Formalizing Coherence and Consistency Applied to Transfer Learning in Neuro-Symbolic Autoencoders

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Abstract

In the study of reasoning in neural networks, recent efforts have sought to improve coherence and consistency of neural sequence models. This is an important development in the study of neuro-symbolic systems. In symbolic AI, however, the concepts of consistency and coherence are defined formally. The provision of such formal definitions is needed to offer a common basis for the quantitative evaluation and systematic comparison of connectionist, neuro-symbolic and transfer learning approaches. In this paper we introduce formal definitions for coherence and consistency of neural systems. To illustrate the usefulness of the definitions, we propose a new dynamic relation-decoder model built around the principles of consistency and coherence. By comparing several existing relation-decoders on a partial relation transfer learning task and novel data set introduced in this paper, our experiments show that relation-decoders that can maintain consistency over unobserved regions of representation space, retain coherence across domains and achieve better transfer learning performance.

1 Introduction

Humans are capable of learning concepts that can be applied to many different scenarios [9, 19, 11]. An important principle is that human-like concepts remain *coherent* across contexts [16]. As an example, consider the concept of *ordinality*, *e.g*. “A is larger than B”, which allows comparisons to be made between ordered sets. Ordinality should apply equally whether A and B are digits or a tower of blocks. It is said that a concept may pertain to a multitude of properties: position, volume, reach, *etc*. As long as one of these properties can be attributed to an object, a set of objects can be compared on that basis. All in all, if the concept of ordinality was to be learned in its most general form, its use should be consistent across objects and coherent across object properties.

In [16], empirical results on story generation and instruction-following have shown that an intuitive use of consistency and coherence can increase the accuracy of neural networks. It is argued in [16] that *System 1* approaches, fast and capable of learning patterns efficiently from data, “are often inconsistent and incoherent", and that “adding *System 2*-inspired logical reasoning" as a logically-consistency, training-free module allows for an improved selection of candidate stories generated by *System 1*. While [16] makes an important contribution by exploring several variations on the theme, in this paper we offer a formal definition for consistency and coherence in the context of neural networks, in particular autoencoders. We also take one step further and apply and evaluate consistency and coherence to transfer learning, where we believe that the theme will have its most practical impact.

We argue that for a concept to be useful during transfer learning, the system of relations that define the concept in the source domain must be coherent with the target domain, whereby logical consistency achieved in the source is retained in the target domain. This is to say that the concept-specific relations

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learned in the source ought to be consistent with a logical theory that defines their semantics, and that such consistency must extend beyond the representations learned in the source domain and, in particular, hold for the embeddings learned in the target domain.

In this paper, we offer a formal definition for consistency and coherence of sub-symbolic learners, inspired by analogous definitions from symbolic AI. This is expected to define the conditions that make a learned concept transfer well across properties and objects. We propose a simple neural-symbolic autoencoder architecture consisting of a neural encoder for objects coupled with consistent and modular relation-decoders, and we show in comparison with alternative popular approaches that this simple architecture is capable of achieving an improved transfer learning performance by being coherent across object properties [20, 5, 1, 23, 14, 4].

Specifically, consistency and coherence metrics are shown to offer a more fine-grained measure for transfer learning than accuracy alone. The proposed architecture is evaluated on a new Partial Relation Transfer (PRT) task and data set introduced in this paper. The application of a set of logical relations to a domain is specified as a model-theoretic structure with an analogous (soft-)structure for non-symbolic learners. Consistency and coherence of soft-structures is then shown to provide a practical score calculation to the evaluation of autoencoders. The benchmark PRT learning task uses a new BlockStacks data set derived from the CLEVR data set rendering agent. This is compared with several existing relation-decoder models on transfer learning tasks from BlockStacks to the MNIST handwritten digits data set, on relations such as isGreater, isEqual..., such that the learning of ordinality among the digits is evaluated against the learning of the relative position of a block in the stack. Our experiments show that relation-decoders which maintain consistency over unobserved regions of representational space retain coherence across domains whilst achieving better transfer learning performance. In summary, the contributions of this paper are:

- A formal definition of consistency and coherence for sub-symbolic learners offering a practical evaluation score for concept coherence;
- A derived model implementation and partial relation transfer experimental setup used to evaluate the interplay between concept coherence and concept transfer;
- A comprehensive critical evaluation of results and comparison of multiple relation-decoder models with varied model capacity, showing that regularisation via model capacity or β-induced disentanglement pressure improves concept coherence.

In Section 2 we provide the required logic background. Section 3 introduces soft-structures and formally defines coherence and consistency. Section 4 describes the neuro-symbolic architecture and its associated practical consistency loss. After detailing the PRT task and data set in Section 5, comparative experimental results are discussed in Section 6. Section 7 concludes the paper with a discussion, including limitations and future work. We discuss related work, experimental setup and data set characteristics, model details and parameterization, and we make the code and additional experimental results available in the Supplementary Material.

2 Preliminaries

Notation: We reserve uppercase calligraphic letters to denote sets, and lowercase versions of the same letter to denote their elements, e.g. $\mathcal{S} = \{s_1, \ldots, s_n\}$ is a set $\mathcal{S}$ of n elements s_i. We indicate with $|\mathcal{S}| = n$ the cardinality of $\mathcal{S}$. We use uppercase roman letters to denote a random variable e.g. S, and use the uppercase calligraphic version of the same letter ($\mathcal{S}$) to denote the set from which the random variable takes values according to some corresponding probability distribution p_{S} , over the elements of the set, such that $\sum_{i=1}^{|\mathcal{S}|} p_{\mathrm{S}}(s_i) = 1$ for a discrete $\mathcal{S}$. For brevity, we may write $p_{\mathrm{S}}(s_i)$ as $p(s_i)$, where the random variable is implied by the argument. We use bold font lowercase letters to denote vector elements, e.g. $\boldsymbol{s}_i \in \mathbb{R}^d$ is an d-dimensional vector element from the set $\mathcal{S} = \mathbb{R}^d$.

