Infinitely wide limits for deep Stable neural networks: sub-linear, linear and super-linear activation functions

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Abstract

There is a recent and growing literature on large-width asymptotic properties of deep Gaussian neural networks (NNs), i.e. deep NNs with Gaussian-distributed parameters or weights, and Gaussian stochastic processes. Motivated by some empirical analyses that show the potential of replacing Gaussian distributions with the more general Stable distributions for the NN's weights, in this paper we investigate large-width asymptotic properties of deep Stable NNs, i.e. deep NNs with Stable-distributed parameters. For sub-linear activation functions, a recent work has characterized the infinitely wide limit of a suitable rescaled deep Stable NN in terms of a Stable stochastic process, which generalize the Gaussian process. Here, we extend such a characterization to a general class of activation functions, which includes sub-linear, linear and super-linear functions. Our results show that in the Stable setting the scaling of the NN may depend on the choice of the activation function, thus bringing out a critical difference with respect to the Gaussian setting. In particular, while in the Gaussian setting the choice of the activation function does not affect the scaling required to achieve the infinitely wide Gaussian process, in the Stable setting the use of a linear activation function in place of a sub-linear or a super-linear activation function results in a change of the scaling, through an additional logarithmic term, in order to achieve the infinitely with Stable process.

1 Introduction

Deep (feed-forward) neural networks (NNs) play a critical role in many domains of practical interest, and nowadays they are the subject of numerous studies investigating their theoretical properties. Of special interest is the study of prior distributions over the NN's parameters or weights, namely random initializations of NNs. In such a context, there is a growing interest on large-width asymptotic properties of deep NNs with Gaussian-distributed parameters, with emphasis on the interplay between infinitely wide limits of such NNs and some classes of Gaussian stochastic processes. In his seminal work, Neal (1996) characterized the infinitely wide limit of a shallow Gaussian NN. In particular, let: i) $\boldsymbol{x} \in \mathbb{R}^d$ be the input of the NN; ii) $\tau : \mathbb{R} \to \mathbb{R}$ be an activation function or nonlinearity; iii) $\theta = \{w_i^{(0)}, w, b_i^{(0)}, b\}_{i \geq 1}$ be the collection of NN's parameters such that $w_{i,j}^{(0)} \overset{iid}{\sim} w_j \overset{iid}{\sim} N(0, \sigma_w^2)$ and $b_i^{(0)} \overset{iid}{\sim} b \overset{iid}{\sim} N(0, \sigma_b^2)$ for $\sigma_w^2, \sigma_b^2 > 0$, with $N(\mu, \sigma^2)$ being the Gaussian distribution with mean μ and variance σ^2. Then, consider a rescaled shallow Gaussian NN defined as

$$\begin{cases} g_j(\boldsymbol{x}) = \langle w_j^{(0)}, \boldsymbol{x} \rangle_{\mathbb{R}^d} + b_j^{(0)} \\ f_{\boldsymbol{x}}(n)[\tau, n^{-1/2}] = b + \frac{1}{n^{1/2}} \sum_{j=1}^n w_j \tau(g_j(\boldsymbol{x})), \end{cases} \tag{1}$$

with $n^{-1/2}$ being the scaling factor. Neal (1996) showed that, as $n \to +\infty$ the NN output $f_{\boldsymbol{x}}(n)[\tau, n^{-1/2}]$ converges in distribution to a Gaussian random variable (RV) with mean zero and a suitable variance. The proof follows by an application of the Central Limit Theorem (CLT), thus relying on minimal assumptions on the activation function τ, as it is sufficient to ensure that $\mathbb{E}[(g_j(\boldsymbol{x}))^2]$ is finite. The result of Neal (1996) has been extended to a general input matrix, i.e. $k > 1$ inputs of dimension d, and deep Gaussian NNs, assuming

both a sequential growth (Der & Lee, 2005) and a joint growth (de G. Matthews et al., 2018) of the width over the NN's layers. See Theorem A.1 in the Appendix. In general, these large-width asymptotic results rely on some minimal assumptions for the function τ, thus allowing to cover the most popular activation functions.

Neal (1996) first discussed the problem of replacing the Gaussian distribution of the NN's parameters with a general Stable distribution, namely a distribution with heavy tails (Samorodnitsky & Taqqu, 1994). Some empirical analyses in Neal (1996) show that while the contribution of Gaussian weights vanishes in the infinitely wide limit, Stable weights retain a non-negligible contribution. Motivated by these empirical analyses, Favaro et al. (2020) characterized the infinitely wide limit of a deep Stable NN in terms of a class of Stable stochastic process, assuming both a joint growth and a sequential growth of the width over the NN's layers. See also Favaro et al. (2021), and references therein, for a detailed discussion. Critical to achieve the infinitely wide Stable process is the assumption of a sub-linear activation function τ, i.e. $|\tau(x)| \leq a + b|x|^\beta$, with $a, b > 0$ and $0 < \beta < 1$. In particular, for a shallow Stable NN, let: $\boldsymbol{x} \in \mathbb{R}^d$ be the input of the NN; ii) $\tau : \mathbb{R} \to \mathbb{R}$ be the sub-linear activation function of the NN; iii) $\theta = \{w_i^{(0)}, w, b_i^{(0)}, b\}_{i\geq 1}$ be the NN's parameters such that $w_{i,j}^{(0)} \overset{iid}{\sim} w_j \overset{iid}{\sim} S_\alpha(\sigma_w)$ and $b_i^{(0)} \overset{iid}{\sim} b \overset{iid}{\sim} S_\alpha(\sigma_b)$ for $\alpha \in (0, 2]$ and $\sigma_w, \sigma_b > 0$, with $S_\alpha(\sigma)$ being the symmetric Stable distribution with stability α and scale σ. Then, consider a rescaled shallow Stable NN defined as

$$\begin{cases} g_j(\boldsymbol{x}) = \langle w_j^{(0)}, \boldsymbol{x}\rangle_{\mathbb{R}^d} + b_j^{(0)} \\ f_{\boldsymbol{x}}(n)[\tau, n^{-1/\alpha}] = b + \frac{1}{n^{1/\alpha}} \sum_{j=1}^n w_j \tau(g_j(\boldsymbol{x})), \end{cases} \tag{2}$$

with $n^{-1/\alpha}$ being the scaling factor. The NN (1) is recovered from (2) by setting $\alpha = 2$. Favaro et al. (2020) showed that, as $n \to +\infty$ the NN output $f_{\boldsymbol{x}}(n)[\tau, n^{-1/\alpha}]$ converges in distribution to a Stable RV with stability α and a suitable scale. See Theorem A.2 in the Appendix. See also Favaro et al. (2021) for some refinements of this result in terms of sup-norm convergence rates. Differently from the Gaussian setting of (de G. Matthews et al., 2018), the result of Favaro et al. (2020) relies on the assumption of a sub-linear activation function. This is a strong assumption, as it does not allow to cover some popular activation functions.

1.1 Our contributions

In this paper, we consider the problem of characterizing the infinitely wide limit of deep Stable NNs with a general activation function, thus removing the assumption of a sub-linear function considered in Favaro et al. (2020). In particular, we assume $\tau : \mathbb{R} \to \mathbb{R}$ to be a continuous activation function belonging to one of these classes:

$$E_1 = \{\tau \in \mathcal{C}(\mathbb{R};\mathbb{R}) : |\tau(z)| = \mathcal{O}(|z|^\beta) \text{ with } 0 \leq \beta < 1\};$$

$$E_2 = \{\tau \in \mathcal{C}(\mathbb{R};\mathbb{R}) : |\tau(z)| \backsim |z|^\gamma \text{ for } z \to \pm + \infty \text{ and } \tau \text{ strictly increasing for } |z| > a, \text{ for some } \gamma, a > 0\};$$

$$E_3 = \{\tau \in \mathcal{C}(\mathbb{R};\mathbb{R}) : \tau(z) \backsim z^\gamma \text{ for } z \to +\infty, |\tau(z)| = \mathcal{O}(|z|^\beta) \text{ with } \beta < \gamma \text{ for } z \to -\infty \text{ and } \tau \text{ strictly increasing for } z > a, \text{ for some } \gamma, a > 0\}.$$

We characterize the infinitely wide limit of a deep Stable NN with an activation function belonging to the above classes, for a d-dimensional input and assuming a sequential growth of the width over the NN's layers. The extension to a general input matrix, i.e. k inputs of dimension d, and the joint growth of the width over the NN's layers are also discussed. The use of activation functions of linear envelope, i.e. E_1, is covered in Favaro et al. (2020), and here we present an alternative proof of the infinitely wide limit of the NN, which relies on the application of a generalized CLT (Uchaikin & Zolotarev, 2011; Otiniano & Gonçalves, 2010). The novelty of our result lies in the use of linear functions, i.e. $E_2 \cup E_3$ choosing $\gamma = 1$, and super-linear functions, i.e. $E_2 \cup E_3$ choosing $\gamma > 1$. For example, the ReLU function, i.e. $\tau(s) = \max\{0, s\}$, which is arguably the most popular linear activation function, belongs to the class E_3 whereas a classical activation like $\tau(s) = s^\beta$, with $\beta > 1$ odd, is contained in E_2. Our results show that in the Stable setting the

scaling factor of the NN may depend on the choice of the activation function τ, thus bringing out a critical difference with respect to the Gaussian setting. While in the Gaussian setting the choice of τ does not affect the scaling $n^{-1/2}$ required to achieve the infinitely wide Gaussian process, in the Stable setting the usage of certain activation functions, e.g. linear activation when the parameters of the NN have the same index α, results in a change of the scaling $n^{-1/\alpha}$, through an additional $(\log n)^{-1/\alpha}$ term, to achieve the infinitely wide Stable process.

