Empirical Gateaux Derivatives for Causal Inference

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Abstract

We study a constructive procedure that approximates Gateaux derivatives for statistical functionals by finite-differencing, with attention to causal inference functionals. We focus on the case where probability distributions are not known a priori but need also to be estimated from data, leading to empirical Gateaux derivatives, and study relationships between empirical, numerical, and analytical Gateaux derivatives. Starting with a case study of counterfactual mean estimation, we verify the exact relationship between finite-differences and the analytical Gateaux derivative. We then derive requirements on the rates of numerical approximation in perturbation and smoothing that preserve statistical benefits of the resulting estimation procedure via one-step adjustment, such as rate-double-robustness. We study more complicated functionals such as dynamic treatment regimes and the linear-programming formulation for policy optimization infinite-horizon Markov decision processes. The newfound ability to approximate bias adjustments in the presence of arbitrary constraints illustrates the usefulness of constructive approaches for Gateaux derivatives. We find that the statistical structure of the functional (rate-double robustness) can permit less conservative rates of finite-difference approximation. This however depends on the functional – it occurs for the counterfactual mean but not the infinite-horizon MDP policy value.

1 Introduction

Inferential targets in causal machine learning are often statistical functionals of the data distribution, for example an average treatment effect or average policy value for infinite-horizon offline reinforcement learning. Estimation of these statistical functionals can be vulnerable to the first-stage bias introduced by estimating nuisance functions such as outcome regressions or transition probabilities. However, one can derive bias-adjusted estimators by leveraging the causal structure and weaken requirements on estimation of these nuisance functions. The celebrated doubly robust estimator is one such bias adjustment, and general frameworks exist to derive and verify influence functions, e.g. double robustness, semiparametric inference, Neyman orthogonality, "debiased/double" machine learning [7, 49, 44, 10, 11]. These general frameworks can also be applied to problems in offline reinforcement learning (also called offline policy evaluation/learning) [58, 36, 55, 5, 37, 54, 38, 39].

Despite extensive literature deriving bias-corrected estimators, these analytic derivations often require significant effort in case-specific analysis, making it challenging to extend to new settings. On the other hand, there are classes of estimands where an analyst may be interested in many variants. For example, in *constrained* MDPs [3], an analyst can easily computationally specify custom linear or convex constraints in the linear programming formulation. In these settings, each possible choice could require a case-specific re-analysis and deduction for deriving bias adjustment. Therefore, complementary to analytic derivations, it may be useful to also develop constructive or algorithmic approaches to obtain some bias adjustment.

Submitted to 36th Conference on Neural Information Processing Systems (NeurIPS 2022). Do not distribute.

To this end, we study algorithmic and constructive procedures for obtaining bias adjustments for statistical functionals. We build on a recent line of work of [FQWD, CLL, IN], which proposes a constructive procedure by numerical approximation of a Gateaux derivative (in the sense of [26, 32]) by finite differences, but with a smoothed perturbation rather than a point mass. In relation to this line of work, we additionally consider interactions between statistical estimation and numerical approximation, where the probability distributions required to evaluate the statistical functional need also to be estimated. Our analysis and concretizations illustrate the exact relationship between the numerical approximation and the analytical derivative. However, in general, it is well-known that the finer-scaled discretization required for lower approximation error eventually incurs fundamental issues with floating-point computation that prohibit further gains in accuracy. Our analysis shows how the statistical structure of causal functionals weakens the required approximation error rates. Therefore we expand the computational regimes where numerical differentiation is applicable.

Our contributions are as follows: We begin with a case study of counterfactual mean estimation. In this setting, we can characterize the exact relationship between finite-differences and the analytical Gateaux derivative. This helps concretize the proposal by allowing us to study rates of *numerical* approximation that preserve *statistical* performance. Finally, with these concretizations in hand, we return to the more complicated motivating examples. We illustrate with the case of policy learning in finite-state, finite-action infinite-horizon Markov decision processes how this approach can directly extend also to the case of custom constraints.

2 Problem Setup: From Numerical to Empirical Gateaux derivatives

Setup. Denote $O \sim P$ as the observations following distribution P belonging to a statistical model $\mathcal{M}$. We will focus on targeting the estimation of statistical functionals $\Psi(P)$, where $\Psi : \mathcal{M} \to \mathbb{R}^q$ is pathwise differentiable. Throughout, we generally assume conditions required for causal identification (i.e. overlap/positivity, unconfoundedness). Examples of functionals include counterfactual means.

Example 1 (Counterfactual mean.)**.** *The observation $O = (X, A, Y)$ includes covariate, treatment $A \in \{0, 1\}$, and outcome $Y \in \mathbb{R}$. The statistical functional is:*

$$\Psi(P) = \mathbb{E}[Y(1)] = \mathbb{E}_P[\mathbb{E}_P[Y \mid A = 1, X]].$$

Influence function and Gateaux derivative. A functional Ψ is Gateaux differentiable if $\Psi'(H; P) \triangleq \frac{d\Psi(P+\epsilon H)}{d\epsilon}|_{\epsilon=0}$ exists and is linear and continuous in H for all H.

[26, 32] defines the *influence function* $\phi(O; P)$ as a Gateaux derivative with respect to perturbations $H = \delta_o - P$ toward a degenerate Dirac measure δ_o satisfying the identity that $\int o\delta_{o'}(o) = o'$,

$$\phi(o; P) = \frac{d\Psi\left(P + \epsilon(\delta_o - P)\right)}{d\epsilon}\Big|_{\epsilon=0}.$$

Related work: Numerical Gateaux derivatives. Although the above characterization is insufficient for continuous distributions, [CLL, IN] show that it suffices to restore absolute continuity in semiparametric models by taking an additional limit as a *smoothed perturbation* converges to the point mass. That is, it suffices to replace the Dirac delta measure δ_o with a smoothed distribution $\tilde{\delta}^{\lambda}_{o'}(o)$,a smoothed delta function perturbed in the direction of a generic observation o', evaluated at o , dominated by P, such that $\int g(u)d\tilde{\delta}^{\lambda}_{o'}(u) = 1$. A common choice for $\tilde{\delta}^{\lambda}_{o'}$ uses kernel functions $\tilde{\delta}^{\lambda}_{o'}(o) = K_\lambda(o - o')$, where K_λ is an appropriately bandwidth-normalized kernel function and notation is overloaded to also designate product kernels. The perturbed distribution in the direction of a generic observation o_i is $P^{o_i}_{\epsilon,\lambda} = P^{i}_{\epsilon,\lambda} = (1 - \epsilon)P + \epsilon\tilde{\delta}^{\lambda}_{o_i}$. [CLL, Thm. 1] establishes that the Gateaux derivative is the limit as the paths and smoothing parameter, ϵ and λ, tend to 0. They generally assume all distributions are known in the analysis and code. [9, Thm. 5] provided generic approximation rates in (ϵ, λ) for multi-point finite difference scheme using generic results. For a two-point approximation with a uniform kernel, the rate from [9, Thm. 5] is $\left(\epsilon\lambda^{-d}\right) + \lambda^{-2}$, which established general validity at conservative rates. Our analysis in this section weakens these approximation error requirements.

[FQWD, CLL] both discuss *numerical* approximation by finite-differences: for a fixed ϵ, λ with $\epsilon \ll \lambda$, evaluate the numerical derivative $\hat{\phi}(o; P) = \epsilon^{-1}(\Psi(P_{\epsilon,\lambda}) - \Psi(P))$. On the other hand, [28, 41] highlight influence function derivations based on the analytic Gateaux derivative.

Empirical Gateaux derivatives: Plug-in estimation of the functional. In the context of statistical estimation, the underlying probability distributions also need to be estimated. Therefore, to ensure validity of the numerical derivative approach for a general class of functionals, we consider *plug-in estimation* of the statistical functional by conducting probability density estimation of the collection of joint probability estimates $\tilde{p}(o)$ that comprise $\tilde{P}$. For example, a *plug-in* representation of the counterfactual mean is $\Psi(\tilde{P}) = \int\int y \frac{\tilde{p}(y, A=1, x)}{\tilde{p}(A=1, x)} dydx$.

Hence, we distinguish between the *empirical Gateaux derivative* computed by numerical approximation with estimated probability densities (e.g. $\tilde{P}(Y, A=1, X)$), the *numerical* Gateaux derivative obtained by finite-differencing but with the true probability densities (e.g. $P(Y, A=1, X)$), and the analytic Gateaux derivative.

$$\tilde{\phi}(O_i) = \epsilon^{-1}\left(\Psi(\tilde{P}^i_\epsilon) - \Psi(\tilde{P})\right) \quad \text{empirical derivative at smoothed and estimated distributions,}$$

$$\hat{\phi}(O_i) = \epsilon^{-1}\left(\Psi(P^i_\epsilon) - \Psi(P)\right) \quad \text{numerical derivative at smoothed and true distributions,}$$

$$\phi(O_i) = \frac{d}{d\epsilon}\Psi(P^i_\epsilon)\Big|_{\epsilon=0} \quad \text{analytical Gateaux derivative.}$$

The *one-step estimator* is an immediate way to use computationally estimated Gateaux derivatives for statistical estimators [47]; justified by the asymptotically linear representation (see Appendix B).

$$\Psi_n = \Psi(\tilde{P}) + \frac{1}{n}\sum_{i=1}^n \tilde{\phi}(O_i), \qquad \text{where } \tilde{\phi}_i(O_i) = \frac{1}{\epsilon}(\Psi(\tilde{P}^i_\epsilon) - \Psi(\tilde{P})) \tag{1}$$

Although for simplicity in the main text we consider first-order finite difference methods, of course multipoint methods for numerical differentiation can also be used.

Problem statement. About the empirical, numerical, and analytical derivatives, we answer:

Q1: How exactly does the empirical Gateaux derivative approximate the analytical derivative?

Q2: What are the required rates of *numerical approximation* in perturbation and smoothing (ϵ, λ) that preserve the beneficial *statistical* properties of the constructed estimator?

In Section 3, we first address these questions in the simple case of our running example: estimation of the counterfactual mean, where we can finely compare the finite-difference estimator and the classical doubly robust estimator to characterize statistical properties. In Section 4, we study more complicated functionals to highlight more general applicability.

Example 2 (Example 1, continued.)**.** *The influence function [49, 25] for a counterfactual mean is:*

$$\phi(O) = \frac{\mathbb{I}[A=1]}{p(A=1|X)}(Y - \mathbb{E}[Y \mid A=1, X]) + \mathbb{E}[Y \mid A=1, X] - \Psi(P).$$

Hence the canonical doubly-robust estimator (augmented inverse propensity weighting, or AIPW) can be viewed as a one-step adjustment.

Remark 1. *We omit discussion of semiparametric efficiency since we focus on computing influence functions without the discussion of model restrictions and tangent spaces. Although [CLL, IN] focus in particular on semiparametric models, we focus on Gateaux derivatives under a nonparametric model; [CLL] suggest handling evaluating a Gateaux derivative and projecting onto the model space directly. Hence our analysis can be viewed as considering the case of nonparametric models; or analyzing the first step of a procedure for semiparametric models.*

Remark 2. *Throughout, we assume the functional is pathwise differentiable. This does warrant caution. Empirical Gateaux derivatives are intended to augment but not supplant or replace analytical expertise. However, if the functional is assumed to be Frechet rather than Gateaux differentiable, previous works give sufficient conditions that imply the validity of these approaches. [33, sec. 4] provides a generic sufficient condition for asymptotic linearity assuming additionally that the Frechet derivative is Lipschitz while [40] show the remainder term is second order; both assume convergence rate conditions on the density inputs.*

3 Case study: Empirical Gateaux derivative of the counterfactual mean

To bolster confidence in the numerical procedure, we work through the running example of the counterfactual mean and concretize precisely how the *numerical differentiation* incurs error relative to the analytical derivative, i.e. the doubly robust or augmented IPW (AIPW) estimator.

3.1 Relating the empirical and analytical Gateaux derivatives

Setup. We target estimation of a counterfactual mean, Example 1. For the smoothed perturbation, we specialize to kernel functions $K_\lambda(u)$ from kernel density estimation satisfying that $\int_{-\infty}^{\infty} K(u)du = 1$ (normalization to a probability density) and $K(-u) = K(u), \forall u$ (symmetry).[1] The counterfactual mean is identified as:

$$\Psi(P) = \mathbb{E}[Y(1)] = \mathbb{E}[\mathbb{E}[Y \mid A = 1, X]] = \int\int yp(y \mid A = 1, x)dydx = \int\int y\frac{p(y, A=1, x)}{p(A=1, x)}p(x)dydx \tag{2}$$

We obtain $\Psi(\tilde{P}_\epsilon)$ by plugging in the $\tilde{p}$ probability density estimates. Note this is distinct from the plug-in estimator in empirical process theory, which plugs in the empirical CDF; we plug-in statistical estimates. See Algorithm 1 for a summary of the procedure.

