Geodesic Graph Neural Network for Efficient Graph Representation Learning

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Abstract

Recently, Graph Neural Networks (GNNs) have been applied to graph learning tasks and achieved state-of-the-art results. However, many competitive methods employ preprocessing on the target nodes, such as subgraph extraction and customized labeling, to capture some information which is hard to be learned by normal GNNs. Such operations are time-consuming and do not scale to large graphs. In this paper, we propose an efficient GNN framework called Geodesic GNN (GD-GNN). It injects conditional relationship between nodes into the model without labeling. Specifically, we view the shortest paths between two nodes as the spatial graph context of the neighborhood around them. The GNN embeddings of nodes on the shortest paths are used to generate geodesic representations. Conditioned on the geodesic representations, GD-GNN is able to generate node, link, and graph representations that carry much richer structural information than plain GNNs. We theoretically prove that GD-GNN is more powerful than plain GNNs, and present experimental results to show that GD-GNN achieves highly competitive performance with state-of-the-art GNN models on link prediction and graph classification tasks while taking significantly less time.

1 Introduction

Graph Neural Network (GNN) is a type of neural networks that learns on relational data. With the emergence of large-scale network data, it can be applied to solve many real-world problems, including recommender systems [38], protein structure modeling [25], and knowledge graph completion [2]. The growing and versatile nature of graph data poses great challenges to GNN algorithms both on their performance and on their efficiency.

GNNs use message passing to propagate features between connected nodes in the graph, and the nodes aggregate their received messages to generate representations that encode the graph structure and feature information around them. These representations can be combined to form multi-node structural representations. Because GNNs are efficient and have great generalizability, they are widely employed in node-level, edge-level and graph-level tasks. A part of the power of GNNs comes from the fact that they resemble the process of 1-dimensional Weisfeiler-Lehman (1-WL) algorithm [17]. The algorithm encodes subtrees rooted from each node through an iterative node coloring process, and if two nodes have the same color, they have the same rooted subtree and should have very similar surrounding graph structures. However, as pointed out by Xu *et al.* [39], GNN's expressive power is also upper-bounded by the 1-WL test. Specifically, GNN is not able to differentiate nodes that have exactly the same subtrees but have different substructures. For example, consider the graph in Figure 1 with two connected components, because all nodes have the same number of degrees, they will have exactly the same rooted subtrees. Nevertheless, the nodes in the left component is clearly different from nodes in the right component, because the left component is a 3-cycle and the right

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one is a 4-cycle. Such cases can not be discriminated by GNNs or 1-WL test. We refer to this type of GNN as plain GNN or basic GNN.

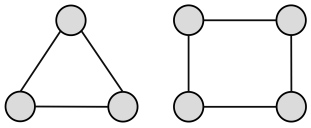


Figure 1: An example where plain GNN fails to distinguish nodes in the graph.

On the other hand, even when a carefully-designed GNN can differentiate all node substructures in a graph, Zhang *et al.* [46] show that the learned node representation still cannot effectively perform structural learning tasks involving multiple nodes, including link predictions and subgraph predictions, because the nodes are not able to learn the relative structural information. We will discuss this with more details in Section 3.1.

These limitations motivate recent research that pushes the limit of GNN's expressiveness. One branch is to use higher-order GNNs that mimic higher-dimensional Weisfeiler-Lehman tests [18, 21]. They extend node-wise message passing to node-tuple-wise and obtain more expressive representations. Another branch is to use labeling methods. By extracting subgraphs around the nodes of interests, they design a set of subgraph-customized labels as additional node features. The labels can be used to distinguish between the nodes of interests and other nodes. The resulting node representations can encode subtructures which plain GNNs cannot, like cycles.

However, the major drawback of these methods is that compared to plain GNNs, their efficiency is significantly compromised. Higher-order GNNs update embeddings over tuples and incur at least cubic complexity. Labeling tricks usually require subgraph extraction on a large graph and running GNNs on mini-batches of subgraphs. Because the subgraphs overlap with each other, the aggregated size of the subgraphs is possibly hundreds times of the large graph. Nowadays GNNs are massive, the increased graph computation cost makes labeling methods inapplicable to real-world problems with millions of nodes.

We observe that a great portion of the labeling methods rely on the geodesics, namely the shortest paths, between nodes. For node representation, Distance-Encoding(DE) [20] labels the target nodes' surrounding nodes with their shortest path distance to the target nodes and achieves higher than 1-WL test expressiveness. Current state-of-the-art methods for link prediction, such as SEAL [43], leverage the labeling tricks in order to capture the shortest path distance along with the number of shortest paths between the two nodes of the link. Our proposed Geodesic-GNN (GD-GNN) model is designed to also capture this characteristic, while addressing the efficiency issue of previous methods.

Specifically, our GD-GNN model learns a function that maps the learning target to a geodesic-augmented target representation (the target can be node, edge, graph, etc). The model consists of two units, a GNN unit and a geodesic pooling unit. The two units are connected to perform end-to-end training. We first apply the GNN to the graph to obtain embeddings of all nodes. Then, for each target, we find the task-specific geodesics and extract the corresponding GNN representations of nodes on the geodesics. The task-specific geodesics have three levels. For node-level tasks, we extract the geodesics between the target node and all of its neighbors. For edge-level tasks, we extract the geodesics between the two target nodes of the edge. For graph-level tasks, we get the geodesic-augmented node representation for each node as in node-level tasks and readout all node representations as the graph representation. Finally, using the geodesic pooling unit, we combine the geodesic representation and target representation to form the final geodesic augmented target representation. Figure 2 summarizes the GD-GNN framework.

All task-specific geodesics are formed by one or multiple **pair-wise** geodesics between two nodes. We propose two methods to extract the pair-wise geodesic information, horizontal or vertical. The horizontal geodesic is obtained by directly extracting one shortest path between the two nodes. The vertical geodesic is obtained by extracting all direct neighbors of the two nodes that are on any of their shortest paths. They focus on the depth and breath respectively, and possess different properties. Horizontal geodesic is good at capturing long distance information not covered by the GNN. Vertical geodesic is *provably more powerful* than plain GNN. Moreover, by incorporating the information of the subgraph induced by the vertical geodesic nodes, we can use GD-GNN to distinguish some *distance-regular graphs*, a type of graph that many current most powerful GNNs cannot distinguish.

