements in row i of $\widetilde{\boldsymbol{\mathcal{A}}}_S$ are more uniformly distributed, Eq. (7) reaches its lower bound. Meanwhile, if the graph is not connected and there is only one element whose value is larger than 0 in row i, Eq. (7) reaches its upper bound. Therefore, the value of $\tau \in [1, n]$ is determined as follows: when the high-order graph connectivity is high, $\tau \to 1$ and when the graph is less connected, $\tau \to n$.

5 Main Theory

In this section, we analyze the denoising effect of NGC. As we have discussed in Sec. 4, since NGC is a general framework, it can also be used to analyze other similar GNN models. Before we present our main theory, we first present our aggregation on noisy feature matrix and formulate four assumptions, which are necessary to construct our theory.

For the convenience of theoretical analysis, we adopt MSE loss[2] for our main theory. Consider $\widetilde{\boldsymbol{\mathcal{A}}}_S$ as our aggregation scheme, the NGC training based on Eq. (4) can be formulated as

$$\min_{\mathbf{W}} f(\mathbf{W}) = \left\|\widetilde{\boldsymbol{\mathcal{A}}}_S \mathbf{X}\mathbf{W} - \mathbf{Y}\right\|_F^2 = \left\|\widetilde{\boldsymbol{\mathcal{A}}}_S (\mathbf{X}^* + \boldsymbol{\eta})\mathbf{W} - \mathbf{Y}\right\|_F^2, \tag{8}$$

where $\mathbf{X}^*$ is the clean feature matrix, $\boldsymbol{\eta}$ denotes the noise added on $\mathbf{X}^*$, and $\mathbf{X} = \mathbf{X}^* + \boldsymbol{\eta}$ is the observed data matrix. Intuitively, if $\widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta}$ is small enough, the added noise will not change the optimization direction on which the parameter is updated under the clean feature matrix $\mathbf{X}^*$. Before we present our main theory, we give four assumptions about noise $\boldsymbol{\eta}$, $\widetilde{\boldsymbol{\mathcal{A}}}_S$, and parameters $\mathbf{W}$.

Assumption 1. *Each entry of the noise matrix $\boldsymbol{\eta}$, i.e., $[\boldsymbol{\eta}]_{ij}$ is i.i.d sub-Gaussian random variable with variance σ and mean $\mu = 0$, i.e.,*

$$\mathbb{E}\left[e^{\lambda([\boldsymbol{\eta}]_{ij} - \mu)}\right] \leq e^{\sigma^2 \lambda^2 / 2} \quad \textit{for all } \lambda \in \mathbb{R}. \tag{9}$$

Note that it is common to assume that the noise follows Gaussian distribution [33, 3, 32], which is also covered by our sub-Gaussian assumption.

Assumption 2. *The high-order graph connectivity factor τ is $\mathcal{O}(n)$, i.e., $\lim_{n\to\infty} \frac{\tau}{n} = 0$.*

As we have discussed in Sec. 4, τ depends on the graph structure. In a well-connected graph, τ is usually relatively small compared with n. Only if all nodes of a graph are isolated, τ reaches its upper bound n.

[1] Note that this result is obtained by using $\widetilde{\boldsymbol{\mathcal{A}}} = \widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}$ for the ease of theoretical analysis while in experiments we adopt more commonly used $\widetilde{\boldsymbol{\mathcal{A}}} = \widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}$. The proof can be found in Appendix C.

[2] We consider MSE loss since it gives easier form of gradient and it can be extended to other losses satisfying certain conditions.

Assumption 3. *The Frobenius norm of the parameter matrix* $\mathbf{W}$ *is bounded by a constant. There exists* $C > 0$ *such that* $\|\mathbf{W}\|_F \leq C$*, which is unrelated to* n*.*

We assume that the Frobenius norm of $\mathbf{W}$ is bounded by a constant. This is reasonable since recent advances in Neural Tangent Kernel [11] indicate that over-parameterized network weights lie in the neighborhood of the small random initialization, which justifies Assumption 3.

Assumption 4. *The loss function in Eq. (8) is* L*-smooth,*

$$\|\nabla f(\mathbf{W}_1) - \nabla f(\mathbf{W}_2)\|_2 \leq L\|\mathbf{W}_1 - \mathbf{W}_2\|_2 \quad \textit{for all } \mathbf{W}_1, \mathbf{W}_2 \in \mathbb{R}^{d\times c}. \tag{10}$$

The L-smoothness of f depends on the largest singular value of $\widetilde{\boldsymbol{\mathcal{A}}}_S\mathbf{X}$. For conciseness, we start with the smooth case.

As the core part of our proof, we first derive the upper bound of the Frobenius norm of $\widetilde{\boldsymbol{\mathcal{A}}}_S\boldsymbol{\eta}$.

Lemma 1. *Suppose we choose* $t = 2\tau\left(1 - \left(\frac{\lambda}{\lambda+1}\right)^S\right)^2 (4\log n + \log 2d)/n$*. Then under Assumptions 1 and 2, with a high probability* $1 - 1/d$*, we have*

$$\left\|\widetilde{\boldsymbol{\mathcal{A}}}_S\boldsymbol{\eta}\right\|_F^2 \leq \frac{2\tau\left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2 \sigma^2 (4\log n + \log 2d)}{n}, \tag{11}$$

where the proof can be found in Appendix D.

Lemma 1 implies that the norm of the aggregated noise matrix $\widetilde{\boldsymbol{\mathcal{A}}}_S\boldsymbol{\eta}$ is bounded by three terms: the number of nodes of a graph n, the expansion order S, the high-order graph connectivity factor τ. Intuitively, as the concentration bounds suggest, if we extract enough samples from the same sub-Gaussian variable, the average of these samples will converge to zero with a high probability. This requires our graph to be large enough and the sum of squares of the elements in the row of $\widetilde{\boldsymbol{\mathcal{A}}}_S$ to be small enough, which depends on the graph structure.

Now we start to present our main theorem for graph denoising. In order to demonstrate the effect of graph denoising, we further consider another loss function $g(\cdot)$ with the clean feature matrix:

$$g(\mathbf{W}) = \left\|\widetilde{\boldsymbol{\mathcal{A}}}_S\mathbf{X}^*\mathbf{W} - \mathbf{Y}\right\|_F^2. \tag{12}$$

Let $\mathbf{W}_g^* = \arg\min_{\mathbf{W}} g(\mathbf{W})$ be the minimizer of clean loss g, we aim to demonstrate that the learned model (from gradient descent on the noisy data $\mathbf{X}$) has essentially the same performance as $\mathbf{W}_g^*$ which is the optimal solution for the clean loss g.

Theorem 1. *Under Assumptions 1, 2, 3, 4 and Lemma 1, let* $\mathbf{W}_f^{(k)}$ *denote the* k*-th step gradient descent solution for* $\min_{\mathbf{W}} f(\mathbf{W})$ *with step size* $\alpha \leq 1/L$*, with probability* $1 - 1/d$ *we have*

$$g\left(\mathbf{W}_f^{(k)}\right) - g\left(\mathbf{W}_g^*\right) \leq \mathcal{O}\left(\frac{1}{2k\alpha}\right) + \mathcal{O}\left(\frac{\tau\log n}{n}\right), \tag{13}$$

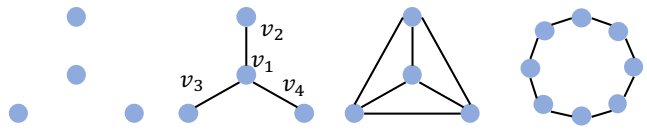
where $\mathbf{W}_g^* = \arg\min_{\mathbf{W}} g(\mathbf{W})$ *is the optimal solution of the clean loss function* $g(\mathbf{W})$*,* τ *is the high-order graph connectivity factor, and* n *is the number of nodes of a graph.*

The proof of Theorem 1 can be found in Appendix E.

Remark 2. ***Denoising Effect.*** *Theorem 1 suggests that the* k*-th step gradient descent solution* $\mathbf{W}_f^{(k)}$ *which is trained using the noisy feature matrix* $\mathbf{X}$ *enjoys a similar performance as the actual clean loss minimizer* $\mathbf{W}_g^*$ *with large enough* k *and* n*. This implies the denoising effect of our proposed solution in Eq. (4).*

Remark 3. ***Effect of graph structure on denoising.*** *Note that the second term in Eq. (13) suggests that the denoising effect is linear with respect to* τ*, which is directly related to the dataset graph structure. Specifically, as will be shown in Sec. 5, refer to Sec. 5 on how graph structure affects the value of* τ*, and thus the denoising effect. A large well-connected graph tend to have a better denoising performance since* n *is large and* τ *is close to 1.*

Graph



Metric	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
#Nodes	4	4	4	8
τ	4	1.14	1	1.03
$\tau \log n / n$	1.39	0.39	0.35	0.27

Figure 1: An illustration of the graph structures on the implicit denoising performances. $\mathcal{G}_1$: nodes are isolated; $\mathcal{G}_2$: a star graph with 4 nodes; $\mathcal{G}_3$: a complete graph with 4 nodes; $\mathcal{G}_4$: a ring graph with 8 nodes. For computing τ, λ and S are set to be 64.