Comparison of empirical Gateaux derivative and known doubly robust adjustment. We have argued that in the context of statistical estimation, approximating Gateaux derivatives requires plug-in estimation of the statistical functional with estimated probabilities. Towards resolving Q1, we observe that the perturbed probability densities induce the functional form of the nuisance functions in the analytical derivative. This is in contrast to the analytical derivative, which describes the estimand in terms of, e.g. conditional expectations, and allows the analyst to choose the functional form for estimating the nuisance function.

We will denote the $\tilde{P}$-induced conditional expectation as

$$\mathbb{E}_{\tilde{P}}[Y \mid A = 1, X = x] = \int y\tilde{p}(y \mid A = 1, x)dy.$$

For example, the density-induced conditional expectation of kernel density estimates is exactly Nadaraya-Watson regression.[2] Given this, we study empirical derivatives with kernel density estimation. We study KDEs because they provide a classic textbook example of density estimates that induce conditional expectation estimators; our analysis is modular. In Appendix E we discuss other choices of density estimates: any "modern" generative modeling approach/density estimation approach can be used for plug-in evaluation of the statistical functional.

Our characterization will show exactly how plug-in evaluation with finite differences is nearly equivalent to evaluating the doubly-robust estimator with a Nadaraya-Watson regressor as the conditional outcome regression for $\mathbb{E}[Y \mid A = 1, X]$ and a kernel density estimate of the propensity score; up to an additive bias that arises due to the smoothed perturbation distribution, and what we will term a *smoothed nuisance evaluation*. We next discuss what this smoothed nuisance function is, and how we study its approximation error relative to estimation error of the induced nuisance.

Integrating the smoothed perturbation induces smoothed nuisance functions. Integrating with respect to the smoothed perturbation distribution can be interpreted as a smoothed evaluation of the resulting nuisance functions. We define the $\tilde{\delta}^\lambda_{x_0}$-smoothed conditional expectation which smooths the evaluation point, i.e. smooths evaluation around x_0 rather than precisely at x_0.

Definition 1 ($\tilde{\delta}^\lambda_{x_0}$-smoothed conditional expectation)**.**

$$\tilde{\mathbb{E}}_P[Y \mid A = 1, X = x_0] = \int \mathbb{E}_P[Y \mid A = 1, X = u]\ \tilde{\delta}^\lambda_{x_0}(u)\ du \tag{3}$$

Similar ideas appeared in the literature on non-regular inference, for example smoothing a non-regular target estimand, such as a dose-response curve or density evaluated at a point x_0, by conducting a smoothed evaluation locally around x_0 [59]. In that work, a function evaluated at x_0, $g(x_0)$, is approximated with a kernel smooth over x in a neighborhood of x_0 with bandwidth h; so $\Psi_\lambda(x_0) = \int K_\lambda(u - x_0)g(u)du$. The following lemma, similar to [59, Lemma 25.1], summarizes convergence rate implications of these corresponding smoothed nuisances; these are direct consequences of typical analyses of kernel density estimation.

[1] Examples include the Gaussian kernel with $K(u) = (2\pi)^{-\frac{1}{2}}e^{-u^2/2}$, or uniform with $K(u) = {}^1/_2\mathbb{I}\left[|u| \leq 1\right]$. We generally consider $K_\lambda(u) = \lambda^{-d}K(u/\lambda), K(x) = K\left(x_1, \ldots, x_d\right)$.

[2] For example,

$$\hat{\mu}^{NW}(x) = \int \frac{y\sum_{i=1}^n K_h(x-X_i)K_h(y-Y_i)}{\sum_{j=1}^n K_h\left(x-X_j\right)}dy = \frac{\sum_{i=1}^n K_h(x-X_i)\int yK_h(y-Y_i)dy}{\sum_{j=1}^n K_h\left(x-X_j\right)} = \frac{\sum_{i=1}^n K_h(x-X_i)y_i}{\sum_{j=1}^n K_h\left(x-X_j\right)}.$$

Algorithm 1 Empirical Gateaux derivatives

1: Inputs: $\tilde{P}$ probability density estimates, data $\{O_i\}_{1:n}$, perturbation and smoothing (ϵ, λ) parameters, functional Ψ
2: **for** $i = 1, \dots n$ **do**
3: $\quad \tilde{P}^i_\epsilon = (1-\epsilon)\tilde{P} + \epsilon \tilde{\delta}^\lambda_{o_i}$
4: $\quad \tilde{\phi}(O_i) \leftarrow \epsilon^{-1}(\Psi(\tilde{P}^i_\epsilon) - \Psi(\tilde{P}))$
5: **end for**
6: Output: one-step adjusted estimate from empirical Gateaux derivatives, $\Psi(P) + \frac{1}{n}\sum_i \tilde{\phi}(o_i)$

Lemma 1. *Let $\tilde{\mu}_\epsilon(X) = \tilde{E}_{\tilde{P}_\epsilon}[Y \mid A = 1, X]$ and $\tilde{e}_\epsilon(X) = \tilde{p}_\epsilon(A = 1, X)/\tilde{p}(X)$ denote the nuisances induced by plug-in estimation of probability distributions $\tilde{p}$. Assume that $\tilde{\delta}^\lambda$ is a product kernel, and $\int |u|^\beta |\tilde{\delta}^\lambda_u| \,\mathrm{d}u < \infty$ and $\int u^s \tilde{\delta}^\lambda_u \,\mathrm{d}u = 0$ for $s \leq \beta$. For any bounded function $g(x)$ in a Hölder class of degree β, $(\int \tilde{\delta}^\lambda_x(u) g(u) \,\mathrm{d}u - g(x))^2 \,\mathrm{d}x = O(\lambda^\beta)$. And, the perturbed nuisances satisfy the following rates:*

$$\mathbb{E}[(\tilde{\mu}_\epsilon(X) - \mu(X))^2] = \mathbb{E}[(\tilde{\mu}(X) - \mu(X))^2] + O(\epsilon^2 \lambda^{-d}),$$
$$\mathbb{E}[(\tilde{e}_\epsilon(X) - e(X))^2] = \mathbb{E}[(\tilde{e}(X) - e(X))^2] + O(\epsilon^2 \lambda^{-d}).$$

The perturbed nuisances are asymptotically consistent, so long as $\tilde{p}$ induces asymptotically consistent nuisances and we fix ϵ as a sequence vanishing in n at appropriate rate to counteract the growth in λ due to "roughness", ie the variability of $\tilde{\delta}^\lambda$. The difference between the function evaluation and its smoothed nuisance arises from the bias analysis of kernel density estimators and the dimension-dependence of ensuring absolute continuity, but it is a non-stochastic, entirely deterministic argument. While the analysis improves mildly in the dimension dependence on λ compared to the sufficient general analysis of [CLL, Thm. 5], this still highlights the generally unfavorable dependence of the numerical approach on the dimension due to smoothing.

Our first proposition establishes the exact approximation error induced by finite-differencing.

Proposition 1 (Gateaux derivative of probability density representation)**.** *Consider perturbations in the direction of $o_i = (x_i, a_i, y_i)$. Let $\tilde{p}(x), \tilde{p}(A = 1, x), \tilde{p}(y, A = 1, x)$ denote kernel density estimates. Let $\Psi(P)$ be as in eq.* (2)*.*

$$\frac{\Psi(\tilde{P}_{\epsilon_i}) - \Psi(\tilde{P})}{\epsilon} = \int \frac{\tilde{p}(x)\mathbb{I}[a_i = 1]\,\tilde{\delta}^\lambda_{x_i}(x)}{\tilde{p}_\epsilon(A = 1, x)} \left\{ \left(\int y \; \tilde{\delta}^\lambda_{y_i}(y) dy \right) - \mathbb{E}_{\tilde{P}}[Y \mid A = 1, X = x] \right\} dx$$
$$+ \left(\tilde{\mathbb{E}}_{\tilde{P}_\epsilon}[Y \mid A = 1, X = x_i] - \Psi(\tilde{P}) \right).$$

We compare our expansion of Proposition 1 to the canonical AIPW estimator in Example 2. We specialized to the case of a uniform kernel for $\tilde{\delta}^\lambda_{y_i}(y)$ so that $\int y \; \tilde{\delta}^\lambda_{y_i}(y) dy = y_i$. We next observe that the convolution with respect to the smoothing kernel introduces additive bias relative to evaluating terms in the expression at the observation o_i. Lemma 1 allows us to further simplify and conclude that the decomposition of Proposition 1 is close to the canonical AIPW estimator up to the $O(\lambda^\beta)$ bias induced by smoothed evaluation.

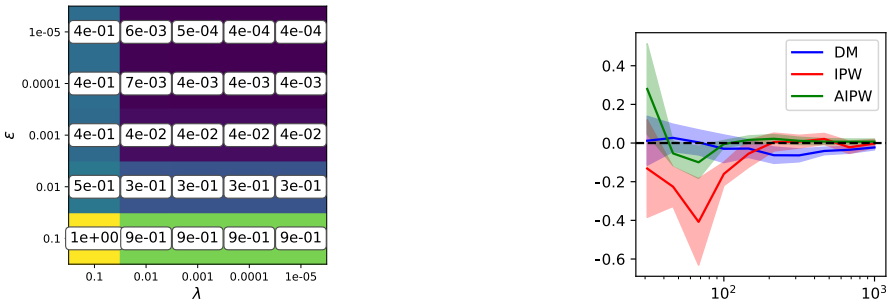
Corollary 1. *When the perturbation observation is the observation datapoint $O_i = (X_i, A_i, Y_i)$,*

$$\tilde{\phi}(O_i) = \frac{\mathbb{I}[A_i = 1]}{\tilde{p}_\epsilon(A = 1 \mid X_i)} \left(Y_i - \mathbb{E}_{\tilde{P}}[Y \mid A = 1, X_i]\right) + \left(\mathbb{E}_{\tilde{P}_\epsilon}[Y \mid A = 1, X_i] - \Psi(\tilde{P}) \right) + O(\lambda^\beta).$$

3.2 Implications for estimation

Next, we use our characterization of Proposition 1, Lemma 1, and Corollary 1 to study the *statistical* properties of our *computational/numerical* approximation. Corollary 1 is deterministic and allows us to study the relationship to the typical "oracle" AIPW estimator; albeit with induced ϵ-perturbed nuisances.

An appealing property of bias-adjusted treatment effect estimation is *rate-double robustness*: because we incur the product of convergence rates of the nuisances, we may enjoy parametric $n^{-\frac{1}{2}}$-rate convergence of the target functional while nuisances converge at slower rates, for example both



(a) (b)

Figure 1: (a) Epsilon-lambda plot for counterfactual mean: MAE in approximation of one-step adjustment. (b) Error in estimation of $\mathbb{E}[Y(1)]$ over n.

at $n^{-\frac{1}{4}}$; see [11, 51] for more detail. We use the rate-double robustness property of the target and Lemma 1 to infer the required rate conditions on (ϵ, λ) that would retain the beneficial statistical properties of the canonical AIPW estimator.

Assumption 1 (Regularity conditions.)**.** *(i) Y is bounded.*

(ii) $p(y \mid A = 1, x), p(A = 1 \mid x)$ are in Hölder classes of minimum degree β.

(iii) The estimates $p(y, a, x), p(a, x), p(x)$ belong to a Donsker function class.

(iv) Assume $\tilde{\mu}, \tilde{e}$ satisfy the product-rate condition:

$$\mathbb{E}[(\tilde{\mu}_\epsilon(X) - \mu(X))^2] \times \mathbb{E}[(\tilde{e}_\epsilon(X) - e(X))^2] = o_p(n^{-1})$$

Assume $\tilde{\mu}, \tilde{e}$ are RMSE-consistent with r_μ, r_e, respectively, so that $r_\mu + r_e \geq \frac{1}{2}$.

Theorem 1 (Rate double-robustness)**.** *Consider the one step-estimator with the empirical Gateaux derivative. Under Assumption 1, when $\epsilon\lambda^{-d/2} = o(n^{-\max(r_\mu, r_e)})$, and $\lambda^\beta = o(n^{-\frac{1}{2}})$:*

$$(\Psi(\tilde{P}) + \tfrac{1}{n}\textstyle\sum_{i=1}^n \tilde{\phi}(O_i)) - \Psi(P) = O_p(n^{-\frac{1}{2}}).$$

We now interpret the qualitative estimation implications of the analysis. Theorem 1 verifies the rate conditions of *numerical approximation* in ϵ, λ that preserve the *statistical property* of rate double robustness. Assuming that the nuisances induced by the probability density estimations satisfy a product-rate condition that the product of their RMSEs is faster than the parametric $O(n^{-\frac{1}{2}})$ rate, the form of Proposition 1 suggests that the rate of RMSE convergence of perturbed nuisances is required to be faster than the rates of the unperturbed nuisance functions, $\max(r_\mu, r_e)$, which can be a slower rate than the $\epsilon = o(n^{-\frac{1}{2}})$ implied by the generic analysis of finite differences.