1.2 Organization of the paper

Section 2 contains the main results of the paper, which are organized as follows: i) the weak convergence of a shallow Stable with an activation function τ in the classes E_1, E_2 and E_3, for an input $x = 1$ and no biases; ii) the weak convergence of a deep Stable NN with an activation function τ in the classes E_1, E_2 and E_3, for an input $x \in \mathbb{R}^d$ and biases; iii) some examples of weak convergence of deep Stable NNs using specific activation functions, and in particular the ReLU activation function, for an input $x \in \mathbb{R}^d$ and biases. In Section 3 we discuss some extensions of our work and directions for future research in the context of deep Stable NNs.

2 Main results

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space, such that the RVs introduced in this paper are defined on such a probability space. Given a RV Z we define the cumulative distribution function (CDF) of Z as $P_Z(z) = \mathbb{P}(Z \leq z)$, the truncated CDF of Z as $\overline{P}_Z(z) = 1 - P_Z(z)$ and the density function (PDF) as $p_Z(z) = \frac{dP_Z(z)}{dz}$. We work with probability measures with respect to the Lebesgue measure, and use the notation $P_Z(dz)$ to indicate $p_Z(z)dz$. A RV Z is symmetric if $Z \overset{d}{=} -Z$, i.e. if Z and $-Z$ has the same distribution, that is $\overline{P}_Z(z) = P_Z(-z)$ for all $z \in \mathbb{R}$. We say that Z_n converges to Z in distribution, as $n \to +\infty$, if for every point of continuity $z \in \mathbb{R}$ of P_Z it holds $P_{Z_n}(z) \to P_Z(z)$ as $n \to +\infty$, in which case we write $Z_n \xrightarrow{d} Z$. Given two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ we write $f \sim g$ if and only if $\lim_{z\to+\infty} \frac{f(z)}{g(z)} = 1$, $f = o(g)$ if and only if $\lim_{z\to+\infty} \frac{f(z)}{g(z)} = 0$ and $f = \mathcal{O}(g)$ if and only if there exists $C > 0$ and z_0 such that $\frac{f(z)}{g(z)} \leq C$ for every $z \geq z_0$.

Definition 2.1. *A $\mathbb{R}$-valued RV X has Stable distribution with stability $\alpha \in (0, 2]$, skewness $\beta \in [-1, 1]$, scale $\sigma > 0$ and shift $\mu \in \mathbb{R}$, and we write $X \sim S_\alpha(\sigma, \beta, \mu)$, if its characteristic function is $\varphi_X(t) = \mathbb{E}[e^{itX}] = e^{\psi(t)}$, for $t \in \mathbb{R}$, where*

$$\psi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha [1 + i\beta \tan(\frac{\alpha\pi}{2}) sign(t)] + i\mu t & \alpha \neq 1 \\ -\sigma |t| [1 + i\beta \frac{2}{\pi} sign(t) \log(|t|)] + i\mu t & \alpha = 1. \end{cases}$$

By means of Samorodnitsky & Taqqu (1994, Property 1.2.16), if $X \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$ then $\mathbb{E}[|X|^r] < +\infty$ for $0 < r < \alpha$, and $\mathbb{E}[|X|^r] = +\infty$ for any $r \geq \alpha$. A $\mathbb{R}$-valued RV X is distributed as the symmetric α-Stable distribution with scale parameter σ, and we write $X \sim S_\alpha(\sigma)$, if $X \sim S_\alpha(\sigma, 0, 0)$, which implies that

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = e^{-\sigma^\alpha |t|^\alpha}, \quad t \in \mathbb{R}.$$

This allows to prove that if $X \sim S_\alpha(\sigma)$, then $aX \sim S_\alpha(|a|\sigma)$; see Samorodnitsky & Taqqu (1994, Property 1.2.3). Furthermore, although the PDF of a Stable RV is not available in a closed form, except for few cases, one has a complete characterization of the tail behaviour of the CDF and PDF of Stable RVs: for a symmetric α-Stable distribution, Samorodnitsky & Taqqu (1994, Proposition 1.2.15) states that, if $X \sim S_\alpha(\sigma)$ with $0 < \alpha < 2$,

$$\overline{P}_X(x) = P_X(-x) \sim \frac{1}{2} C_\alpha \sigma^\alpha |x|^{-\alpha},$$

where

$$C_\alpha = \Big(\int_0^{+\infty} x^{-\alpha} \sin(x) dx \Big)^{-1} = \frac{2}{\pi} \Gamma(\alpha) \sin\Big(\alpha \frac{\pi}{2}\Big) = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\frac{\alpha}{2})} & \alpha \neq 1 \\ \frac{2}{\pi} & \alpha = 1. \end{cases}$$

Analogously to what written before, if $X \sim S_\alpha(\sigma)$ with $0 < \alpha < 2$, then $p_X(x) = p_X(-x) \sim \frac{\alpha}{2} C_\alpha \sigma^\alpha |x|^{-\alpha-1}$ holds true.

For an activation function belonging to the classes E_1, E_2 and E_3, we characterize the infinitely wide limit of a deep Stable NN, assuming a sequential growth of the width over the NN's layers. Critical is the use of the following Generalized Central Limit Theorem (GCLT) (Uchaikin & Zolotarev, 2011; Otiniano & Gonçalves, 2010).

Theorem 2.1 (GCLT)**.** *Let Z be a RV such that $\overline{P}_Z(z) \sim cz^{-p}L(z)$ and $P_Z(-z) \sim dz^{-p}L(z)$ for some $c, d > 0$, $0 < p < 2$ and with L being a slow varying function. Moreover, let $(Z_n)_{n\geq 1}$ be a sequence of RVs iid as Z. If*

$$a_n = \begin{cases} 0 & 0 < p < 1 \\ (c-d)\log n & p = 1 \\ \mathbb{E}[Z] & 1 < p < 2, \end{cases}$$

then, as $n \to +\infty$

$$\frac{1}{(nL(n))^{1/p}} \sum_{i=1}^{n} (Z_i - a_n) \xrightarrow{d} S_p\left(\left[\frac{c+d}{C_p}\right]^{\frac{1}{p}}, \frac{c-d}{c+d}, 0\right).$$

2.1 Shallow Stable NNs: large-width asymptotics for an input $x = 1$ and no biases

We start by considering a shallow Stable NN, for an input $x = 1$ and no biases. Let $w^{(0)} = [w_1^{(0)}, w_2^{(0)}, \dots]^T$ and $w = [w_1, w_2, \dots]^T$ independent sequences of RVs such that $w_j^{(0)} \overset{iid}{\sim} S_{\alpha_0}(\sigma_0)$ and $w_j \overset{iid}{\sim} S_{\alpha_1}(\sigma_1)$. Then, we set $Z_j = w_j \tau(w_j^{(0)})$, where $\tau : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function, and define the shallow Stable NN

$$f(n)[\tau, p] = \frac{1}{n^{1/p}} \sum_{j=1}^{n} Z_j \tag{3}$$

with p being a positive real number. From the definition of the shallow Stable NN (3), being $Z_1, Z_2, \dots$ iid according to a certain RV Z, it is sufficient to study the tail behaviour of $\overline{P}_Z(z)$ and $P_Z(-z)$ in order to obtain the convergence in distribution of $f(n)[\tau, p]$. As a general strategy, we proceed as follows: i) we study the tail behaviour of $X \cdot \tau(Y)$ where $X \sim S_{\alpha_x}(\sigma_x)$, $Y \sim S_{\alpha_x}(\sigma_y)$, $X \perp\!\!\!\perp Y$ and $\tau \in E_1$, $\tau \in E_2$ and $\tau \in E_3$; ii) we make use of the GCLT, i.e. Theorem 2, in order to characterize the infinitely wide limit of the shallow Stable NN (3).