Case Study: the Influence of Graph Structure on Implicit Denoising. Remark 3 suggests that τ plays an important role on the implicit denoising effect. In this case study, we give four illustration samples in Figure 1. G_1, G_2, and G_3 have the same number of nodes. But the nodes on G_1 are isolated. G_2 has only one connected component and has a center node v_1 on the graph. G_3 is a complete graph such that there is an edge between any two nodes. In addition, we give a larger illustration graph G_4 to understand the influence of graph size.

From Figure 1, we can extract the following insights: 1) There is no denoising effect (the value of $\tau \log n/n$ is quite large) on G_1 since the nodes are isolated. And GNNs will degrade to MLP under such graph structure, leading to no aggregation. 2) The complete graph G_3 has the best denoising effect among the graphs of the same size if we only consider the influence of graph structure. Since the values of elements in each row are distributed uniformly, leading to the lower bound of τ. 3) Although G_2 has only one connected component, there is a center node v_1 on the graph. The existence of the center node makes the value of elements in each row imbalanced, which means that τ tends to have a larger value compared with G_3. 4) The decentralized graph like G_4 also can get a smaller τ. 5) In terms of graph size, the graph with a larger size has a better denoising effect.

6 Robust Neumann Graph Convolution

To further improve the graph denoising performances with noisy node features, in this section, we propose a new graph signal denoising problem - adversarial graph signal denoising (AGSD) problem based on which we propose the roboust graph convolution model.

6.1 Adversarial Graph Signal Denoising Problem

Note that the second term in the GSD problem (Eq. (1)) which controls the smoothness of the feature matrix over graphs, is related to both the graph Laplacian and the node features. Therefore, the slight changes in the graph Laplacian matrix could lead to an unstable denoising effect. Inspired by the recent studies in adversarial training [18], we formulate the adversarial graph signal denoising problem as a min-max optimization problem:

$$\min_{\mathbf{F}} \left[\|\mathbf{F} - \mathbf{X}\|_F^2 + \lambda \cdot \max_{\mathbf{L}'} \operatorname{tr}\left(\mathbf{F}^\top \mathbf{L}' \mathbf{F}\right) \right] \quad \text{s.t.} \quad \left\| \mathbf{L}' - \widetilde{\mathbf{L}} \right\|_F \leq \varepsilon. \tag{14}$$

Intuitively, the inner maximization on the Laplacian $\mathbf{L}'$ generates perturbations on the graph structure[3], and enlarges the distance between the node representations of connected neighbors. Such maximization finds the worst case perturbations on the graph Laplacian that hinders the global smoothness of $\mathbf{F}$ over the graph. Therefore, by training on those worse case Laplacian perturbations, one could obtain a robust graph signal denoising solution. Ideally, through solving Eq. (14), the smoothness of the node representations as well as the implicit denoising effect can be enhanced.

[3]Here we do not need exact graph structure perturbations as in graph adversarial attacks [35, 36] but a virtual perturbation that could lead to small changes in the Laplacian.

6.2 Minimization of the Optimization Problem

The min-max formulation in Eq. (14) also makes the adversarial graph signal denoising problem much harder to solve. Fortunately, unlike adversarial training [17] where we need to first adopt PGD to solve the inner maximization problem before we solve the outer minimization problem, here inner maximization problem is simple and has a closed form solution. In other words, we do not need to add random perturbations on the graph structure at each training epoch and can find the largest perturbation which maximizes the inner adversarial loss function. Denote the perturbation as $\boldsymbol{\delta}$, and $\mathbf{L}' = \widetilde{\mathbf{L}} + \boldsymbol{\delta}$. Directly solving[4] the inner maximization problem, we get $\boldsymbol{\delta} = \varepsilon \nabla h(\boldsymbol{\delta}) = \frac{\varepsilon \mathbf{F}\mathbf{F}^\top}{\|\mathbf{F}\mathbf{F}^\top\|_F}$. Plugging this solution into Eq. (14), we can rewrite the outer optimization problem as follows:

$$\rho(\mathbf{F}) = \min_{\mathbf{F}} \left[\|\mathbf{F} - \mathbf{X}\|_F^2 + \lambda \max \operatorname{tr}\left(\mathbf{F}^\top \widetilde{\mathbf{L}} \mathbf{F}\right) + \lambda \varepsilon \operatorname{tr} \frac{\mathbf{F}^\top \mathbf{F}\mathbf{F}^\top \mathbf{F}}{\|\mathbf{F}\mathbf{F}^\top\|_F} \right]. \tag{15}$$

Taking the gradient of $\rho(\mathbf{F})$ to zero, we get the solution of the outer optimization problem as follows:

$$\mathbf{F} = \left(\mathbf{I} + \lambda \widetilde{\mathbf{L}} + \lambda \varepsilon \frac{\mathbf{F}\mathbf{F}^\top}{\|\mathbf{F}\mathbf{F}^\top\|_F} \right)^{-1} \mathbf{X}. \tag{16}$$

Both sides of Eq. (16) contains $\mathbf{F}$, directly computing the solution is difficult. Note that in Eq. (14) we also require $\mathbf{F}$ to be close to $\mathbf{X}$, we can approximate Eq. (16) by replacing the $\mathbf{F}$ with $\mathbf{X}$ in the inverse matrix on the right hand side. With the Neumann series expansion of the inverse matrix, we get the final approximate solution as

$$\mathbf{H} \approx \frac{1}{\lambda + 1} \sum_{s=0}^{S} \left[\frac{\lambda}{\lambda + 1} \left(\widetilde{\boldsymbol{\mathcal{A}}} - \frac{\varepsilon \mathbf{X}\mathbf{X}^\top}{\|\mathbf{X}\mathbf{X}^\top\|_F} \right) \right]^s \mathbf{X}\mathbf{W}. \tag{17}$$

The difference between Eq. (17) and Eq. (4) is that there is one more term in Eq. (17) derived from solving the inner optimization problem of Eq. (14). Based on this, we proposed our robust Neumann graph convolution (RNGC).

Scalability. Although RNGC introduces extra computational burdens for large graphs due to the $\mathbf{X}\mathbf{X}^\top$ term, if the feature matrix is sparse, the extra computational effort is minimal as the $\mathbf{X}\mathbf{X}^\top$ term can also be sparse. For the scalability of RNGC on large graphs with dense feature matrix, we only compute the inner product of feature vectors $(\mathbf{X}_i, \mathbf{X}_{j|j \in \mathcal{N}_i})$ between adjacent neighbors like masked attention in GAT. Compared with NGC, the additional computation cost is $\mathcal{O}(|\mathcal{E}|)$.

7 Experiments

In this section, we conduct a comprehensive empirical study to understand the influence of different factors on the denoising effect of various models. To quantify the denoising effect, we test the model accuracy on noisy data on various GNN architectures for standard node classification tasks, where the noisy data is synthesized by mixing Gaussian noise with the original feature matrix. We also synthesize noisy data by flipping individual feature with a small Bernoulli probability on three citation datasets with binary features.

7.1 Denoising Effectiveness Comparison of Various GNN Models

In this section, we compare the denoising effectiveness of different GNN models through their test accuracy by training on the noisy feature matrix with Gaussian noise.

Datasets. In our experiments, we utilize three public citation network datasets Cora, Citeseer, and Pubmed [20] which are homophily graphs for semi-supervised node classification. For the semi-supervised learning experimental setup, we follow the standard fixed splits employed in [30],

[4] More details on how to solve the inner maximization problem can be found in Appendix A.

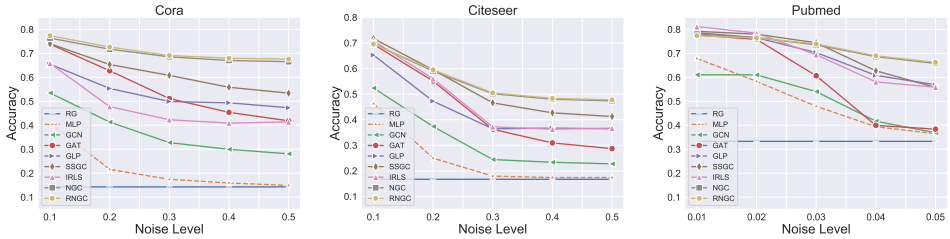


Figure 2: Comparison of classification accuracy v.s. noise level for semi-supervised node classification tasks. The noise level ξ controls the magnitude of the Gaussian noise we add to the feature matrix: $\mathbf{X} + \xi\boldsymbol{\eta}$ where $\boldsymbol{\eta}$ is sampled from standard i.i.d., Gaussian distribution.

with 20 nodes per class for training, 500 nodes for validation, and 1,000 nodes for testing. We also use 4 datasets: Cornell, Texas, Wisconsin, and Actor which are heterophily graphs for full-supervised node classification. For each dataset, we randomly split nodes into 60%, 20%, and 20% for training, validation, and testing as suggested in [19]. Moreover, we utilize three large-scale graph datasets: Coauthor-CS, Coauthor-Phy [21], and ogbn-products [10] for evaluation. For Coauthor datasets, we split nodes into 60%, 20%, and 20% for training, validation, and testing. For ogbn-products dataset, we follow the experimental setup as in OGB [10].