Logic and model-theoretic background: We assume a formal language $\mathcal{L}$ composed of variables, predicates (i.e. relations), logical connectives $\neg$ (negation), $\vee$ (disjunction), $\wedge$ (conjunction), $\rightarrow$ (implication), and universal quantification $\forall$ (for all) with their conventional meaning (see [21]). Relations express knowledge over the elements of a domain. For instance, $r(s_1, s_2)$ states that elements s_1 and s_2 are related through the binary relation r. The meaning of a relation is defined by an *interpretation* $I_{\mathcal{S}_\sigma}$ over elements of an non-empty *domain* $\mathcal{S}$.

Definition 2.1 (Signature, Interpretation, Structure)**.** The *signature* of a language $\mathcal{L}$ is a set of relations $\sigma = \{r \in \mathcal{L}\}$ whose elements have *arity* given by $\mathsf{ar} : \sigma \to \mathcal{N}$, where $\mathcal{N}$ is the set of natural numbers. Given a signature σ and a non-empty domain $\mathcal{S}$, an *interpretation* $I_{\mathcal{S}_\sigma}$ of σ over elements of $\mathcal{S}$ assigns to each relation $r \in \sigma$ a set $I_{\mathcal{S}_\sigma}(r) \subseteq \mathcal{S}^{\mathsf{ar}(r)}$. A *structure* is a tuple $\mathcal{S}_\sigma = (\mathcal{S}, I_{\mathcal{S}_\sigma})$.

Note that for a fixed domain $\mathcal{S}$ and signature σ, different interpretations yield different structures. We construct universally quantified first-order formulae (called sentences) using the signature σ of $\mathcal{L}$, whose truth-value is defined with respect to a given structure $\mathcal{S}_\sigma$. To do so, we first consider *ground* instances of a formula. These are given by replacing all the variables in the formula with elements from the domain $\mathcal{S}$. For example, $r(s_1, s_2)$, where s_1 and s_2 are elements of $\mathcal{S}$, is a *ground* instance of an atomic formula $r(i, j)$ where i and j are variables in $\mathcal{L}$. Given a structure $\mathcal{S}_\sigma = (\mathcal{S}, I_{\mathcal{S}_\sigma})$, a relation r, and a tuple $(s_1, \ldots, s_{\mathsf{ar}(r)}) \in \mathcal{S}^{\mathsf{ar}(r)}$, a ground instance $r(s_1, \ldots, s_{\mathsf{ar}(r)})$ is true in the structure $\mathcal{S}_\sigma$ if and only if $(s_1, \ldots, s_{\mathsf{ar}(r)}) \in I_{\mathcal{S}_\sigma}(r)$. The truth value of a sentence in a given structure $\mathcal{S}_\sigma$ depends on the truth value of its respective ground instances. Specifically, a sentence is true in a structure $\mathcal{S}_\sigma$ if and only if all of its ground instances are true in $\mathcal{S}_\sigma$. When a sentence, τ, is true in a structure, $\mathcal{S}_\sigma$, we say that the structure *satisfies* τ, denoted as $\mathcal{S}_\sigma \models \tau$. A set of sentences form a *theory*, $\mathcal{T}$. A *model* of $\mathcal{T}$ is a structure that satisfies every sentence in $\mathcal{T}$.

Definition 2.2 (Model of a theory)**.** Let $\mathcal{T}$ be a theory written in a language $\mathcal{L}$ and let $\mathcal{S}_\sigma = (\mathcal{S}, I_{\mathcal{S}_\sigma})$ be a structure, where σ is the signature of $\mathcal{L}$. $\mathcal{S}_\sigma$ is a *model of* $\mathcal{T}$ if and only if $\mathcal{S}_\sigma \models \tau$ for every sentence $\tau \in \mathcal{T}$.

Example 1. Suppose we have the structure $\mathcal{S}_\sigma = (\mathcal{S}, I_{\mathcal{S}_\sigma})$, where $\mathcal{S}$ is a domain of images of handwritten digits and σ the signature of binary relations $\sigma = \{\mathsf{isGreater}, \mathsf{isEqual}, \mathsf{isLess}, \mathsf{isSuccessor}, \mathsf{isPredecessor}\}$, or for short $\sigma = \{\mathsf{G}, \mathsf{E}, \mathsf{L}, \mathsf{S}, \mathsf{P}\}$. Let $\mathcal{T}$ be the theory that defines ordinality including, for instance, the sentence $\forall i, j.\, \mathsf{G}(i, j) \to \neg\mathsf{E}(i, j)$ (if a digit is greater than another then they are not equal). Any structure $\mathcal{S}_\sigma = (\mathcal{S}, I_{\mathcal{S}_\sigma})$ with interpretations $I_{\mathcal{S}_\sigma}$ of σ that captures a total order over the elements of $\mathcal{S}$ is a model of $\mathcal{T}$.

3 A Formalization of Consistency and Coherence

In this section we turn our attention to the challenge of learning a model of a theory over a real-world domain given a signature. Here a learner must determine an appropriate interpretation over real-world data, such as images or other perceptions. This can be challenging because, firstly, we may only have a partial description of the interpretation, and secondly data may be noisy and contain information that is not relevant to the theory. For example, the handwritten digits in the MNIST dataset contain stylistic details such as line thickness and digit skew that are irrelevant to the notion of ordinality, which makes learning the structure from Example 1 non-trivial.