In general, the dimension dependence on λ in Lemma 1 will prevent setting a slower rate for ϵ for nontrivial dimensions. Note that without the smoothing requirement, the potential improvement in the rate of ϵ could be on the order of generic rate improvements implied by a central difference scheme. That is: we would have attained the approximation benefits of generically approximation-improved finite difference schemes via the specialized *statistical* structure of the target adjustments.

Our analysis did use our characterization in Corollary 1, i.e. a deterministic approximation error from AIPW with induced nuisance functions.

4 Applications: counterfactual mean and dynamic settings

We first show an epsilon-lambda plot in the simple case of the counterfactual mean as a numerical validation. Then, with some of the tools we derived in the previous section, we analyze more complicated settings: longitudinal data (dynamic treatment regime) and the policy linear program for an infinite-horizon Markov decision with linear constraints, with attention to how different functionals have different interactions between approximation error and statistical properties.

4.1 Counterfactual mean

We first conduct a validation of our decomposition and theoretical characterization. Our data generating process includes a piecewise linear outcome model (such that kernel regression is misspecified). We conduct plug-in evaluation of the counterfactual mean with kernel density estimates. In Figure 1a we include an (ϵ, λ)-plot for the simple case of AIPW (see [9] for more discussion and examples). We consider a one-dimensional case with uniformly distributed X, piecewise-linear Y, and smooth propensity scores that are logistic in $\sin(X)$. We use $n = 500$ and fix the bandwidth $h = 0.05$. We use colors to denote magnitude of the mean absolute error, included in text on the heatmap. Without loss of generality, we study the estimation of a mean under missingness, $\mathbb{E}[Y(1)]$. Figure 1b illustrates the estimation error of various strategies with the comparable kernel-based estimates (DM is regression adjustment): even in this simple setting AIPW offers benefits and sample efficiency relative to DM. We include further numerical experiments in the Appendix.

4.2 Dynamic treatment regimes

We study the estimation properties of the empirical Gateaux derivative for multi-stage DTR from the probability density representation. In T-stage dynamic treatment regimes, the causal quantity of interest is the counterfactual mean outcome $\mathbb{E}\left[Y(\bar{a})\right]$, where $\bar{a} = (a_0, \ldots, a_T)$ is the (deterministic) treatment strategy. Assuming $Y(\bar{a})$ is sequentially ignorable given treatment and covariate history $(\bar{A}_t, \bar{X}_t)$ at each time t, this causal quantity can be identified by the g-formula,

$$\mathbb{E}\left[Y(\bar{a})\right] = \int \mathbb{E}\left[Y \mid \bar{A} = \bar{a}, \bar{X} = \bar{x}\right] \prod_{t=1}^{T} \tilde{p}(x_t \mid \bar{a}_{t-1}, \bar{x}_{t-1}) \, \mathrm{d}\bar{x},$$

where $\bar{A} = (A_0, \ldots, A_t), \bar{X} = (X_0, \ldots, X_t)$. To derive its influence function, we take the empirical Gateaux derivative by considering the perturbation in the direction of $o_i = (\bar{x}_i, \bar{a}_i, y_i)$, where $\bar{x}_i = (x_{i0}, \ldots, x_{it})$ and $\bar{a}_i = (a_{i0}, \ldots, a_{it})$,

$$\Psi(\tilde{P}_\epsilon^i) = \int \left(\int y \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})} \, \mathrm{d}y \right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \tilde{p}_\epsilon(x_0) \, \mathrm{d}\bar{x}. \tag{4}$$

Below we characterize how this empirical Gateaux derivative at the smoothed distribution differs from the one at the (unsmoothed) estimated distribution. The following result is analogous to Proposition 1 and corollary 1 but is extended to the dynamic treatment regimes.

Proposition 2.

$$\begin{aligned}
&\epsilon^{-1}(\Psi(\tilde{P}_\epsilon^i) - \Psi(\tilde{P})) \\
&= \left(\mathbb{E}_{\tilde{p}(x_{1:T})} \left[\mathbb{E}_{\tilde{P}_\epsilon} \left[Y \mid \bar{A}, \bar{X} \right] \mid \bar{x}_0 \right] - \Psi(\tilde{P}) \right) + \frac{\mathbb{I}\left[\bar{A} = \bar{a}_i\right]}{\tilde{p}_\epsilon(\bar{a} \mid \bar{x})} \left[Y_i - \mathbb{E}_{\tilde{p}(\bar{x})} \left[\mathbb{E}_{\tilde{P}} \left[Y \mid \bar{A}, \bar{X} \right] \right] \right] \\
&\quad + \sum_{s=1}^{T} \frac{\mathbb{I}\left[\bar{A} = \bar{a}_{i,T-s}\right]}{\tilde{p}_\epsilon(\bar{a}_{T-s} \mid \bar{x}_{T-s})} \left\{ \mathbb{E}_{\tilde{P}} \left[\mathbb{E}_{\tilde{P}_\epsilon} \left[Y \mid \bar{A}, \bar{X} \right] \mid \bar{a}_{i,T-s}, \bar{x}_{i,T-s+1} \right] \right. \\
&\qquad \left. - \mathbb{E}_{\tilde{P}} \left[\mathbb{E}_{\tilde{P}_\epsilon} \left[Y \mid \bar{A}, \bar{X} \right] \mid \bar{a}_{i,T-s}, \bar{x}_{i,T-s} \right] \right\} + O(\epsilon^2) + O(\lambda^\beta)
\end{aligned} \tag{5}$$

The expansion verifies that the requirements for numerical approximation are similar in ϵ as in the case of a single-timestep counterfactual mean, despite the additional complexity of the dynamic treatment regime function as a nested expectation. Again, smoothing incurs overall unfavorable dependence in the dimension, as the extension of Lemma 1 to $\mathbb{E}_{\tilde{P}_\epsilon}[Y \mid \bar{A}, \bar{X}]$ also incurs unfavorable dependence on the dimension. The history-dependent nuisances imply that the dimension additionally grows in time horizon.

4.3 Perturbations for (constrained) infinite-horizon off-policy evaluation

We show how this framework can be applied to derive generic bias corrections for infinite-horizon off-policy optimization in offline reinforcement learning, via the canonical *linear programming* characterization of [48, 14]. Our application is relevant to a recent line of work on off-policy evaluation and learning (OPE/L) [36, 55, 5, 37, 54, 38, 39].

Setup. We recall the classical linear programming formulation of finding an optimal policy in a tabular (finite-state, finite-action) Markov decision process, due to [48, 14]. Recent work

has revisited this formulation with interest in developing primal-dual algorithms [61, 43, 43, 1, 52]. The dynamics follow an infinite-horizon Markov decision process with discount factor γ. The offline dataset comprises of trajectories of state, action, next state observations: $\{(s_i^0, a_i^0, \ldots, s_i^t, \ldots, s_i^T)\}_{1:N}$. We derive our bias adjustment under a statistical model where the stationary distribution factorizes as $p(s, a, s') = d(s, a)P(s \mid s, a)$. From the joint state-action-state occupancy distribution p(s,a,s'), we estimate the marginalization $d(s, a)$, the stationary state-action occupancy distribution, which we use to estimate the transition probability $P(s' \mid a, s) = \frac{p(s,a,s')}{d(s,a)}$. Let $\mu_0(s)$ denote the initial state distribution, estimated from offline data and assumed to be equivalent in the original confounded and evaluation setting. Because we focus on the tabular case with discrete distributions, we do not require smoothing.

The optimal policy linear program. The *optimal* value function solves the following linear program in a finite-state and finite-action setting; the objective value at optimality is the policy value. Here, $P_a \in \mathbb{R}^{|S|\times|S|}$ is transition matrix at action a, with $P_a(s, s') = P(s' \mid s, a)$, $\mu_0(s)$ is the marginal occupancy distribution of the initial state. We assume, as is fairly common in theoretical analyses of reinforcement learning, that the reward function is a known function of state and action, $r(s, a)$. Let $\mathbf{r}_a = \{r(s, a)\}_{s\in\mathcal{S}}$ be the vector of reward values per state, for fixed action a.

$$\Psi_D(P) = \min_V \{(1-\gamma)\mu_{\mathbf{0}}{}^\top \mathbf{V} \colon\ (I - \gamma P_a)\,\mathbf{V} - \mathbf{r}_a \geq 0, \quad \forall a \in \mathcal{A}\} \tag{6}$$

The dual formulation is well-known to parametrize the stationary occupancy probabilities of the optimal policy (at optimality):

$$\Psi_P(P) = \max_\mu \left\{\textstyle\sum_{a\in\mathcal{A}} \mu_a^\top \mathbf{r_a} \colon\ \sum_{a\in\mathcal{A}} (I - \gamma P_a^\top)\mu_a = (1-\gamma)\mu_{\mathbf{0}}, \quad \mu_a \geq 0, \quad \forall a \in \mathcal{A}\right\} \tag{7}$$

We verify the bias correction from the Gateaux derivative $\frac{d}{d\epsilon}\Psi_D(P_\epsilon)\Big|_{\epsilon=0}$. Note that evaluating this functional at the perturbed distribution corresponds to evaluating the linear program with perturbed left-hand side coefficients, i.e. perturbing the constraints of the coefficient matrix as well as the objective (marginalizing over a perturbed initial state distribution. We evaluate the Gateaux derivative in the direction of $(\tilde{s}, \tilde{a}, \tilde{s}')$. We assume nondegeneracy (unique optimal solution); this can be relaxed by using more sophisticated perturbation analysis.

Assumption 2. *The optimal solution is unique; there is no degeneracy.*

Proposition 3. *Perturb in the direction of a generic observation* $(\tilde{s}, \tilde{a}, \tilde{s}')$.

$$\frac{d}{d\epsilon}\Psi_D(P_\epsilon)\Big|_{\epsilon=0} = (1-\gamma)V^*(\tilde{s}) - \frac{\mu^*(\tilde{s}, \tilde{a})}{d(\tilde{s}, \tilde{a})}\left(r(\tilde{s}, \tilde{a}) + \gamma V^*(\tilde{s}') - V^*(\tilde{s})\right) - \Psi_D(P) \tag{8}$$

Sketch of argument. The statement of proposition 3 is not surprising per se, since we can immediately verify the mean-zero sample version of the Bellman residual, and so one might plausibly derive this form from double robustness. Rather, we provide an alternative argument that applies perturbation analysis of optimization programs with the specific perturbations from the finite difference. We build on the sensitivity analysis characterization of [20] which studies perturbations of linear programs with respect to constraint matrix perturbations. For ϵ small enough, the *active basis* remains the same. When the perturbation matrix is $P + \epsilon G$ for some matrix G, [20] notes that the derivative of the optimal value is $d'(\epsilon) = -\overline{\pi} G \overline{x}$ (where $(\overline{\pi}, \overline{x})$ are the dual and primal optimal solutions, respectively) can be obtained from the derivative of a matrix inverse, and obtains higher order expansions. We generalize this argument for the specific form of the perturbed transition probabilities.

We next compute the approximation error that arises from evaluating finite differences, so long as ϵ is small enough to maintain the same active basis, which is empirically verifiable.

Proposition 4 (Error analysis from finite-differencing)**.** *Perturb in the direction of a generic observation* $(\tilde{s}, \tilde{a}, \tilde{s}')$. *Then:*

$$\epsilon^{-1}(\Psi(P_\epsilon) - \Psi(P)) = (1-\gamma)V_\epsilon^*(\tilde{s}) - \frac{\mu^*(\tilde{s}, \tilde{a})}{d_\epsilon(\tilde{s}, \tilde{a})}\left(r(\tilde{s}, \tilde{a}) + \gamma V(\tilde{s}') - V_\epsilon(\tilde{s})\right) - \Psi_D(P) + O(\epsilon) \tag{9}$$

In this setting, unlike the earlier case of the treatment mean in Corollary 1 and Theorem 1, even in a favorable discrete-state case without smoothing requirements, it does not appear that rate-double robustness would admit weaker numerical requirements on ϵ.

Generalization to constraints. More importantly, we argue the main benefit of this approach is that by computing these finite differences, we can obtain generic bias corrections for generalizations building upon this formulation. For example, consider a relevant subclass of constrained Markov decision processes comprising of additional constraints on the policy variables: the linear programming formulation is particularly appealing because linear constraints can be directly added [3]. An advantage is that bias correction can be obtained by evaluating the functional at feasible points (i.e. optimizing with respect to feasible probability distributions).