During inference, the GNN unit is applied **only once** to the graph, after which we feed nodes and geodesic information to the efficient geodesic pooling unit. This reduces the number of GNN runs from the query number (or number of nodes for graph-level tasks) to a constant of one, which largely reduces the computation cost. *Compared to labeling methods, GD-GNN only requires one GNN run*

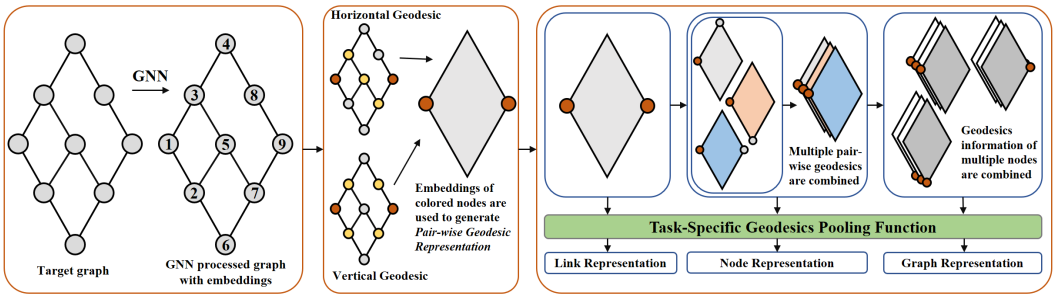


Figure 2: The overall pipeline of GD-GNN. GNN is applied once to generate node embeddings. We then use horizontal or vertical geodesic to extract pair-wise geodesic representation. One or more such representations are collected to generate task-specific representation.

and is much more efficient. Compared to plain GNN methods, GD-GNN uses geodesic information and thus is more powerful. We conducted experiment on link prediction and graph classification across datasets with different scales and show that GD-GNN is able to significantly improve the performance of basic GNNs and achieves very competitive results compared to state-of-the-art methods. We also demonstrate GD-GNN's runtime superiority over other more-expressive GNNs.

2 Preliminaries

A graph can be denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{R})$, where $\mathcal{V} = \{v_1, ..., v_n\}$ is the node set, $\mathcal{E} \subseteq \{(v_i, r, v_j) | v_i, v_j \in \mathcal{V}, r \in \mathcal{R}\}$ is the edge set and $\mathcal{R}$ is the set of edge types in the graph. When $|\mathcal{R}| = 1$, the graph can be referred to as homogeneous graph. When $|\mathcal{R}| > 1$, the graph can be referred to as heterogeneous graph. Knowledge graphs are usually heterogeneous graphs. The nodes can be associated with features $\mathcal{X} = \{\boldsymbol{x}_v | \forall v \in \mathcal{V}\}$. For simplicity, we use homogeneous graphs to illustrate our ideas and methods.

We follow the definition of message passing GNNs in [39] with MESSAGE and AGGREGATE functions. Such GNNs iteratively update the node embeddings by receiving and combining the neighbor nodes' embeddings. The k-th layer of a GNN can be expressed as:

$$\boldsymbol{m}_u^{(k)} = \text{MESSAGE}^{(K)}(\boldsymbol{h}_u^{(k-1)}), \quad \boldsymbol{h}_v^{(k)} = \text{AGGREGATE}(\{\boldsymbol{m}_u^{(k)}, u \in \mathcal{N}(v)\}, \boldsymbol{h}_v^{(k-1)}) \tag{1}$$

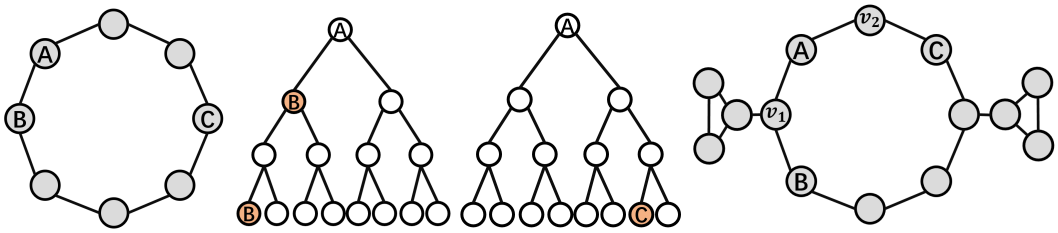
where $\boldsymbol{h}_v^{(k)}$ is the node representation after k iterations, $\boldsymbol{h}_v^{(0)} = \boldsymbol{x}_v$, $\mathcal{N}(v)$ is the set of direct neighbors of v, $\boldsymbol{m}_u$ is the message embedding. Different GNNs vary mainly by the design choice of the MESSAGE and AGGREGATE functions. In this paper, we use $\boldsymbol{h}_v$ to represent the final node representation of v from the plain GNN. Throughout this paper, we use bold letters to represent vectors, capital letters to represent sets, and $\mathcal{N}^k(v)$ to refer to nodes within k-hop neighborhood of v.

3 Geodesic Graph Neural Network

Geodesic is conventionally defined as the shortest path between two nodes and indeed in homogeneous graph it encodes only the distance. However, we found that when combined with GNNs and node representations, geodesics express information much richer than shortest path distance alone. In this section, we first discuss the intuition behind GD-GNN and its effectiveness. Then, we describe the GD-GNN framework and how its GNN unit and geodesic pooling unit are combined to make predictions. Finally, we provide theoretical results of the better expressiveness of GD-GNN.

3.1 Intuition and Overview

The key weakness of directly using plain GNNs is that the embeddings are generated without conditional information. For node-level tasks, as pointed out in [42], a plain GNN is unaware of the position of the target node in the rooted subtree, and hence the target node is oblivious about the conditional information about itself. For edge-level tasks, Zhang *et al.* [46] show that despite a node-most-expressive-GNN, without node labeling, the embedding of one node in the edge is not able to preserve the conditional information of the other node. Concretely, take link prediction as



(a) (b) (c) (d)

Figure 3: (a) A circular graph where plain GNN cannot distinguish link AB and AC. (b) and (c) are the GNN computation graphs of A if B and C, respectively, are labelled. (d) An example where geodesic representation can distinguish links AB and AC, but distance fails to do so.

an example, in Figure 3a, we have two target links AB and AC. Since AB is already connected by a link and AC is not, they should be represented differently. On the other hand, All nodes in the graph are symmetric and should be assigned the same embeddings using any GNN. Hence, if we simply concatenate the node representations of AB and AC, they will have the same link representations, and cannot be differentiated by any downstream learner. To overcome this problem, previous methods rely on labeling tricks to augment the computation graphs with conditional information. When node B and C are labelled respectively in two separate subgraphs, the resulting computation graphs of A is different as shown in Figure 3b and 3c.

Notice that, for the labeling methods to take effect, subgraph extraction is necessary. However, relabeling the graph and applying a GNN to every target to predict is extremely costly. Hence, we wonder whether we can also incorporate the conditional information between nodes without the customized labeling? Observe that in the example, the computation graph captures the number of layers before node A first receives a message from a labelled node, which is essentially the shortest path distance d between A and the other node. This implies that even when we **only** have node representations from a plain GNN, by adding the distance information (which is 1 and 3 for AB and AC respectively) independent of the computation graph, we can still distinguish the links.

While the previous example demonstrates the effectiveness of a simple shortest path distance metric, we notice that distance alone is not sufficient in many times. For example, in Figure 3d, node B and node C are symmetric and hence should be assigned the same node embedding when the graph is not labelled. However, the links AB and AC are apparently different as v_1 connecting AB connects to another group of nodes whereas v_2 connecting AC does not connect to any other node. If we only use shortest path distance between AB and AC to assist the prediction, we still cannot differentiate them. However, if we inject the **entire geodesic** expressed by the corresponding **node embeddings along the shortest path** to the model, we are able to tell AB and AC apart due to v_1 and v_2's different embeddings.

