ON INSTAHIDE, PHASE RETRIEVAL, AND SPARSE MATRIX FACTORIZATION

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ABSTRACT

In this work, we examine the security of InstaHide, a scheme recently proposed by Huang et al. (2020b) for preserving the security of private datasets in the context of distributed learning. To generate a synthetic training example to be shared among the distributed learners, InstaHide takes a convex combination of private feature vectors and randomly flips the sign of each entry of the resulting vector with probability 1/2. A salient question is whether this scheme is secure in any provable sense, perhaps under a plausible complexity-theoretic assumption.

The answer to this turns out to be quite subtle and closely related to the average-case complexity of a multi-task, missing-data version of the classic problem of phase retrieval that is interesting in its own right. Motivated by this connection, under the standard distributional assumption that the public/private feature vectors are isotropic Gaussian, we design an algorithm that can actually recover a private vector using only the public vectors and a sequence of synthetic vectors generated by InstaHide.

1 INTRODUCTION

In distributed learning, where decentralized parties each possess some private local data and work together to train a global model, a central challenge is to ensure that the security of any individual party's local data is not compromised. Huang et al. (2020b) recently proposed an interesting approach called *InstaHide* for this problem. At a high level, InstaHide is a method for aggregating local data into synthetic data that can hopefully preserve the privacy of the local datasets and be used to train good models.

Informally, given a collection of *public feature vectors* (e.g. a publicly available dataset like ImageNet Deng et al. (2009)) and a collection of *private feature vectors* (e.g. the union of all of the private datasets among learners), InstaHide produces a synthetic feature vector as follows. Let integers $k_{\mathsf{pub}}, k_{\mathsf{priv}}$ be sparsity parameters.

1. Form a random convex combination of k_{pub} public and k_{priv} private vectors.
2. Multiply every coordinate of the resulting vector by an independent random sign in $\{\pm 1\}$, and define this to be the synthetic feature vector.

The hope is that by removing any sign information from the vector obtained in Step 1, Step 2 makes it difficult to discern which public and private vectors were selected in Step 1. Strikingly, Huang et al. (2020b) demonstrated on real-world datasets that if one trains a ResNet-18 or a NASNet on a dataset consisting of synthetic vectors generated in this fashion, one can still get good *test accuracy* on the underlying private dataset for modest sparsity parameters (e.g. $k_{\mathsf{pub}} = k_{\mathsf{priv}} = 2$). [1]

The two outstanding *theoretical* challenges that InstaHide poses are understanding:

- **Utility**: What property, either of neural networks or of real-world distributions, lets one tolerate this kind of covariate shift between the synthetic and original datasets?

[1] We did not describe how the labels for the synthetic vectors are assigned, but this part of InstaHide will not be important for our theoretical results and we defer discussion of labels to Section 4.

- **Security**: Can one rigorously formulate a refutable security claim for InstaHide, under a plausible average-case complexity-theoretic assumption?

In this paper we consider the latter question. One informal security claim implicit in Huang et al. (2020b) is that given a synthetic dataset of a certain size, no efficient algorithm can recover a private image to within a certain level of accuracy (see Problem 1 for a formal statement of this recovery question). On the one hand, it is a worthwhile topic of debate whether this is a satisfactory guarantee from a security standpoint. On the other, even this kind of claim is quite delicate to pin down formally, in part because it seems impossible for such a claim to hold for arbitrary private datasets.

Known Attacks and the Importance of Distributional Assumptions If the private and public datasets consisted of natural images, for example, then attacks are known Jagielski (2020); Carlini et al. (2020). At a high level, the attack of Jagielski (2020) crucially leverages local Lipschitz-ness properties of natural images and shows that when $k_{\mathsf{priv}} + k_{\mathsf{pub}} = 2$, even a single synthetic image can reveal significant information. The very recent attack of Carlini et al. (2020), which was independent of the present work and appeared a month after this submission appeared online, is more sophisticated and bears interesting similarities to the algorithms we consider. We defer a detailed discussion of these similarities to Appendix A in the supplement.