Example 3 (Constrained policy optimization)**.**

$$\Psi_P(P) = \max\left\{\sum_{a\in\mathcal{A}} \mu_a^\top \mathbf{r_a} \colon \sum_{a\in\mathcal{A}}(I-\gamma P_a^\top)\mu_a = (1-\gamma)\mu_{\mathbf{0}},\ \ \mu_a \geq 0,\ \ \forall a \in \mathcal{A},\ \mu \in \mathcal{P}\right\}$$

The additional constraint $\mu \in \mathcal{P}$ reflects linearly representable constraints on state-occupancy under the optimal policy. For example, relevant constraints could require the occupancy in certain risky states to be bounded. [46] develops a framework with a "caution function", a convex function on the state-action occupancy measures; this analysis applies to linear caution functions.

To compare to related work, we note that double robustness has been studied (for policy evaluation) in [54]. Our derivation via perturbation analysis is similar to [54, Thm. 4.1]'s illustration via the Lagrangian. This is unsurprising since the Lagrangian provides the dual characterization that also underlies perturbation analysis. Relative to the doubly robust estimators derived elsewhere, this influence function may be higher variance.

5 Related Work

In this section, we discuss the most closely related directions but do not attempt to provide a survey or overview of influence functions/semiparametric statistics; see [7, 56, 17, 41] for a broader overview or the appendix for more detailed discussions.

Numerical approximation of Gateaux derivatives. Recent pedagogical reviews [17, 28, 41] emphasize the Gateaux derivative and its use as a heuristic tool for deriving influence functions. [9] derive high-level sufficient conditions for approximation error via general results for finite-difference approximations. Relative to those general sufficient conditions, we focus on specific improvements in approximation error in different functionals. For the specific case of the counterfactual mean/average treatment effect [19] use additional structural knowledge. Namely, they change orders of integration to obtain functional evaluations in terms of the *conditional expectation*. In general, it may be difficult to deduce the form of the nuisance functions without closely using knowledge of the estimand. The approach in Appendix B of [8] considers the specific case of debiasing moment conditions with respect to first stages but also implicitly leverages this change in order of integration.

Other work on computerizing causal inference with different approaches. Other recent work in causal inference builds towards "automatic" or "computerized" semiparametric inference via different approaches: via Riesz representation or generalized regression residuals [12, 13, 29, 16, 40]. Our last application to optimization is not a generalized regression residual.

Numerical derivatives, statistical machine learning, and optimization. Independently of causal inference, other works study the use of numerical gradients in the context of machine learning and stochastic gradients [45], and statistical estimation and inference [31, 30]. Analysis of finite-difference estimators has been of interest in (stochastic) zeroth-order optimization and derivative-free optimization [18, 15]. More broadly in machine learning, influence functions have been used for qualitative sensitivity analysis or bias correction [42, 21, 22, 62]; further connections could be interesting, although the causal inference setting is different because one needs to compute influence functions of the statistical functional with respect to perturbations in the distribution (rather than perturbing an estimator). Recently, finite-difference approaches for Gateaux derivatives have been used in data-driven optimization for debiasing in-sample bias, see e.g. [24]. [23] builds on the finite difference method of [34] for debiasing optimization with causal outcome coefficients.

Concluding remarks. Our analysis suggests that the current major technical limitations include the dimension-dependence due to smoothing. Regarding Q2, our analysis of different functionals surfaced fine-grained differences in opportunities for statistical properties to weaken requirements of computational approximation: Theorem 1 and Proposition 4 have different, estimand-dependent implications. It would be interesting to generalize this further, such as to Z- and M-estimators. Our analysis also relied on a priori knowledge of beneficial statistical properties: it would be interesting to verify these adaptively.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See limitations and potential negative impacts paragraph.
   (c) Did you discuss any potential negative societal impacts of your work? [Yes] See limitations and potential negative impacts paragraph.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] Assumptions are typically stated before they are used.
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3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] Yes, included in supplemental materials.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]
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   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
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   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

A Additional related work

Other work on computerizing semiparametric inference. We briefly expand our discussion on other work in causal inference towards "automatic" or "computerized" semiparametric inference. [12, 13] build upon Riesz representation characterizations of influence functions and show that the functionals that can be represented as generalized regression residuals can be written as the target of a least-squares regression problem that depends on derivatives of the functional. Later on we focus on functionals such as stochastic optimization that do not fit into this previous framework. [29] studies minimax estimation of linear functionals. [16] studies an influence function in a particular structural model of heterogeneity similar to a functional coefficient model. [40] discuss Gateaux derivatives of statistical functionals with attention to i; their focus is on developing valid leave-one-out estimators (rather than relying on sample splitting). Other works consider total effect estimation and develop "automatic semiparametric inference" based on graphical criteria, including [53, 50, 4], but generally require knowledge of the graph. [2] use influence functions instead for validation.

B Overview and background for influence functions

For a broader overview of influence functions and their role in causal inference, see [56, 28, 41].

We focus on functionals that admit an asymptotically linear representation, which is relevant because the influence function describes the first-order bias behavior:

$$\Psi(\tilde{P}) - \Psi(P) = -\int \phi(u; \tilde{P}) dP(u) + R_2\left(\tilde{P}, P\right)$$

That is, we focus on functionals that are functionally smooth enough to admit the above *von Mises expansion* with $\phi(o; P)$ the influence function a mean-zero, finite-variance function and a second-order remainder term $R_2\left(P_1, P\right)$.

C Proofs for Section 3

C.0.1 Supplementary lemmas

Lemma 2 (Intermediate decomposition)**.**

$$\begin{aligned}
&\frac{1}{\epsilon}\int\int y\left(\frac{\tilde{p}_\epsilon(y, A=1, x)}{\tilde{p}_\epsilon(A=1, x)} - \frac{\tilde{p}(y, A=1, x)}{\tilde{p}(A=1, x)}\right)\tilde{p}(x) dy\, dx \\
&\quad = \int \frac{\tilde{p}(x)}{\tilde{p}_\epsilon(A=1, x)} \mathbb{I}\left[a_i = 1\right] \tilde{\delta}^{\lambda}_{(x_i)}(x) \left\{\left(\int y\, \tilde{\delta}^{\lambda}_{y_i}(y) dy\right) - \mathbb{E}_{\tilde{p}}[Y \mid A=1, x]\right\} dx
\end{aligned}$$

C.0.2 Proofs

Proof of Proposition 1.

$$\begin{aligned}
\frac{\Psi(\tilde{P}_{\epsilon_i}) - \Psi(\tilde{P})}{\epsilon} &= \frac{1}{\epsilon}\int\left\{\int y\left[\frac{\tilde{p}_\epsilon(y, A=1, x)}{\tilde{p}_\epsilon(A=1, x)}\left((1-\epsilon)\tilde{p}(x) + \epsilon\tilde{\delta}^{\lambda}_{x_i}\right) - \frac{\tilde{p}(y, A=1, x)}{\tilde{p}(A=1, x)}\tilde{p}(x)\right] dy\right\} dx \\
&= \left[\int \tilde{p}(x)\int y\left(\frac{\tilde{p}_\epsilon(y, A=1, x)}{\tilde{p}_\epsilon(A=1, x)} - \frac{\tilde{p}(y, A=1, x)}{\tilde{p}(A=1, x)}\right) dy\, dx\right] \\
&\quad + \cancel{\frac{1}{\epsilon}}\cancel{\epsilon}\int\int y\left[\frac{\tilde{p}_\epsilon(y, A=1, x)}{\tilde{p}_\epsilon(A=1, x)}\tilde{\delta}^{\lambda}_{x_i} - \frac{\tilde{p}(y, A=1, x)}{\tilde{p}(A=1, x)}\tilde{p}(x)\right] dy\, dx
\end{aligned}$$

The first term of the above decomposition can be expanded using the decomposition of Lemma 2. We recognize the second term as $\left(\tilde{\mathbb{E}}_{\tilde{P}_\epsilon}[Y \mid A=1, X=x_i] - \Psi(\tilde{P})\right)$ by definition of the perturbed

nuisances. Therefore

$$\begin{aligned}
\frac{\Psi(\tilde{P}_{\epsilon_i}) - \Psi(\tilde{P})}{\epsilon} &= (1-\epsilon)\int \tilde{p}(x) \int y \left(\frac{\tilde{p}_\epsilon(y, A=1, x)}{\tilde{p}_\epsilon(A=1,x)} - \frac{\tilde{p}(y, A=1, x)}{\tilde{p}(A=1,x)} \right) dy\, dx \\
&= \int \frac{\tilde{p}(x)}{\tilde{p}_\epsilon(A=1,x)} \mathbb{I}\left[a_i = 1\right] \tilde{\delta}^{\lambda}_{x_i}(x) \left\{ \left(\int y\, \tilde{\delta}^{\lambda}_{y_i}(y) dy \right) - \mathbb{E}_{\tilde{P}}[Y \mid A=1, x] \right\} dx \\
&+ \left(\tilde{\mathbb{E}}_{\tilde{P}_\epsilon}[Y \mid A=1, X=x_i] - \Psi(\tilde{P}) \right)
\end{aligned}$$

□

Proof of Lemma 2. Verify by direct computation.

For simplicity we consider a single summand in the sum (the claim follows by applying the same argument for every summand). Recall that $\tilde{P}_\epsilon(A=1, X_j) = (1-\epsilon)\tilde{p}(A=1, X_j) + \epsilon \tilde{\mathbb{I}}\left[(o_{\setminus y})_{\tilde{i}}\right]$

$$\begin{aligned}
&= \int \frac{\tilde{p}(x)}{\epsilon} \int y \left[\frac{(1-\epsilon)\tilde{p}(y, A=1, x) + \epsilon \tilde{\delta}^{\lambda}_{o_i}}{\tilde{p}_\epsilon(A=1,x)} - \frac{\tilde{p}(y, A=1, x)}{\tilde{p}(A=1,x)} \right] dy\, dx \\
&= \int \frac{\tilde{p}(x)}{\epsilon} \int y \left[\frac{\left\{ \cancel{(1-\epsilon)} \tilde{p}(y, A=1, x) + \epsilon \tilde{\delta}^{\lambda}_{o_i} \right\} \tilde{p}(A=1,x)}{\tilde{p}_\epsilon(A=1,x)\tilde{p}(A=1,x)} - \frac{\tilde{p}(y, A=1, x) \left\{ \cancel{(1-\epsilon)} \tilde{p}(A=1,x) + \epsilon \tilde{\delta}^{\lambda}_{(a_i, x_i)} \right\}}{\tilde{p}(A=1,x)\tilde{p}_\epsilon(A=1,x)} \right] dy\, dx \\
&= \int \frac{\tilde{p}(x)}{\cancel{\epsilon}} \int y \left[\cancel{\epsilon} \frac{\tilde{\delta}^{\lambda}_{o_i}}{\tilde{p}_\epsilon(A=1,x)} - \cancel{\epsilon} \frac{\tilde{p}(y, A=1, x)\tilde{\delta}^{\lambda}_{(a_i, x_i)}}{\tilde{p}(A=1,x)\tilde{p}_\epsilon(A=1,x)} \right] dy\, dx \\
&= \int \frac{\tilde{p}(x)}{\tilde{p}_\epsilon(A=1,x)} \tilde{\delta}^{\lambda}_{(a_i, x_i)} \left\{ \int y \tilde{\delta}^{\lambda}_{y_i} dy - \mathbb{E}_{\tilde{p}}[Y \mid A=1, x] \right\} dx \\
&= \int \frac{\tilde{p}(x)}{\tilde{p}_\epsilon(A=1,x)} \mathbb{I}\left[a_i = 1\right] \tilde{\delta}^{\lambda}_{(x_i)}(x) \left\{ \int y \tilde{\delta}^{\lambda}_{y_i} dy - \mathbb{E}_{\tilde{p}}[Y \mid A=1, x] \right\} dx
\end{aligned}$$

□

Proof of Lemma 1. To bound $\mathbb{E}[(\tilde{e}_\epsilon(X) - e(X))^2]$, argue similarly and observe that since $\tilde{e}_\epsilon = \tilde{p}_\epsilon(A=1 \mid x) = \tilde{p}_\epsilon(A=1,x)/\tilde{p}(x)$,

$$\begin{aligned}
\mathbb{E}[(\tilde{e}_\epsilon(X) - \tilde{e}(X))^2] &= \int (((1-\epsilon)\tilde{p}(A=1,x) + \epsilon \tilde{\delta}^{\lambda}_{a,\tilde{x}}(x)) - \tilde{p}(A=1,x))^2 / \tilde{p}(x)^2 \,\mathrm{d}x \cdot \tilde{p}(x) \\
&= \epsilon^2 \int (\tilde{\delta}^{\lambda}_{a,\tilde{x}}(x) - \tilde{p}(A=1,x))^2 / \tilde{p}(x) \,\mathrm{d}x \\
&= \epsilon^2 \int ((\tilde{\delta}^{\lambda}_{a,\tilde{x}}(x)^2 - 2\tilde{\delta}^{\lambda}_{a,\tilde{x}}(x)\tilde{p}(A=1,x) + \tilde{p}(A=1,x)^2)/\tilde{p}(x) \,\mathrm{d}x \\
&\leq \epsilon^2 \int \nu(\tilde{\delta}^{\lambda}_{a,\tilde{x}}(x)^2 + 1) \,\mathrm{d}x \\
&= \epsilon^2 (\nu \lambda^{-d} R(\tilde{\delta}^{\lambda}) + \nu) = O(\epsilon^2 \lambda^{-d})
\end{aligned}$$

In the second-to-last inequality we lower bound $\tilde{p}(x) \geq \nu$ and upper bound $\tilde{p}(A=1,x) \leq 1$. In the last inequality we apply a change of variables to identify the term that arises in analysis of variance of kernel density estimates. We incur the λ^{-d} dependence since we assumed that multivariate kernels are product kernels. The roughness term $R(\tilde{\delta}^{\lambda}) = \int \tilde{\delta}^2(u) du$ depends on the particular $\tilde{\delta}$ kernel function and is a kernel-dependent constant. Hence the term scales overall as λ^{-d} as in standard analyses of variance of kernel density estimates; see e.g. [57, pg. 23-26].