Finding the shortest path can be seen as extracting a chain of nodes whose embeddings are all one, whereas in the neural setting, the nodes on the same chain can have different representations by encoding their own neighborhood. The extension of shortest path to the neural setting grants stronger expressive power to the geodesics. Note that the geodesic generation process and the GNN process is still *decoupled*, because we select the geodesic nodes after all node representations are generated. Dwelling on this decoupling and conditioning philosophy, GD-GNN can run the GNN **only once** to generate generic node embeddings, and then use geodesic information to inject conditional information into the model in a much **more efficient** way.

3.2 Pair-wise Geodesic Representation

GD-GNN has different ways to leverage geodesic information for different levels of tasks, but they all have the same basic component, pair-wise geodesic representation. It consists of structural information of nodes on the shortest paths of the pair. Directly extracting all such nodes is time-consuming and sometimes not effective, hence we present two primary ways to learn the geodesic representation, *horizontal* and *vertical*.

3.2.1 Horizontal Geodesic Representation

Horizontal Geodesic Representation is designed to directly capture the distinctiveness as described in the previous example shown in the Figure 3d. It first finds a shortest path between two nodes as the geodesic. If there are multiple paths between them, we randomly choose one. The nodes on the shortest path is denoted by $P^{(H)}_{(u,v)} = \{u, n_1, n_2, ..., n_{d-1}, v\}$ with a length of d. Using $P^{(H)}_{(u,v)}$ we can construct the geodesic representation $\boldsymbol{g}_{(u,v)}$ by a geodesic pooling layer.

$$\boldsymbol{g}^{(H)}_{(u,v)} = R^{(H)}_{gd}(\{\boldsymbol{h}_w, \forall w \in P^{(H)}_{(u,v)}\}) \tag{2}$$

To prevent extremely long geodesics which may cause overfitting, we limit the maximum length to a manageable constant d_{max}. Any pair with distance larger than d_{max} will have infinite distance and a zero geodesic representation.

Because of the high computation cost of GNNs, the number of GNN layers is usually very small. Using horizontal geodesics, we are able to connect nodes that are *far away* and do not coexist in a GNN's limited-height computation graph.

3.2.2 Vertical Geodesics Representation

Horizontal geodesics seem like an intuitive choice to directly connect the pair of nodes by a sequence of adjacent nodes. However, a natural question to ask is whether a single path is enough to represent the conditional information between the pair? Realizing this limitation, we propose vertical geodesic representation that includes all geodesic information in an efficient way.

For a pair of nodes u and v, vertical geodesic extracts the **direct neighbors** of them that are on **any** of their shortest paths. Specifically, let the distance between u and v be $d(u, v)$, we have the nodes on their shortest paths to be $W_{(u,v)} = \{w | d(u, w) + d(w, v) = d(u, v), \forall w \in \mathcal{V}\}$. The geodesics can be represented as,

$$P^{(V)}_{(u,v)} = P^{(V)}_{(u,v),u} \cup P^{(V)}_{(u,v),v}, \quad \text{where } P^{(V)}_{(u,v),i} := W \cap \mathcal{N}(i), i \in \{u, v\}. \tag{3}$$

Here $P^{(V)}_{(u,v),i}$ represents the nodes in the geodesic that are also the direct neighbors of i. For some tasks, we only consider one $P^{(V)}_{(u,v),i}$ as the geodesics instead of both. We use a pooling layer to generate vertical geodesic

$$\boldsymbol{g}^{(V)}_{(u,v)} = R^{(V)}_{gd}(\{\boldsymbol{h}_w, \forall w \in P^{(V)}_{(u,v)}\}). \tag{4}$$

While the vertical geodesic does not explicitly encode a long shortest path, it can still capture short shortest path information due to pooling the embeddings of the nodes in it. Compared with the horizontal geodesic that takes one random shortest path, the vertical geodesic focuses more on breadth by incorporating all direct neighbors on any shortest paths. Furthermore, when combined with an injective graph pooling function, the vertical geodesic can help distinguish a great portion of regular graphs that plain GNNs cannot. We will demonstrate this in Section 3.4.

Unlike nodes in horizontal geodesic, nodes in vertical geodesic induce a more complex subgraph. Consider only $P^{(V)}_{(u,v),v}$ (we will use P_v here for simplicity) and the subgraph $\mathcal{G}_v$ induced by nodes in P_v. We can have a pooling function which encodes the graph structure of $\mathcal{G}_v$,

$$\boldsymbol{g}^{(V)}_{v} = R^{(G)}_{gd}(\{\boldsymbol{h}_w, \forall w \in P_v\}, \mathcal{G}_v). \tag{5}$$

The pooling layer can be a small plain GNN itself, and the initial embedding of $\mathcal{G}_v$ can take the pooled node embeddings from the main GNN. However, to keep our method efficient, we only add the degree of nodes in the subgraph to the node embeddings. And the vertical geodesic representation becomes,

$$\boldsymbol{g}^{(V)}_{v} = R^{(V)}_{gd}(\{\boldsymbol{h}_w \oplus deg_{\mathcal{G}_v}(w), \forall w \in P_v\}), \tag{6}$$

where $\oplus$ means concatenation. Note that a node's degree in the subgraph is different from that in the main graph. This can be seen as a 1-layer GNN applied onto $\mathcal{G}_v$ with identical initial node embeddings. Even with this simple pooling design, the vertical geodesic is able to help distinguish some distance regular graphs, which we cover in section 3.4.

Because vertical geodesics do not encode distance directly, we concatenate distance to the vertical geodesics as an additional feature. Like in horizontal geodesics, when $d(u, v)$ is greater than d_{max}, we do not generate geodesic representation for the pair, and set their distance to infinity.

3.3 Task specific geodesic representation

Task specific geodesic representation is a collection of pair-wise geodesic representations. Here, we focus on three levels of tasks, node-level, edge-level, and graph-level.

For **node-level** tasks, the goal is to generate a representation $\boldsymbol{z}_v^{(n)}$ for a target node v. We first find v's k-hop neighbor nodes $S_v = \mathcal{N}^k(v)$. Then for each node s_i in S_v, we find the pair-wise geodesic, $\boldsymbol{g}_{v,s_i}$. Note that in node level tasks, vertical geodesic only contains the geodesic nodes close to s_i, $P^{(V)}_{(v,s_i),s_i}$. Then the geodesic representations of the neighborhood is combined by a pooling layer,

$$\boldsymbol{z}_v^{(n)} = R^{(n)}(\boldsymbol{g}_{(v,s_i)}|s_i \in S_v). \tag{7}$$

While the GNN node embeddings of the neighborhood is not conditioned on v, by adding the geodesic information to every neighbor, we inject the conditional information of v to its neighbors. Then the collective information of the neighbors helps distinguish v.