Note that to find the tail behaviour of $X \cdot \tau(Y)$ it is sufficient to find the tail behaviour of $|X \cdot \tau(Y)|$, and then use the fact that $\mathbb{P}[X \cdot \tau(Y) > z] = (1/2) \cdot \mathbb{P}[|X \cdot \tau(Y)| > z]$ for every $z \geq 0$, since $X \cdot \tau(Y)$ is symmetric as X is so. Then, to find the asymptotic behaviour of the truncated CDF of $|X \cdot \tau(Y)|$ we make use of some results in the theory of convolution tails and domain of attraction of Stable distributions. Hereafter, we recall some basic facts. Given two CDFs F and G, the convolution $F * G$ is defined as $F * G(t) = \int F(t-y)dG(y)$, which inherits the linearity of the convolution and the commutativity of the convolution from properties of the integral operator. Recall that a function F on $[0, +\infty]$ has exponential tails with rate α ($F \in \mathcal{L}_\alpha$) if and only if

$$\lim_{t \to +\infty} \frac{\overline{F}(t-y)}{\overline{F}(t)} = e^{\alpha y}, \quad \text{for all real } y \in \mathbb{R}.$$

Then,

$$\bar{F}(t) = a(t) \exp\left[-\int_0^t \alpha(v)dv\right], \text{ where } a(t) \to a > 0, \alpha(t) \to \alpha, \text{ as } t \to +\infty.$$

A complimentary definition is the following: a function U on $[0, +\infty]$ is regularly varying with exponent ρ ($U \in \mathcal{RV}_\rho$) if and only if

$$\lim_{t \to +\infty} \frac{U(yt)}{U(t)} = y^\rho, \quad \text{for all } y > 0.$$

Then,

$$U(t) = a(t)\exp\left[\int_0^t \frac{\rho(v)}{v}dv\right], \text{ where } a(t) \to a > 0, \rho(t) \to \rho, \text{ as } t, \to +\infty,$$

i.e. the Karamata's representation of U. Clearly $F \in \mathcal{L}_\alpha$ if and only if $\bar{F}(\ln t) \in \mathcal{RV}_{-\alpha}$. The next lemma provides the tail behaviour of the convolution of F and G, assuming that they have exponential tails with the same rates.

Lemma 2.2 (Theorem 4 of Cline (1986))**.** *Let $F, G \in \mathcal{L}_\alpha$ for some $\alpha > 0$, $f \in \mathcal{RV}_\beta$ and $g \in \mathcal{RV}_\gamma$ where $f(t) = e^{\alpha t}\overline{F}(t)$ and $g(t) = e^{\alpha t}\overline{G}(t)$ and $\beta > -1$ and $\gamma > -1$. Then*

$$\overline{F * G}(t) \backsim \frac{\Gamma(1+\beta)\Gamma(1+\gamma)}{\Gamma(1+\beta+\gamma)}\alpha t e^{\alpha t}\overline{F}(t)\overline{G}(t), \qquad \text{as } t \to +\infty$$

We make use of Lemma 2.2 to find the tail behaviour of $|X \cdot \tau(Y)|$ when $|X|$ and $|\tau(Y)|$ have regularly varying truncated CDFs with same rates. If $|X|$ and $|\tau(Y)|$ have regularly varying truncated CDFs with different rates, then we make use of the next lemma, which describes the tail behaviour of $U \cdot W$, where U and W are two independent non-negative RVs such that $\mathbb{P}[U > u]$ is regularly varying of index $-\alpha \leq 0$ and $\mathbb{E}[W^\alpha] < +\infty$.

Lemma 2.3 (Breiman's theorem)**.** *Suppose U and W are two independent non-negative RVs such that*

$$\mathbb{E}[W^\alpha] < +\infty \text{ and } \mathbb{P}[W > u] = \mathrm{o}(\mathbb{P}[U > u]).$$

If $\mathbb{P}[U > u] \backsim cu^{-\alpha}$, with $c > 0$, then

$$\mathbb{P}[UW > u] \backsim \mathbb{E}[W^\alpha] \cdot \mathbb{P}[U > u].$$

Lemma 2.3 was first stated in Breiman (1965) for $\alpha \in [0, 1]$, and then extended by Cline & Samorodnitsky (1994) for all values of α, still under the hypothesis that $\mathbb{E}[W^{\alpha+\epsilon}] < +\infty$ for some $\epsilon > 0$. Lemma 2.3 provides a further extension of these results in the case $\mathbb{P}[U > x] \backsim cx^{-\alpha}$, with $c > 0$, and has been proved in Denisov & Zwart (2005).

Based on Lemma 2.2 and Lemma 2.3, it remains to find the tail behaviour of $|X|$ and $|\tau(Y)|$. For the former, it is immediate to derive that $\overline{P}_{|X|}(t) := \mathbb{P}[|X| > t] \backsim C_{\alpha_x}\sigma_x^{\alpha_x}t^{-\alpha_x}$, while, for the latter, we have the following lemma.

Lemma 2.4 (Tail behaviour of $\tau(Y)$, $\tau \in E_2 \cup E_3$)**.** *If $Y \sim S_\alpha(\sigma)$, then*

i) $\mathbb{P}[|\tau(Y)| > t] \backsim C_\alpha\sigma^\alpha t^{-\alpha/\gamma}, \quad$ *if* $\tau \in E_2$

ii) $\mathbb{P}[|\tau(Y)| > t] \backsim \frac{1}{2}C_\alpha\sigma^\alpha t^{-\alpha/\gamma}, \quad$ *if* $\tau \in E_3$

Proof. If τ is strictly increasing for $x > a$ and $\tau(z) \sim z^\gamma$ for $x \to +\infty$ with $\gamma > 0$, then $\tau^{-1}(y) \sim y^{1/\gamma}$. Analogously at $-\infty$. We refer to Theorem 5.1 of Olver (1974) for the case $\gamma = 1$. Now, starting with $\tau \in E_2$ and defining the inverse of τ where the activation is strictly increasing, we can write for a sufficiently large t:

$$\begin{aligned} P(|\tau(Y)| > t) = P(\tau(Y) > t) + P(\tau(Y) < -t) &\backsim \frac{1}{2}C_\alpha\sigma^\alpha|\tau^{-1}(t)|^{-\alpha} + \frac{1}{2}C_\alpha\sigma^\alpha|\tau^{-1}(-t)|^{-\alpha} \\ &\backsim \frac{1}{2}C_\alpha\sigma^\alpha t^{-\alpha/\gamma} + \frac{1}{2}C_\alpha\sigma^\alpha|t|^{-\alpha/\gamma} = C_\alpha\sigma^\alpha t^{-\alpha/\gamma}. \end{aligned}$$

Instead, if $\tau \in E_3$, then there exits $b > 0$ and $y_0 < 0$ such that $|\tau(y)| < b|y|^\beta$ for $y \leq y_0$. Then, for t sufficiently large,

$$P(|\tau(Y)| > t, Y < 0) \leq P(b|Y|^\beta > t, Y < 0) \sim \frac{1}{2}C_\alpha\sigma^\alpha(bt)^{-\alpha/\beta}.$$

Furthermore,

$$P(|\tau(Y)| > t, Y > 0) = P(Y > \tau^{-1}(t)) \backsim \frac{1}{2}C_\alpha\sigma^\alpha|\tau^{-1}(t)|^{-\alpha} \backsim \frac{1}{2}C_\alpha\sigma^\alpha z^{-\alpha/\gamma},$$

hence, since $\beta < \gamma$, it holds that $P(|\tau(Y)| > t) \backsim \frac{1}{2}C_\alpha\sigma^\alpha z^{-\alpha/\gamma}$, which concludes the proof. □

Based on the previous results, it is easy to derive the tail behaviour of $|X \cdot \tau(Y)|$, which is stated in the next theorems.

Theorem 2.5 (Tail behaviour of $|X \cdot \tau(Y)|$)**.** *Let $|Z| = |X \cdot \tau(Y)|$ where X and Y are independent and distributed respectively as $S_{\alpha_x}(\sigma_x)$ and $S_{\alpha_y}(\sigma_y)$. If $\tau \in E_1$ and $\beta\alpha_x < \alpha_y$, then*

$$\overline{P}_{|Z|}(t) \backsim C_{\alpha_x}\sigma_x^{\alpha_x}\mathbb{E}[|\tau(Y)|^{\alpha_x}]t^{-\alpha_x}.$$

For $\tau \in E_2 \cup E_3$, define $\underline{\alpha} = \min(\alpha_x, \alpha_y/\gamma)$ and $c_\tau = \frac{1}{2}$ if $\tau \in E_3$ and $c_\tau = 1$ otherwise. Then

$$\overline{P}_{|Z|}(z) \sim \begin{cases} c_\tau C_{\underline{\alpha}\gamma}\sigma_y^{\underline{\alpha}\gamma}\mathbb{E}[|X|^{\underline{\alpha}}]z^{-\underline{\alpha}} & \text{if} \quad \gamma > \alpha_y/\alpha_x \\ c_\tau C_{\underline{\alpha}}C_{\underline{\alpha}\gamma}\sigma_x^{\underline{\alpha}}\sigma_y^{\underline{\alpha}\gamma}z^{-\underline{\alpha}}\log z & \text{if} \quad \gamma = \alpha_y/\alpha_x \\ C_{\underline{\alpha}}\sigma_x^{\underline{\alpha}}\mathbb{E}[|\tau(Y)|^{\underline{\alpha}}]z^{-\underline{\alpha}} & \text{if} \quad \gamma < \alpha_y/\alpha_x. \end{cases}$$