Next observe that:

$$
\begin{aligned}
&\mathbb{E}[(\tilde{\mu}_\epsilon(X) - \mu(X))^2] \\
&\leq \mathbb{E}[(\tilde{\mu}_\epsilon(X) - \tilde{\mu}(X))^2] + \mathbb{E}[(\tilde{\mu}(X) - \mu(X))^2] \\
&= \int \left\{ \int \int y \left(\frac{(1-\epsilon)\tilde{p}(y, A=1, x') + \epsilon \tilde{\delta}^\lambda_o}{(1-\epsilon)\tilde{p}(A=1, x') + \epsilon \tilde{\delta}^\lambda_{a,x}} - \frac{\tilde{p}(y, A=1, x')}{\tilde{p}(A=1, x')} \right) dy dx' \right\}^2 p(x) dx + \mathbb{E}[(\tilde{\mu}(X) - \mu(X))^2] \\
&= \int \epsilon^2 \left\{ \frac{\tilde{p}(x)}{\tilde{p}_\epsilon(A=1, x)} \mathbb{I}\left[a_i = 1\right] \tilde{\delta}^\lambda_{(x_i)}(x) \left\{ y_i - \mathbb{E}_{\tilde{p}}[Y \mid A = 1, x] \right\} \right\}^2 p(x) dx + \mathbb{E}[(\tilde{\mu}(X) - \mu(X))^2] \\
&= \mathbb{E}[(\tilde{\mu}(X) - \mu(X))^2] + O(\epsilon^2 \lambda^{-d})
\end{aligned}
$$

where the first inequality comes from the triangle inequality, and the second equality from similar arguments as in the previous bound. □

Proof of Theorem 1. Let

$$\tilde{e}(X) = \tilde{p}(A = 1, X)/\tilde{p}(X) = \tilde{p}(A = 1 \mid X), \tilde{\mu}(X)$$

denote the propensity and conditional outcome nuisances arising from plug-in estimation of $\tilde{P}$.

Let

$$\tilde{e}_\epsilon(X) = \tilde{p}_\epsilon(A = 1 \mid X) = \tilde{p}_\epsilon(A = 1, X)/\tilde{p}(X)$$

denote the nuisance arising from plug-in estimation at the perturbed distribution.

First observe by Corollary 1:

$$\frac{\Psi(\tilde{P}^i_\epsilon) - \Psi(\tilde{P})}{\epsilon} = (1-\epsilon) \frac{\mathbb{I}[A_i = 1]}{\tilde{p}_\epsilon(A = 1 | X_i)} \left(Y_i - \mathbb{E}_{\tilde{P}}[Y \mid A = 1, X_i]\right) + \left(\mathbb{E}_{\tilde{P}_\epsilon}[Y \mid A = 1, X_i] - \Psi(\tilde{P})\right) + O(\lambda^J)$$

This decomposition studies convergence of an AIPW estimator with possibly biased but asymptotically consistent nuisances $\tilde{\mu}, \tilde{e}_\epsilon$ to the AIPW estimator with oracle nuisances – a standard analysis – and an additional term that vanishes on the order of ϵ (assuming strong overlap and boundedness). Define $\Gamma^*(O)$ as the doubly-robust score for AIPW with *oracle nuisances* e^*, μ^*, and let Γ^*_j denote its empirical counterpart at $O_j = (X_k, A_k, Y_k)$:

$$\Gamma^*(O) = \mathbb{I}\left[A = 1\right] \frac{Y - \mu^*(X)}{e^*(X)} + \mu^*(X)$$

We will study the decomposition relative to the AIPW estimand with oracle nuisance estimates, with $\Gamma(O; e, \mu)$ denoting the above score but with nuisance functions e, μ:

$$
\begin{aligned}
&\Psi_n(\tilde{P}) - \mathbb{E}[\Gamma^*(O; e^*, \mu^*)] \\
&= (\Psi(\tilde{P}) + \tfrac{1}{n} \textstyle\sum_{i=1}^n \tilde{\phi}_i) - \mathbb{E}[\Gamma^*(O; e^*, \mu^*)] \\
&= \mathbb{E}_n[\Gamma(O; \tilde{e}_\epsilon, \tilde{\mu})] - \mathbb{E}[\Gamma^*(O; e^*, \mu^*)] + \mathbb{E}_n[\tilde{\mu}_\epsilon(X) - \tilde{\mu}(X)] + O(\lambda^\beta) \\
&= (\mathbb{E}_n - \mathbb{E})[\Gamma(O; \tilde{e}_\epsilon, \tilde{\mu})] + \mathbb{E}[\Gamma(O; \tilde{e}_\epsilon, \tilde{\mu}) - \Gamma^*(O; e^*, \mu^*)] + \mathbb{E}_n[\tilde{\mu}_\epsilon(X) - \tilde{\mu}(X)] + O(\lambda^\beta)
\end{aligned}
$$

where the second equality follows by Proposition 1, and the third by adding/subtracting $\mathbb{E}[\Gamma(O; \tilde{e}_\epsilon, \tilde{\mu})]$.

The first term, $(\mathbb{E}_n - \mathbb{E})[\Gamma(O; \tilde{e}_\epsilon, \tilde{\mu})]$ is the standard empirical process term of the decomposition: by the Donsker assumption and classical stability results of Donsker classes [60],

$$(\mathbb{E}_n - \mathbb{E})[\Gamma(O; \tilde{e}_\epsilon, \tilde{\mu})] = (\mathbb{E}_n - \mathbb{E})[\Gamma(O; e^*, \mu^*)] + o_p(n^{-\frac{1}{2}}).$$

Hence

$$\Psi_n(\tilde{P}) - \mathbb{E}[\Gamma^*(O; e^*, \mu^*)] = \mathbb{E}[\Gamma(O; \tilde{e}_\epsilon, \tilde{\mu}) - \Gamma^*(O; e^*, \mu^*)] + \mathbb{E}_n[\tilde{\mu}_\epsilon(X) - \tilde{\mu}(X)] + O(\lambda^\beta) + o_p(n^{-\frac{1}{2}}) \tag{10}$$

We obtain the product error conditions by standard decomposition of the remaining first term.

$$
\begin{aligned}
&\mathbb{E}[\Gamma(O;\tilde{e}_\epsilon,\tilde{\mu}) - \Gamma^*(O;e^*,\mu^*)] \\
&= \mathbb{E}\left[\left(\mathbb{I}\left[A=1\right]\frac{Y-\tilde{\mu}(X)}{\tilde{e}_\epsilon(X)} + \tilde{\mu}(X)\right)\right] - \mathbb{E}\left[\left(\mathbb{I}\left[A=1\right]\frac{Y-\mu^*(X)}{e^*(X)} + \mu^*(X)\right)\right] \\
&= \mathbb{E}\left[\mathbb{I}\left[A=1\right]\left(\frac{1}{\tilde{e}_\epsilon(X)} - \frac{1}{e^*(X)}\right)(Y-\mu^*(X))\right] - \mathbb{E}\left[\left(\frac{1}{e^*(X)} - 1\right)(\tilde{\mu}(X)-\mu^*(X))\right] \\
&\qquad + \mathbb{E}\left[\mathbb{I}\left[A=1\right]\left(\frac{1}{\tilde{e}_\epsilon(X)} - \frac{1}{e^*(X)}\right)(\tilde{\mu}(X)-\mu^*(X))\right] \\
&= \mathbb{E}\left[\mathbb{I}\left[A=1\right]\left(\frac{1}{\tilde{e}_\epsilon(X)} - \frac{1}{e^*(X)}\right)(\tilde{\mu}(X)-\mu^*(X))\right] \\
&\leq \nu\|\tilde{e}_\epsilon(X) - e^*(X)\|_{2,P} \times \|\tilde{\mu}(X) - \mu^*(X)\|_{2,P}
\end{aligned}
$$

The second equality follows by adding/subtracting $\mathbb{I}\left[A=1\right]/\tilde{e}_\epsilon, \mathbb{I}\left[A=1\right]\tilde{\mu}/e^*, \mathbb{I}\left[A=1\right]\mu^*/e^*$). The third equality follows from iterated expectations. The last inequality follows from the Cauchy-Schwarz inequality and overlap. Therefore, ϵ must satisfy a rate so that $\tilde{e}_\epsilon$ converges at typical rate-double robustness fast rates, such as $n^{-1/4}$; under a suitable sample splitting scheme.

For the last term of Equation (10), note that by Lemma 2, the numerical error ϵ is multiplied by error rate of the outcome nuisance.

$$
\begin{aligned}
\mathbb{E}[\tilde{\mu}_\epsilon(X) - \tilde{\mu}(X)] &= \epsilon \int\int \frac{\tilde{p}(x)}{\tilde{p}_\epsilon(A=1,x)}\mathbb{I}\left[a_i=1\right]\tilde{\delta}^{\lambda}_{(x_i)}(x)\ \{y_i - \mathbb{E}_{\tilde{p}}[Y \mid A=1,x]\}\,dx \\
&= \epsilon\,\mathbb{E}\left[\frac{\mathbb{I}\left[A=1\right]}{\tilde{p}_\epsilon(A=1\mid X)}\ \{Y - \mathbb{E}_{\tilde{p}}[Y \mid A=1,X]\}\right] + O(\lambda^\beta) \\
&\leq \|\mathbb{I}\left[A=1\right]/\tilde{p}_\epsilon(A=1\mid X)\|_{2,P}\|Y - \mathbb{E}_{\tilde{p}}[Y \mid A=1,X]\|_{2,P} + O(\lambda^\beta) \\
&= O(\lambda^\beta) + O_p(n^{-r_\mu})\epsilon
\end{aligned}
$$

by application of Cauchy-Schwarz inequality and strong overlap. The claim follows by application of Lemma 1 to deduce implications on ϵ from standard product-error rate requirements on $\tilde{\mu}, \tilde{e}$. $\square$

D Proofs for Section 4

D.1 Empirical Gateaux derivative for multi-stage dynamic treatment regimes

We study the estimation properties of the empirical Gateaux derivative for multi-stage DTR from the probability density representation. In T-stage dynamic treatment regimes, the causal quantity of interest is the counterfactual mean outcome $\mathbb{E}\left[Y^{\bar{a}}\right]$, where $\bar{a} = (a_0, \ldots, a_T)$ is the (deterministic) treatment strategy. Assuming $Y^{\bar{a}}$ is sequentially ignorable given treatment and covariate history $(\bar{A}_t, \bar{X}_t)$ at each time t, this causal quantity can be identified by the g-formula

$$\mathbb{E}\left[Y^{\bar{a}}\right] = \int \mathbb{E}\left[Y \mid \bar{A} = \bar{a}, \bar{X} = \bar{x}\right] \prod_{t=1}^{T} \tilde{p}(x_t \mid \bar{a}_{t-1}, \bar{x}_{t-1}) \, \mathrm{d}\bar{x}$$

where $\bar{A} = (A_0, \ldots, A_t), \bar{X} = (X_0, \ldots, X_t)$.