For **edge-level** tasks, the goal is to generate an edge embedding $\boldsymbol{z}^{(e)}_{(u,v)}$ for nodes u and v. Here, we directly use the pair-wise geodesic for the two nodes, $\boldsymbol{z}^{(e)}_{(u,v)} = \boldsymbol{g}_{(u,v)}$. The advantage of this setup is that, when the downstream task is to rank one link against N other links with one same node v, we only need to compute the distance vector from v once and find the geodesic nodes efficiently. This allows us to perform 1-vs-N link prediction on large scale knowledge graphs, where subgraph-based methods may not be applicable.

For **graph-level** tasks, we need to generate a graph embedding $\boldsymbol{z}_{\mathcal{G}}^{(g)}$ for $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We use another pooling layer to summarize the geodesic-augmented representation of every node in the graph,

$$\boldsymbol{z}_{\mathcal{G}}^{(g)} = R^{(g)}(\boldsymbol{z}_v|v \in \mathcal{V}). \tag{8}$$

GD-GNN can also be easily generalized to other levels of tasks such as subgraph-level.

3.4 Expressive Power of GD-GNN

Distinguishing power on graph-level tasks is often used to characterize the expressiveness of a GNN model. For GNN models to be more expressive than 1-WL test, we can test whether they are able to distinguish regular graphs. In the following theorem, we show that given an injective $R^{(g)}$, the graph embedding $\boldsymbol{z}_G$ generated based on vertical geodesics can distinguish most regular graphs.

Theorem 1. *Let the graph pooling layer $R^{(g)}$ be injective given input from a countable space. Consider all pairs of n-sized r-regular graphs, where $3 \leq r < \sqrt{2\log n}$, n is the number of nodes in the graph. For any small constant $\epsilon > 0$, there exists a GD-GNN using vertical geodesic representations with $d_{max} = \lceil (\frac{1}{2} + \epsilon)\frac{\log n}{\log(r-1-\epsilon)} \rceil$ which distinguishes almost all $(1 - o(1))$ such pairs of graphs.*

We include the proof of Theorem 1 in Appendix A. This theorem implies that GD-GNN with vertical geodesics is more expressive than basic GNNs (and 1-WL) and can discriminate almost all r-regular graphs. Moreover, GD-GNN only needs to search a small neighborhood for geodesics to achieve the higher expressiveness. This guarantees the good efficiency and performance of GD-GNN.

Moreover, when the vertical geodesic pooling layer encodes the subgraph induced by the geodesic nodes (using Equation 6 for geodesic pooling), GD-GNN is able to distinguish some distance regular graphs. Figure 4 shows two distance regular graphs, Shrikhande graph and 4x4 rook's graph. They have the same intersection array and number of nodes. As proved by Li *et al.* [20], distance encoding is not able to distinguish this pair of graphs, because nodes in the two graphs will be assigned the same embedding. However, vertical geodesic is able to discriminate them. Suppose the black nodes are the target nodes, the brown nodes are one distance away from the black nodes and the light blue nodes are two distance away from the black nodes. Consider the light blue nodes in both graphs, they each have a pair of brown common neighbors with a black node. The brown pair is the vertical geodesics of one light blue node. If we count the number of edges in the subgraph formed by each brown pair, we will see that four out of nine pairs form a one-edge graph in the Shrikhande graph, while zero out of nine pairs form a one-edge graph in the 4x4 rook's graph. Since Equation 6 directly captures this and will give four light blue nodes in Shrikhande graph different pair-wise geodesics with respect to

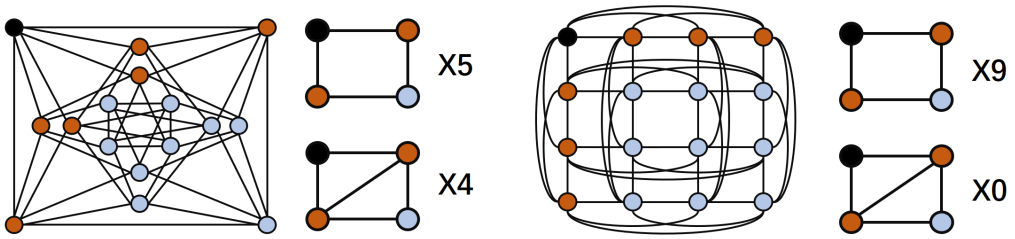


Figure 4: An example of two graphs where GD-GNN with vertical geodesics and subgraph degrees is able distingshui, while distance encoding cannot.

the black node, an injective graph pooling layer will yield different node representations for the two black nodes, and in turn discriminates the two graphs.

Complexity. The main computation advantage of GD-GNN comes from the fact that, for GD-GNN, the number of GNN runs stays as a constant of one, while subgraph-based methods incur a number of GNN runs which increases linearly with respect to the number of queries. Note that the overhead of geodesic finding has the same complexity as that for subgraph extraction. Hence, GD-GNN maintains the good efficiency of basic GNN methods, while injecting the high-expressiveness of subgraph methods into the model. A detailed complexity analysis can be found in Appendix B.

4 Related Works

GNNs beyond 1-WL test. Designing GNNs with stronger representation power is a fundamental task in GNN research. Xu *et al.* [39] first proved that the expressiveness of basic GNNs, such as GCN [15], GraphSAGE [13], and GAT [36], is upper-bounded by the 1-WL test. Following works conducted extensive research to break this limit. They can be sorted into several directions: (1) Earlier works proposed higher-order GNNs that are as expressive as k-WL algorithm, such as k-GNN [18], RingGNN[6], and PPGN[21]. These methods require message passing on node tuples that grows exponentially with respect to k. The increased computation cost makes them not scalable to large graphs. (2) Another type of methods, like GSN [4] and MotifNet [22], incorporates isomorphism counting on certain graph substructures like cycles, triangles, and cliques, similar to earlier kernel-based methods [40, 11]. They essentially augment the models with features undetectable by GNNs. However, a properly-design substructure usually requires expert knowledge which is not available for all data. (3) Some other works use identity or random node labeling within the plain GNN framework. ID-GNN [42] extracts a subgraph around the target node and perform heterogeneous message passing. DE-GNN [20] also requires subgraph extraction but labels nodes in the subgraph with distance to the targets. They both have stronger expressiveness than basic GNNs. Abboud *et al.* [1] shows that GNNs possess universality with random node feature initialization (RNI). Nevertheless, RNI also induces a huge sample space, and GNNs adopting RNI is very hard to train. (4) Some recent works propose to use additional subgraph structures to augment the node embeddings. GraphSNN [37] incorporates properties of overlapping subgraphs between adjacent nodes to be more expressive than 1-WL. Nested-GNN [45] applies GNN to ego-subgraphs of every node and proves that it is able to discriminate regular graphs. Sharing the same idea of Nested-GNN, GNN-AK[47] uses a different subgraph pooling strategy and uses subgraph dropping to improve the model scalability. Our work is fundamentally different from existing models. It does not require subgraph extraction and inject the conditional information between nodes via geodesic selection. This keeps our model both efficient and more expressive than basic GNNs.