While the original InstaHide paper Huang et al. (2020b) focused on image data, their general approach has the potential to be applicable to other forms of real-valued data, and it is an interesting mathematical question whether the above attacks remain viable. For instance, for distributions over private vectors where individual features are nearly independent, one cannot hope to leverage the kinds of local Lipschitz-ness properties that the attack of Jagielski (2020) exploits. Additionally, if the individual features are identically distributed, then it is information theoretically impossible to discern anything from just a single synthetic vector. For instance, if a synthetic vector $\widetilde{v}$ is given by the entrywise absolute value of $\frac{1}{2}v_1 + \frac{1}{2}v_2$ for private vectors v_1, v_2, then an equally plausible pair of private vectors generating $\widetilde{v}$ would be v_1', v_2' given by swapping the i-th entry of v_1 with that of v_2 for any collection of indices $i \in [d]$. In other words, there are 2^d pairs of private vectors which are equally likely under the Gaussian measure and give rise to the exact same synthetic vector.

Gaussian Images, and Our Results A natural candidate for probing whether such properties can make the problem of recovering private vectors more challenging is the case where the public and private vectors are sampled from the standard Gaussian distribution over $\mathbb{R}^d$. While this distribution does not capture datasets in the real world, it avoids some properties of distributions over natural images that might make InstaHide more vulnerable to attack and is thus a clean testbed for stress-testing candidate security claims for InstaHide. Furthermore, in light of known hardness results for certain learning problems over Gaussian space Diakonikolas et al. (2017); Bruna et al. (2020); Diakonikolas et al. (2020b); Goel et al. (2020a); Diakonikolas et al. (2020a); Klivans & Kothari (2014); Goel et al. (2020b); Bubeck et al. (2019); Regev & Vijayaraghavan (2017), one might hope that when the vectors are Gaussian, one could rigorously establish some lower bounds, e.g. on the size of the synthetic dataset (information-theoretic) and/or the runtime of the attacker (computational), perhaps under an average-case assumption, or in some restricted computational model like SQ.

Orthogonally, we note that the recovery task the attacker must solve appears to be an interesting inverse problem in its own right, namely a multi-task, missing-entry version of phase retrieval with an intriguing connection to sparse matrix factorization (see Section 2.2 and Section 3). The assumption of Gaussianity is a natural starting point for understanding the average-case complexity of this problem, and in this learning-theoretic context it is desirable to give algorithms with provable guarantees. Gaussianity is often a standard starting point for developing guarantees for such inverse problems Moitra & Valiant (2010); Netrapalli et al. (2013); Candes et al. (2015); Hardt & Price (2015); Zhong et al. (2017b;a); Li & Yuan (2017); Ge et al. (2018); Li & Liang (2018); Zhong et al. (2019); Chen et al. (2020); Kong et al. (2020); Diakonikolas et al. (2020b).

Our main result is to show that when the private and public data is Gaussian, we can use the synthetic and public vectors to recover a subset of the private vectors.

Theorem 1.1 (Informal, see Theorem B.1)**.** *If there are n_{priv} private vectors and n_{pub} public vectors, each of which is an i.i.d. draw from $\mathcal{N}(0, \mathsf{Id}_d)$, then as long as $d = \Omega(\mathrm{poly}(k_{\mathsf{pub}}, k_{\mathsf{priv}}) \log(n_{\mathsf{pub}} + n_{\mathsf{priv}}))$, there is some $m = o(n_{\mathsf{priv}}^{k_{\mathsf{priv}}})$ such that, given a sample of m random synthetic*

vectors independently generated as above, one can exactly recover $k_{\mathsf{priv}} + 2$ *private vectors in time* $O(d(m^2 + n_{\mathsf{pub}}^2)) + \mathrm{poly}(n_{\mathsf{pub}})$ *with probability 9/10 over the randomness of the private and public vectors and the randomness of the selection vectors.*[2]

We emphasize that we can take $m = o(n_{\mathsf{priv}}^{k_{\mathsf{priv}}})$, meaning we can achieve recovery even with access to a vanishing fraction of all possible combinations of private vectors among the synthetic vectors generated. For instance, when $k_{\mathsf{priv}} = 2$, we show that $m = O(n_{\mathsf{priv}}^{4/3})$ suffices (see Theorem B.1). See Remark B.2 for additional discussion.

Additionally, to ensure we are not working in an uninteresting setting where InstaHide has zero *utility*, we empirically verify that in the setting of Theorem 1.1, one can train on the synthetic vectors and get reasonable test accuracy on the original Gaussian dataset (see Section 4).