To derive its influence function, we take the empirical Gateaux derivative by considering the perturbation in the direction of $o_i = (\bar{x}_i, \bar{a}_i, y_i)$, where $\bar{x}_i = (x_{i0}, \ldots, x_{it})$ and $\bar{a}_i = (a_{i0}, \ldots, a_{it})$.

$$\Psi(\tilde{P}_\epsilon^i) = \int \left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A} = \bar{a}, \bar{X} = \bar{x})}{\tilde{p}_\epsilon(\bar{A} = \bar{a}, \bar{X} = \bar{x})} \, \mathrm{d}y \right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \cdot \tilde{p}_\epsilon(x_0) \, \mathrm{d}\bar{x}. \tag{11}$$

Below we characterize how this empirical Gateaux derivative at the smoothed distribution differs from the one at the (unsmoothed) estimated distribution. The following result is analogous to Proposition 1 but is extended to the dynamic treatment regimes.

Proposition 5.

$$\begin{aligned}
&\frac{\Psi(\tilde{P}_\epsilon^i) - \Psi(\tilde{P})}{\epsilon} \\
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A} = \bar{a}, \bar{X}_{1:T} = \bar{x}_{1:T}, X_0 = x_{i0}\right] \prod_{t=1}^{T} \tilde{p}(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}) \, \mathrm{d}\bar{x} - \Psi(\tilde{P}) \right] \\
&\quad + \int \tilde{p}(x_0) \left\{ \left[\frac{\tilde{\delta}^\lambda_{(\bar{a}_i, \bar{x}_i)}}{\tilde{p}_\epsilon(\bar{A} = \bar{a}, \bar{X} = \bar{x})} \left[\int y \cdot \tilde{\delta}^\lambda_{y_i} \, \mathrm{d}y - \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A} = \bar{a}, \bar{X} = \bar{x}\right] \right] \right] \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right\} \mathrm{d}\bar{x} \\
&\quad + \sum_{s=1}^{T} \int \tilde{p}(x_0) \left\{ \left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A} = \bar{a}, \bar{X} = \bar{x})}{\tilde{p}_\epsilon(\bar{A} = \bar{a}, \bar{X} = \bar{x})} \, \mathrm{d}y \right) \left[\left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right.\right. \\
&\qquad \left.\left. \times \frac{1}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} \left[\tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))} \tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})} \right] \right] \right\} \mathrm{d}\bar{x} \qquad (12) \\
&\quad + \epsilon \sum_{s=1}^{T} \int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A} = \bar{a}, \bar{X}_{1:T} = \bar{x}_{1:T}, X_0 = x_{i0}\right] \left[\left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \right.\right. \\
&\qquad \left.\left. \times \frac{1}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0}))} \left[\tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0}))} \tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})} \right] \right] \right] \mathrm{d}\bar{x} \qquad (13)
\end{aligned}$$

For comparison, the exact efficient influence function under the non-Markovian dynamic treatment regime is

$$
\begin{aligned}
&\Big[\mathbb{E}_{\tilde{p}(x_{1:T})}\left[\mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right]\right]-\Psi(\tilde{P})\Big] \\
&\quad+\frac{\mathbb{I}(\bar{a}_i)}{\tilde{p}(\bar{A}=\bar{a} \mid \bar{X}=\bar{x})}\left[Y_i-\mathbb{E}_{\tilde{p}(\bar{x})}\left[\mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A}=\bar{a}, \bar{X}=\bar{x}\right]\right]\right] \\
&\quad+\sum_{s=1}^{T} \frac{\mathbb{I}(a_{i,T-s}, \bar{x}_{i,T-s+1})}{\tilde{p}_\epsilon(\bar{a}_{T-s} \mid \bar{x}_{T-s})}\left[\int \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A}=\bar{a}, \bar{X}=\bar{x}\right] \prod_{t=T-s+1}^{T-1} \tilde{p}(x_t \mid \bar{a}_{t-1}, \bar{x}_{t-1}) \,\mathrm{d}\bar{x}_{(T-s+1):(T-1)}\right] \\
&\quad-\sum_{s=1}^{T} \frac{\mathbb{I}(a_{i,T-s}, \bar{x}_{i,T-s})}{\tilde{p}(\bar{a}_{T-s} \mid \bar{x}_{T-s})}\left[\int \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A}=\bar{a}, \bar{X}=\bar{x}\right] \prod_{t=T-s}^{T-1} \tilde{p}(x_t \mid \bar{a}_{t-1}, \bar{x}_{t-1}) \,\mathrm{d}\bar{x}_{(T-s):(T-1)}\right]
\end{aligned}
\tag{14}
$$

This proposition performs a decomposition of the empirical derivative. It quantifies the additive bias due to the smoothed perturbation compared to the (raw) perturbation at the observation o_i.

Proof. We first decompose the empirical Gateaux derivative.

$$
\begin{aligned}
&\frac{\Psi(\tilde{P}_\epsilon^i)-\Psi(\tilde{P})}{\epsilon} \\
&=\frac{1}{\epsilon}\left[\int\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})} \,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \cdot\left[(1-\epsilon) \tilde{p}(x_0)+\epsilon \tilde{\delta}_{x_{i0}}^{\lambda}\right] \mathrm{d}\bar{x}\right. \\
&\quad\left.-\int\left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})} \,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \cdot \tilde{p}(x_0) \,\mathrm{d}\bar{x}\right] \qquad (15) \\
&=\int\left[\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})} \,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \cdot \tilde{\delta}_{x_{i0}}^{\lambda}\right. \\
&\quad\left.-\left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})} \,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \cdot \tilde{p}(x_0)\right] \mathrm{d}\bar{x} \\
&\quad+\frac{1}{\epsilon}\left[\int \tilde{p}(x_0)\left[\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})} \,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right.\right. \\
&\quad\left.\left.-\left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})} \,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1})}\right] \mathrm{d}\bar{x}\right] \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_T=\bar{x}_T\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{t-1}) \tilde{p}(x_0) \tilde{\delta}^{\lambda}_{x_{i0}} \,\mathrm{d}\bar{x} - \Psi(\tilde{P})\right] \\
&\quad + \frac{1}{\epsilon}\left[\int \tilde{p}(x_0)\left[\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right.\right. \\
&\quad \left.\left. - \left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right] \mathrm{d}\bar{x}\right] \\
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}) \,\mathrm{d}\bar{x} - \Psi(\tilde{P})\right] \\
&\quad + \frac{1}{\epsilon}\left[\int \tilde{p}(x_0)\left[\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right.\right. \\
&\quad \left.\left. - \left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right] \mathrm{d}\bar{x}\right] \qquad (17)
\end{aligned}
$$

Equation (15) is due to the definition of the smoothed perturbation. Equation (16) separates the effect of smoothed perturbation into one in the evaluation of y and the other in the evaluation of the density ratios. Equation (17) is due to the definition of $\mathbb{E}_{\tilde{p}_\epsilon}[\cdot]$.

Calculating the second term of Equation (17). We first decompose this term into two difference terms.

$$
\begin{aligned}
&\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \\
&= \left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) \left[\prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right] \\
&\quad + \left[\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) - \left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right)\right] \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}. \qquad (18)
\end{aligned}
$$

The second difference term can be decomposed in a similar way as in Lemma 2.

$$
\begin{aligned}
&\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) - \left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) \\
&= \int y \cdot \left[\frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})} - \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\right]\mathrm{d}y \\
&= \int y \cdot \left[\frac{(1-\epsilon)\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x}) + \epsilon\tilde{\delta}^{\lambda}_{o_i}}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})} - \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\right]\mathrm{d}y \\
&= \int y \cdot \left[\frac{((1-\epsilon)\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x}) + \epsilon\tilde{\delta}^{\lambda}_{o_i})\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\right. \\
&\qquad \left. - \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})((1-\epsilon)\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x}) + \epsilon\tilde{\delta}^{\lambda}_{(\bar{a}_i, \bar{x}_i)})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\right]\mathrm{d}y \qquad (19) \\
&= \int y \cdot \left[\frac{(\epsilon\tilde{\delta}^{\lambda}_{o_i})\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})} - \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})(\epsilon\tilde{\delta}^{\lambda}_{(\bar{a}_i, \bar{x}_i)})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\right]\mathrm{d}y \\
&= \frac{\epsilon\tilde{\delta}^{\lambda}_{(\bar{a}_i, \bar{x}_i)}}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\int y \cdot \left[(\tilde{\delta}^{\lambda}_{y_i}) - \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\right]\mathrm{d}y \\
&= \frac{\epsilon\tilde{\delta}^{\lambda}_{(\bar{a}_i, \bar{x}_i)}}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\left[\int y \cdot \tilde{\delta}^{\lambda}_{y_i}\,\mathrm{d}y - \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A}=\bar{a}, \bar{X}=\bar{x}\right]\right] \qquad (20)
\end{aligned}
$$

We next analyze the first difference term by induction.

$$
\begin{aligned}
&\prod_{t=1}^{T}\frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \prod_{t=1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \\
&= \prod_{t=1}^{T-1}\frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))}\left[\frac{\tilde{p}_\epsilon(x_T, \bar{a}_{T-1}, \bar{x}_{T-1})}{\tilde{p}_\epsilon(\bar{a}_{T-1}, \bar{x}_{T-1}))} - \frac{\tilde{p}(x_T, \bar{a}_{T-1}, \bar{x}_{T-1})}{\tilde{p}(\bar{a}_{T-1}, \bar{x}_{T-1}))}\right] \\
&\quad + \frac{\tilde{p}(x_T, \bar{a}_{T-1}, \bar{x}_{T-1})}{\tilde{p}(\bar{a}_{T-1}, \bar{x}_{T-1}))}\left[\prod_{t=1}^{T-1}\frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \prod_{t=1}^{T-1}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right] \qquad (21)
\end{aligned}
$$

Next we evaluate the key term in this induction $\frac{\tilde{p}_\epsilon(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))}$.

$$
\begin{aligned}
&\frac{\tilde{p}_\epsilon(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))} \\
&= \frac{((1-\epsilon)\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1}) + \epsilon\tilde{\delta}^{\lambda}_{o_i})\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))} - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})((1-\epsilon)\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1})) + \epsilon\tilde{\delta}^{\lambda}_{o_i})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} \\
&= \frac{(\epsilon\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})})}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})(\epsilon\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} \\
&= \frac{\epsilon}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))}\left[\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))}\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})}\right] \qquad (22)
\end{aligned}
$$

Thus we have the following inductive relationship in T,

$$
\begin{aligned}
&\prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \\
&= \prod_{t=1}^{T-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \frac{\epsilon}{\tilde{p}_\epsilon(\bar{a}_{T-1}, \bar{x}_{T-1}))} \left[\tilde{\delta}^{\lambda}_{(\bar{a}_{i,T-1}, \bar{x}_{i,T})} - \frac{\tilde{p}(x_T, \bar{a}_{T-1}, \bar{x}_{T-1})}{\tilde{p}(\bar{a}_{T-1}, \bar{x}_{T-1}))} \tilde{\delta}^{\lambda}_{(\bar{a}_{i,T-1}, \bar{x}_{i,T-1})} \right] \\
&\quad + \frac{\tilde{p}(x_T, \bar{a}_{T-1}, \bar{x}_{T-1})}{\tilde{p}(\bar{a}_{T-1}, \bar{x}_{T-1}))} \left[\prod_{t=1}^{T-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \prod_{t=1}^{T-1} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right] \qquad (23) \\
&= \sum_{s=T-1}^{T} \left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \frac{\epsilon}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} \right. \\
&\qquad \left. \times \left[\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})} - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))} \tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})} \right] \right\} \\
&\quad + \prod_{t=T-1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \left[\prod_{t=1}^{T-2} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \prod_{t=1}^{T-2} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right] \qquad (24) \\
&= \sum_{s=1}^{T} \left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \frac{\epsilon}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} \right. \\
&\qquad \left. \times \left[\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})} - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))} \tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})} \right] \right\} \qquad (25)
\end{aligned}
$$