Proof. We start from the proof of the first case, i.e. $\tau \in E_1$. Here, $|\tau(Y)| < b|Y|^\beta$ for certain $\beta \in (0,1)$ and $b > 0$, when $|Y|$ is larger than some $y_0 > 0$, hence there exists $c > 0$ such that

$$\mathbb{E}[|\tau(Y)|^{\alpha_x}] \leq c + \mathbb{E}[b|Y|^{\beta\cdot\alpha_x}] < +\infty,$$

being $\beta\alpha_x < \alpha_y$ by hypothesis. The thesis then follows from Lemma 2.3. An analogous strategy can be used in the case $\alpha_x \neq \alpha_y/\gamma$. Indeed, $\mathbb{E}[|X|^{\alpha_y/\gamma}] < +\infty$ if $\alpha_x > \alpha_y/\gamma$ and $\mathbb{E}[|\tau(Y)|^{\alpha_x}] < +\infty$ if $\alpha_x < \alpha_y/\gamma$. Hence Breiman's theorem allows to conclude. A different situation arises when $\alpha_x = \alpha_y/\gamma$. In this case, consider the RVs $\log|X|$ and $\log|\tau(Y)|$ and observe that

$$\overline{P}_{\log|X|}(t) := \mathbb{P}[\log|X| > t] = \mathbb{P}[|X| > e^t] \backsim C_{\alpha_x}\sigma_x^{\alpha_x}e^{-\alpha_x t} \in \mathcal{L}_{\alpha_x},$$

i.e. $\mathbb{P}[\log|X| > t]$ has an exponential tail with index α_x, and the same has $\overline{P}_{\log|\tau(Y)|} := \mathbb{P}[\log|\tau(Y)| > t]$ since $\alpha_x = \alpha_y/\gamma$. Furthermore, $e^{\alpha_x t} \cdot \overline{P}_{\log|X|}(t) \in \mathcal{RV}_0$ and $e^{\alpha_x t} \cdot \overline{P}_{\log|\tau(Y)|} \in \mathcal{RV}_0$, hence we apply Lemma 2.2 with $\beta = \gamma = 0$, and obtain that

$$\begin{aligned} \mathbb{P}[\log|X \cdot \tau(Y)| > t] &= \mathbb{P}[\log|X| + \log|\tau(Y)| > t] = \overline{P}_{\log|X|} * \overline{P}_{\log|\tau(Y)|}(t) \\ &= \alpha_x t e^{\alpha_x t}\overline{P}_{\log|X|}(t)\overline{P}_{\log|\tau(Y)|}(t) \\ &= \alpha_x C_{\alpha_x}C_{\alpha_y}\sigma_x^{\alpha_x}\sigma_y^{\alpha_y}te^{-\alpha_x t}. \end{aligned}$$

It is sufficient to evaluate this expression in $\log t$ to obtain the thesis. As for the case $\tau \in E_3$, the proof is the same except for an extra $\frac{1}{2}$ in the tail behaviour of $\overline{P}_{|\tau(Y)|}(t)$. □

Based on Theorem 2.5, the next theorem is an application of the generalized CLT 2 that provides the infinitely wide limit of the shallow Stable NN (3), with the activation function τ belonging to the classes E_1, E_2, E_3.

Theorem 2.6 (Shallow Stable NN, $\tau \in E_1, E_2, E_3$)**.** *Consider $f(n)[\tau, p]$ defined in (3). If $\tau \in E_1$ and $\beta\alpha_1 < \alpha_0$, then*

$$f(n)[\tau, \alpha_1] \xrightarrow{d} S_{\alpha_1}\left(\sigma_1\left(\mathbb{E}_{Z\sim S_{\alpha_0}(\sigma_0)}[|\tau(Z)|^{\alpha_1}]\right)^{1/\alpha_1}\right).$$

If $\tau \in E_2 \cup E_3$, define $\underline{\alpha} = \min(\alpha_1, \alpha_0/\gamma)$, $c_\tau = \frac{1}{2}$ if $\tau \in E_3$ and $c_\tau = 1$ otherwise, and $m_n(\gamma) = \log n$ if $\gamma = \alpha_0/\alpha_1$ and $m_n(\gamma) = 1$ otherwise. Then

$$m_n(\gamma)^{-1/\underline{\alpha}}f(n)[\tau, \underline{\alpha}] \xrightarrow{d} S_{\underline{\alpha}}(\sigma),$$

where

$$\sigma = \begin{cases} \sigma_0^\gamma\sigma_1\left(c_\tau\frac{C_{\underline{\alpha}\gamma}}{C_{\underline{\alpha}}}\mathbb{E}_{Z\sim S_{\alpha_1}(1)}[|Z|^{\underline{\alpha}}]\right)^{1/\underline{\alpha}} & \text{if} \quad \gamma > \alpha_0/\alpha_1 \\ \sigma_0^\gamma\sigma_1\left(c_\tau\underline{\alpha}C_{\gamma\underline{\alpha}}\right)^{1/\underline{\alpha}} & \text{if} \quad \gamma = \alpha_0/\alpha_1 \\ \sigma_1\left(\mathbb{E}_{Z\sim S_{\underline{\alpha}}(\sigma_0)}[|\tau(Z)|^{\underline{\alpha}}]\right)^{1/\underline{\alpha}} & \text{if} \quad \gamma < \alpha_0/\alpha_1 \end{cases}$$

Proof. Observe that, since $w_j \cdot \tau(w_j^{(0)})$ is symmetric, then

$$\mathbb{P}[w_j \cdot \tau(w_j^{(0)}) > t] = \mathbb{P}[w_j \cdot \tau(w_j^{(0)}) < -t] = \frac{1}{2}\mathbb{P}[|w_j \cdot \tau(w_j^{(0)})| > t].$$

Hence, the proof of this theorem follows by combining Theorem 2.5 and the GCLT, i.e. Theorem 2, after observing that $a_n = 0$ due to the symmetry of $w_j \cdot \tau(w_j^{(0)})$. □

The term $(\log n)^{-1/\underline{\alpha}}$ in the scaling in the case $\tau \in E_2 \cup E_3$ and $\alpha_1 = \alpha_0/\gamma$, is a novelty with respect to the Gaussian setting. That is, NNs with Gaussian-distributed parameters are not affected by the presence of one activation in place of another as the scaling is always $n^{-1/2}$, while this is not true for Stable NNs as shown above.

2.2 Deep Stable NNs: large-width asymptotics for an input $x \in \mathbb{R}^d$ and biases

The above results can be extended to deep Stable NNs, assuming a sequential growth of the width over the NN's layers, for an input $\boldsymbol{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ and biases. In particular, let $\theta = \{w_i^{(0)}, ..., w_i^{(L-1)}, w, b_i^{(0)}, ..., b_i^{(L-1)}, b\}_{i\geq 1}$ the set of all parameters and $\boldsymbol{x} \in \mathbb{R}^d$ be the input. Define $\forall i \geq 1$ and $\forall l = 1, ..., L-1$

$$\begin{cases} w_i^{(0)} = [w_{i,1}^{(0)}, w_{i,2}^{(0)}, \dots w_{i,d}^{(0)}] & \in \mathbb{R}^d \\ w_i^{(l)} := [w_{i,1}^{(l)}, w_{i,2}^{(l)}, \dots w_{i,n}^{(l)}] & \in \mathbb{R}^n \\ w = [w_1, w_2, \dots w_n] & \in \mathbb{R}^n \\ w_{i,j}^{(0)}, w_{i,j}^{(l)}, w_i, b_i^{(l)}, b & \in \mathbb{R} \\ w_{i,j}^{(0)} \overset{iid}{\sim} w_{i,j}^{(l)} \overset{iid}{\sim} w_j \overset{iid}{\sim} b_i^{(l)} \overset{iid}{\sim} b \overset{iid}{\sim} S_\alpha(1). \end{cases} \tag{4}$$

Then, we define the deep Stable NN as

$$\begin{cases} g_j^{(1)}(\boldsymbol{x}) = \sigma_w \langle w_j^{(0)}, \boldsymbol{x}\rangle_{\mathbb{R}^d} + \sigma_b b_j^{(0)} \\ g_j^{(l)}(\boldsymbol{x}) = \sigma_b b_j^{(l-1)} + \sigma_w \nu(n)^{-\frac{1}{\alpha}} \sum_{i=1}^n w_{j,i}^{(l-1)} \tau(g_i^{(l-1)}(\boldsymbol{x})), \quad \forall l = 2, ..., L \\ f_{\boldsymbol{x}}(n)[\tau, \alpha] = g_1^{(L+1)}(\boldsymbol{x}) = \sigma_b b + \sigma_w \nu(n)^{-\frac{1}{\alpha}} \sum_{j=1}^n w_j \tau(g_j^{(L)}(\boldsymbol{x})) \end{cases} \tag{5}$$

where $\nu(n) = n \cdot \log(n)$ if $\tau \in E_2 \cup E_3$ with $\gamma = 1$ and $\nu(n) = n$ otherwise, and $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ denotes the Euclidean inner product in $\mathbb{R}^d$. Note that the definition (5) coincides with the definition (3) provided that $L = 1$, $\sigma_b = 0$, $d = 1$ and $x = 1$. For the sake of simplicity and readability of the results, we have restricted ourselves to the case where all the parameters are Stable-distributed with same index α, but this setting can be further generalized.

The next theorem provides the infinitely wide limit of the deep Stable NN (5), assuming a sequential growth of the width over the NN's layers. In particular, if we expand the width of the hidden layers to infinity one at the time, from $l = 1$ to $l = L$, then it is sufficient to apply Theorem 2.6 recursively through the NN's layers.