Qualitatively, the main takeaway of Theorem 1.1 is that to prove meaningful security guarantees for InstaHide, we must be careful about the properties we posit about the underlying distribution generating the public and private data, even in challenging settings where this data does not possess the nice properties of natural images that have made other attacks possible.

1.1 Connections and Extensions to Phase Retrieval

Our algorithm is based on connections and extensions to the classic problem of *phase retrieval*. At a high level, this can be thought of as the problem of linear regression where the signs of the linear responses are hidden. More formally, this is a setting where we get pairs $(x_1, y_1), ..., (x_N, y_N) \in \mathbb{C}^n \times \mathbb{R}$ for which there exists a vector $w \in \mathbb{C}^n$ satisfying $|\langle w, x_i \rangle| = y_i$ for all $i = 1, ..., N$, and the goal is to recover w. Without distributional assumptions on how $x_1, ..., x_N$ are generated, this problem is NP-hard Yi et al. (2014), and in the last decade, there has been a huge body of work, much of it coming from the machine learning community, on giving algorithms for recovering w under the assumption that $x_1, ..., x_N$ are i.i.d. Gaussian, see e.g. Candes et al. (2013; 2015); Conca et al. (2015); Netrapalli et al. (2013).

To see the connection between InstaHide and phase retrieval, first imagine that InstaHide only works with public vectors (in the notation of Theorem 1.1, $n_{\mathsf{priv}} = k_{\mathsf{priv}} = 0$). Now, consider a synthetic vector $y \in \mathbb{R}^d$ generated by InstaHide, and let the vector $w \in \mathbb{R}^{n_{\mathsf{pub}}}$ be the one specifying the convex combination of public vectors that generated y. The basic observation is that for any feature $i \in [d]$, if $p_i \in \mathbb{R}^{n_{\mathsf{pub}}}$ is the vector consisting of i-th coordinates of all the public vectors, then $|\langle w, x_i \rangle| = y_i$.

In other words, if InstaHide only works with public vectors, then the problem of recovering which public vectors generated a given synthetic vector is formally *equivalent* to phase retrieval. In particular, if the public dataset is Gaussian, then we can leverage the existing algorithms for Gaussian phase retrieval. Huang et al. (2020b) already noted this connection but argued that if InstaHide also uses private vectors, the existing algorithms for phase retrieval fail. Indeed, consider the extreme case where InstaHide only works with *private* vectors (i.e. $n_{\mathsf{pub}} = 0$), so that the only information we have access to is the synthetic vector $(y_1, ..., y_d)$ generated by InstaHide. As noted above in the discussion about private distributions where the features are identically distributed, it is clearly information-theoretically impossible to recover anything about w or the private dataset.

As we will see, the key workaround is to exploit the fact that InstaHide ultimately generates *multiple* synthetic vectors, each of which is defined by a *random* sparse convex combination of public/private vectors. And as we will make formal in Section 2.2, the right algorithmic question to study in this context can be thought of as a *multi-task*, *missing-data* version of phase retrieval (see Problem 2) that we believe to be of independent interest.

Lastly, we remark that in spite of this conceptual connection to phase retrieval, and apart from one component of our algorithm (see Section B.1) which draws upon existing techniques for phase retrieval, the most involved parts of our algorithm and its analysis utilize techniques that are quite different from the existing ones in the phase retrieval literature. We elaborate upon these techniques in Section 3.

[2] See Problem 1 and Remark 2.7 for what exact recovery precisely means in this context.

2 TECHNICAL PRELIMINARIES

Miscellaneous Notation Given a subset T, let $\mathcal{C}_T^k$ denote the set of all subsets of T of size exactly k. Given a vector $v \in \mathbb{R}^n$ and a subset $S \subseteq [n]$, let $[v]_S \in \mathbb{R}^{|S|}$ denote the restriction of v to the coordinates indexed by S.

Definition 2.1. Given a Gaussian distribution $\mathcal{N}(0, \Sigma)$, let $\mathcal{N}^{\mathsf{fold}}(0, \Sigma)$ denote the *folded Gaussian distribution* defined as follows: to sample from $\mathcal{N}^{\mathsf{fold}}(0, \Sigma)$, sample $g \sim \mathcal{N}(0, \Sigma)$ and output $|g|$.