These calculations imply that

$$
\begin{aligned}
&\frac{\Psi(\tilde{P}^i_\epsilon) - \Psi(\tilde{P})}{\epsilon} \\
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon} \left[Y \mid \bar{A} = \bar{a}, \bar{X}_{1:T} = \bar{x}_{1:T}, X_0 = x_{i0} \right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}) \, \mathrm{d}\bar{x} - \Psi(\tilde{P}) \right] \\
&\quad + \frac{1}{\epsilon} \left[\int \tilde{p}(x_0) \left[\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A} = \bar{a}, \bar{X} = \bar{x})}{\tilde{p}_\epsilon(\bar{A} = \bar{a}, \bar{X} = \bar{x})} \, \mathrm{d}y \right) \prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right. \right. \\
&\quad \left. \left. - \left(\int y \cdot \frac{\tilde{p}(y, \bar{A} = \bar{a}, \bar{X} = \bar{x})}{\tilde{p}(\bar{A} = \bar{a}, \bar{X} = \bar{x})} \, \mathrm{d}y \right) \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right] \mathrm{d}\bar{x} \right] \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x} - \Psi(\tilde{P})\right] \\
&\quad + \frac{1}{\epsilon}\int \tilde{p}(x_0)\left\{\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right)\left[\prod_{t=1}^{T}\frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \prod_{t=1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right]\right. \\
&\quad \left. + \left[\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right) - \left(\int y \cdot \frac{\tilde{p}(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right)\right]\prod_{t=1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right\}\mathrm{d}\bar{x}
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x} - \Psi(\tilde{P})\right] \\
&\quad + \frac{1}{\epsilon}\int \tilde{p}(x_0)\left\{\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right)\left[\prod_{t=1}^{T}\frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} - \prod_{t=1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right]\right\}\mathrm{d}\bar{x} \\
&\quad + \frac{1}{\epsilon}\int \tilde{p}(x_0)\left\{\left[\frac{\epsilon\tilde{\delta}^{\lambda}_{(\bar{a}_i, \bar{x}_i)}}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\left[\int y \cdot \tilde{\delta}^{\lambda}_{y_i}\,\mathrm{d}y - \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A}=\bar{a}, \bar{X}=\bar{x}\right]\right]\right]\prod_{t=1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right\}\mathrm{d}\bar{x}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x} - \Psi(\tilde{P})\right] \\
&\quad + \frac{1}{\epsilon}\int \tilde{p}(x_0)\left\{\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right)\left[\sum_{s=1}^{T}\left\{\prod_{t=s+1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\prod_{t=1}^{s-1}\frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right.\right. \\
&\qquad \left.\left. \times \frac{\epsilon}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))}\left[\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))}\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})}\right]\right\}\right]\right\}\mathrm{d}\bar{x} \\
&\quad + \frac{1}{\epsilon}\int \tilde{p}(x_0)\left\{\left[\frac{\epsilon\tilde{\delta}^{\lambda}_{(\bar{a}_i, \bar{x}_i)}}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\left[\int y \cdot \tilde{\delta}^{\lambda}_{y_i}\,\mathrm{d}y - \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A}=\bar{a}, \bar{X}=\bar{x}\right]\right]\right]\prod_{t=1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right\}\mathrm{d}\bar{x}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x} - \Psi(\tilde{P})\right] \\
&\quad + \int \tilde{p}(x_0)\left\{\left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A}=\bar{a}, \bar{X}=\bar{x})}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\,\mathrm{d}y\right)\left[\sum_{s=1}^{T}\left\{\prod_{t=s+1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\prod_{t=1}^{s-1}\frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right.\right. \\
&\qquad \left.\left. \times \frac{1}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))}\left[\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))}\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})}\right]\right\}\right]\right\}\mathrm{d}\bar{x} \\
&\quad + \int \tilde{p}(x_0)\left\{\left[\frac{\tilde{\delta}^{\lambda}_{(\bar{a}_i, \bar{x}_i)}}{\tilde{p}_\epsilon(\bar{A}=\bar{a}, \bar{X}=\bar{x})}\left[\int y \cdot \tilde{\delta}^{\lambda}_{y_i}\,\mathrm{d}y - \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A}=\bar{a}, \bar{X}=\bar{x}\right]\right]\right]\prod_{t=1}^{T}\frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))}\right\}\mathrm{d}\bar{x}
\end{aligned} \tag{30}
$$

Calculating the first term of Equation (30). The first term of Equation (30) can be decomposed as follows.

$$\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x}$$

$$= \int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x} \tag{31}$$

$$+ \int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \left[\prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} - \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))}\right] \mathrm{d}\bar{x} \tag{32}$$

Following a similar calculation as in Equation (25), we have that

$$\prod_{t=1}^{T} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} - \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \tag{33}$$

$$= \sum_{s=1}^{T} \left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \frac{\epsilon}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0}))} \right.$$

$$\left. \times \left[\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0}))} \tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})}\right] \right\} \tag{34}$$

Thus the first term of Equation (30) is

$$\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x}$$

$$= \int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \prod_{t=1}^{T} \tilde{p}(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x} \tag{35}$$

$$+ \int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A}=\bar{a}, \bar{X}_{1:T}=\bar{x}_{1:T}, X_0=x_{i0}\right] \left[\sum_{s=1}^{T} \left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \right.\right.$$

$$\left.\left. \times \frac{\epsilon}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0}))} \left[\tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0}))} \tilde{\delta}^{\lambda}_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})}\right] \right\}\right] \mathrm{d}\bar{x} \tag{36}$$

Plugging this calculation into $\frac{\Psi(\tilde{P}^i_\epsilon) - \Psi(\tilde{P})}{\epsilon}$:

$$
\begin{aligned}
&\frac{\Psi(\tilde{P}^i_\epsilon) - \Psi(\tilde{P})}{\epsilon} \\
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A} = \bar{a}, \bar{X}_{1:T} = \bar{x}_{1:T}, X_0 = x_{i0}\right] \prod_{t=1}^{T} \tilde{p}_\epsilon(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x} - \Psi(\tilde{P})\right] \\
&\quad + \int \tilde{p}(x_0) \left\{ \left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A} = \bar{a}, \bar{X} = \bar{x})}{\tilde{p}_\epsilon(\bar{A} = \bar{a}, \bar{X} = \bar{x})}\, \mathrm{d}y \right) \left[\sum_{s=1}^{T} \left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right.\right. \\
&\qquad \left.\left. \times \frac{1}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} \left[\tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))} \tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})} \right] \right] \right\} \mathrm{d}\bar{x} \\
&\quad + \int \tilde{p}(x_0) \left\{ \left[\frac{\tilde{\delta}^\lambda_{(\bar{a}_i, \bar{x}_i)}}{\tilde{p}_\epsilon(\bar{A} = \bar{a}, \bar{X} = \bar{x})} \left[\int y \cdot \tilde{\delta}^\lambda_{y_i}\, \mathrm{d}y - \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A} = \bar{a}, \bar{X} = \bar{x}\right] \right] \right] \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right\} \mathrm{d}\bar{x}
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&= \left[\int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A} = \bar{a}, \bar{X}_{1:T} = \bar{x}_{1:T}, X_0 = x_{i0}\right] \prod_{t=1}^{T} \tilde{p}(x_t \mid \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})\, \mathrm{d}\bar{x} - \Psi(\tilde{P})\right] \\
&\quad + \int \tilde{p}(x_0) \left\{ \left[\frac{\tilde{\delta}^\lambda_{(\bar{a}_i, \bar{x}_i)}}{\tilde{p}_\epsilon(\bar{A} = \bar{a}, \bar{X} = \bar{x})} \left[\int y \cdot \tilde{\delta}^\lambda_{y_i}\, \mathrm{d}y - \mathbb{E}_{\tilde{p}}\left[Y \mid \bar{A} = \bar{a}, \bar{X} = \bar{x}\right] \right] \right] \prod_{t=1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right\} \mathrm{d}\bar{x} \\
&\quad + \sum_{s=1}^{T} \int \tilde{p}(x_0) \left\{ \left(\int y \cdot \frac{\tilde{p}_\epsilon(y, \bar{A} = \bar{a}, \bar{X} = \bar{x})}{\tilde{p}_\epsilon(\bar{A} = \bar{a}, \bar{X} = \bar{x})}\, \mathrm{d}y \right) \left[\left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{t-1}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{t-1})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{t-1}))} \right.\right. \\
&\qquad \left.\left. \times \frac{1}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{s-1}))} \left[\tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{s-1})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{s-1}))} \tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})} \right] \right] \right\} \mathrm{d}\bar{x} \\
&\quad + \epsilon \sum_{s=1}^{T} \int \mathbb{E}_{\tilde{p}_\epsilon}\left[Y \mid \bar{A} = \bar{a}, \bar{X}_{1:T} = \bar{x}_{1:T}, X_0 = x_{i0}\right] \left[\left\{ \prod_{t=s+1}^{T} \frac{\tilde{p}(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \prod_{t=1}^{s-1} \frac{\tilde{p}_\epsilon(x_t, \bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0})}{\tilde{p}_\epsilon(\bar{a}_{t-1}, \bar{x}_{1:t-1}, x_{i0}))} \right.\right. \\
&\qquad \left.\left. \times \frac{1}{\tilde{p}_\epsilon(\bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0}))} \left[\tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s})}) - \frac{\tilde{p}(x_s, \bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0})}{\tilde{p}(\bar{a}_{s-1}, \bar{x}_{1:s-1}, x_{i0}))} \tilde{\delta}^\lambda_{(\bar{a}_{i,s-1}, \bar{x}_{i,s-1})} \right] \right] \right] \mathrm{d}\bar{x}
\end{aligned}
\tag{38}
$$

$$
\tag{39}
$$

□

D.2 Proofs for infinite-horizon evaluation

Proof of Proposition 3. **Background on sensitivity analysis of linear programs to coefficient matrix perturbations.** Our proof strategy broadly follows from that of [20], which studies perturbations to linear programs of the constraint coefficient matrix. However our perturbation is entrywise upon the transition probability (a conditional transition probability); hence nonlinear in ϵ; we compute the entrywise matrix derivative directly using an alternative technique described in the proof of [20, Thm.1]. But, the overall argument for sensitivity analysis subject to left-hand-side perturbations based on active set identification is similar, so we recount the argument here as a preface.

Namely, note that for ϵ small enough, the *active basis* remains the same when solving with respect to unperturbed or perturbed distributions, P_0 or P_ϵ. Consider the primal program in standard form,

$$d(\epsilon) = \max\{c^\top x \colon A^\epsilon x = b\}.$$

[20] considers perturbations where $A^\epsilon = A + \epsilon G$. Letting β denote an index vector for the optimal basis, note that

$$d(\epsilon) = c_\beta x^*(\epsilon) = c_\beta (A^\epsilon_\beta)^{-1} b_\beta$$

by nondegeneracy and properties of the basic feasible solution for linear programming. [20] notes that the derivative $d'(\epsilon) = -\overline{\pi} G \overline{x}$ (where $(\overline{\pi}, \overline{x})$ are the dual and primal optimal solutions, respectively) can be obtained from the well-known matrix inverse fact

$$\frac{dM^{-1}(\epsilon)}{d\epsilon} = -M^{-1}(\epsilon)(\frac{dM(\epsilon)}{d\epsilon})M^{-1}(\epsilon). \tag{40}$$

We simply compute $\frac{dM^{-1}(\epsilon)}{d\epsilon}$ for the perturbations to conditional probabilities, rather than linear perturbations restricted to the form $A + \epsilon G$, for a matrix G independent of ϵ.

Perturbations for the optimal policy. Define the (s, s') entry of the perturbed transition matrix, in the direction of a generic observation $(\tilde{s}, \tilde{a}, \tilde{s}')$, as $P^\epsilon_a(s, s') = \frac{(1-\epsilon)p(s,a,s') + \epsilon \mathbb{I}\left[s,a,s'\right]}{(1-\epsilon)d(s,a) + \epsilon \mathbb{I}\left[s,a\right]}$.

Define the perturbed version of Equation (6):

$$\Psi_D(P_\epsilon) = \min_V \left\{ (1-\gamma) \sum_s \left((1-\epsilon)\mu_0(s) + \epsilon \mathbb{I}\left[\tilde{s}\right]\right) V^*_\epsilon(s) \colon \; (I - \gamma P^\epsilon_a)\, \mathbf{V} - \mathbf{r}_a \geq 0, \quad \forall a \in \mathcal{A} \right\} \tag{41}$$

Let $V^*_\epsilon(s) \in \arg\min \Psi_D(P_\epsilon)$ achieve the optimal solution of $\Psi_D(P_\epsilon)$.

We will repeatedly use the fact that $\frac{d}{d\epsilon}\{(1-\epsilon)p(o) + \epsilon \mathbb{I}\left[o\right]\} = \epsilon(\mathbb{I}\left[o\right] - p(o))$.