Theorem 2.7 (Deep Stable NN, $\tau \in E_1$ and $\tau \in E_2 \cup E_3$ with $\gamma \leq 1$)**.** *Consider $g_j^{(l)}(\boldsymbol{x})$ for fixed $j = 1, ..., n$ and $l = 2, ..., L+1$ as defined in (5). Then, as the width goes to infinity sequentially over the NN's layers,*

$$g_j^{(l)}(\boldsymbol{x}) \xrightarrow{d} S_\alpha(\sigma_x^{(l)}),$$

where

$$\sigma_x^{(1)} = \left(\sigma_w^\alpha \sum_{j=1}^d |x_j|^\alpha + \sigma_b^\alpha\right)^{1/\alpha},$$

and, for $l = 2, \dots, L+1$

$$\sigma_x^{(l)} = \begin{cases} \left(\sigma_w^\alpha \mathbb{E}_{Z\sim S_\alpha(\sigma_x^{(l-1)})}[|\tau(Z)|^\alpha] + \sigma_b^\alpha\right)^{1/\alpha} & \text{if} \quad \tau \in E_1 \text{ or } \tau \in E_2 \cup E_3 \text{ with } \gamma < 1 \\ \left(c_\tau \alpha C_\alpha \sigma_w^\alpha (\sigma_x^{(l-1)})^\alpha + \sigma_b^\alpha\right)^{1/\alpha} & \text{if} \quad \tau \in E_2 \cup E_3, \text{ with } \gamma = 1., \end{cases}$$

with $c_\tau = 1/2$ *if* $\tau \in E_3$ *and* $c_\tau = 1$ *otherwise.*

Proof. The case $L = 1$ deals again with a shallow Stable NN but considering non-null Stable biases and a more complex type of input. The result follows from Theorem 2.6 by replacing $w_j^{(0)}$ with $g_j^{(1)}(\boldsymbol{x})$ and σ_0 with $\sigma_x = (\sigma_b^\alpha + \sigma_w^\alpha \sum_{j=1}^d |x_j|^\alpha)^{\frac{1}{\alpha}}$ thanks to the fact that $g_j(\boldsymbol{x}) \overset{iid}{\sim} S_\alpha(\sigma_x)$ for $j = 1, ..., n$. This can be easily proved using the following properties of the Stable distribution (Samorodnitsky & Taqqu, 1994, Chapter 1):

i) if $X_1 \perp\!\!\!\perp X_2$ and $X_i \sim S_\alpha(\sigma_i)$ then $X_1 + X_2 \sim S_\alpha([\sigma_1^\alpha + \sigma_2^\alpha]^{\frac{1}{\alpha}})$;

ii) if $c \neq 0$ and $X_1 \sim S_\alpha(\sigma_1)$ then $c \cdot X_1 \sim S_\alpha(|c|\sigma_1)$.

The proof for the case $L > 1$ is based on the fact that the $g_i^{(l-1)}(\boldsymbol{x})$'s are independent and identically distributed as $S_\alpha(\sigma_x^{(l-1)})$ since they inherit these properties from the iid initialization of weights and biases: the thesis then follows applying the result for $L = 1$ layer after layer and substituting $\sigma_x^{(l-1)}$ in place of σ_x. □

Theorem 2.7 includes the limiting behaviour of $f_{\boldsymbol{x}}(n)[\tau, \alpha]$ in the case $l = L + 1$. In general, it is possible to write an explicit form of the scale parameter by recursively expanding the scale parameters of the hidden layers. See Subsection 2.3 for an example in the case of the ReLU activation function. Before concluding, we point out that when using a sub-linear activation, i.e. $\tau \in E_1$ or $\tau \in E_2 \cup E_3$ with $\gamma \in (0, 1)$, or a linear activation, i.e. $\tau \in E_2 \cup E_3$ with $\gamma = 1$, the index α of the limiting Stable distribution does not change as the depth of a node increases so that, even for a very deep NN, the limiting output is distributed as a α-Stable distribution. Such a behaviour is not preserved for super-linear activation functions, i.e. $\tau \in E_2 \cup E_3$ with $\gamma > 1$. When $\alpha_1 < \alpha_0/\gamma$, the convergence result of Theorem 2.6 involves a Stable RV with index equal to α_0/γ, and not α_0. In case $\alpha_x = \alpha_y = \alpha$, this is the case when $\gamma > 1$, which corresponds to a super-linear activation in $E_2 \cup E_3$. The fact that the limiting RV takes a factor $1/\gamma$ prevent us from writing a theorem in the setting of Definition (5) because we would not be able to apply the property i) above as it describes the distribution of the sum of independent Stable RVs with different scales but same index. We are then forced to adjust the initialization of the biases and to this purpose we define a new setting. Let $\theta = \{w_i^{(0)}, ..., w_i^{(L-1)}, w, b_i^{(0)}, ..., b_i^{(L-1)}, b\}_{i\geq 1}$ the set of all parameters and $\boldsymbol{x} \in \mathbb{R}^d$ be the input. Define $\forall i \geq 1$ and $\forall l = 0, ..., L-1$

$$\begin{cases} w_i^{(0)} = [w_{i,1}^{(0)}, w_{i,2}^{(0)}, \dots w_{i,d}^{(0)}] & \in \mathbb{R}^d \\ w_i^{(l)} = [w_{i,1}^{(l)}, w_{i,2}^{(l)}, \dots w_{i,d}^{(l)}] & \in \mathbb{R}^n \\ w = [w_1, w_2, \dots w_n] & \in \mathbb{R}^n \\ w_{i,j}^{(0)}, w_{i,j}^{(l)}, w_i, b_i^{(l)}, b & \in \mathbb{R} \\ w_{i,j}^{(l)} \overset{iid}{\sim} w_j \overset{iid}{\sim} S_\alpha(1) \\ b_i^{(l)} \overset{iid}{\sim} S_{\frac{\alpha}{\gamma^l}}(1) \\ b \overset{iid}{\sim} S_{\frac{\alpha}{\gamma^L}}(1). \end{cases} \tag{6}$$

Then, we define the deep Stable NN as

$$\begin{cases} g_j^{(1)}(\boldsymbol{x}) = \sigma_w \langle w_j^{(0)}, \boldsymbol{x} \rangle_{\mathbb{R}^d} + \sigma_b b_j^{(0)} \\ g_j^{(l)}(\boldsymbol{x}) = \sigma_b b_j^{(l-1)} + \sigma_w n^{-\frac{1}{\alpha}} \sum_{i=1}^n w_{j,i}^{(l-1)} \tau(g_i^{(l-1)}(\boldsymbol{x})), \quad \forall l = 2, ..., L \\ f_{\boldsymbol{x}}(n)[\tau, \alpha] = g_1^{(L+1)}(\boldsymbol{x}) = \sigma_b b + \sigma_w n^{-\frac{1}{\alpha}} \sum_{j=1}^n w_j \tau(g_j^{(L)}(\boldsymbol{x})). \end{cases} \tag{7}$$

The next theorem provides the counterpart of Theorem 2.7 for the deep Stable NN (5). It provides the infinitely wide limit of the deep Stable NN (7), assuming a sequential growth of the width over the NN's layers.

Theorem 2.8 (Deep Stable NN, $\tau \in E_2 \cup E_3$ with $\gamma > 1$)**.** *Consider $g_j^{(l)}(\boldsymbol{x})$ for fixed $j = 1, ..., n$ and $l = 2, ..., L+1$ as defined in (7). As the width goes to infinity sequentially over the NN's layers,*

$$g_j^{(l)}(\boldsymbol{x}) \xrightarrow{d} S_{\alpha/\gamma^{l-1}}\left(\sigma_x^{(l)}\right)$$

where

$$\sigma_x^{(1)} = \left(\sigma_w^\alpha \sum_{j=1}^d |x_j|^\alpha + \sigma_b^\alpha\right)^{1/\alpha},$$

and

$$\sigma_x^{(l)} = \left(\frac{c_\tau C_{\alpha/\gamma^{l-2}}}{C_{\alpha/\gamma^{l-1}}} \sigma_w^{\alpha/\gamma^{l-1}} (\sigma_x^{(l-1)})^{\alpha/\gamma^{l-2}} \mathbb{E}_{Z\sim S_\alpha(1)}[|Z|^{\alpha/\gamma^{l-1}}] + \sigma_b^{\alpha/\gamma^{l-1}}\right)^{\gamma^{l-1}/\alpha},$$

with $c_\tau = 1/2$ if $\tau \in E_3$ and $c_\tau = 1$ otherwise.

Proof. The proof is along lines similar to the proof of Theorem 2.7. Notice that the fact that $b_i^{(l-1)} \overset{iid}{\sim} S_{\alpha/\gamma^{l-1}}(1)$ is critical to conclude the proof. □

As a corollary of Theorem 2.8, the limiting distribution of $f_{\boldsymbol{x}}(n)[\tau, \alpha]$, as $n \to +\infty$, follows a (α/γ^L)-Stable distribution with scale parameter that can be computed recursively. That is, for a large number L of layers, the stability parameter of the limiting distribution is close to zero. As we have pointed out for a shallow Stable NN, this is a peculiar feature of the class $\tau \in E_2 \cup E_3$ with $\gamma > 1$, and it can be object of a further analysis.