Baselines. For the baselines, we consider graph neural networks derived from graph signal denoising, including GLP [14], S^2GC [34], and IRLS [29]; popular GNN architectures, such as GCN [13] and GAT [24]; and MLP which has no aggregation operation.

Experimental Setup and Implementations. We assume that the original feature matrix is clean and do not have noise and we synthesize the noise from the standard Gaussian distribution and add them on the original feature matrix. By default, we apply row normalization (except for ogbn-products) for data after adding the Gaussian noise[5], and train all the models based on these noisy feature matrix. For the hyper-parameters of each model, we follow the optimal setting that reported in their original papers. To eliminate the effect of randomness, we repeat such experiment for multiple times and report the mean accuracy. Note that in each repeated run, we add different Gaussian noises while for the same run, we apply the same noisy feature matrix for training all the models. For our NGC and RNGC model, we set $S = 16$ and $\lambda = 32$ for three citation datasets by default. On heterophily graphs, we add two-layer MLP after aggregating features with NGC or RNGC and set $S = 16$ or $S = 2$, and $\lambda = 1$ or $\lambda = 1024$ for different datasets and noise setting. On Coauthor datasets, we also set $S = 16$, and $\lambda = 1$ or $\lambda = 128$ for different noise setting. On ogbn-products dataset, we set $S = 128$, and $\lambda = 32$ or $\lambda = 256$ for different noise setting. We also two-layer MLP for NGC and RNGC on ogbn-products.

Results on Semi-supervised Node Classification. Figure 2 illustrates the comparison of classification accuracy against the various noise levels for semi-supervised node classification tasks. The noise level ξ controls the magnitude of the Gaussian noise we add to the feature matrix: $\mathbf{X} + \xi\boldsymbol{\eta}$ where $\boldsymbol{\eta}$ is sampled from standard i.i.d., Gaussian distribution. For Cora and Citeseer, we test $\xi \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and for Pubmed, we test $\xi \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$. From Figure 2, we can observe that the test accuracy of MLP is close to randomly guessing (RG) when the noise level is relatively large. This implies the weak denoising effect of MLP models. For shallow GNN models, such as GCN and GAT (which usually contain 2 layers), their denoising performance is limited especially on Pubmed since they do not aggregate information (features and noise) from higher-order neighbors. For models with deep layers[6], such as IRLS ($\geq$ 8 layers), the denoising performance is much better compared to shallow models. Lastly, our NGC and RNGC model with 16

[5]We also perform an analysis on the effect of row normalization in noisy feature matrix in Appendix H.1.

[6]We also perform an analysis on the denoising effect of depth in NGC and RNGC in Appendix H.2.

Table 1: Summary of results (10 runs) on heterophily graphs in terms of classification accuracy (%)

Noise Level	Cornell		Texas		Wisconsin		Actor	
	0.01	1	0.01	1	0.01	1	0.01	1
MLP	69.7±8.6	55.3±7.6	69.7±8.6	55.3±7.6	**78.6±6.5**	44.6±6.2	33.3±1.1	**25.1±1.0**
GCN	56.9±8.4	51.7±14.9	56.6±8.1	52.5±11.7	48.0±6.1	41.2±9.0	26.4±1.0	23.8±3.0
GAT	55.8±8.9	55.0±7.5	56.4±8.1	54.7±8.2	53.4±7.2	**48.2±7.2**	27.3±1.2	24.3±0.7
GLP	65.3±8.6	54.2±7.6	60.0±9.3	52.8±8.0	59.0±5.4	42.6±5.4	31.0±1.3	**25.1±0.8**
S^2GC	60.6±9.3	48.6±10.4	56.4±7.2	50.3±8.0	47.4±4.5	37.2±3.7	27.2±1.1	23.5±1.3
IRLS	48.1±8.5	46.7±6.2	65.6±8.4	42.8±15.9	65.2±6.0	37.4±8.5	**36.1±0.9**	21.5±4.1
NGC	**72.8±8.7**	**56.4±8.1**	**73.9±6.9**	**56.4±8.1**	74.8±6.8	**46.8±6.6**	34.0±1.6	**25.1±1.0**
RNGC	**75.8±7.9**	**56.4±8.1**	**74.2±6.1**	**56.4±8.1**	**76.4±5.3**	**46.8±6.6**	**34.3±1.6**	**25.1±1.0**

Table 2: Summary of results (10 runs) on Coauthor-CS and Coauthor-Phy in terms of classification accuracy (%)

Noise Level	Coauthor-CS		Coauthor-Phy	
	0.1	1	0.1	1
MLP	82.5±1.8	22.3±0.1	81.6±8.1	47.0±10.0
GCN	87.3±0.5	61.3±14.3	94.2±0.4	78.6±10.6
GAT	86.8±3.6	57.9±20.2	94.0±0.4	63.7±16.7
GLP	91.3±0.4	52.4±17.3	93.3±2.5	81.3±10.6
S^2GC	86.1±0.2	79.6±10.2	92.6±1.3	89.4±4.3
IRLS	78.8±5.1	62.1±17.8	89.2±3.4	87.0±4.5
NGC	**95.3±0.2**	**87.1±3.1**	**95.7±0.2**	**93.1±1.4**
RNGC	**95.4±0.2**	**87.8±1.5**	**95.7±0.2**	**93.6±0.8**

Table 3: Summary of results (10 runs) on ogbn-products in terms of classification accuracy (%)

Noise Level	ogbn-products	
	0.1	1
MLP	59.68±0.16	38.08±0.10
GCN	75.60±0.19	72.76±0.20
S^2GC	74.95±0.13	63.17±0.12
NGC	**77.56±0.15**	**73.36±0.11**
RNGC	**77.54±0.15**	**73.66±0.13**

layers ($S = 16$) achieve significantly better denoising performance compared with other baseline methods, which backup our theoretical analyses. In most cases, NGC and RNGC achieve very similar denoising performance but in general, RNGC still slightly outperforms NGC, suggesting that we indeed gain more benefits by solving the adversarial graph denoising problem.

Results on Full-supervised Node Classification. Table 1 reports the comparison of classification accuracy against the various noise levels for full-supervised node classification tasks on heterophily graphs. The first- and second-highest accuracies are highlighted in bold. For these datasets, we test $\xi \in \{0.01, 1\}$. From Table 1, we can observe that MLP is better than most GNN models in most cases due to the heterophily properties of these graphs. However, our proposed RNGC (+MLP) achieves significantly better or matches denoising performance compared with other baseline methods, which demonstrates the superiority of our RNGC by solving the AGSD.

For ogbn-products, we only choose MLP, GCN, and S^2GC as baselines, since the results are sensitive concerning model size and various tricks such as mini-batch training from the OGB leaderboard. For fair comparison, the size of parameters for these baselines and RNGC is the same. Table 2 and 3 report the comparison of classification accuracy against the various noise levels. Compared with the above small datasets, the node degree on these three datasets is larger, which means they have better connectivity. From Table 2 and 3, we can observe the test accuracy of GCN is slightly smaller than RNGC on these datasets since they are well-connected and have a large graph size and we can achieve a good denoising performance with shallow-layer GNN models. For the scalability of RNGC on large graphs such as ogbn-products, we use the acceleration method mentioned in Sec. 6.2.

8 Conclusion

Our work conducts a comprehensive study on the implicit denoising effect of graph neural networks. We theoretical show that the denoising effect of GNNs are largely influenced by the connectivity and the size of the graph structure, as well as the GNN architectures. Motivated by our analysis, we also propose a robust graph convolution model by solving the robust graph signal denoising problem which enhances the smoothness of node representations and the implicit denoising effect.

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A The details on how to solve the inner maximization problem in Sec. 6.2

Different from the non-concave inner maximization problem in the adversarial attack, our inner maximization problem is indeed a convex optimization problem. Hence, we do not need to add random perturbations on the graph structure at each training epoch and can find the largest perturbation which maximizes the inner adversarial loss function. Denote the perturbation as $\boldsymbol{\delta}$, and $\mathbf{L}' = \widetilde{\mathbf{L}} + \boldsymbol{\delta}$. We can rewrite the inner maximization problem as

$$\max_{\mathbf{L}'} \operatorname{tr}\left(\mathbf{F}^\top \mathbf{L}' \mathbf{F}\right) = \langle \widetilde{\mathbf{L}}, \mathbf{F}^\top \mathbf{F} \rangle + \max_{\boldsymbol{\delta}} \langle \boldsymbol{\delta}, \mathbf{F}^\top \mathbf{F} \rangle \quad \text{s.t.} \quad \|\boldsymbol{\delta}\|_F \leq \varepsilon. \tag{18}$$

We denote $h(\boldsymbol{\delta}) = \langle \boldsymbol{\delta}, \mathbf{F}^\top \mathbf{F} \rangle$. Obviously, $h(\boldsymbol{\delta})$ reaches the largest value when $\boldsymbol{\delta}$ has the same direction with the gradient of $h(\boldsymbol{\delta})$, e.g. $\boldsymbol{\delta} = \varepsilon \nabla h(\boldsymbol{\delta}) = \frac{\varepsilon \mathbf{F}\mathbf{F}^\top}{\|\mathbf{F}\mathbf{F}^\top\|_F}$, which is illustrated in Fig. 3.