Link prediction using GNNs. Earlier works [16, 15, 7, 35] applied basic GNNs to link prediction using an auto-encoder framework, where the GNN serves as the encoder of the nodes, and edges are decoded by their nodes' encoding vectors. Another line of works, including SEAL [43], IGMC [44], and GraIL [32], use GNNs to encode a labelled subgraph to form the link representation. Later work [30, 46] reveals that a good node-level/graph-level representation does not necessarily lead to a good edge-level representation. Zhang *et al.* [46] systematically show that the labeling trick bridges the gap between node-level and edge-level GNN expressiveness. Sharing the spirits of labeling trick, NBFNet [49] extends the generalized Bellman-Ford algorithm to the neural setting, and can encode many traditional path-based methods. NBFNet has a better scalability when there are multiple links sharing one same node, but has the same complexity as other labeling methods when all links in the

query set have different nodes. On the contrary, our GD-GNN maintains the same good efficiency in any scenario. Some other works also propose to use path information between links to enhance the prediction. WalkPooling [23] computes the random-walk probability to form a random walk profile for link prediction. However, this method is still subgraph-based, whereas GD-GNN is also able to encode path information and remains computationally friendly in large graph context.

5 Experimental Results

5.1 Link Prediction

Datasets. We use several types of datasets for link prediction. (1) Knowledge Graph (KG) inductive link prediction datasets. Inductive link prediction means that the model is trained on a training graph while tested on a different graph. This requires the model to generalize to unseen entities, which is a property preserved by a limited number of models. We follow the standard inductive split of WN18RR[9] and FB15K237[34] as in [32]. We rank each positive link against 50 randomly sampled negative links with the same head/tail as the positive link, and report Hit@10 ratio. (2) OGB large-scale link prediction dataset [14], including OGBL-COLLAB and OGBL-PPA. We use the official split provided by the OGB team. We rank each positive link against a set of provided negative links in the dataset, and report Hit@N ratio. (3) Results of citation datasets can be found in Appendix D. (4) Full results on the knowledge graph datasets are included in Appendix C. These datasets vary by graph size and demonstrate GD-GNN's capability across scales.

Baselines. We compare GD-GNN with Rules Mining methods, DRUM [26] and NeuralLP [41], and GNN-based methods, NBFNet [49] and GraIL [32] for the knowledge graph tasks. For the OGB dataset, we compare GD-GNNs with other GNN-based methods achieving top places on the OGB link prediction leaderboard, including Deep Walk [24], Resource Allocation (RA) [48], SEAL, Deeper GCN [19].

Implementation Details. For link prediction, we use GCN (RGCN for knowledge graph) as the basic GNN. For all the datasets, we search the number of GNN layers in $\{2, 3, 4, 5\}$, and the max search distance for geodesic, d_{max}, is same as the GNN layers. We use 32 hidden dimensions for all fully-connected layers in the model. We train 50 epochs with a batch size of 64 for the knowledge graph datasets, and we train 50 epochs with a batch size of 4096 for the OGB datasets. We record the test score using the model with the best set of hyperparameters chosen based on the performance on the validation set. The experiment is repeated 5 times and we take the average of the scores.

Table 1: Link prediction results (%) on KG datasets.

Method	FB15K-237		WN18RR	
	v1	v3	v1	v3
DRUM	52.9	52.9	74.4	46.2
NeuralLP	52.9	52.9	74.4	46.2
GraIL	64.2	82.8	82.5	58.4
NBFNet	83.4	95.1	94.8	89.3
GD-GNN-Vert	**85.4**	**98.2**	**95.7**	87.7
GD-GNN-Hor	**85.4**	93.4	93.1	**89.5**

Table 2: Link prediction results (%) on OGB datasets.

	OBGL-COLLAB (H@50)	OGBL-PPA (H@100)
GCN	44.75 ± 1.45	18.67 ± 1.32
DEEP WALK	50.37 ± 0.34	28.88 ± 1.63
RA	-	**49.33** ± 0.00
DeeperGCN	52.73 ± 0.47	-
SEAL	54.37 ± 0.49	48.80 ± 3.16
GD-GNN-Vert	**54.68** ± 0.51	45.61 ± 2.58
GD-GNN-Hor	51.42 ± 0.72	32.17 ± 3.70

Results and discussion. We achieved SOTA performance on the inductive datasets. GD-GNN is able to significantly improve over rule-mining methods and subgraph based method, GraIL. We also outperform NBFNet on most of the datasets. Note that NBFNet is only efficient when the query links contain many nodes in common, whereas GD-GNN's efficiency is not bounded by such limitation. On the OGBL-COLLAB dataset (Table 2), vertical GD-GNN achieved SOTA performance. On the OGBL-PPA dataset, despite being outperformed by top performance methods, GD-GNN is able to keep a very competitive result. On both OGB datasets, GD-GNN improves significantly over its basic GNN, GCN (a relative increase of 22% for OGBL-COLLAB and 144% for OGBL-PPA). Moreover, as shown in Table 3, GD-GNN has running-time superiority over methods achieving comparable performance, and has memory advantage to handle large graphs where NBFNet can not. To make fair running time comparison, we run all models on 32 CPUs and 1 Nvidia GeForce 1080Ti GPU. We also include ablation studies on all components of GD-GNN in Appendix E.

Table 3: Link prediction running time results.

Method	OGBL-COLLAB	OGBL-PPA
GCN	27 sec	17 min
SEAL	90 sec	2 hr 20 min
NBFNet	OOM	OOM
GD-GNN-Vert	36 sec	33 min

Table 4: Graph classification running time results.

Method	OGBG-MOLHIV	OGBG-MOLPCBA
GIN	20 sec	2 min
GIN-AK	70 sec	12 min
Nested-GIN	149 sec	16 min
GD-GNN-Vert	25 sec	2.5 min

5.2 Graph Classification

Datasets. We use two types of datasets for graph classification. (1) TU datasets contain D&D[10], MUTAG[8], PROTEINS[10], PTC_MR[33], ENZYMES[28]. The evaluation metric is Accuracy (%). We follow the evaluation protocol in [45], where we use 10-fold cross-validation to compute the score. (2) OGB datasets[14], including OGBG-MOLHIV and OGBG-MOLPCBA. The task of OGBG-MOLHIV dataset is to predict whether a molecule inhibits HIV virus. We use AUC to evaluate OGBG-MOLHIV dataset. OGBG-MOLPCBA dataset has 128 classification tasks. We follow the evaluation protocol provided by the OGB team and use AP averaged across all classes to evaluate the dataset.