2.3 Some examples

As Theorem 2.6 is quite abstract, we present some concrete examples using well-known activations functions. First consider the case when $\tau = \tanh \in E_1$, since it is bounded. Then, the output of a shallow Stable NN (3) is such that

$$f(n)[\tanh, \alpha_1] \xrightarrow{d} S_{\alpha_1}\left(\sigma_1 \left(\mathbb{E}_{Z\sim S_{\alpha_0}(\sigma_0)}[|\tanh(Z)|^{\alpha_1}]\right)^{1/\alpha_1}\right).$$

See also Favaro et al. (2020). As for the new classes of activations introduced here, we can start considering the super-linear activation $\tau(z) = z^3 \in E_2$ with $\gamma = 3$, in the case of a shallow NN with $\alpha_1 = \alpha_0 = \alpha$ and obtain that

$$f(n)[(\cdot)^3, \alpha/3] \xrightarrow{d} S_{\alpha/3}\left(\sigma_0^3 \sigma_1 \left(\frac{C_\alpha}{C_{\alpha/3}} \mathbb{E}_{Z\sim S_\alpha(1)}[|Z|^{\alpha/3}]\right)^{3/\alpha}\right),$$

with the novelty here lying in the fact that the index of the limiting output is $\alpha/3$ instead of α. As for linear activations, if you take $\tau = id$, i.e. the identity function, again under the hypothesis of a shallow NN with $\alpha_1 = \alpha_0 = \alpha$, you obtain that

$$(\log n)^{-1/\alpha} f(n)[id, \alpha] \xrightarrow{d} S_\alpha\left(\left[\alpha C_\alpha\right]^{1/\alpha} \sigma_0 \sigma_1\right),$$

which shows the presence of an extra logarithmic factor of $(\log n)^{-1/\alpha_1}$ in the scaling for the first time. Beware that this behaviour, which is a critical difference with the Gaussian case, does not show up only with linear activation: for example, if you take $\alpha_1 = 1$, i.e. Cauchy distribution, $\alpha_0 = 3/2$, i.e. Holtsmark distribution, and $\tau(z) = z^{3/2}$ then

$$(\log n)^{-1} f(n)[(\cdot)^{3/2}, 1] \xrightarrow{d} S_1\left(C_1\sigma_0\sigma_1\right).$$

Finally, we consider the ReLU activation function, which is one of the most popular activation functions. The following theorems deal with shallow Stable NNs and deep Stable NNs, respectively, with a ReLU activation function.

Theorem 2.9 (Shallow Stable NN, ReLU)**.** *Consider $Z = X \cdot ReLU(Y)$ where $X \sim S_\alpha(\sigma_x)$, $Y \sim S_\alpha(\sigma_y)$, $X \perp\!\!\!\perp Y$. Then*

$$\overline{P}_Z(z) = P_Z(-z) \sim \frac{\alpha}{4} C_\alpha^2 \sigma_y^\alpha \sigma_x^\alpha z^{-\alpha} \log z.$$

Furthermore, if $f(n)[ReLU, p] = n^{-\frac{1}{p}} \sum_{j=1}^n w_j ReLU(w_j^{(0)})$, where $ReLU(t) = \max\{0, t\}$ and $w_j^{(0)} \overset{iid}{\sim} S_\alpha(\sigma_0) \perp\!\!\!\perp w_j \overset{iid}{\sim} S_\alpha(\sigma_1)$, then

$$(\log n)^{-\frac{1}{\alpha}} f(n)[\tau, \alpha] \xrightarrow{d} S_\alpha\left(\left[\frac{1}{2}\alpha C_\alpha\right]^{\frac{1}{\alpha}} \sigma_0\sigma_1\right)$$

Proof. The theorem follows easily from Theorem 2.5 and Theorem 2.6, after noticing that ReLU $\in E_3$ with $\beta = 0$ and $\gamma = 1$. In addition to that, we also provide an alternative proof which can be useful in other applications. So, first, let us derive the distribution of $Q := \mathrm{ReLU}(Y)$

$$\mathbb{P}[Q \le q] = \mathbb{P}(\max\{0, Y\} \le q) = \begin{cases} 0 & \text{if} \quad q < 0 \\ \mathbb{P}[Y \le q] & \text{if} \quad q \ge 0, \end{cases}$$

from which we observe that Q is neither discrete nor absolutely continuous with respect to the Lebesgue measure as it has a point mass of $\frac{1}{2}$ at $z = 0$ while the remaining $\frac{1}{2}$ of the mass is concentrated on $\mathbb{R}_+$ accordingly to the Stable law of Y on $(0, +\infty)$. Hence, having in mind the shape of $P_Q(q) = \mathbb{P}(Q \le q)$, we derive the approximation for the tails of the distribution of $X \cdot \mathrm{ReLU}(Y)$ and, as usual, we make use of the GCLT to prove the following theorem. We prove the tail behaviour of $\overline{P}_Z(z)$ first. For any $z > 0$ we can write that

$$\overline{P}_Z(z) = \int_0^{+\infty} \mathbb{P}\left[Q > \frac{z}{x}\right] p_X(x)dx = \int_0^{+\infty} \mathbb{P}\left[Y > \frac{z}{x}\right] p_X(x)dx,$$

since Y and Q have the same distribution on $(0, +\infty)$. Now, observe that

$$\mathbb{P}[XY \ge z] = \int_{\mathbb{R}} \mathbb{P}[xY \ge z] P_X(dx) = 2\int_0^{+\infty} \mathbb{P}\left[Y \ge \frac{z}{x}\right] p_X(x)dx$$

where the second equality holds by splitting the integral on $\mathbb{R}$ into the sum of the integrals on $(-\infty, 0)$ and $(0, +\infty)$ and using the fact that Y is symmetric. It follows that, for every $z > 0$, $\overline{P}_Z(z) = \frac{1}{2}\mathbb{P}[XY \ge z]$. Applying the results for $\tau = id$, we find that

$$\overline{P}_Z(z) = \frac{1}{2}\mathbb{P}[XY \ge z] \backsim \frac{\alpha}{4} C_\alpha^2 \sigma_y^\alpha \sigma_x^\alpha z^{-\alpha} \log z.$$

The proof for the asymptotic behaviour of $P_Z(z)$ works in the same way after fixing $z < 0$ and using a change of variable while the convergence in distribution of $(\log n)^{-\frac{1}{\alpha}} f(n)[\tau, \alpha]$ follows by a direct application of the GCLT. □

Theorem 2.9 can be extended to deep Stable NN with input $\boldsymbol{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ and considering the biases.

Theorem 2.10 (Deep Stable NN, ReLU)**.** *Consider the deep Stable NN with ReLU activation defined as follows*

$$\begin{cases} g_j^{(1)}(\boldsymbol{x}) = \sigma_w \langle w_j^{(0)}, \boldsymbol{x} \rangle_{\mathbb{R}^d} + \sigma_b b_j^{(0)} \\ g_j^{(l)}(\boldsymbol{x}) = \sigma_b b_j^{(l-1)} + \sigma_w (n \log n)^{-\frac{1}{\alpha}} \sum_{i=1}^n w_{j,i}^{(l-1)} ReLU(g_i^{(l-1)}(\boldsymbol{x})), \forall l = 2, ..., L \\ f_{\boldsymbol{x}}(n)[ReLU, \alpha] = g_1^{(L+1)}(\boldsymbol{x}) = \sigma_b b + \sigma_w (n \log n)^{-\frac{1}{\alpha}} \sum_{j=1}^n w_j ReLU(g_j^{(L)}(\boldsymbol{x})) \end{cases} \tag{8}$$

Then, under the hypothesis of Stable initialization for weights and biases as in (5), as the width of the previous layers goes to infinity sequentially,

$$g_j^{(l)}(\boldsymbol{x}) \xrightarrow{d} S_\alpha \left(\sigma_x^{(l)} \right),$$

where

$$\sigma_x^{(1)} = \left(\sigma_w^\alpha \sum_{j=1}^d |x_j|^\alpha + \sigma_b^\alpha \right)^{1/\alpha},$$

and, for $l = 2, \ldots, L+1$,

$$\sigma_x^{(l)} = \left(\frac{1}{2} \alpha C_\alpha (\sigma_x^{(l-1)})^\alpha \sigma_w^\alpha + \sigma_b^\alpha \right)^{1/\alpha}.$$

Proof. The proof is along lines similar to the proof of Theorem 2.7 with $\gamma = 1$ and $\tau \in E_3$. □

Then, as a corollary of Theorem 2.10, the limiting distribution of $f_{\boldsymbol{x}}(n)[\text{ReLU}, \alpha]$, as $n \to +\infty$, is the distribution of a α-Stable RV whose scale can be computed recursively. In particular, we can write the following statement.