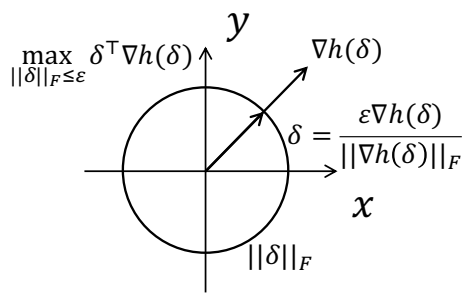

Figure 3: The illustration of the inner maximization problem. The adversarial loss function reaches the largest value when the direction of $\boldsymbol{\delta}$ is the same with $\nabla h(\boldsymbol{\delta})$

B Additional Details on the Neumann Series

We provide additional details and derivations on how to obtain the Neumann Series which leads to our Neumann Graph Convolution (NGC) method.

Before we derive the Neumann Series, we first introduce the following lemmas which are crucial to the derivation of the Neumann Series.

Lemma B.1. (Gelfand formula) [2] *Given any matrix norm* $|||\cdot|||$*, then* $\rho(\mathbf{A}) = \lim_{k\to\infty} |||\mathbf{A}^k|||^{1/k} = \inf_{k\geq 1} |||\mathbf{A}^k|||^{1/k} \leq |||\mathbf{A}|||$.

Lemma B.1 describes the relationship between the spectral radius of a matrix and its matrix norm, *i.e.* $\rho(\mathbf{A}) = \lim_{k\to\infty} |||\mathbf{A}^k|||^{1/k}$.

Lemma B.2. *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, the spectral radius* $\rho(\mathbf{A}) = \max(\operatorname{abs}(\operatorname{spec}(\mathbf{A})))$*, if* $\rho(\mathbf{A}) < 1$*, then* $\sum_{k=0}^{\infty} \mathbf{A}^k$ *converges to* $(\mathbf{I} - \mathbf{A})^{-1}$.

Proof. We first prove that $(\mathbf{I} - \mathbf{A})^{-1}$ exists as follows: Based on the definition of eigenvalues of $\mathbf{A}$, we have $|\lambda \mathbf{I} - \mathbf{A}| = 0$ and the solution is the eigenvalue of $\mathbf{A}$. Since $\rho(\mathbf{A}) < 1$, if $\lambda \geq 1$, then $|\lambda \mathbf{I} - \mathbf{A}| \neq 0$, so $|\mathbf{I} - \mathbf{A}| \neq 0$, which means $(\mathbf{I} - \mathbf{A})^{-1}$ exists.

Since $\rho(\mathbf{A}) < 1$ and by Lemma B.1, we have $\lim_{k\to\infty} |||\mathbf{A}^k||| = \rho(\mathbf{A})^k = 0$. Let $\mathbf{S}_k = \mathbf{A}^0 + \mathbf{A}^1 + \cdots + \mathbf{A}^k$, then we have

$$\begin{aligned}
\lim_{k\to\infty} (\mathbf{S}^k - \mathbf{A}\mathbf{S}^k) &= \lim_{k\to\infty} (\mathbf{I} - \mathbf{A})\mathbf{S}^k \\
&= \lim_{k\to\infty} (\mathbf{I} - \mathbf{A}^{k+1}) \\
&= \mathbf{I}
\end{aligned}$$

Since $(\mathbf{I}-\mathbf{A})^{-1}$ exists, so we have $(\mathbf{I}-\mathbf{A})\lim_{k\to\infty}\mathbf{S}^k=\mathbf{I}$, and $\lim_{k\to\infty}\mathbf{S}^k=(\mathbf{I}-\mathbf{A})^{-1}$, which finishes the proof. □

Lemma B.2 describes the convergence of Neumann Series and the condition to get the convergence.

Lemma B.3. (Gerschgorin Disc) [2] *Let $\mathbf{A}\in\mathbb{C}^{n\times n}$, with entries a_{ij}. For any eigenvalue λ, there exits i and the corresponding Gerschgorin disc $D\left(a_{ii},R_i\right)\subseteq\mathbb{C}$ such that λ lies in this disc, i.e.*

$$|\lambda-a_{ii}|\leq\sum_{j\neq i}^{n}|a_{ij}|.$$

Lemma B.3 describes the estimated range of eigenvalues.

Now we start to derive the Neumann Series expansion of the solution of GSD as follows.

Lemma B.4. *Let $\mathbf{A}\in\{0,1\}^{n\times n}$ be the adjacency matrix of a graph and $\widetilde{\boldsymbol{\mathcal{A}}}=\widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}$ or $\widetilde{\boldsymbol{\mathcal{A}}}=\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}$, then*

$$(\mathbf{I}-\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}})^{-1}=\sum_{k=0}^{\infty}\left(\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)^{k}.$$

Proof. We first prove that $\rho(\widetilde{\boldsymbol{\mathcal{A}}})\leq 1$ where $\widetilde{\boldsymbol{\mathcal{A}}}=\widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}$. Let λ be the eigenvalue of $\widetilde{\boldsymbol{\mathcal{A}}}$, and $\mathbf{v}$ be the corresponding eigenvector. Then we have

$$\begin{aligned}\left(\widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}\right)\mathbf{v}=\lambda\mathbf{v}&\Longrightarrow\widetilde{\mathbf{D}}^{-\frac{1}{2}}\left(\widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}\right)\mathbf{v}=\lambda\widetilde{\mathbf{D}}^{-\frac{1}{2}}\mathbf{v}\\&\Longrightarrow\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)\widetilde{\mathbf{D}}^{-\frac{1}{2}}\mathbf{v}=\lambda\widetilde{\mathbf{D}}^{-\frac{1}{2}}\mathbf{v},\end{aligned}$$

which means $(\lambda,\widetilde{\mathbf{D}}^{-\frac{1}{2}}\mathbf{v})$ is the eigen-pair of $\widetilde{\mathbf{D}}^{-1}\mathbf{A}$. By Lemma B.3, there exists i, such that

$$\begin{aligned}&\left|\lambda-\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ii}\right|\leq\sum_{j\neq i}\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|\\&\Longrightarrow\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ii}-\sum_{j\neq i}\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|\leq\lambda\leq\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ii}+\sum_{j\neq i}\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|.\end{aligned}$$

Since $\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}>0$ and $\sum_j\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|=\sum_j\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}=1$, obviously

$$-1<\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ii}-\sum_{j\neq i}\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|\leq\lambda\leq\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ii}+\sum_{j\neq i}\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|=1.$$

So if $\widetilde{\boldsymbol{\mathcal{A}}}=\widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}$, we have $\rho(\widetilde{\boldsymbol{\mathcal{A}}})\leq 1$. When $\widetilde{\boldsymbol{\mathcal{A}}}=\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}$, we denote $(\lambda,\mathbf{v})$ as the eigen-pair of $\widetilde{\mathbf{D}}^{-\mathbf{1}}\mathbf{A}$. Similarly, by Lemma B.3, there exists i, such that

$$\begin{aligned}&\left|\lambda-\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ii}\right|\leq\sum_{j\neq i}\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|\\&\Longrightarrow\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ii}-\sum_{j\neq i}\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|\leq\lambda\leq\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ii}+\sum_{j\neq i}\left|\left(\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right)_{ij}\right|.\end{aligned}$$

Obviously, we can get the same conclusion for $\widetilde{\boldsymbol{\mathcal{A}}}=\widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}$. So it is true for $\rho\left(\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)\leq\frac{\lambda}{\lambda+1}<1$

By Lemma B.2, we get the result $(\mathbf{I}-\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}})^{-1}=\sum_{k=0}^{\infty}\left(\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)^{k}$, which finishes the proof. □

By Lemma B.4, we approximate the inverse matrix $(\mathbf{I}+\lambda\tilde{\mathbf{L}})^{-1}$ up to S-th order with

$$\left(\mathbf{I}+\lambda\widetilde{\mathbf{L}}\right)^{-1}=\frac{1}{\lambda+1}\left(\mathbf{I}-\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)^{-1}\approx\frac{1}{\lambda+1}\sum_{s=0}^{S}\left(\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}\right)^{s}.$$

C The Row Summation of the Neumann Series

We provide the derivations of the row sum of $\widetilde{\boldsymbol{\mathcal{A}}}_S$ in this section. Before we derive the row summation of $\widetilde{\boldsymbol{\mathcal{A}}}_S$, we first derive the row summation of $\widetilde{\boldsymbol{\mathcal{A}}}^k$.