Following the convention from the disentanglement literature [2, 10, 8, 7], we make the assumption that real-world observations S are drawn from some conditional distribution $p_{\mathrm{S}|\mathrm{Z}}$, where Z is a latent random variable, itself drawn from prior p_{Z}. It is therefore useful to define a domain *encoding* of the form:

$$\psi_{\mathcal{S}} : \mathcal{S} \to \mathcal{Z}, \tag{1}$$

tasked with approximating the conditional expectation of the posterior, *i.e.* $\psi_{\mathcal{S}}(s) = \mathbb{E}[p_{\mathrm{Z}|\mathrm{S}}(\mathrm{Z}|s)]$. Since obtaining an interpretation from domain encodings, for a given signature, may require dealing with noise, we express the interpretation of relations over real-world data by belief functions over the space $\mathcal{Z}$ [18, 17], and refer to these as *relation-decoders*:

$$\phi_r : \mathcal{Z}^{\mathsf{ar}(r)} \to (0, 1) \tag{2}$$

with $\phi = \{\phi_r : r \in \sigma\}$. Concretely, for a binary relation r and ordered pair $(s_i, s_j) \in \mathcal{S}^2$, $\phi_r(\psi_{\mathcal{S}}(s_i), \psi_{\mathcal{S}}(s_j))$ describes the belief that $(s_i, s_j) \in I_{\mathcal{S}_\sigma}(r)$. A belief $\phi_r(\psi_{\mathcal{S}}(s_i), \psi_{\mathcal{S}}(s_j)) \approx 1$ signifies a strong belief that $(s_i, s_j) \in I_{\mathcal{S}_\sigma}(r)$ and $\phi_r(\psi_{\mathcal{S}}(s_i), \psi_{\mathcal{S}}(s_j)) \approx 0$ signifies a strong belief that $(s_i, s_j) \notin I_{\mathcal{S}_\sigma}(r)$. Together, $\psi_{\mathcal{S}}$ and ϕ allow us to define a belief-based analogue to a structure.

Definition 3.1 (Soft-Structure/Soft-Substructure)**.** Given signature σ, a possibly infinite set $\mathcal{Z}$ and relation-decoders ϕ, a *soft-structure* is a tuple $\tilde{\mathcal{Z}}_\sigma = (\mathcal{Z}, \phi)$. For (finite) domain $\mathcal{S}$ and encoding $\psi_{\mathcal{S}} : \mathcal{S} \to \mathcal{Z}$, $\tilde{\mathcal{S}}_\sigma = (\psi_{\mathcal{S}}(\mathcal{S}), \phi)$ is a (finite) *soft-substructure* of $\tilde{\mathcal{Z}}_\sigma$, with sub-domain $\psi_{\mathcal{S}}(\mathcal{S}) = \{\psi_{\mathcal{S}}(s) | s \in \mathcal{S}\} \subseteq \mathcal{Z}$.

A soft-structure can be used to learn a (logical) structure over a real-world domain through learning $\psi_{\mathcal{S}}$ and ϕ. Clearly, a finite soft-substructure is a soft-structure. To determine the degree to which a soft-structure *supports* any given structure, we introduce the following measure:

$$p(\mathcal{S}_\sigma|\tilde{\mathcal{S}}_\sigma) = \prod_{r\in\sigma} \prod_{O\in\mathcal{S}^{\mathsf{ar}(r)}} f(\phi_r, \psi_{\mathcal{S}}, O, \gamma^r_{O,\mathcal{S}_\sigma}) \tag{3}$$

with $f(\phi_r, \psi_{\mathcal{S}}, O, \gamma^r_{O,\mathcal{S}_\sigma}) = (\phi_r(\psi_{\mathcal{S}}(O)))^{\gamma^r_{O,\mathcal{S}_\sigma}} \cdot (1-\phi_r(\psi_{\mathcal{S}}(O)))^{1-\gamma^r_{O,\mathcal{S}_\sigma}}$, where $\gamma^r_{O,\mathcal{S}_\sigma} = 1$ if $O \in I_{\mathcal{S}_\sigma}(r)$, and 0 otherwise; we use $\phi_r(\psi_{\mathcal{S}}(O))$ as shorthand for $\phi_r(\psi_{\mathcal{S}}(s_1), \ldots, \psi_{\mathcal{S}}(s_n))$ for $n = \mathsf{ar}(r)$. Eqn. 3 expresses the assumption that, given a finite soft-structure, the beliefs in what constitutes the different interpretations of a relation are independent of one another. It is straightforward to show that $\sum_{\mathcal{S}_\sigma} p(\mathcal{S}_\sigma|\tilde{\mathcal{S}}_\sigma) = 1$ (summed over all possible structures with domain $\mathcal{S}$ and signature σ) and so it can be treated as a probability measure, where $p(\mathcal{S}_\sigma|\tilde{\mathcal{S}}_\sigma) \approx 1$ means that there is a high probability that the interpretation sampled from $\tilde{\mathcal{S}}_\sigma$ will be $I_{\mathcal{S}_\sigma}$. If we have a theory $\mathcal{T}$ over σ then it is natural to ask with what weight $\tilde{\mathcal{S}}_\sigma$ supports any given structure that is a model of $\mathcal{T}$. In the following, we use *model weight*, $\Gamma^{\tilde{\mathcal{S}}_\sigma}_{\mathcal{T}}$, to describe the support given by $\tilde{\mathcal{S}}_\sigma$ to models of $\mathcal{T}$:

$$\Gamma^{\tilde{\mathcal{S}}_\sigma}_{\mathcal{T}} = \sum_{\mathcal{S}_\sigma\in\mathcal{M}^{\mathcal{T}}_{\mathcal{S}}} p(\mathcal{S}_\sigma|\tilde{\mathcal{S}}_\sigma) \tag{4}$$

where $\mathcal{M}^{\mathcal{T}}_{\mathcal{S}}$ is the set of all structures with domain $\mathcal{S}$ that are models of $\mathcal{T}$. This lets us compare soft-structures, wherein a good soft-structure will be one that has a high model weight.

Definition 3.2 (ϵ-Consistency of Soft-Structure)**.** Given a finite soft-structure $\tilde{\mathcal{S}}_\sigma$, if $1 - \Gamma^{\tilde{\mathcal{S}}_\sigma}_{\mathcal{T}} \leq \epsilon$ then we say that the soft-structure is *ϵ-consistent* with theory $\mathcal{T}$.