Baselines. For the TU dataset, we compare with another plug-and-play GNN framework, NGNN[45]. Both GD-GNN and NGNN use GIN and GCN as their basic GNNs. For the OGB dataset, we use GIN as the basic GNN, we compare GD-GNN with methods achieving top places on the OGB leaderboard, including GCN[15], GIN[39], GSN[4], GINE [5], GIN-AK [47], Nested GNN [45].

Implementation details. We use vertical geodesic in the graph-level tasks because of its theoretical advantage over horizontal GD-GNN. For TU datasets, we search the number of GNN layers in {2,3,4}, and keep the geodesics searching distance the same as the GNN layers. We train on the TU dataset for 100 epochs with a batch size of 64. For OGB datasets, we search the number of GNN layers in {2,3,4,5}, and train for 100 epochs with a batch size of 256. Following previous successful practice, we adopt virtual node technique [12] to increase the connectivity of graphs for the OGBG-MOLPCBA dataset.

Table 5: Graph classification results (%) on TU datasets.

	D&D	MUTAG	PROTEINS	PTC_MR	ENZYMES
Avg. #nodes	284.32	17.93	39.06	14.29	32.63
GCN	71.6 ± 2.8	73.4 ± 10.8	71.7 ± 4.7	56.4 ± 7.1	27.3 ± 5.5
GIN	71.6 ± 3.0	74.0 ± 8.8	71.2 ± 5.2	57.0 ± 5.5	30.7 ± 6.3
Nested GCN	76.3 ± 3.8	82.9 ± 11.1	73.3 ± 4.0	57.3 ± 7.7	31.2 ± 6.7
Nested GIN	**77.8** ± 3.9	87.9 ± 8.2	**73.9** ± 5.1	54.1 ± 7.7	29.0 ± 8.0
GD-GCN-Vert	77.6 ± 4.0	88.4 ± 6.6	73.7 ± 3.4	**60.3** ± 4.5	**32.3** ± 5.1
GD-GIN-Vert	**77.8** ± 3.6	**89.4** ± 7.1	73.6 ± 2.5	57.9 ± 3.4	31.6 ± 3.3

Table 6: Graph classification results (%) on OGB datasets.

Method	OGBG-MOLHIV (AUC)		OGBG-MOLPCBA (AP)	
	Validation	Test	Validation	Test
GCN	82.04 ± 1.41	76.06 ± 0.97	24.95 ± 0.42	24.24 ± 0.34
GIN	82.32 ± 0.90	75.58 ± 1.40	27.98 ± 0.25	27.03 ± 0.23
GSN	86.58 ± 0.84	77.99 ± 1.00	-	-
GINE	-	-	30.65 ± 0.3	29.17 ± 0.15
GIN-AK	-	78.22 ± 0.75	30.47 ± 0.07	**29.30** ± 0.44
Nested GIN	83.17 ± 1.99	78.34 ± 1.86	29.15 ± 0.35	28.32 ± 0.41
GD-GIN-Vert	82.41 ± 1.13	**78.43** ± 1.02	29.09 ± 0.38	28.17 ± 0.34

Results and discussions. On the TU dataset (Table 5), GD-GNNs consistently outperform the basic GNNs with an average relative performance increase of 11.8%. Compared to NGNN (7.9% relative performance increase), GD-GNN is able to bring larger performance increase to the basic GNN. This shows GD-GNN's enhancing power. On the OGB dataset (Table 6), note that the GNN we use for GD-GNN, GIN, is not the most powerful GNN, but GD-GNN is still able to achieve very competitive results compared to usually heavily-tuned methods on the OGB leaderboard. Similarly, GD-GNN is able to increase the performance of GIN by a significant amount. We also note the efficiency advantage of GD-GNN in Table 4. GD-GNN is able to maintain a running time very close to the base GNN and significantly improve over other similar performance methods. To study the effect of different components of GD-GNN, we include ablation studies of graph-level geodesics in Appendix E.

6 Conclusions

In this paper, we propose a general GNN framework which incorporates geodesic information to generate node, link, and graph representations. Our framework is more powerful than basic GNNs and is more efficient than current more-expressive GNNs. Theoretical results show that our models can distinguish almost all regular graphs (which 1-WL always fails to) and even some distance regular graphs (which distance encoding fails to). Experimental results show that our framework achieved very competitive performance to state-of-the-art models with significantly less time. Our framework has the potential to make GNNs encoding conditional information really applicable to large-scale real-world graphs.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] We faithfully summarize the paper in the abstract.
   (b) Did you describe the limitations of your work? [Yes] As described in the Section 5 and Appendix C.
   (c) Did you discuss any potential negative societal impacts of your work? [No] Our work only involves existing datasets and solves general graph learning problem without negative societal impact.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] Assumption is described in the Theorem 1
   (b) Did you include complete proofs of all theoretical results? [Yes] In Appendix A
3. If you ran experiments...

(a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]

(b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] As described in the implementation details of Section 5

(c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] As shown in the experiment result tables.

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] In the discussion of Section 5

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

(a) If your work uses existing assets, did you cite the creators? [Yes] The datasets and model source code repositories are all cited.

(b) Did you mention the license of the assets? [N/A]

(c) Did you include any new assets either in the supplemental material or as a URL? [No]

(d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [No] The datasets and source code are open source information allowed for research use.

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [No] The data we are using is known to be de-identified.

5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

Appendix

A Proof of Theorem 1

The proof of Theorem 1 generally follows the proof of Nested-GNN's expressiveness[45]. Let $Q^k_{v,\mathcal{G}}$ be the set of nodes in $\mathcal{G}$ that are exactly k distance away from node v. We can have the following definition,

Definition A.1. *The edge configuration bewteen $Q^k_{v,\mathcal{G}}$ and $Q^{k+1}_{v,\mathcal{G}}$ is a list $C^k_{v,\mathcal{G}} = (a^{1,k}_{v,\mathcal{G}}, a^{2,k}_{v,\mathcal{G}}, ...)$ where $a^{i,k}_{v,\mathcal{G}}$ denotes the number of nodes in $Q^{k+1}_{v,\mathcal{G}}$ of which each has exactly i edges from $Q^k_{v,\mathcal{G}}$.*

The proof leverages the fact that nodes in randomly sampled regular graphs are very likely to have different edge configuration within their small neighborhood, and GD-GNN directly captures this distinction to discriminate the graphs. Formally, we have the following excerpted Lemma from [45],

Lemma 2. *For two graphs $\mathcal{G}^{(1)} = (\mathcal{V}^{(1)}, \mathcal{E}^{(1)})$ and $\mathcal{G}^{(2)} = (\mathcal{V}^{(2)}, \mathcal{E}^{(2)})$ that are uniformly independently sampled from all n-node r-regular graphs, where $3 \leq r < \sqrt{2\log n}$, we pick any two nodes, each from one graph, denoted by v_1 and v_2 respectively. Then there is at least one $i \in (\frac{1}{2}\frac{\log n}{\log(r-1-\epsilon)}, (\frac{1}{2}+\epsilon)\frac{\log n}{\log(r-1-\epsilon)})$ with probability $1 - o(n^{-1})$ such that $C^i_{v_1,\mathcal{G}^{(1)}} \neq C^i_{v_2,\mathcal{G}^{(2)}}$.*

Proof. To prove Theorem 1, we first assume, without loss of generality, that the hidden dimension of GD-GNN is one. We know plain message passing GNN will generate the same embeddings for every node in the regular graph, because all nodes have the same degree/subtree. Hence, we can normalize all the node representations to a constant of one. For simplicity, we also reduce vertical geodesic pooling function $R^{(V)}_{gd}$ in Equation 4 to a simple sum of the representations of the geodesic nodes.