Lemma C.1. *Consider a probability matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$, where $\mathbf{P}_{ij} \geq 0$. Besides, for all i, we have $\sum_{j=1}^n \mathbf{P}_{ij} = 1$. Then for any $s \in \mathbb{Z}_+$, we have $\sum_{j=1}^n \mathbf{P}_{ij}^s = 1$,*

Proof. We give a proof by induction on k.
Base case: When $k = 1$, the case is true.
Inductive step: Assume the induction hypothesis that for a particular k, the single case n = k holds, meaning $\mathbf{P}^k$ is true:

$$\forall i, \sum_{j=1}^n \mathbf{P}_{ij}^k = 1.$$

As $\mathbf{P}^{k+1} = \mathbf{P}^k\mathbf{P}$, so we have

$$\sum_{j=1}^n \mathbf{P}_{ij}^{k+1} = \sum_{j=1}^n\sum_{k=1}^n \mathbf{P}_{ik}^k\mathbf{P}_{kj} = \sum_{k=1}^n\sum_{j=1}^n \mathbf{P}_{ik}^k\mathbf{P}_{kj} = \sum_{k=1}^n \mathbf{P}_{ik}^k\left(\sum_{j=1}^n \mathbf{P}_{kj}\right) = \sum_{k=1}^n \mathbf{P}_{ik}^k = 1,$$

which finishes the proof. □

Lemma C.1 describes the row summation of $\widetilde{\boldsymbol{\mathcal{A}}}^k$ is 1. Now we can obtain the row summation for $\widetilde{\boldsymbol{\mathcal{A}}}_S$. Then for any i, we have

$$\begin{aligned}\sum_{j=1}^n \left[\widetilde{\boldsymbol{\mathcal{A}}}_S\right]_{ij} &= \frac{1}{\lambda+1}\sum_{s=0}^S\left(\frac{\lambda}{\lambda+1}\left[\widetilde{\boldsymbol{\mathcal{A}}}\right]_{ij}\right)^s \\ &= \frac{1}{\lambda+1}\sum_{s=0}^S\left(\frac{\lambda}{\lambda+1}\right)^s \\ &= 1-\left(\frac{\lambda}{\lambda+1}\right)^{S+1}.\end{aligned} \tag{19}$$

D Proof of Lemma 1

We provide the details of proof of Lemma 1. We first introduce the General Hoeffding Inequality [9], which is essential for bounding $\left\|\widetilde{\boldsymbol{\mathcal{A}}}_S\boldsymbol{\eta}\right\|_F^2$.

Lemma D.1. (General Hoeffding Inequality [9]) *Suppose that the variables $X_1, \cdots, X_n$ are independent, and X_i has mean μ_i and sub-Gaussian parameter σ_i. Then for all $t \geq 0$, we have*

$$\mathbb{P}\left[\sum_{i=1}^n (X_i-\mu_i) \geq t\right] \leq \exp\left\{-\frac{t^2}{2\sum_{i=1}^n \sigma_i^2}\right\}. \tag{20}$$

Now let's prove Lemma 1.

Proof of Lemma 1. For any entry $\left[\widetilde{\boldsymbol{\mathcal{A}}}_S\boldsymbol{\eta}\right]_{ij} = \sum_{p=1}^n \left(\widetilde{\boldsymbol{\mathcal{A}}}_S\right)_{ip}\boldsymbol{\eta}_{pj}$, where $\boldsymbol{\eta}_{pj}$ is a sub-Gaussian variable with parameter σ^2. By the General Hoeffding inequality D.1, we have

$$\mathbb{P}\left(\left|\left[\frac{1}{\lambda+1}\sum_{s=0}^S\left(\frac{\lambda}{\lambda+1}\widetilde{\boldsymbol{\mathcal{A}}}_S\right)^s\boldsymbol{\eta}\right]_{ij}\right| \geq t\right) \leq 2\exp\left\{-\frac{nt^2}{2\tau\left(1-\left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2\sigma^2}\right\}. \tag{21}$$

where $\tau = \max_i \tau_i$ and $\tau_i = n \sum_{j=1}^{n} \left[\widetilde{\boldsymbol{\mathcal{A}}}_S\right]_{ij}^2 \Big/ \left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2$.

Applying union bound [25] to all possible pairs of $i \in [n]$, $j \in [n]$, we get

$$\mathbb{P}\left(\left\|\widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta}\right\|_{\infty,\infty} \geq t\right) \leq \sum_{i,j} \mathbb{P}\left(\left[\widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta}\right]_{ij} \geq t\right) \leq 2n^2 \exp\left\{-\frac{nt^2}{2\tau\left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2 \sigma^2}\right\}. \tag{22}$$

Applying union bound again, we have

$$\mathbb{P}\left(\left\|\widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta}\right\|_F^2 \geq t\right) \leq \sum_{i,j} \mathbb{P}\left(\left\|\widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta}\right\|_{\infty,\infty} \geq \sqrt{t}\right) \leq 2n^4 \exp\left\{-\frac{nt}{2\tau\left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2 \sigma^2}\right\}. \tag{23}$$

Choose $t = 2\tau\left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2 (4\log n + \log 2d)/n$ and with probability $1 - 1/d$, we have

$$\left\|\widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta}\right\|_F^2 \leq \frac{2\tau\left(1 - \left(\frac{\lambda}{\lambda+1}\right)^{S+1}\right)^2 \sigma^2 (4\log n + \log 2d)}{n}, \tag{24}$$

which finishes the proof. □

E Proof of the Main Theorem 1

We provide the details of proof of main theorem 1.

[Restatement of Theorem 1] *Under Assumptions 1,2,3,4, let $\mathbf{W}_f^{(k)}$ denote the k-th step gradient descent solution for $\min_{\mathbf{W}} f(\mathbf{W})$ with step size $\alpha \leq 1/L$, with probability $1 - 1/d$ we have*

$$g\left(\mathbf{W}_f^{(k)}\right) - g\left(\mathbf{W}_g^*\right) \leq \mathcal{O}\left(\frac{1}{2k\alpha}\right) + \mathcal{O}\left(\frac{\tau \log n}{n}\right), \tag{25}$$

where $\mathbf{W}_g^* = \arg\min_{\mathbf{W}} g(\mathbf{W})$ is the optimal solution of the clean loss function $g(\mathbf{W})$, τ is the high-order graph connectivity factor, and n is the number of nodes of a graph.

Proof. By the definition of L-smooth, we can obtain the following inequality:

$$f(\mathbf{W}_f^{'}) \leq f(\mathbf{W}_f) + \langle \nabla f(\mathbf{W}_f), \mathbf{W}_f^{'} - \mathbf{W}_f \rangle + \frac{1}{2} L \|\mathbf{W}_f^{'} - \mathbf{W}_f\|_F^2. \tag{26}$$

Let's use the gradient descent algorithm with $\mathbf{W}_f^{'} = \mathbf{W}_f^{+} = \mathbf{W}_f - \alpha \nabla f(\mathbf{W}_f)$. We then get:

$$\begin{aligned}
f\left(\mathbf{W}_f^{+}\right) &\leq f(\mathbf{W}_f) + \langle \nabla f(\mathbf{W}_f), \mathbf{W}_f^{+} - \mathbf{W}_f \rangle + \frac{1}{2} L \left\|\mathbf{W}_f^{+} - \mathbf{W}_f\right\|_F^2 \\
&= f(\mathbf{W}_f) + \langle \nabla f(\mathbf{W}_f), \mathbf{W}_f - \alpha \nabla f(\mathbf{W}_f) - \mathbf{W}_f \rangle + \frac{1}{2} L \|\mathbf{W}_f - \alpha \nabla f(\mathbf{W}_f) - \mathbf{W}_f\|_F^2 \\
&= f(\mathbf{W}_f) - \langle \nabla f(\mathbf{W}_f), \alpha \nabla f(\mathbf{W}_f \rangle + \frac{1}{2} L \|\alpha \nabla f(\mathbf{W}_f)\|_F^2 \\
&= f(\mathbf{W}_f) - \alpha \|\nabla f(\mathbf{W}_f)\|_F^2 + \frac{1}{2} L \alpha^2 \|\nabla f(\mathbf{W}_f)\|_F^2 \\
&= f(\mathbf{W}_f) - \left(1 - \frac{1}{2} L \alpha\right) \alpha \|\nabla f(\mathbf{W}_f)\|_F^2.
\end{aligned} \tag{27}$$

With the fixed step size $\alpha \leq 1/L$, we know that $-(1-\frac{1}{2}L\alpha) = \frac{1}{2}L\alpha - 1 \leq \frac{1}{2}L(1/L) - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$. Plugging this into Eq. (27), we have the following inequality:

$$f\left(\mathbf{W}_f^+\right) \leq f(\mathbf{W}_f) - \frac{1}{2}\alpha\|\nabla f(\mathbf{W}_f)\|_F^2. \tag{28}$$

If we choose t to be small enough such that $t \leq 1/L$, this inequality implies that the loss function value strictly decreases under each iteration of gradient descent since $\|\nabla f(\mathbf{W}_f)\|$ is positive unless $\nabla f(\mathbf{W}_f) = 0$ e.g. $\mathbf{W}_f = \mathbf{W}_f^*$, where $\mathbf{W}_f$ reaches $\mathbf{W}_f^*$.