We propose ϵ-consistency as an appropriate quantified measure of the notion of consistency presented in [16]. A consistent soft-structure $\tilde{\mathcal{S}}_\sigma$ ensures that ϕ gives high belief only to interpretations that satisfy, and therefore are logically consistent with, $\mathcal{T}$. As expected, consistency pertains to the domain encodings of $\tilde{\mathcal{S}}_\sigma$, *i.e.* $\psi_{\mathcal{S}}(\mathcal{S})$. For a concept to be learned in a manner comparable to what a human might learn, we would expect that this consistency carries over to new domains with their corresponding soft-structures, which gives our definition of coherence between soft-structures, as follows. Consider a situation where a deep network has already learned a soft-structure that has high model weight given the relations $\{\mathsf{G}, \mathsf{E}, \mathsf{L}, \mathsf{S}, \mathsf{P}\}$ from Example 1. Now suppose that we are given a new domain of images, $\mathcal{Y}$, showing single block stacks of different heights, and we wish to re-use the signature of ordinal relations and $\mathcal{T}$ from Example 1. Lastly, let $I_{\mathcal{Y}_\sigma}$ be a interpretation in the new domain that orders images according to block stack height and is a model of $\mathcal{T}$. We can summarise this with the following two structures:

$$\mathcal{X}_\sigma = (\mathcal{X}, I_{\mathcal{X}_\sigma}) \in \mathcal{M}^{\mathcal{T}}_{\mathcal{X}} \quad \text{and} \quad \mathcal{Y}_\sigma = (\mathcal{Y}, I_{\mathcal{Y}_\sigma}) \in \mathcal{M}^{\mathcal{T}}_{\mathcal{Y}}, \tag{5}$$

where $\mathcal{X}_\sigma$ is the structure from Example 1 with a domain of handwritten digits and $\mathcal{Y}_\sigma$ is our new structure, with a domain of block stack images. These can be learned by soft-structures:

$$\tilde{\mathcal{X}}_\sigma = (\psi_{\mathcal{X}}(\mathcal{X}), \phi) \qquad \text{and} \qquad \tilde{\mathcal{Y}}_\sigma = (\psi_{\mathcal{Y}}(\mathcal{Y}), \phi), \tag{6}$$

which use domain-specific encoders, $\psi_{\mathcal{X}}$ and $\psi_{\mathcal{Y}}$, but share the same relation-decoders. As we know that $\tilde{\mathcal{X}}_\sigma$ has a high model weight and since ϕ is shared with $\tilde{\mathcal{Y}}_\sigma$, a natural question to ask is: under what conditions will a ϕ that is consistent over domain-encodings $\psi_{\mathcal{X}}(\mathcal{X})$ also be consistent over $\psi_{\mathcal{Y}}(\mathcal{Y})$? Concretely, we are interested in when the following *coherence* condition holds.

Definition 3.3 (ϵ-Coherence across soft-structures)**.** Two soft-structures, $\tilde{\mathcal{X}}_\sigma$ and $\tilde{\mathcal{Y}}_\sigma$ that share relation-decoders ϕ, are said to be *ϵ-coherent* with respect to a theory $\mathcal{T}$, if $\tilde{\mathcal{X}}_\sigma$ is ϵ_1-consistent with $\mathcal{T}$, $\tilde{\mathcal{Y}}_\sigma$ is ϵ_2-consistent with $\mathcal{T}$, $\epsilon_1 \leq \epsilon$, and $\epsilon_2 \leq \epsilon$.

Coherence between $\tilde{\mathcal{X}}_\sigma$ and $\tilde{\mathcal{Y}}_\sigma$ as defined above means that the concept of ordinality that applies to digit ordering can also be applied to block stack height ordering. It is desirable that learning ordinality on the domain of digits produces a coherent concept of ordinality with respect to other ordinal properties, such as height. Since it is possible that $\psi_{\mathcal{S}}(\mathcal{X})$ and $\psi_{\mathcal{S}}(\mathcal{Y})$ produce unique encodings, coherence relies on ϕ's ability to generalise over possibly disjoint subsets of $\mathcal{Z}$[1].

[1] If soft-structure $\tilde{\mathcal{Z}}_\sigma$ defined over the full space $\mathcal{Z}$ is consistent then coherence is guaranteed between all possible soft-substructures.

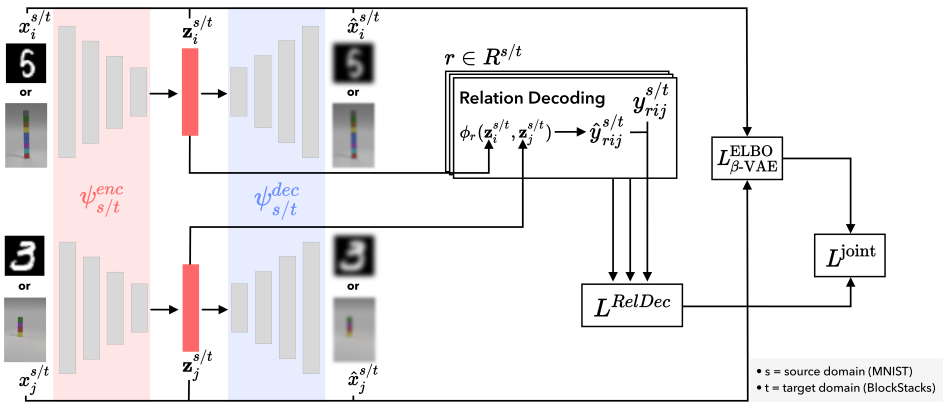

Figure 1: Network architecture used for PRT task.Relational learning is performed on the source MNIST data set (to learn e.g. that digit 5 is greater than 3). Moving to the target data set (to learn that a stack of blocks is greater than another) involves training a new encoder-decoder together with a subset of the relation-decoders (with fixed parameters) from MNIST. The remaining relations are held-out to evaluate zero-shot transfer learning performance.