Let $d_{max} = \lceil(\frac{1}{2}+\epsilon)\frac{\log n}{\log(r-1-\epsilon)}\rceil$, the maximum distance in which GD-GNN searches for a target node's neighbors. We then consider node v_i and it's d_{max}-hop rooted subgraph $\mathcal{G}^{d_{max}}_{v_i}$, the vertical geodesic pooling process is essentially counting the number of geodesic nodes of each node v_j in $\mathcal{G}_{v_i}$. And the pair-wise vertical geodesics between the v_i and v_j becomes,

$$\boldsymbol{g}^{(V)}_{v_i,v_j} = |\{P^{(V)}_{v_j}\}| \oplus d(v_i, v_j) \tag{9}$$

where $d(v_i, v_j)$ is the distance between v_i and v_j, and hence $v_j \in Q^{d(v_i,v_j)}_{v_i,\mathcal{G}_{v_i}}$ and $\boldsymbol{g}^{(V)}_{v_i,v_j}$ records the number of edges of v_j that connects to nodes in $Q^{d(v_i,v_j)-1}_{v_i,\mathcal{G}_{v_i}}$, and its distance to v_i. We also have the node-level pooling function to be a distance sorted bin count function on the set of pair-wise vertical geodesic representations. It first sorts the geodesic representations by their second dimension (distance), and perform bin count on their first dimension (number of geodesic nodes) separately for each distance. Then GD-GNN produces an mapping of $\mathcal{G}_{v_i}$ to its exact set of edge configurations, $\mathcal{C}_{v_i} = (C^1_{v_i,\mathcal{G}_{v_i}}, C^2_{v_i,\mathcal{G}_{v_i}}, ...)$.

Let $\mathcal{G}^{(1)} = (\mathcal{V}^{(1)}, \mathcal{E}^{(1)})$ and $\mathcal{G}^{(2)} = (\mathcal{V}^{(2)}, \mathcal{E}^{(2)})$ be two graphs uniformly independently sampled from all n-node r-regular graphs. We consider two nodes $v_i \in \mathcal{V}^{(1)}$ and $v_j \in \mathcal{V}^{(2)}$. Because $d_{max} = \lceil(\frac{1}{2}+\epsilon)\frac{\log n}{\log(r-1-\epsilon)}\rceil$, the sets of edge configurations of the rooted subgraphs, $\mathcal{C}_{v_i}$ and $\mathcal{C}_{v_j}$ are different with probability $1 - o(n^{-1})$, according to Lemma 2. In such case, GD-GNN also maps their rooted subgraphs to different edge configurations, and hence the node-level geodesic representation $\boldsymbol{z}^{(n)}_{v_i}$ and $\boldsymbol{z}^{(n)}_{v_j}$ generated by GN-GNN will be different with the same probability.

Consider v_i and all node $v_j \in \mathcal{V}^{(2)}$, by union bound, $\boldsymbol{z}_{v_i} \notin \{\boldsymbol{z}_{v_j} | v_j \in \mathcal{V}^{(2)}\}$ with a probability of $1 - o(1)$. □

B Complexity Analysis

The complexity of GD-GNN differs by tasks. On almost all tasks, GD-GNN demonstrates superiority in efficiency. Take graph-level task as an example, to evaluate a graph with N nodes, GD-GNN needs to find node-level geodesics N times. For each node-level geodesic, GD-GNN computes the shortest distances between the node and its local subgraph, this takes $O(ae)$ where a is the average number of

nodes in the subgraph and e is the average degree of nodes. In the local subgraph. The total number of nodes included in the geodesic is also bounded by the number of edges $O(ae)$. For each node-level geodesic, we apply a pooling layer with complexity linear to the number of geodesic nodes, and the complexity of geodesic pooling unit is $O(aec)$, where c is the GNN hidden dimension. The overall complexity of evaluating a N nodes graph is $O(T|\mathcal{E}|c + N \cdot aec)$, where $O(T|\mathcal{E}|c)$ is the complexity of one run of the GCN. Since $|\mathcal{E}| = O(Ne)$, we can rewrite the complexity as $O(NTec + Naec)$. T is almost always smaller than a, we have the overall complexity to be $O(Naec)$. Consider a labeling based method, its complexity is $O(NTaec)$, for applying GNN N times to N extracted subgraphs. Comparing the two complexity, GD-GNN amortizes the expensive computation cost of GNN. In reality, the architecture of GNN layer is more complex with hidden dimension larger than the output dimension and even has internal MLP layers. This will exacerbates the already heavy computation load of the labeling method, whereas GD-GNN can apply the same complex GNN once, and enjoy the efficient inference complexity using a control-sized embedding. The same argument applies to edge-level and node-level tasks in a large graph, the more prediction targets we have, the better the amortized complexity of GD-GNN. Hence, GD-GNN has great scalability.

C Knowledge graph link prediction

For knowledge graphs, we use RGCN[27] as our base GNN. We evaluate GD-GNN on FB15K237[34] and WN18RR[9] knowledge graphs for both inductive and transductive link prediction tasks. We use the standard split in [34] for transductive link prediction, and follow [32] for inductive link prediction. We compare with Rules Mining methods, DRUM [26] and NeuralLP [41], and GNN-based methods, NBFNet [49] and GraIL [32], for the inductive link prediction tasks. For transductive tasks, we additionally compared to general knowledge graph link prediction methods that are not inductive, including TransE [3], RotatE [31], and RGCN [27]. For inductive dataset, we randomly sample 50 negative links that have either the same head or tail as the positive link, we report the H@10 against the 50 negative links. For the transductive dataset, we follow the filtered ranking protocol [3], where we rank a positive link against all negative links that have the same head (or tail) as the positive links. We report H@N and Mean Reciprocal Rank (MRR) for the transductive setting.

Table 7: Knowledge graph inductive link prediction results.

Method	FB15K-237				WN18RR			
	v1	v2	v3	v4	v1	v2	v3	v4
DRUM	52.9	58.7	52.9	55.9	74.4	68.9	46.2	67.1
NeuralLP	52.9	58.9	52.9	55.9	74.4	68.9	46.2	67.1
GraIL	64.2	81.2	82.8	89.3	82.5	78.7	58.4	73.4
NBFNet	83.4	94.9	95.1	96.0	94.8	**90.5**	89.3	**89.0**
GD-GNN-Vert	**85.4**	95.6	**98.2**	**97.8**	**95.7**	89.8	87.7	85.6
GD-GNN-Hor	**85.4**	**96.7**	93.4	**97.7**	93.1	88.0	**89.5**	88.5

Table 8: Knowledge graph transductive link prediction results.