Corollary 2.10.1. *Under the setting of Theorem 2.10 with a generic depth L,*

$$f(n)[ReLU, \alpha] \xrightarrow{d} S_\alpha \left(\left[\left(\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha \right)^L \sigma_x^\alpha + \sum_{i=0}^{L-1} (\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha)^i \sigma_b^\alpha \right]^{\frac{1}{\alpha}} \right)$$

Proof. The claim is true for $L = 1$, which can be proved using the standard two properties of the Stable distribution. Moreover, for a NN with depth of $L+1$, using Theorem 2.10, the scale is

$$\left[\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha \left[\left(\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha \right)^L \sigma_x^\alpha + \sum_{i=0}^{L-1} \left(\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha \right)^i \sigma_b^\alpha \right] + \sigma_b^\alpha \right]^{\frac{1}{\alpha}} =$$

$$\left[\left(\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha \right)^{L+1} \sigma_x^\alpha + \sum_{i=1}^{L} \left(\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha \right)^i \sigma_b^\alpha + \sigma_b^\alpha \right]^{\frac{1}{\alpha}} =$$

$$\left[\left(\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha \right)^{L+1} \sigma_x^\alpha + \sum_{i=0}^{L} \left(\frac{1}{2} \alpha C_\alpha \sigma_w^\alpha \right)^i \sigma_b^\alpha \right]^{\frac{1}{\alpha}}$$

which concludes the proof. □

3 Discussion

In this paper, we have characterized the infinitely wide limit of deep Stable NNs with a general activation function, thus removing the strong assumption of a sub-linear function considered in Favaro et al. (2020) and Favaro et al. (2021). For $\alpha_1 = \alpha_0/\gamma$, and in particular for the choices $\tau = id$ and $\tau = \text{ReLU}$ with $\alpha_0 = \alpha_1$, Theorem 2.6 shows that the right scaling of the NN is $(n \log n)^{-1/\alpha}$, thus including the extra factor $(\log n)^{-1/\alpha}$ with respect to sub-linear activation functions. For $\alpha_1 > \alpha_0/\gamma$, and in particular for the choice

of a super-linear activations with $\alpha_0 = \alpha_1$, Theorem 2.8 shows that the distribution of the limiting output is (α_0/γ)-Stable, with $\gamma > 1$, and this may have undesirable consequences for posterior estimates in case of a very deep NN.

We conclude our work by presenting some open challenges and some directions for future research in the context of deep Stable NNs with $\tau \in E_i$, for some $i = 1, 2, 3$. A natural problem consists in extending our results to the case of deep Stable NNs with $k > 1$ inputs of dimension d, i.e. the input consists of a $d \times k$ matrix $\boldsymbol{X}$. Recently, this problem has been considered in Favaro et al. (2022) for the sole special case of shallow Stable NNs with a ReLU activation function. A further natural problem consists in extending our results to the case of a joint growth of the width over the NN's layers (Favaro et al., 2021). In general, under the setting specified in definition (5) or definition (7), one may consider a deep NN defined recursively as follows:

$$f_i^{(1)}(\mathbf{X}) = \sum_{j=1}^{d} w_{i,j}^{(1)} \mathbf{x}_j + b_i^{(1)} \mathbf{1}^T$$

and

$$f_i^{(l)}(\mathbf{X}, n) = \frac{1}{f(n)^{1/\alpha}} \sum_{j=1}^{n} w_{i,j}^{(l)} \left(\tau \circ f_j^{(l-1)}(\mathbf{X}, n) \right) + b_i^{(l)} \mathbf{1}^T,$$

with $f_i^{(1)}(\mathbf{X}, n) = f_i^{(1)}(\mathbf{X})$, where $\mathbf{1}$ is the k-dimensional unit (column) vector, $\circ$ denotes the element-wise application and $f(n) = n \cdot \log n$ if $\tau \in E_2$ and $f(n) = n$ otherwise. Then, the goal is to extend our results to $f_i^{(l)}(\mathbf{X}, n)$, assuming a joint growth of the width over the NN's layers. The case $\tau \in E_1$ was already tackled by Favaro et al. (2020) but the other two cases are missing. Clearly, for the case $\tau = id$, it is immediate to obtain the spectral measure of the limiting multivariate Stable distribution using Levy's theorem and the linearity of the scalar product. Less trivial would be to adapt these computations when $\tau = \text{ReLU}$ or when $\tau \in E_2 \cup E_3$ with $\gamma > 1$.

An interesting research direction consists in extending to the Stable setting some results of the popular works of Jacot et al. (2020) and Arora et al. (2019), which established the equivalence between a specific training setting of deep Gaussian NNs and kernel regression. In particular, they considered a deep Gaussian NN where the hidden layers are trained jointly under quadratic loss and gradient flow, i.e. gradient descent with infinitesimal learning rate, and it was shown that, as the width of the NN goes to infinity simultaneously, the point predictions are arbitrarily close to those given by a kernel regression with respect to the so-called neural tangent kernel (NTK). It would be interesting to study if such an equivalence holds also in the case of Stable weights and biases. For shallow Stable NNs with ReLU activation function, and input $\boldsymbol{X}$, this problem has been considered in Favaro et al. (2022). In particular, their result characterized the large-width training dynamics of the shallow Stable NN with respect to a kernel regression with an $(\alpha/2)$-Stable random kernel.

The main concept at the basis of the equivalence between overparameterized Gaussian NNs and kernel regression is the fact that a wide NN, whose output is relatively close to zero, behaves like its linearization around the initialization. Intuitively, when the NN is very wide, each individual parameter needs to move a tiny amount from its initialization in order to have a non-negligible change in the NN's output. This behaviour is known in the literature as the "lazy training" phenomenon, and it is one of the hottest topic in the field of machine learning since it is a phenomenon which can affect any model, not only NNs. More precisely, Chizat et al. (2018) showed that lazy training is caused by an implicit choice of the scaling and that every parametric model can be trained in the lazy regime provided that its output is initialized close to zero. Furthermore, coming back to NNs, they considered a two layers NN with Gaussian weights and proved a sufficient condition for achieving lazy training, provided that $\mathbb{E}[w_i^{(0)} \tau(\langle w_i \cdot x \rangle)] = 0$. Clearly, the theorem applies also in the case of symmetric Stable weights and biases when $\alpha > 1$, but not when $\alpha \leq 1$ as the expectation of such RVs is undefined. It would be then interesting to study what happens in that case in order to find a new theoretical result which leads to a suitable scaling for which we have the lazy training phenomenon.

References

Sanjeev Arora, Simon S Du, Wei Hu, Zhiyuan Li, Ruslan Salakhutdinov, and Ruosong Wang. On exact computation with an infinitely wide neural net. *Advances in Neural Information Processing Systems*, 32, 2019.

L. Breiman. On some limit theorems similar to the arc-sin law. *Theory of Probability & Its Applications*, 10 (2):323–331, 1965. doi: 10.1137/1110037. URL https://doi.org/10.1137/1110037.

Lenaic Chizat, Edouard Oyallon, and Francis Bach. On lazy training in differentiable programming. 2018. doi: 10.48550/ARXIV.1812.07956. URL https://arxiv.org/abs/1812.07956.

D. B. H. Cline and G. Samorodnitsky. Subexponentiality of the product of independent random variables. *Stochastic Processes and their Applications*, 49(1):75–98, January 1994. URL https://ideas.repec.org/a/eee/spapps/v49y1994i1p75-98.html.

Daren BH Cline. Convolution tails, product tails and domains of attraction. *Probability Theory and Related Fields*, 72(4):529–557, 1986.

Alexander G. de G. Matthews, Jiri Hron, Mark Rowland, Richard E. Turner, and Zoubin Ghahramani. Gaussian process behaviour in wide deep neural networks. In *International Conference on Learning Representations*, 2018. URL https://openreview.net/forum?id=H1-nGgWC-.

Denis Denisov and Bert Zwart. On a theorem of breiman and a class of random difference equations. *Journal of Applied Probability*, 44, 01 2005. doi: 10.1239/jap/1197908822.

Ricky Der and Daniel Lee. Beyond gaussian processes: On the distributions of infinite networks. In Y. Weiss, B. Schölkopf, and J. Platt (eds.), *Advances in Neural Information Processing Systems*, volume 18. MIT Press, 2005. URL https://proceedings.neurips.cc/paper/2005/file/e2c61965b5e23b47b77d7c51611b6d7f-Paper.pdf.

Stefano Favaro, Sandra Fortini, and Stefano Peluchetti. Stable behaviour of infinitely wide deep neural networks. In *International Conference on Artificial Intelligence and Statistics*, pp. 1137–1146. PMLR, 2020.

Stefano Favaro, Sandra Fortini, and Stefano Peluchetti. Deep stable neural networks: large-width asymptotics and convergence rates. *Preprint arXiv:2108.02316*, 2021.

Stefano Favaro, Sandra Fortini, and Stefano Peluchetti. Neural tangent kernel analysis of shallow alpha-stable relu neural networks. *Preprint arXiv:2206.08065*, 2022.

Arthur Jacot, Franck Gabriel, and Clément Hongler. Neural tangent kernel: Convergence and generalization in neural networks, 2020.

Radford M. Neal. *Bayesian Learning for Neural Networks*. Springer-Verlag, Berlin, Heidelberg, 1996. ISBN 0387947248.

F. W. J. Olver. *Asymptotics and special functions*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. Computer Science and Applied Mathematics.

CEG Otiniano and CR Gonçalves. Domain of attraction of-stable distributions under finite mixture models. *TEMA-Tendências em Matemática Aplicada e Computacional*, 11(1):69–76, 2010.

Gennady Samorodnitsky and Murad S. Taqqu. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Routledge, 1994. ISBN 9780412051715.

Vladimir V Uchaikin and Vladimir M Zolotarev. *Chance and stability: stable distributions and their applications*. Walter de Gruyter, 2011.