4 A Consistent and Coherent Neuro-Symbolic Autoencoder

In order to ground our definitions of consistency (3.2) and coherence (3.3) into a real system and evaluate their practical value, in this section we propose a simple autoencoder neuro-symbolic architecture intended to satisfy our definitions. To derive an efficient loss function, we introduce an estimate measure for a soft-structure's ϵ-consistency and coherence with a given theory when access to every logical model is not available or computationally feasible.[2]

Suppose there is a fixed domain $\mathcal{S}$ and theory $\mathcal{T}$ whose sentences use relations from a signature σ. Let $k \in \{1, ..., K_0\}$ denote the index associated with each unique ground instance of the relations in $\mathcal{T}$. Take $B_\mathcal{T}$ to be a Boolean random variable. The probability of $\mathcal{T}$ being satisfied under a soft-structure $\tilde{\mathcal{S}}_\sigma$ is expressed as $p(b_\mathcal{T}|\tilde{\mathcal{S}}_\sigma, k)$, where $b_\mathcal{T} = 1$ if $\mathcal{T}$ is satisfied (i.e. $true$), or 0 otherwise (denoting $false$). By definition, $p(b_\mathcal{T} = 1|\mathcal{S}_\sigma, k) = 1$ if $\mathcal{S}_\sigma \in \mathcal{M}^\mathcal{T}_\mathcal{S}$, where $\mathcal{M}^\mathcal{T}_\mathcal{S}$ denotes the set of models of $\mathcal{T}$. When $\tilde{\mathcal{S}}_\sigma$ is consistent with $\mathcal{T}$ then we should also find that $p(b_\mathcal{T} = 1|\tilde{\mathcal{S}}_\sigma, k) \approx 1$. Hence, we define a loss function as the expectation of the binary cross-entropy between $p(B_\mathcal{T}|\mathcal{S}_\sigma, k)$ and $p(B_\mathcal{T}|\tilde{\mathcal{S}}_\sigma, k)$, which simplifies to the expected negative log-likelihood of satisfying $\mathcal{T}$ under a random sampling from the set of ground instances:

$$L(\mathcal{T}, \tilde{\mathcal{S}}_\sigma) = \mathbb{E}_{k\sim p(k)}[-\ln p(b_\mathcal{T} = 1|\tilde{\mathcal{S}}_\sigma, k)]. \tag{7}$$

where $p(k) = \frac{1}{K_0}$ is taken to be uniform distribution over the set of unique groundings. A measure based on this loss is required to enable the practical evaluation of coherence. To achieve this, we define $\bar{\Gamma}^{\tilde{\mathcal{S}}_\sigma}_\mathcal{T} = \exp(-L(\mathcal{T}, \tilde{\mathcal{S}}_\sigma))$ and use its relationship with the already defined $\Gamma^{\tilde{\mathcal{S}}_\sigma}_\mathcal{T}$ to obtain a bound on the loss function:

$$\ln \frac{1}{1-\bar{\epsilon}} \geq L(\mathcal{T}, \tilde{\mathcal{S}}_\sigma) \tag{8}$$

where $\bar{\epsilon} \geq 1 - \bar{\Gamma}^{\tilde{\mathcal{S}}_\sigma}_\mathcal{T}$. We take the coherence to be the upper value of $\ln \frac{1}{1-\bar{\epsilon}}$ between domains.[3]

Figure 1 outlines the main components of our autoencoder: a domain-encoder $\psi_\mathcal{S}$ and modular relation-decoders ϕ form an autoencoding architecture that, given a domain of images $\mathcal{S} \subset \mathbb{R}^{W\times H}$ and a d-dimensional latent space $\mathcal{Z} = \mathbb{R}^d$, converts sub-symbolic encodings from $\psi_\mathcal{S}$ into a modular relational representation via decoding for each $\phi_r, r \in \sigma$. Additionally, to retain information in $\mathcal{Z}$ pertaining to $\mathcal{S}$ which is beyond the requirements of ϕ, a domain-decoder produces domain reconstructions $\hat{\mathcal{S}}$. In Figure 1, we use ψ^{enc}_S to refer to the domain-encoder and ψ^{dec}_S for the domain-decoder. To train the model, ground-truth interpretations $I_{\mathcal{S}_\sigma}$ are given, allowing us to directly

[2]Calculating Eqn. 4 can become intractable as it involves computing ϕ beliefs for every grounding.

[3]The complete derivation of loss function and bounds is presented in the Supplementary Material.

maximise Eqn. 3 via the negative log-likelihood loss:

$$L^{\tilde{\mathcal{S}}_\sigma} = -\log p(\mathcal{S}_\sigma | \tilde{\mathcal{S}}_\sigma), \tag{9}$$

To obtain informative latent representations for $\mathcal{S}$, we use a Variational AutoEncoder (VAE), specifically the β-VAE, given its simplicity and demonstrated ability to separate distinct factors in the latent representation, known as disentanglement (although disentanglement is not seen here as a requirement for consistency and coherence) [8, 3, 10]. We therefore combine the ELBO objective with an additional β scalar hyperparameter that seeks to achieve disentanglement ($L^{\text{ELBO}}_{\beta\text{-VAE}}$) with $L(\mathcal{T}, \tilde{\mathcal{S}}_\sigma)$ over each component of the autoencoder architecture to obtain the following aggregate objective (we provide the full ELBO derivation with a detailed explanation in the Supplementary):

$$L^{\text{joint}} = L^{\text{ELBO}}_{\beta\text{-VAE}} - \lambda L^{\tilde{\mathcal{S}}_\sigma} \tag{10}$$

where λ is a scalar weighting parameter.

Together with the $L^{\text{ELBO}}_{\beta\text{-VAE}}$, the choice of relation-decoder can shape the domain-encodings [6]. In our evaluation, the following choices are made. We propose a Dynamic Comparator (DC) model composed of two modes, a distance-based measure, $\phi^\dagger_r$, to measures the distance between two inputs relative to a reference point, and a step-function, $\phi^\ddagger_r$, that determines the sign of the difference between two points, optionally with an offset. Although any function can be used that has the required characteristics for $\phi^\dagger$ and $\phi^\ddagger$, in this paper we use the following implementation:

$$\phi^{DC}_r(\boldsymbol{z}_i, \boldsymbol{z}_j) = a_{r,0} \cdot \phi^\dagger_r + a_{r,1} \cdot \phi^\ddagger_r \tag{11}$$

where,

$$\phi^\dagger_r = f_0\big(-\eta_{r,0}(\|\boldsymbol{u}_r \odot (\boldsymbol{z}_i - \boldsymbol{z}_j + \boldsymbol{b}^\dagger_r)\|_2)\big) \tag{12}$$

$$\phi^\ddagger_r = f_1\big(\eta_{r,1} \cdot \boldsymbol{u}_r^\top (\boldsymbol{z}_i - \boldsymbol{z}_j + \boldsymbol{b}^\ddagger_r)\big). \tag{13}$$