Then

$$\begin{aligned}
\frac{d}{d\epsilon}\Psi_D(P_\epsilon)\Big|_{\epsilon=0} &= (1-\gamma)\sum_s \frac{d}{d\epsilon}\left\{(1-\epsilon)\mu_0(s) + \epsilon \mathbb{I}\left[\tilde{s}\right]\right\}\Big|_{\epsilon=0} V^*(s) + (1-\gamma)\sum_s \mu_0(s)\frac{d}{d\epsilon}\{V^*_\epsilon(s)\}\Big|_{\epsilon=0} \\
&= (1-\gamma)\sum_s (\mathbb{I}\left[s = \tilde{s}\right] - \mu_0(s))V^*(s) + (1-\gamma)\sum_s \mu_0(s)\frac{d}{d\epsilon}\{V^*_\epsilon(s)\}\Big|_{\epsilon=0}.
\end{aligned}$$

where $V^*(s)$ is the unperturbed solution. Note that, letting P_{a^*} denote the transition matrix under $a(s) \sim \pi^*$, we have that $V^*_\epsilon = (I - \gamma P^\epsilon_{a^*})^{-1} r_{a^*}$.

Entrywise, where we overload notation for $a^* = a^*(s)$ (the optimal action at s, i.e. under active basis), and $\mathbb{I}\left[\tilde{s}, \tilde{a}, \tilde{s'}\right] = \mathbb{I}\left[(s, a^*, s') = (\tilde{s}, \tilde{a}, \tilde{s'})\right]$, etc:

$$\begin{aligned}
\frac{d(I - \gamma P^\epsilon_{a^*})_{(s,s')}}{d\epsilon}\Big|_{\epsilon=0} &= -\gamma \frac{d}{d\epsilon}\left\{\frac{(1-\epsilon)p(s,a^*,s') + \epsilon \mathbb{I}\left[\tilde{s},\tilde{a},\tilde{s}'\right]}{(1-\epsilon)d(s,a^*) + \epsilon \mathbb{I}\left[\tilde{s},\tilde{a}\right]}\right\}\Big|_{\epsilon=0} \\
&= \gamma\left(\frac{\mathbb{I}\left[\tilde{s},\tilde{a},\tilde{s}'\right] - p(s,a^*,s')}{d(s,a^*)} - p(s,a^*,s')\frac{\mathbb{I}\left[\tilde{s},\tilde{a}\right] - d(s,a^*)}{d(s,a^*)^2}\right) \\
&= \gamma\left(\frac{\mathbb{I}\left[\tilde{s},\tilde{a},\tilde{s}'\right]}{d(s,a^*)} - p(s' \mid s, a^*)\frac{\mathbb{I}\left[\tilde{s},\tilde{a}\right]}{d(s,a^*)}\right)
\end{aligned}$$

Hence, applying eq. (40),

$$\begin{aligned}\frac{d}{d\epsilon}\Psi_D(P_\epsilon)\Big|_{\epsilon=0} &= (1-\gamma)\sum_s(\mathbb{I}\left[s=\tilde{s}\right]-\mu_0(s))V^*(s) + (1-\gamma)\sum_s \mu_0(s)\frac{d}{d\epsilon}\{V^*_\epsilon(s)\}\Big|_{\epsilon=0} \\ &= (1-\gamma)\sum_s(\mathbb{I}\left[s=\tilde{s}\right]-\mu_0(s))V^*(s) \\ &\quad + (1-\gamma)\mu_{\mathbf{0}}(I-\gamma P_{a^*})^{-1}\left(\frac{d(I-\gamma P^\epsilon_{a^*})}{d\epsilon}\Big|_{\epsilon=0}\right)(I-\gamma P_{a^*})^{-1}\mathbf{r}.\end{aligned}$$

Note that, as in [20], $(1-\gamma)\mu_{\mathbf{0}}(I-\gamma P_{a^*})^{-1} = \mu^*$ is the dual (unperturbed) optimal solution of example 3 and $(I-\gamma P_{a^*})^{-1}\mathbf{r}$ is the primal (unperturbed) optimal solution V^*. Therefore,

$$\frac{d}{d\epsilon}\Psi_D(P_\epsilon)\Big|_{\epsilon=0} \tag{42}$$

$$\begin{aligned}&= (1-\gamma)(V^*(\tilde{s}) - \sum_s \mu_0(s)V^*(s)) \\ &\quad - \gamma\sum_s \mu^*(s,a)\sum_{s'} V^*(s')\left(\frac{\mathbb{I}\left[\tilde{s},\tilde{a},\tilde{s}'\right]-p(s,a^*,s')}{d(s,a^*)} - p(s,a^*,s')\frac{\mathbb{I}\left[\tilde{s},\tilde{a}\right]-d(s,a^*)}{d(s,a^*)^2}\right)\end{aligned} \tag{43}$$

$$= (1-\gamma)(V^*(\tilde{s}) - \sum_s \mu_0(s)V^*(s)) - \gamma\frac{\mu^*(\tilde{s},\tilde{a})}{d(\tilde{s},\tilde{a})}\left(V(\tilde{s}') - (P^\pi V^*)(\tilde{s})\right)$$

where $(P^\pi V^*)(\tilde{s}) = \sum_{s'} p(s' \mid \tilde{s}, a)V^*(\tilde{s})$ is the Bellman operator evaluated starting from the state $\tilde{s}$.

Further note that $\gamma(P^\pi V^*)(\tilde{s}) = (V(\tilde{s}) - r(\tilde{s},\tilde{a}))$, (ie a Bellman residual term), and $\Psi_D(P) = (1-\gamma)\sum_s \mu_0(s)V^*(s))$.

Hence

$$\frac{d}{d\epsilon}\Psi_D(P_\epsilon)\Big|_{\epsilon=0} = (1-\gamma)V^*(\tilde{s}) - \frac{\mu^*(\tilde{s},\tilde{a})}{d(\tilde{s},\tilde{a})}\left(r(\tilde{s},\tilde{a}) + \gamma V(\tilde{s}') - V(\tilde{s})\right) - \Psi_D(P) \tag{44}$$

□

Proof of Proposition 4. We analyze the error introduced by finite differencing. Consider the dual formulation of eq. (6), and the perturbed formulation of eq. (41). Fix ϵ small enough such that the active basis remains the same under P and P_ϵ; let P^π denote the submatrix of the transition matrix P (stacked for the optimization problem) corresponding to the active basis (equivalently, optimal policy).

Define the entries of G_ϵ such that $P_\epsilon = P + \epsilon G_\epsilon$, i.e. (analogous to Lemma 2)

$$(G_\epsilon^a)_{s,s'} = \frac{\mathbb{I}\left[\tilde{s}, \tilde{a}, \tilde{s}'\right]}{d_\epsilon(s,a)} - \frac{\mathbb{I}\left[\tilde{s}, \tilde{a}\right] P(s' \mid s, a)}{d_\epsilon(s,a)}$$

Let $V_\epsilon^\pi(s), V^\pi(s)$ be the active subvector of the optimal solution achieving the optimal values of $\Psi(P_\epsilon), \Psi(P)$, respectively.

Then

$$\begin{aligned}
\epsilon^{-1}(\Psi_D(P_\epsilon) - \Psi_D(P)) &= \epsilon^{-1}(1-\gamma)\left(\textstyle\sum_s \left((1-\epsilon)\mu_0(s) + \epsilon \mathbb{I}\left[\tilde{s}\right]\right) V_\epsilon^\pi(s) - \sum_s \mu_0(s) V^\pi(s)\right) \\
&= \epsilon^{-1}(1-\gamma)\left(\textstyle\sum_s \epsilon\left(\mathbb{I}\left[\tilde{s}\right] - \mu_0(s)\right) V_\epsilon^\pi(s) + \sum_s \mu_0(s)(V_\epsilon^\pi(s) - V^\pi(s))\right) \\
&= (1-\gamma)\left(\textstyle\sum_s \left(\mathbb{I}\left[\tilde{s}\right] - \mu_0(s)\right) V_\epsilon^\pi(s) + \epsilon^{-1} \sum_s \mu_0(s)(V_\epsilon^\pi(s) - V^\pi(s))\right)
\end{aligned} \tag{45}$$

The optimal solutions V_ϵ^π, V^π differ in the perturbation matrix; inverting the basic feasible solution, where invertibility is assured by nondegeneracy, gives that:

$$V_\epsilon^\pi = (I - \gamma P_\epsilon^\pi)^{-1} r_\pi, \qquad V^\pi = (I - \gamma P^\pi)^{-1} r_\pi.$$

We analyze $V_\epsilon^\pi - V^\pi = ((I - \gamma P_\epsilon^\pi)^{-1} - (I - \gamma P^\pi)^{-1}) r_\pi$ via a general result about perturbed matrices. Note that for two matrices A, B, clearly $(A+B)^{-1} = A^{-1} - A^{-1}B(A+B)^{-1}$. Applying this recursively we obtain the power series expansion

$$(A+B)^{-1} = A^{-1} - A^{-1} \sum_{k=0}^{\infty} (B)^k (A^{-1})^k.$$

Applying this result to our setting with $A = (I - \gamma P_\epsilon^\pi)$ and $B = -\gamma \epsilon G_\epsilon$, we obtain

$$V_\epsilon^\pi - V^\pi = \epsilon\gamma (I - \gamma P^\pi)^{-1} G_\epsilon^\pi (I - \gamma P^\pi)^{-1} r_\pi + O(\epsilon^2)$$

The statement follows by substituting in the above expression, evaluating based on definition of the entries of G_ϵ^π, and substituting the above and eq. (45) into eq. (43) of the proof of Proposition 3 and following that proof onwards. □

E Practical considerations

Monte Carlo sampling for uniform kernel for $\tilde{\delta}^\lambda$. For computational purposes, it is helpful to evaluate the final integral over a uniform kernel in x by Monte Carlo sampling.[3] Hence for perturbations in the direction of (x_i, a_i, y_i), for computational purposes we evaluate over N_{MC} Monte Carlo samples:

$$\begin{aligned}
\hat{\phi}_{\epsilon,\lambda}^{MC}(o_i) = \tfrac{1}{N_{MC}} \textstyle\sum_k & \frac{\left((1-\epsilon)\left(\sum_{j:A_j=1} K(X_j - \tilde{x}_k) Y_j\right) P(A=1) + \epsilon y_i \cdot \mathbb{I}[a_i=1] \cdot 1\right)}{(1-\epsilon) p(A=1, \tilde{x}_k) + \epsilon \mathbb{I}[a_i=1]} \\
& + (1-\epsilon) \tfrac{1}{N_{MC}} \textstyle\sum_k \frac{\tilde{p}(\tilde{x}_k)}{\tilde{p}_\epsilon(A=1, \tilde{x}_k)} \mathbb{I}(a_i = 1) \left\{y_i - \mathbb{E}_{\tilde{P}}[Y \mid A = 1, \tilde{x}_k]\right\}
\end{aligned} \tag{46}$$

Bandwidth for kernel density estimation of functionals. [6] discuss optimal bandwidth selection for estimation of functionals via plug-in of nonparametric density estimation. They show that asymptotic linearity/root-n consistency of $\hat{\beta}$ is possible under the optimal bandwidth for density estimation when the number of derivatives of the influence function is more than half the dimension of observation o; requiring a higher order kernel. (See [33] for more discussion).

[3] In computational experiments not reported here, we found scipy.integrate.quad to be quite sensitive in evaluating the smoothed perturbation. We leave more extensive numerical evaluation for further work.

Other density estimation approaches. Although we have discussed estimation of the components of $\tilde{P}$ by nonparametric Nadaraya-Watson/kernel-density estimation for simplicity, other estimates of probability densities or generative models can be similarly used; we describe a few below. These approaches may work well empirically; we leave a thorough evaluation for future work.

Example 4 (Bayesian Dirichlet Process mixture models)**.** *[27] develops Dirichlet Process mixtures of Generalized Linear Models with covariate density given by a mixture of exponential-family distributions, conditional response density modeled via a generalized lienar model (with a set of GLM parameters for each mixture component). This specification of the probability densities implies a regression estimate.*

Example 5 (Generative modeling with Gaussian mixture models)**.** *[35] uses generative modeling with Gaussian mixture networks, and sampling from the generative model, to compute the integral in order to scale-up Nadaraya-Watson regression to high-dimensional settings in sensitivity analysis. Similar approaches may be used to compute the density estimates in this setting.*

F Details on simulation

We consider a piecewise linear outcome mean specification which is hard for nonparametric estimation.

$$\begin{aligned}\mathbb{E}[Y \mid A, X] = \mathbb{I}\left[X < 0.25\right] \cdot (-5AX + 3(2A-1)) + \mathbb{I}\left[X \in [0.25, 0.5]\right] \cdot (5AX + 3) \\ + \mathbb{I}\left[X \in [0.5, 0.75]\right] \cdot -5AX + \mathbb{I}\left[X \in [0.75, 1]\right] \cdot 5AX\end{aligned}$$

The propensity score is drawn as $A \sim \mathrm{Bern}(\sin(20X) + 0.5)$.

Details on computation Experiments were conducted on a MacBook Pro with 16g RAM.