Method	FB15K-237				WN18RR			
	MRR	H@1	H@3	H@10	MRR	H@1	H@3	H@10
DRUM	34.3	25.5	37.8	51.6	48.6	42.5	51.3	58.6
NeuralLP	24.0	-	-	36.2	43.5	37.1	43.4	56.6
TransE	29.4	-	-	46.5	22.6	-	-	50.1
RotatE	33.8	24.1	37.5	55.3	47.6	42.8	49.2	57.1
RGCN	27.3	18.2	30.3	45.6	40.2	34.5	43.7	49.4
NBFNet	**41.5**	**32.1**	**45.4**	**59.9**	**55.1**	**49.7**	**57.3**	**66.6**
GD-GNN-Vert	23.2	15.8	26.4	45.1	46.2	39.3	48.6	59.1
GD-GNN-Hor	25.1	16.2	28.7	44.9	43.2	35.2	47.2	58.0

We achieved SOTA performance on the inductive datasets. GD-GNN is able to significantly improve over rule-mining methods and subgraph based method, GraIL. We also outperform NBFNet on most of the datasets. We note that horizontal geodesic is able to exceed the performance of vertical geodesic on some datasets. We suspect this is because horizontal geodesics implicitly capture the inter-node translation process as in the embedding methods, and this piece of information is more important than graph structure in the knowledge graph setting. We also acknowledge that GD-GNN does not perform as well in the transductive setting. However, GD-GNN is still able to greatly improve the performance of RGCN on WN18RR dataset, and RGCN requires trainable node embeddings while GD-GNN does not. We suspect the performance discrepancy between the homogeneous setting and the knowledge graph setting is due to the increased number of edge types. WN18RR dataset has 11 types of relations, FB15K237 has 237. Note that GNN methods like NBFNet[49] train a set of weights for every target relation type, meaning that their message passing is conditioned on the relation, while GD-GNN only uses one set of weights. We notice that GD-GNN is able to outperform some embedding methods on WN18RR dataset, while not performing as good on FB15K237, possibly due to the fact that FB15K237 dataset contains much more relations than WN18RR dataset. This aligns with our hypothesis. A potential solution is to also train multiple GD-GNNs that each handles one target relation type, note that we still only need to run each relation-specific GD-GNN once, and keep the good amortized property of GD-GNN. We leave this to future work.

D Citation dataset results

Citation datasets[29] include Cora, Citeseer, and Pubmed datasets. We follow the experimental setting in [49] and use 85:5:10 for the train/valid/test links split. The evaluation metric is ROC-AUC (AUC) and Average Precision (AP) scores. We compare GD-GNN with popular link prediction methods VGAE [16], SEAL[43], NBFNet[49] for the citation datasets. Note that for citation data, we do not use the pre-trained node embeddings from the publications' content, incorporating such information only increases the power of GD-GNN.

We use GCN as the basic GNN. We search the number of GNN layers in $\{2, 3, 4, 5\}$, and the max search distance for geodesic, d_{max}, is same as the GNN layers. We use 32 hidden dimensions for all fully-connected layers in the model. We train 100 epochs with a batch size of 64 for the citation datasets.

Table 9: Link prediction results (%) on Citation dataset.

Method	Cora		Citeseer		PubMed	
	AUC	AP	AUC	AP	AUC	AP
VGAE	91.40	92.60	90.80	92.00	94.40	94.70
SEAL	93.32	94.21	90.52	92.43	97.78	97.90
NBFNet	**95.61**	**96.17**	**92.28**	**93.74**	**98.30**	**98.15**
GD-GNN-Vert	94.93	**96.01**	91.72	93.29	98.06	**98.14**
GD-GNN-Hor	94.30	95.22	91.18	92.34	97.74	97.76

On the citation dataset (Table 9), GD-GNN achieved very competitive results compared to the SOTA method NBFNet, and GD-GNN is more efficient than NBFNet as demonstrated in Table 3.

E Ablation study

GD-GNN has many components, and here we present ablation study of GD-GNN to demonstrate the effect of each component. For link prediction dataset, we consider 4 settings, plain GCN (GCN), GCN and distance between the two target nodes of the link (GCN+Dist), GD-GNN-Vertical without geodesic degree (GD-GNN-Vert), and GD-GNN-Vertical with geodesics node degree (GD-GNN-Vert-Deg). Note that the results we report in Section 5 include geodesic node degree.

From the experiment results (Table 10), we can see that distance is already able to assist link prediction, especially on sparser datasets like Cora and OGBL-COLLAB. However, on OGBL-PPA, where the connectivity of the graph is much larger, distance does not help, whereas geodesic representation is able to significantly improve the expressive power of basic GNNs. Also, when the graph is sparse, it

Table 10: Ablation study on link prediction datasets.

Method	Cora (AUC)	OGBL-COLLAB (H@50)	OGBL-PPA (H@100)
GCN	81.79	44.75	18.67
GCN+DIST	92.75	51.82	21.48
GD-GNN-Vert	94.13	54.19	43.91
GD-GNN-Vert-Deg	94.26	54.68	45.61

is rather unlikely for the geodesic nodes to form subgraphs, meaning that most of nodes on the vertical geodesic have a degree of zero, and hence geodesic degree does not improve much. Meanwhile, in OBGL-PPA, geodesic degree is able to increase the performance of GD-GNN, which aligns with the intuition and example we present.

For graph classification tasks, we study two simpler version of GD-GNN. We reduce the vertical pair-wise geodesic function (Equation 4) to be a pooling function only on the embeddings of the pair of nodes. Then, the node-level geodesic is essentially combining the node embedding and its k-hop neighbor embedding, we refer to this as Nei. Another version is we append the pair-wise distance to the neighbor embeddings, we refer to this as Dist.

Table 11: Ablation study on graph classification datasets.

Method	MUTAG (Accuracy)	PROTEINS (Accuracy)	OGBG-MOLHIV (AUC)
GIN	74.0	71.2	75.5
GIN+Nei	81.2	71.9	76.2
GIN+Nei+Dist	88.1	71.8	76.0
GD-GNN-Vert	89.0	73.3	77.6
GD-GNN-Vert-Deg	89.4	73.6	78.4

From Table 11, we can see that the neighbor embedding alone is not very effective, and we need to assist it with pair-wise distance, a explicit form of geodesic to improve the performance. By incorporating the geodesic representation, we are able to consistently improve over the 'Nei+Dist' version, which aligns with Theorem 1. Geodesic degree does not give significant improvement on TU dataset possibly due to its high testing variance, but we can still see notable improvement on OGBG-MOLHIV dataset.