Here $\boldsymbol{a}_r = \texttt{Softmax}(\boldsymbol{A}_r) \in (0,1)^2$ is an attention weighting between the two modes, $\phi^\dagger_r$ and $\phi^\ddagger_r$; f_0 and f_1 are an exp and sigmoid function, respectively; $\boldsymbol{u}_r = \texttt{Softmax}(\boldsymbol{U}_r) \in (0,1)^m$ is an attention mask which is applied to m-dimensional embeddings; $\boldsymbol{b}^\dagger_r, \boldsymbol{b}^\ddagger_r \in \mathbb{R}^m$ are learnable bias terms that enables an offset to each mode; and $\eta_{r,0} \in \mathbb{R}^+$ are non-negative and $\eta_{r,1} \in \mathbb{R}$ any-valued scalar terms, respectively. Lastly, $\odot$ denotes the Hadamard product and $\|\cdot\|_2$ is the $L2$-norm. The key innovation behind DC is its ability to model each of the ordinal relations whilst encouraging generalised consistency across the full latent subspace, as defined by each $\boldsymbol{u}_r$. This is achieved without explicit weight sharing, wherein relation-decoders discover parametric relationships between relations from the data. Further details are provided in the Supplementary Material.

5 Relational Transfer Learning Experiment Design

We now describe an experimental design to compare coherence of different relation-decoders.

Partial Relation Transfer (PRT): We evaluate a novel PRT task across two soft-structures $\tilde{\mathcal{X}}_\sigma$ and $\tilde{\mathcal{Y}}_\sigma$. They share a common signature σ and relation-decoders ϕ but have disjoint domains $\mathcal{X}$ and $\mathcal{Y}$, respectively. The experimental design involves first learning ϕ on source domain $\mathcal{X}$, together with its domain-specific autoencoder. In the second phase, we train a new domain-specific autoencoder on the target domain, $\mathcal{Y}$, alongside a selection of the now learned ϕ relation-decoders but with fixed-parameters. The selected relation-decoders are expected to help guide training of $\psi^{\text{enc}}_{\mathcal{Y}}$. Held-out relation-decoders are then evaluated in the new domain on zero-shot transfer learning performance. For domain $\mathcal{X}$ we use the MNIST handwritten digits data set [12], and for domain $\mathcal{Y}$ we use a proposed BlockStacks data set, which includes a single stack of multi-colored cubes of differing heights, each containing one randomly positioned red cube (see Supplementary Material for details and examples). The shared signature includes the ordinal relations $\sigma = \{\mathsf{G}, \mathsf{E}, \mathsf{L}, \mathsf{S}, \mathsf{P}\}$ and is applied to digit ordering in MNIST and red cube position ordering in BlockStacks. We provide results against a theory of ordinality, as explored in Example 1. We provide a formal specification of the theory in the Supplementary Material. When transferring relations from $\psi^{\text{enc}}_{\mathcal{X}}$ to $\psi^{\text{enc}}_{\mathcal{Y}}$, one could use the full set ϕ of relation-decoders. However, this is not necessary from a logical standpoint because the entire system of relations can be expressed in terms of isSuccessor (e.g. the successor of a number is larger than that number). We therefore only employ the isSuccessor relation-decoder as

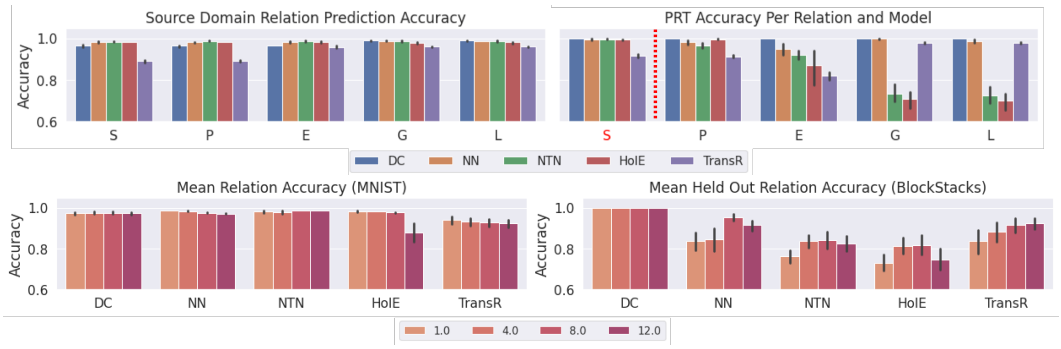

Figure 2: **[Top]** Relation-decoder prediction accuracy per model (DC, NN, NTN, HoIE, TransR) and relation (abbreviated on the x-axis by {S: isSuccessor, P: isPredecessor, E: isEqual, G: isGreater, L: isLess}), in the source domain (MNIST, left) and target domain (BlockStack, right). A red highlighted S and dotted line (top right) indicates that relation isSuccessor is included in training the target domain autoencoder, but none of the other relations are. Both DC and NN retain a good performance while all other models show a decrease of accuracy in the target domain for one or more of the relations not included in training. **[Bottom]** Impact of different values of $\beta \in \{1, 4, 8, 12\}$ for each relation-decoder averaged across all relations in the source domain (left) and held-out relations $\{P, E, G, L\}$ in the target domain (right). It can be seen that DC is not impacted by changes in β and it maintains performance in the target domain. All other models show a decrease of accuracy for the held-out relations in the target domain.