Now, let's bound the loss function value $f(\mathbf{W}_f^+)$. Since f is convex, we can write

$$f(\mathbf{W}_f) \leq f\left(\mathbf{W}_g^*\right) + \langle \nabla f(\mathbf{W}_f), \mathbf{W}_f - \mathbf{W}_g^* \rangle. \tag{29}$$

Introducing this inequality into Eq. (28), we can obtain the following:

$$\begin{aligned}
f\left(\mathbf{W}_f^+\right) - f\left(\mathbf{W}_g^*\right) &\leq \langle \nabla f(\mathbf{W}_f), \mathbf{W}_f - \mathbf{W}_g^* \rangle - \frac{\alpha}{2}\|\nabla f(\mathbf{W}_f)\|_F^2 \\
&\leq \frac{1}{2\alpha}\left(2\alpha\langle \nabla f(\mathbf{W}_f), \mathbf{W}_f - \mathbf{W}_g^* \rangle - \alpha^2\|\nabla f(\mathbf{W}_f)\|_F^2\right) \\
&\leq \frac{1}{2\alpha}\left(2\alpha\langle \nabla f(\mathbf{W}_f), \mathbf{W}_f - \mathbf{W}_g^* \rangle - \alpha^2\|\nabla f(\mathbf{W}_f)\|_F^2 - \left\|\mathbf{W}_f - \mathbf{W}_g^*\right\|_F^2\right) \\
&\quad + \frac{1}{2\alpha}\left\|\mathbf{W}_f - \mathbf{W}_g^*\right\|_F^2 \\
&\leq \frac{1}{2\alpha}\left(\left\|\mathbf{W}_f - \mathbf{W}_g^*\right\|_F^2 - \left\|\mathbf{W}_f - \alpha\nabla f(\mathbf{W}_f) - \mathbf{W}_g^*\right\|_F^2\right).
\end{aligned} \tag{30}$$

Notice that by the definition of gradient descent update, we have $\mathbf{W}_f^+ = \mathbf{W}_f - \alpha\nabla f(\mathbf{W}_f)$. Plugging this into the final inequality of Eq. (30), we can get:

$$f\left(\mathbf{W}_f^+\right) - f\left(\mathbf{W}_g^*\right) \leq \frac{1}{2\alpha}\left(\left\|\mathbf{W}_f - \mathbf{W}_g^*\right\|_F^2 - \left\|\mathbf{W}_f^+ - \mathbf{W}_g^*\right\|_F^2\right). \tag{31}$$

This inequality holds for $\mathbf{W}_f^+$ on every iteration of gradient descent. Summing over iterations, we get:

$$\begin{aligned}
\sum_{i=1}^{k}\left[f\left(\mathbf{W}_f^{(i)}\right) - f\left(\mathbf{W}_g^*\right)\right] &\leq \sum_{i=1}^{k}\frac{1}{2\alpha}\left(\left\|\mathbf{W}_f^{(i-1)} - \mathbf{W}_g^*\right\|_F^2 - \left\|\mathbf{W}_f^{(i)} - \mathbf{W}_g^*\right\|_F^2\right) \\
&= \frac{1}{2\alpha}\left(\left\|\mathbf{W}_f^{(0)} - \mathbf{W}_g^*\right\|_F^2 - \left\|\mathbf{W}_f^{(k)} - \mathbf{W}_g^*\right\|_F^2\right) \\
&\leq \frac{1}{2\alpha}\left(\left\|\mathbf{W}_f^{(0)} - \mathbf{W}_g^*\right\|_F^2\right).
\end{aligned} \tag{32}$$

With the inequality of Eq. (29), we know that $f(\mathbf{W}_f)$ strictly decreases over each iteration. So we have following:

$$\begin{aligned}
f\left(\mathbf{W}_f^{(k)}\right) - f\left(\mathbf{W}_g^*\right) &\leq \frac{1}{k}\left[\sum_{i=1}^{k} f\left(\mathbf{W}_f^{(i)}\right) - f\left(\mathbf{W}_g^*\right)\right] \\
&\leq \frac{1}{2k\alpha}\left(\left\|\mathbf{W}_f^{(0)} - \mathbf{W}_g^*\right\|_F^2\right)
\end{aligned} \tag{33}$$

Equivalently, we have the inequality for the loss function $g(\mathbf{W}_f)$:

$$
\begin{aligned}
g\left(\mathbf{W}_f^{(k)}\right) - g\left(\mathbf{W}_g^*\right) &= f\left(\mathbf{W}_f^{(k)}\right) - f\left(\mathbf{W}_g^*\right) \\
&\quad + 2\langle \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \mathbf{W}_g^*, \widetilde{\boldsymbol{\mathcal{A}}}_S \mathbf{X}^* \mathbf{W}_g^* - \mathbf{Y} \rangle + \langle \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \mathbf{W}_g^*, \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \mathbf{W}_g^* \rangle \\
&\quad - 2\langle \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \mathbf{W}_f^{(k)}, \widetilde{\boldsymbol{\mathcal{A}}}_S \mathbf{X}^* \mathbf{W}_f^{(k)} - \mathbf{Y} \rangle - \langle \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \mathbf{W}_f^{(k)}, \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \mathbf{W}_f^{(k)} \rangle \\
&\leq \frac{1}{2k\alpha} \left(\left\| \mathbf{W}_f^{(0)} - \mathbf{W}_g^* \right\|_F^2 \right) \\
&\quad + \left\| \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \right\|_F^2 \left\| \mathbf{W}_g^* \right\|_F^2 \left(2 \left\| \widetilde{\boldsymbol{\mathcal{A}}}_S \mathbf{X}^* \mathbf{W}_g^* - \mathbf{Y} \right\|_F^2 + \left\| \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \right\|_F^2 \left\| \mathbf{W}_g^* \right\|_F^2 \right) \\
&\quad + \left\| \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \right\|_F^2 \left\| \mathbf{W}_f^{(k)} \right\|_F^2 \left(2 \left\| \widetilde{\boldsymbol{\mathcal{A}}}_S \mathbf{X} \mathbf{W}_f^{(k)} - \mathbf{Y} \right\|_F^2 + \left\| \widetilde{\boldsymbol{\mathcal{A}}}_S \boldsymbol{\eta} \right\|_F^2 \left\| \mathbf{W}_f^{(k)} \right\|_F^2 \right) \\
&\leq \mathcal{O}\left(\frac{1}{2k\alpha} \right) + \mathcal{O}\left(\frac{\tau \log n}{n} \right),
\end{aligned}
\tag{34}
$$

which finishes the proof. □

F More Details on Equation (1).

We provide more details on how to obtain Equation (1).

Note that if we set $\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-\frac{1}{2}} \widetilde{\mathbf{A}} \widetilde{\mathbf{D}}^{-\frac{1}{2}}$, we have $\operatorname{tr}\left(\mathbf{F}^\top \widetilde{\mathbf{L}} \mathbf{F}\right) = \operatorname{tr}\left(\mathbf{F}^\top (\mathbf{I} - \widetilde{\mathbf{D}}^{-\frac{1}{2}} \widetilde{\mathbf{A}} \widetilde{\mathbf{D}}^{-\frac{1}{2}}) \mathbf{F}\right) = \operatorname{tr}\left(\mathbf{F}^\top \mathbf{F}\right) - \operatorname{tr}\left(\mathbf{F}^\top \widetilde{\mathbf{D}}^{-\frac{1}{2}} \widetilde{\mathbf{A}} \widetilde{\mathbf{D}}^{-\frac{1}{2}} \mathbf{F}\right) = \operatorname{tr}\left(\mathbf{F}\mathbf{F}^\top\right) - \operatorname{tr}\left(\widetilde{\mathbf{D}}^{-\frac{1}{2}} \widetilde{\mathbf{A}} \widetilde{\mathbf{D}}^{-\frac{1}{2}} \mathbf{F}\mathbf{F}^\top\right)$. On the other hand, if we set $\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-1} \widetilde{\mathbf{A}}$, we have $\operatorname{tr}\left(\mathbf{F}^\top \widetilde{\mathbf{L}} \mathbf{F}\right) = \operatorname{tr}\left(\mathbf{F}^\top (\mathbf{I} - \widetilde{\mathbf{D}}^{-1} \widetilde{\mathbf{A}}) \mathbf{F}\right) = \operatorname{tr}\left(\mathbf{F}^\top \mathbf{F}\right) - \operatorname{tr}\left(\mathbf{F}^\top \widetilde{\mathbf{D}}^{-1} \widetilde{\mathbf{A}} \mathbf{F}\right) = \operatorname{tr}\left(\mathbf{F}\mathbf{F}^\top\right) - \operatorname{tr}\left(\widetilde{\mathbf{D}}^{-1} \widetilde{\mathbf{A}} \mathbf{F}\mathbf{F}^\top\right)$. We denote $\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_n \end{bmatrix}$ and $\mathbf{F}^\top = \left[\mathbf{F}_1^\top \cdots \mathbf{F}_n^\top\right]$, where $\mathbf{F}_i = [\mathbf{F}_{i1} \cdots \mathbf{F}_{id}]$, then we have $\operatorname{tr}\left(\mathbf{F}\mathbf{F}^\top\right) = \sum_{i=1}^n \mathbf{F}_i \mathbf{F}_i^\top$.