A Complementary statements and proofs

Definition A.1 (Multivariate Stable distribution)**.** *Let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^k : \mathbb{S} = \left\{u \in \mathbb{R}^k : |u| = 1\right\}$. A random vector, X, has a multivariate Stable distribution, denoted as $X \sim St_k(\alpha, \Gamma, \delta)$, if the joint characteristic function of X is*

$$\mathbb{E}[\exp\left(iu^T X\right)] = \exp\left\{-\int_{s\in\mathbb{S}} \left\{\left|u^T s\right|^\alpha + i\nu\left(u^T s, \alpha\right)\right\}\Gamma(ds) + iu^T\delta\right\}$$

where $0 < a < 2$, and for $y \in \mathbb{R}$

$$\nu(y,\alpha) = \begin{cases} -\operatorname{sign}(y)\tan(\pi\alpha/2)|y|^\alpha & \alpha \neq 1 \\ (2/\pi)y\ln|y| & \alpha = 1 \end{cases}$$

The case with $\delta = 0$ is denoted by $St_k(\alpha, \Gamma)$.

The next two theorems characterize the infinitely wide limits of a deep NN with Gaussian and Stable parameters respectively, under the setting of joint growth and taking a matrix as input.

Theorem A.1 ((de G. Matthews et al., 2018))**.** *For any $d \geq 1$ and $k \geq 1$ let $\mathbf{X}$ denote a $d \times k$ (input signal) matrix, with $\mathbf{x}_j$ being the j-th (input signal) row, and for any $D \geq 1$ and $n \geq 1$ let: i) $\left(\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(D)}\right)$ be i.i.d. random (weight) matrices, such that $\mathbf{W}^{(1)} = \left(w_{i,j}^{(1)}\right)_{1\leq i\leq n, 1\leq j\leq d}$ and $\mathbf{W}^{(l)} = \left(w_{i,j}^{(l)}\right)_{1\leq i\leq n, 1\leq j\leq n}$ for $2 \leq l \leq D$, where the $w_{i,j}^{(l)}$'s are i.i.d. as $N\left(0, \sigma_w^2\right)$ for $l = 1, \ldots, D$; ii) $\left(\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(D)}\right)$ be i.i.d. random (bias) vectors, such that $\mathbf{b}^{(l)} = \left(b_1^{(l)}, \ldots, b_n^{(l)}\right)$ where the $b_i^{(l)}$'s are i.i.d. as $N\left(0, \sigma_b^2\right)$ for $l = 1, \ldots, D$. Now, let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous activation function (nonlinearty) such that*

$$|\phi(s)| \leq a + b|s|$$

for every $s \in \mathbb{R}$ and for any $a, b \geq 0$, and consider a (fully connected) feed-forward $NN\left(f_i^{(l)}(\mathbf{X}, n)\right)_{1\leq i\leq n, 1\leq l\leq D}$ of depth D and width n defined as follows

$$f_i^{(1)}(\mathbf{X}) = \sum_{j=1}^{d} w_{i,j}^{(1)}\mathbf{x}_j + b_i^{(1)}\mathbf{1}^T$$

and

$$f_i^{(l)}(\mathbf{X}, n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} w_{i,j}^{(l)}\left(\phi \circ f_j^{(l-1)}(\mathbf{X}, n)\right) + b_i^{(l)}\mathbf{1}^T,$$

with $f_i^{(1)}(\mathbf{X}, n) = f_i^{(1)}(\mathbf{X})$, where 1 is the k-dimensional unit (column) vector, and o denotes the element-wise application. For any $l = 1, \ldots, D$, if $\left(f_i^{(l)}(\mathbf{X}, n)\right)_{i\geq 1}$ is the sequence obtained by extending $\left(\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(D)}\right)$ and $\left(\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\bar{D})}\right)$ to infinite i.i.d. arrays, then as $n \to +\infty$ jointly over the first l NN 's layers

$$\left(f_i^{(l)}(\mathbf{X}, n)\right)_{i\geq 1} \xrightarrow{w} \left(f_i^{(l)}(\mathbf{X})\right)_{i\geq 1},$$

where $\left(f_i^{(l)}(\mathbf{X})\right)_{i\geq 1}$ is distributed as the product measure $\otimes_{i\geq 1} N_k\left(\mathbf{0}, \Sigma^{(l)}\right)$, and the covariance matrix $\Sigma^{(l)}$ has the (u, v)-th entry defined recursively as follows:

$$\Sigma_{u,v}^{(1)} = \sigma_b^2 + \sigma_w^2\left\langle \mathbf{x}_u, \mathbf{x}_v\right\rangle$$

and

$$\Sigma_{u,v}^{(l)} = \sigma_b^2 + \sigma_w^2\mathbb{E}[\phi(f)\phi(g)],$$

where $(f, g) \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{u,u}^{(l-1)} & \Sigma_{u,v}^{(l-1)} \\ \Sigma_{v,u}^{(l-1)} & \Sigma_{v,v}^{(l-1)} \end{pmatrix}\right)$.

Theorem A.2 (Theorem 2 of Favaro et al. (2020))**.** *For any $d \geq 1$ and $k \geq 1$ let $\mathbf{X}$ denote a $d \times k$ (input signal) matrix, with $\mathbf{x}_j$ being the j-th (input signal) row, and for any $D \geq 1$ and $n \geq 1$ let: i) $\left(\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(D)}\right)$ be i.i.d. random (weight) matrices, such that $\mathbf{W}^{(1)} = \left(w_{i,j}^{(1)}\right)_{1 \leq i \leq n, 1 \leq j \leq d}$ and $\mathbf{W}^{(l)} = \left(w_{i,j}^{(l)}\right)_{1 \leq i \leq n, 1 \leq j \leq n}$ for $2 \leq l \leq D$, where the $w_{i,j}^{(l)}$'s are i.i.d. as $S_\alpha(\sigma_w)$ for $l = 1, \ldots, D$; ii) $\left(\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(D)}\right)$ be i.i.d. random (bias) vectors, such that $\mathbf{b}^{(l)} = \left(b_1^{(l)}, \ldots, b_n^{(l)}\right)$ where the $b_i^{(l)}$'s are i.i.d. as*

$$S_\alpha(\sigma_b)$$

for $l = 1, \ldots, D$. Now, let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous activation function (nonlinearty) such that

$$|\phi(s)| \leq \left(a + b|s|^\beta\right)^\gamma$$

for every $s \in \mathbb{R}$ and for any $a, b > 0, \gamma < \alpha^{-1}$ and $\beta < \gamma^{-1}$, and consider a (fully connected) feed-forward NN $\left(f_i^{(l)}(\mathbf{X}, n)\right)_{1 \leq i \leq n, 1 \leq l \leq D}$ of depth D and width n defined as follows

$$f_i^{(1)}(\mathbf{X}) = \sum_{j=1}^{d} w_{i,j}^{(1)} \mathbf{x}_j + b_i^{(1)} \mathbf{1}^T$$

and

$$f_i^{(l)}(\mathbf{X}, n) = \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} w_{i,j}^{(l)} \left(\phi \circ f_j^{(l-1)}(\mathbf{X}, n)\right) + b_i^{(l)} \mathbf{1}^T$$

with $f_i^{(1)}(\mathbf{X}, n) = f_i^{(1)}(\mathbf{X})$, where $\mathbf{1}$ is the k-dimensional unit (column) vector, and o denotes the element-wise application. For any $l = 1, \ldots, D$, if $\left(f_i^{(l)}(\mathbf{X}, n)\right)_{i \geq 1}$ is the sequence obtained by extending $\left(\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(D)}\right)$ and $\left(\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\bar{D})}\right)$ to infinite i.i.d. arrays, then as $n \to +\infty$ jointly over the first lNN' s layers

$$\left(f_i^{(l)}(\mathbf{X}, n)\right)_{i \geq 1} \xrightarrow{w} \left(f_i^{(l)}(\mathbf{X})\right)_{i \geq 1}$$

where $\left(f_i^{(l)}(\mathbf{X})\right)_{i \geq 1}$ is distributed as the product measure $\otimes_{i \geq 1} \mathrm{St}_k\left(\alpha, \Gamma^{(l)}\right)$, with $\alpha \in (0, 2)$, and the spectral measure $\Gamma^{(l)}$ being defined recursively as follows:

$$\Gamma^{(1)} = \left\|\sigma_b \mathbf{1}^T\right\|^\alpha \zeta_{\frac{\mathbf{1}}{\|\mathbf{1}^T\|}} + \sigma_w^\alpha \sum_{j=1}^{d} \|\mathbf{x}_j\|^\alpha \zeta_{\frac{\mathbf{x}_j}{\|\mathbf{x}_j\|}}$$

and

$$\Gamma^{(l)} = \left\|\sigma_b \mathbf{1}^T\right\|^\alpha \zeta_{\frac{\mathbf{1}^T}{\|\mathbf{1}^T\|}} + \int \|\sigma_w(\phi \circ f)\|^\alpha \zeta_{\frac{\phi \circ f}{\|\phi \circ f\|}} q^{(l-1)}(df)$$

where

$$\zeta_{\frac{h}{\|h\|}} = \frac{1}{2} \begin{cases} \delta_{\frac{h}{\|h\|}} + \delta_{-\frac{h}{\|h\|}} & \text{if } \|h\| > 0 \\ 0 & \text{otherwise,} \end{cases}$$

with δ being the Dirac measure, and $q^{(l-1)}$ is the distribution of $f_i^{(l-1)}(\mathbf{X})$. The limiting SP $\left(f_i^{(l)}(\mathbf{X})\right)_{i \geq 1}$ is referred to as the Stable SP with parameter (α, Γ).