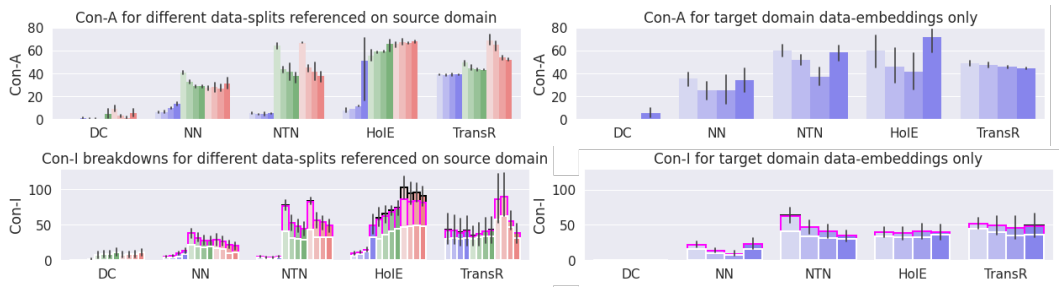

Figure 3: Consistency losses (lower values are better) for the models (DC, NN, NTN, HoIE, TransR) using the MNIST data set (source domain $\mathcal{X}$) **[left]** and BlockStacks (target domain $\mathcal{Y}$) **[right]**. The blue bars show the consistency loss of the data embeddings, with darker shades corresponding to models trained with higher β (disentanglement pressure). Two additional data splits are shown: interpolation (in green) with samples coming from the MNIST data-embedding cluster, and extrapolation (in red) with samples drawn from outside the cluster. Results are further divided into consistency across relations (Con-A) **[top]** and consistency of individual relations (Con-I) **[bottom]**. The following relations are used (see stacked bars at the bottom graphs): transitivity (in white), asymmetry (in magenta) and reflexivity (in black). Notice the large difference in MNIST between data-embedding Con-A vs. interpolated and extrapolated Con-A results, wherein BlockStacks data-embedding Con-A results are similar to the MNIST interpolated/extrapolated Con-A results.

a fixed-parameter guide for $\psi_{\mathcal{Y}}^{\text{enc}}$. If coherence, as defined in this paper, is carried across domains, we would expect the transferring of isSuccessor to be sufficient to produce an improved performance.

Neural model components and Hyperparameters: Together with DC, existing relation-decoder models evaluated here are: TransR [13], HolE [15], NTN [22]. We additionally include a basic feedforward neural-network baseline, NN. To produce domain-encodings, all experiments use a β-VAE. We provide further details for all models, including details about training regimen and implementation in the Supplementary Material. In the source domain we explore β values between $\{1, 4, 8, 12\}$, and set $\lambda = 10^3$ and in the target domain we first normalise losses and set $\beta = 10^{-4}$ and $\lambda = 10^{-2}$ as these produced good reconstructions whilst also ensuring optimisation against $L^{\tilde{\mathcal{Y}}_\sigma}$. In all experiments, we fix $\mathcal{Z} = \mathbb{R}^{10}$.

Table 1: Coherence comparison with respect to source and target data-embeddings. Results are reported with the corresponding $\beta = \beta^*$ value (in parenthesis). The consistency loss abbreviations refer to: (A)cross, (tr)ansitivity, (asym)metry, (refl)exivity and (Aggr)egate, which gives the best obtained aggregate consistencies. DC outperforms all other approaches in coherence scores.

ϕ	Aggr.	(β^*)	Con-A	(β^*)	Con-I-tr	(β^*)	Con-I-asym	(β^*)	Con-I-refl	(β^*)
TransR	90.33	(8)	44.34	(12)	35.30	(8)	9.94	(8)	0.55	(8)
HolE	82.06	(8)	41.18	(8)	32.15	(4)	5.96	(1)	0.07	(8)
NTN	79.54	(8)	38.91	(8)	30.08	(12)	4.49	(12)	0.09	(12)
NN	34.09	(8)	24.78	(8)	7.24	(8)	3.88	(8)	**0.04**	(4)
DC	**0.34**	(1)	**0.07**	(1)	**0.18**	(1)	**0.00**	(1)	0.09	(1)

6 Main Experimental Results and Comparative Evaluation

In this section, experimental results demonstrate the relevance of a model-theoretic perspective on the learning of concepts with neural networks. Results show that transfer learning performance is positively correlated with measures for consistency within and consistency across domains, i.e. coherence. This holds particularly true for embeddings that are close by but different from source domain embeddings. As we have argued, for a neural model to perform well on concept transfer, its representations must maintain high probability of consistency with a logical theory that can provide a semantics for the concept. We further argue that the most robust way of doing this is to maintain consistency across regions of embedding space, rather than relying exclusively on the specific data-points observed at training time in the source domain.

Figure 2 shows standard PRT prediction accuracies per relation in both the source and target domain. Figure 3 then presents consistency losses for three color-coded data splits: data-embeddings (blue), where all inputs are encodings of a domain's test data; interpolation (green), where we obtain an empirical mean and variance for the domain's data-embeddings and sample from a corresponding Gaussian distribution; and extrapolation (red), where we sample from regions strictly outside the smallest, axis-aligned hyper-rectangle that encloses all data-points. Finally, Table 1 offers a direct coherence comparison between relation-decoders, using the derived coherence measure (Eqn. 8)[4].

Relation-decoder PRT accuracy performance: Figure 2-top provides relation-decoder prediction accuracy in both the source MNIST (left), and target BlockStacks (right), domains. Key observations are that DC produces excellent PRT performance, whilst NN, NTN and HolE all see some degradation from their source accuracies on relations other than isSuccessor. TransR seems to maintain an target accuracy profile similar to its performance in the source domain, but this is significantly below the performance of other models in the source domain.We include the impact of adjusting β (disentanglement pressure) in Figure 2-bottom. Barring DC which has little discernible change in either domain, PRT performance is significantly impacted by β in all models, but has little effect in the source domain. TransR shows a strong positive correlation between target domain accuracy and β, whereas the remaining models produce their best PRT performances with intermediate disentanglement pressure.