When $\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-\frac{1}{2}} \widetilde{\mathbf{A}} \widetilde{\mathbf{D}}^{-\frac{1}{2}}$, we have

$$
\begin{aligned}
&\operatorname{tr}\left(\widetilde{\mathbf{D}}^{-\frac{1}{2}} \widetilde{\mathbf{A}} \widetilde{\mathbf{D}}^{-\frac{1}{2}} \mathbf{F}\mathbf{F}^\top\right) \\
&= \operatorname{tr}\left(\begin{bmatrix} \frac{\mathbf{A}_{11}}{\sqrt{d_1+1}\sqrt{d_1+1}} & \frac{\mathbf{A}_{12}}{\sqrt{d_1+1}\sqrt{d_2+1}} & \cdots & \frac{\mathbf{A}_{1n}}{\sqrt{d_1+1}\sqrt{d_n+1}} \\ \frac{\mathbf{A}_{21}}{\sqrt{d_2+1}\sqrt{d_1+1}} & \frac{\mathbf{A}_{22}}{\sqrt{d_2+1}\sqrt{d_2+1}} & \cdots & \frac{\mathbf{A}_{2n}}{\sqrt{d_2+1}\sqrt{d_n+1}} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\mathbf{A}_{n1}}{\sqrt{d_n+1}\sqrt{d_1+1}} & \frac{\mathbf{A}_{n2}}{\sqrt{d_n+1}\sqrt{d_2+1}} & \cdots & \frac{\mathbf{A}_{nn}}{\sqrt{d_n+1}\sqrt{d_n+1}} \end{bmatrix} \begin{bmatrix} \mathbf{F}_1\mathbf{F}_1^\top & \mathbf{F}_1\mathbf{F}_2^\top & \cdots & \mathbf{F}_1\mathbf{F}_n^\top \\ \mathbf{F}_2\mathbf{F}_1^\top & \mathbf{F}_2\mathbf{F}_2^\top & \cdots & \mathbf{F}_2\mathbf{F}_n^\top \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{F}_n\mathbf{F}_1^\top & \mathbf{F}_n\mathbf{F}_2^\top & \cdots & \mathbf{F}_n\mathbf{F}_n^\top \end{bmatrix} \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \frac{\mathbf{A}_{ij}}{\sqrt{d_i+1}\sqrt{d_j+1}} \mathbf{F}_j \mathbf{F}_i^\top.
\end{aligned}
$$

On the other hand, when $\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-1} \widetilde{\mathbf{A}}$, we have

$$
\begin{aligned}
&\operatorname{tr}\left(\widetilde{\mathbf{D}}^{-1} \widetilde{\mathbf{A}} \mathbf{F}\mathbf{F}^\top\right) \\
&= \operatorname{tr}\left(\begin{bmatrix} \frac{\mathbf{A}_{11}}{d_1+1} & \frac{\mathbf{A}_{12}}{d_1+1} & \cdots & \frac{\mathbf{A}_{1n}}{d_1+1} \\ \frac{\mathbf{A}_{21}}{d_2+1} & \frac{\mathbf{A}_{22}}{d_2+1} & \cdots & \frac{\mathbf{A}_{2n}}{d_2+1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\mathbf{A}_{n1}}{d_n+1} & \frac{\mathbf{A}_{n2}}{d_n+1} & \cdots & \frac{\mathbf{A}_{nn}}{d_n+1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_1\mathbf{F}_1^\top & \mathbf{F}_1\mathbf{F}_2^\top & \cdots & \mathbf{F}_1\mathbf{F}_n^\top \\ \mathbf{F}_2\mathbf{F}_1^\top & \mathbf{F}_2\mathbf{F}_2^\top & \cdots & \mathbf{F}_2\mathbf{F}_n^\top \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{F}_n\mathbf{F}_1^\top & \mathbf{F}_n\mathbf{F}_2^\top & \cdots & \mathbf{F}_n\mathbf{F}_n^\top \end{bmatrix} \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \frac{\mathbf{A}_{ij}}{d_i+1} \mathbf{F}_j \mathbf{F}_i^\top.
\end{aligned}
$$

So when $\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}$, we have

$$
\begin{aligned}
&\operatorname{tr}\left(\mathbf{F}^\top \widetilde{\mathbf{L}}\mathbf{F}\right) \quad \left(\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}\right) \\
&= \operatorname{tr}\left(\mathbf{F}^\top (\mathbf{I} - \widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}})\mathbf{F}\right) \\
&= \operatorname{tr}\left(\mathbf{F}\mathbf{F}^\top\right) - \operatorname{tr}\left(\widetilde{\mathbf{D}}^{-\frac{1}{2}}\widetilde{\mathbf{A}}\widetilde{\mathbf{D}}^{-\frac{1}{2}}\mathbf{F}\mathbf{F}^\top\right) \\
&= \sum_{i=1}^{n} \mathbf{F}_i\mathbf{F}_i^\top - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}}{\sqrt{d_i+1}\sqrt{d_j+1}}\mathbf{F}_j\mathbf{F}_i^\top \\
&= \frac{1}{2}\sum_{i=1}^{n} \mathbf{F}_i\mathbf{F}_i^\top + \frac{1}{2}\sum_{j=1}^{n} \mathbf{F}_j\mathbf{F}_j^\top - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}}{\sqrt{d_i+1}\sqrt{d_j+1}}\mathbf{F}_j\mathbf{F}_i^\top \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \mathbf{F}_i\mathbf{F}_i^\top + \sum_{j=1}^{n} \mathbf{F}_j\mathbf{F}_j^\top - 2\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}}{\sqrt{d_i+1}\sqrt{d_j+1}}\mathbf{F}_j\mathbf{F}_i^\top\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}\mathbf{F}_i\mathbf{F}_i^\top}{d_i+1} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}\mathbf{F}_j\mathbf{F}_j^\top}{d_j+1} - 2\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}}{\sqrt{d_i+1}\sqrt{d_j+1}}\mathbf{F}_j\mathbf{F}_i^\top\right) \text{ undirected graph} \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\mathbf{A}_{ij}\mathbf{F}_i\mathbf{F}_i^\top}{d_i+1} + \frac{\mathbf{A}_{ij}\mathbf{F}_j\mathbf{F}_j^\top}{d_j+1} - \frac{\mathbf{A}_{ij}}{\sqrt{d_i+1}\sqrt{d_j+1}}\mathbf{F}_j\mathbf{F}_i^\top - \frac{\mathbf{A}_{ij}}{\sqrt{d_i+1}\sqrt{d_j+1}}\mathbf{F}_i\mathbf{F}_j^\top\right)\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{A}_{ij}\left(\frac{\mathbf{F}_i\mathbf{F}_i^\top}{d_i+1} + \frac{\mathbf{F}_j\mathbf{F}_j^\top}{d_j+1} - \frac{\mathbf{F}_j\mathbf{F}_i^\top}{\sqrt{d_i+1}\sqrt{d_j+1}} - \frac{\mathbf{F}_i\mathbf{F}_j^\top}{\sqrt{d_i+1}\sqrt{d_j+1}}\right)\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{A}_{ij}\left(\frac{\mathbf{F}_i}{\sqrt{d_i+1}} - \frac{\mathbf{F}_j}{\sqrt{d_j+1}}\right)\left(\frac{\mathbf{F}_i^\top}{\sqrt{d_i+1}} - \frac{\mathbf{F}_j^\top}{\sqrt{d_j+1}}\right)\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{A}_{ij}\left\|\frac{\mathbf{F}_i}{\sqrt{d_i+1}} - \frac{\mathbf{F}_j}{\sqrt{d_j+1}}\right\|_2^2\right) = \sum_{(i,j)\in\mathcal{E}} \mathbf{A}_{ij}\left\|\frac{\mathbf{F}_i}{\sqrt{d_i+1}} - \frac{\mathbf{F}_j}{\sqrt{d_j+1}}\right\|_2^2.
\end{aligned}
$$