To gain deeper insight as to which underlying characteristics can explain the observed PRT accuracy profiles, Figure 3-top presents consistency losses against formulae that constrain truth-value assignments across relations under a theory of ordinality, referred to as consistency-across (Con-A).[5]. Results refer to both source (left) and target domain embeddings (right). We note that DC shows excellent Con-A in the target domain in all regions. Most other models have worse interpolation and extrapolation consistency. Increasing β appears to improve interpolation and extrapolation performance for models NN, NTN and TransR, but there are indications that this trend does not persist into the largest $\beta = 12$ value. On the other hand, HolE shows a negative correlation between β and Con-A performance, across all data-splits. DC sustains strong Con-A results for target domain data-embeddings (right). Results for all other models are notably worse with respect to their source data-embedding performances and are instead comparable with their interpolation or extrapolation results in the source domain. Together, these results paint a picture wherein it may be possible to antic-

[4]We take ϕ_r prediction values above 0.5 to signify a truth prediction and those below 0.5 to signify falsity. An alternative, left as future work, would be to sample the space of ϕ values to produce a confidence measure

[5]Truth-tables for each consistency formula are given in the Supplementary Material

ipate poor transfer performance by evaluating interpolation and extrapolation consistency in the source domain. This would indeed be expected, since source and target domain data-embeddings are unlikely to perfectly overlap, and so retained consistency on regions outside the source data-embeddings should increase the probability of consistency over target domain data-embeddings.

Next, Figure 3-bottom presents consistency values for each individual relation-decoder model (Con-I). Stacked bars show the results for logical sentences defining: transitivity (white), asymmetry (magenta) and reflexivity (black). Results are averaged over individual relations and are grouped under label Con-I w.r.t. source domain (left) and target domain (right).We firstly observe that DC and NN share the best overall Con-I performance profiles, with TransR following closely. DC and TransR both show comparable data-embedding versus interpolation/extrapolation performance, whereas NN, NTN and HolE suffer from degradation across these splits. Interestingly, these results show that: DC only suffers on transitivity, NN and TransR mainly struggle to model transitivity but show additional loss for asymmetry and HolE demonstrates difficulty in modelling each of the Con-I sub-stack. With regards to β's impact, it is not possible to determine a correlation for DC. However, NN and NTN demonstrate a negative correlation of β against overall Con-I, with comparable response for each underlying sub-stack. TransR shows a significant Con-I extrapolation improvement with increased β and HolE is for the most part adversely impacted as β is increased. Similar trends can be seen for target Con-I performance.

Lastly, Table 1 provides a comparison between optimal coherences achieved for each relation-decoder model, as defined in Section 4. Results are partitioned according to each consistency type (transitivity, asymmetry and reflexivity) and an aggregate value. DC clearly outperforms all other models on coherence. NN achieves strong aggregate coherence compared with NTN, HolE and TransR. Although NTN and HolE have similar aggregate coherence, TransR performs generally worse. This may be caused by TransR producing weaker belief scores in comparison to other models, as this can result in a worse overall consistency level. Looking at β^* profiles, we see that most models achieve optimum aggregate coherence at $\beta = 8$, other than DC which performs better at $\beta = 1$. Overall, this is in agreement with the β profiles given by Figure 2-bottom (right). However, we can see that β^* profiles for Con-A based coherence are in more direct agreement - as TransR achieves its best at $\beta = 12$.

Our results indicate that increasing regularisation over relation-decoder models, either in the form of disentanglement pressure or relation-decoder model capacity, improves their ability to learn coherent concepts. Firstly, strong PRT transfer for DC and NN (given an appropriately high β setting) showed that both relation-decoder models are able to minimise Eqn. 9 in the source domain and retain good performance in the target domain. Consistency profiles over partial theories (subsets of the sentences that comprise the overall theory of ordinality), covering multiple data-splits, then further suggested that a relation-decoder's ability to retain consistency over interpolated/extrapolated regions with respect to the observed data-encodings during training, i.e. coherence, is key.

7 Conclusion and Future Work

This paper introduced formal definitions of consistency and coherence for neuro-symbolic systems. As a result, a sub-symbolic model can have consistency and coherence measured with respect to a logical theory. We defined a neural model based on domain-encoders coupled with modular relation-decoders and experimental procedure that together allowed the investigation of how concept coherence differs for various implementations of relation-decoders applied to transfer learning. Consistency results and a comparison of coherence scores showed that the models that can achieve excellent coherence also achieve high accuracy at partial relational transfer learning tasks. The empirical evaluations in this paper only considered binary relations and a fixed signature which is learned "all at once" in a source domain. In practical applications, however, it should be possible to discover concepts gradually, e.g. as part of a curriculum or through gradual refinement of pre-learned relations after progressive exposure to different contexts. This necessitates an adaptation of the approach presented here and further evaluations as part of future work. Additionally, we only explored a signature for ordinality, whereas other fundamental properties should be investigated such as periodic (*e.g.* rotation) and unordered categorical (*e.g.* shape) properties. Further evaluations of the formalization introduced here should consider the use of different models, theories and scenarios/data sets in the evaluation of consistency and coherence metrics.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See Section 7
   (c) Did you discuss any potential negative societal impacts of your work? [Yes] These have been included in the Supplementary - the main societal impact of coherent concept learning is that inferences will uphold logical consistency. If the logic does not include biases, the inferences themselves should not be biased, providing that the feature extraction has been properly disentangled. All in all, coherent concepts will be easier to trust.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Section 3 and Section 4
   (b) Did you include complete proofs of all theoretical results? [Yes] In particular, a rigorous proof for the consistency loss (Eqn. 8) is provided in the Supplementary.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] A current zip of the code used for this paper will be included in the paper's supplementary. A URL to public facing (refined and minimised) code will be provided in the camera ready version of the paper.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 5 and we provide further model details in the Supplementary.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] We provide error bars in bar plots. However we did not provide errors in the tabular results. These are obtained from the bar plots, so can be evaluated, but to improve clarity we will include the numeric errors in the tabular results for the camera ready version.

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No] This is a difficult number to extract as experiments were run on different GPU models and with early stopping. We will endeavour to provide an estimate for the camera ready.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] The MNIST data set is cited properly.
   (b) Did you mention the license of the assets? [No]
   (c) Did you include any new assets either in the supplemental material or as a URL? [Yes] The necessary code to generate the BlockStacks data set is included in the Supplementary. We will include the actual data set providing there is space.
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [No]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [No]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]