On the other hand, when $\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}$, we have

$$
\begin{aligned}
&\operatorname{tr}\left(\mathbf{F}^\top \widetilde{\mathbf{L}}\mathbf{F}\right) \quad \left(\widetilde{\mathbf{L}} = \mathbf{I} - \widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}}\right) \\
&= \operatorname{tr}\left(\mathbf{F}^\top (\mathbf{I} - \widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{A}})\mathbf{F}\right) \\
&= \sum_{i=1}^{n} \mathbf{F}_i\mathbf{F}_i^\top - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}}{d_i+1}\mathbf{F}_j\mathbf{F}_i^\top \\
&= \frac{1}{2}\sum_{i=1}^{n} \mathbf{F}_i\mathbf{F}_i^\top + \frac{1}{2}\sum_{j=1}^{n} \mathbf{F}_j\mathbf{F}_j^\top - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}}{d_i+1}\mathbf{F}_j\mathbf{F}_i^\top \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}\mathbf{F}_i\mathbf{F}_i^\top}{d_i+1} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}\mathbf{F}_j\mathbf{F}_j^\top}{d_i+1} - 2\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\mathbf{A}_{ij}}{\sqrt{d_i+1}\sqrt{d_i+1}}\mathbf{F}_j\mathbf{F}_i^\top\right) \text{ undirected graph} \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{A}_{ij}\left(\frac{\mathbf{F}_i}{\sqrt{d_i+1}} - \frac{\mathbf{F}_j}{\sqrt{d_i+1}}\right)\left(\frac{\mathbf{F}_i^\top}{\sqrt{d_i+1}} - \frac{\mathbf{F}_j^\top}{\sqrt{d_i+1}}\right)\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{A}_{ij}\left\|\frac{\mathbf{F}_i}{\sqrt{d_i+1}} - \frac{\mathbf{F}_j}{\sqrt{d_i+1}}\right\|_2^2\right) = \sum_{(i,j)\in\mathcal{E}} \mathbf{A}_{ij}\left\|\frac{\mathbf{F}_i}{\sqrt{d_i+1}} - \frac{\mathbf{F}_j}{\sqrt{d_i+1}}\right\|_2^2.
\end{aligned}
$$

G Datasets Details

Cora, Citeseer, and Pubmed are standard citation network benchmark datasets [20]. Coauthor-CS and Coauthor-Phy are extracted from Microsoft Academic Graph [21]. Cornell, Texas, Wisconsin, and Actor are constructed by [19]. ogbn-products is a large-scale product, constructed by [10].

Table 4: Datasets statistics

Dataset	# Nodes	# Edges	# Features	# Classes
Cora	2708	5429	1433	7
Citeseer	3327	4732	3703	6
Pubmed	19717	44338	500	3
Cornell	183	295	1703	5
Texas	183	309	1703	5
Wisconsin	251	499	1703	5
Actor	7600	33544	931	5
Coauthor-CS	18333	81894	6805	15
Coauthor-Phy	34493	247962	8415	5
ogbn-products	2449029	61859140	100	42

H Additional Experiments

H.1 Analysis on Row Normalization

Table 5: Summary of results of NGC w/o raw normalization on three datasets in terms of classification accuracy (%)

Noise Level	Cora			Citeseer			Pubmed		
	1	10	100	1	10	100	1	10	100
w/o RN	68.3	59.7	56.1	43.5	40.4	37.6	43.1	38.8	37.4
w RN	66.1	65.5	66.2	45.3	45.1	44.8	62.3	62.7	62.1

In this section, we analyze the influence of row normalization on denoising performance. The noise level ξ controls the magnitude of the Gaussian noise we add to the feature matrix: $\mathbf{X} + \xi\boldsymbol{\eta}$ where $\boldsymbol{\eta}$ is sampled from standard i.i.d., Gaussian distribution. For Cora, Citeseer, and Pubmed, we test $\xi \in \{1, 10, 100\}$. From Table 5, we can observe that the denoising performance of w/ row normalization is better than w/o row normalization. Since row normalization can shrink the value of elements in $\boldsymbol{\eta}$, thus reducing the variance σ. In other words, row normalization make $\left\|\widetilde{\boldsymbol{\mathcal{A}}}_S\boldsymbol{\eta}\right\|_F^2$ converge to zero faster.

H.2 Analysis on the Depth of NGC and RNGC

In this section, we analyze the influence of the depth of NGC and RNGC model on denoising performance by testing the classification accuracy on semi-supervised node classification tasks. We conduct two sets of experiments: with/without noise in feature matrix. For experiment with feature noise, we simple fix the noise level $\xi = 1$. In each set of experiments, we evaluate the test accuracy with respect to NGC and RNGC model depth, which corresponding to the value of S in $\widetilde{\mathcal{A}}_S$. From Figure 4 and 5, we can observe that the test accuracy barely changes with depth if the model is trained on the clean features on Cora and Pubmed but changes greatly if the model is trained on the clean feature on Citeseer. In this regard, the over-smoothing issue exists in RNGC model on citeseer. However, the denoising performance of shallow RNGC is not good as deeper RNGC models,

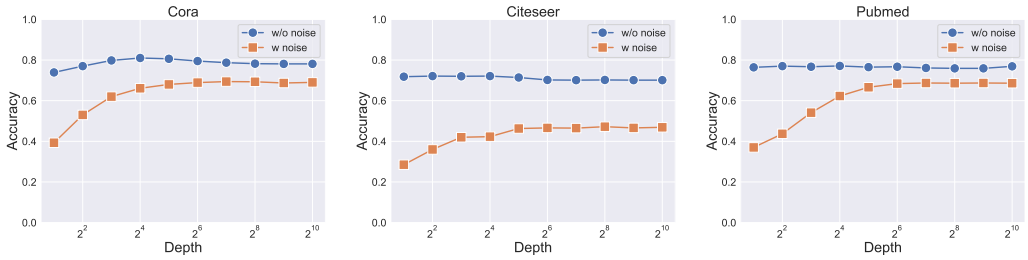

Figure 4: Comparison of classification accuracy v.s. NGC model depth on semi-supervised node classification tasks. The experiments are conducted on clean and noisy features.

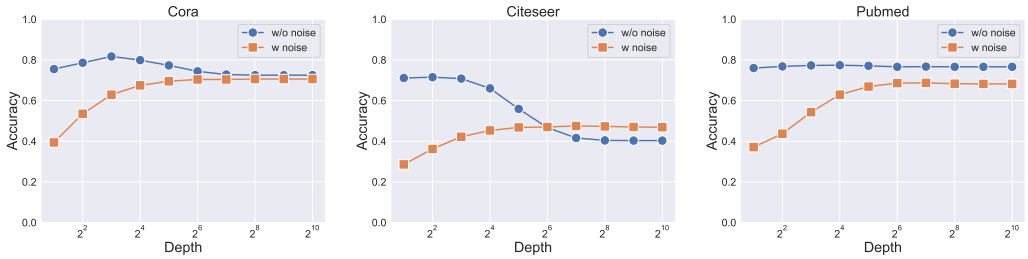

Figure 5: Comparison of classification accuracy v.s. RNGC model depth on semi-supervised node classification tasks. The experiments are conducted on clean and noisy features.

especially on the large graph like Pubmed. This suggests that we do need to increase the depth of GNN model to include more higher-order neighbors for better denoising performances.

Table 6: Denoising performance over 100 runs against flipping perturbation

Flipping probability	Cora			Citeseer			Pubmed		
	0.1	0.2	0.4	0.1	0.2	0.4	0.1	0.2	0.4
MLP	21.2	21.1	23.3	19.3	18.9	18.9	38.0	39.0	40.6
GCN	22.9	19.0	19.0	18.6	18.6	18.5	37.8	38.1	37.6
GAT	70.1	65.6	60.0	45.3	39.3	26.0	43.3	49.5	60.0
GLP	32.3	30.8	29.0	19.7	18.9	18.8	42.1	41.5	40.7
S^2GC	75.0	71.5	63.8	49.9	46.4	43.4	50.4	60.2	69.3
IRLS	66.4	61.0	54.7	50.3	45.9	43.8	51.4	60.0	69.0
NGC	**77.5**	**75.3**	**65.7**	**54.9**	**51.9**	**48.5**	**53.0**	**62.3**	**70.4**
RNGC	**77.6**	**75.2**	**72.8**	**55.0**	**51.8**	**48.7**	**54.3**	**63.9**	**71.6**

Table 7: Defense performance over 100 runs against structure attck

Model	Cora	Citeseer	Pubmed
GCN	47.53	56.94	75.50
GAT	54.78	61.85	65.41
RobustGCN	50.51	55.35	67.95
GCN-Jaccard	60.82	59.89	83.66
GCN-SVD	52.06	57.18	82.72
S^2GC	51.60	54.11	64.04
RNGC	**63.16**	**65.64**	**84.04**

H.3 Denoising Performance on Feature Flipping Perturbation

In this section, we compare the denoising effectiveness of different models through their test accuracy by training on the noisy feature matrix which is perturbated through flipping the individual feature with a small Bernoulli probability on three citation datasets.

Setting and Results. We flip the individual feature on three citation datasets: Cora, Citeseer, and Pubmed as the noise. And we compare the denoising performance of RNGC with MLP and GCN. From Table 6, we can observe that the denoising performance of RNGC is much better than baselines when the flip probability is 0.4. In fact, the added perturbations by flipping the individual feature approximately follow a Bernoulli distribution, which is also a Sub-Gaussian distribution. The results verify our theoretical analysis further.

H.4 Defense Performance of RNGC against Graph Structure Attack

Although we do not perform actual graph structure perturbations as in graph adversarial attacks [35, 36] but a virtual perturbation in the Laplacian. Therefore, it's not clear how much perturbations on the Laplacian correspond to the actual perturbations on graph structure. Nevertheless, we still conduct the experiments of RNGC against graph structure meta-attack where the ptb rate is 25%. As shown in the Table 7, our RNGC model outperforms than GCN, GAT, RobustGCN [37], GCN-Jaccard [27], GCN-SVD [6], and S^2GC on Cora, Citeseer, and Pubmed.