2.1 THE GENERATIVE MODEL

Definition 2.2 (Image matrix notation)**.** Let *image matrix* $\mathbf{X} \in \mathbb{R}^{d \times n}$ be a matrix whose columns consist of vectors $x_1, ..., x_n$ corresponding to n images each with d pixels taking values in $\mathbb{F}$.[3] It will also be convenient to refer to the rows of $\mathbf{X}$ as $p_1, ..., p_d \in \mathbb{R}^n$.

Definition 2.3 (Public/private notation)**.** Let $S \subset \{1, ..., n\}$ be some subset. We will refer to S and $S^c \triangleq \{1, ..., n\} \backslash S$ as the set of *public* and *private* images respectively, and given a vector $w \in \mathbb{R}^n$, we will refer to $\mathrm{supp}(w) \cap S$ and $\mathrm{supp}(w) \cap S^c$ as the *public* and *private coordinates* of w respectively.

Definition 2.4 (Synthetic images)**.** Given sparsity levels $k_{\mathsf{pub}} \leq |S|, k_{\mathsf{priv}} \leq |S^c|$, image matrix $\mathbf{X}$ and a *selection vector* $w \in \mathbb{R}^n$ for which $[w]_S$ and $[w]_{S^c}$ are k_{pub}- and k_{priv}-sparse respectively, the corresponding *synthetic image* is the vector

$$y^{\mathbf{X},w} \triangleq |\mathbf{X}w|, \tag{1}$$

where $|\cdot|$ denotes entrywise absolute value. We say that $\mathbf{X}$ and a sequence of selection vectors $w_1, ..., w_m \in \mathbb{R}^n$ give rise to a *synthetic dataset* consisting of the images $\{y^{\mathbf{X},w_1}, ..., y^{\mathbf{X},w_m}\}$.

Note that instead of the entrywise absolute value of $\mathbf{X}w$, InstaHide in Huang et al. (2020b) randomly flips the sign of every entry of $\mathbf{X}w$, but these two operations are interchangeable in terms of information; it will be slightly more convenient to work with the former.

We will work with the following distributional assumption on the entries of $\mathbf{X}$:

Definition 2.5 (Gaussian images)**.** We say that $\mathbf{X}$ is a random *Gaussian image matrix* if its entries are sampled i.i.d. from $\mathcal{N}(0, 1)$.

We will also work with the following simple notion of "random convex combination" as our model for how the selection vectors $w_1, \ldots, w_m$ are generated:

Definition 2.6 (Distribution over selection vectors)**.** Let $\mathcal{D}$ be the distribution over selection vectors defined as follows. To sample once from $\mathcal{D}$, draw random subset $T_1 \subset S, T_2 \subseteq S^c$ of size k_{pub} and k_{priv} and output the unit vector whose i-th entry is $\frac{1}{\sqrt{k_{\mathsf{pub}}}}$ if $i \in T_1$, $\frac{1}{\sqrt{k_{\mathsf{priv}}}}$ if $i \in T_2$, and zero otherwise.[4]

The main algorithmic question we study is the following:

> *Problem* 1 (Private (exact) image recovery). Let $\mathbf{X} \in \mathbb{R}^{d \times n}$ be a Gaussian image matrix. Given access to the public images $\{x_s\}_{s \in S}$ and the synthetic dataset $\{y^{\mathbf{X},w_1}, \ldots, y^{\mathbf{X},w_m}\}$, where $w_1, ..., w_m \sim \mathcal{D}$ are unknown selection vectors, output a vector $x \in \mathbb{R}^d$ for which there exists private image x_s (where $s \in S^c$) satisfying $|x_i| = |(x_s)_i|$ for all $i \in [d]$.

Remark 2.7. Note that it is information-theoretically impossible to guarantee that $x_i = (x_s)_i$. This is because the distribution over $\mathbf{X}$ and the distribution over matrices given by sampling $\mathbf{X}$ and multiplying every private image by -1 are both Gaussian. And if the selection vectors $w_1, ..., w_m$ generated the synthetic images in the former case, then the selection vectors $w'_1 \ldots, w'_m$, where w'_j is obtained by multiplying the private coordinates of w_j by -1, would generate the exact same synthetic images.

[3] We will often refer to public/private/synthetic feature vectors as *images*, and their coordinates as *pixels*, in keeping with the original applications of InstaHide to image datasets in Huang et al. (2020b)

[4] Note that any such vector does not specify a convex combination, but this choice of normalization is just to make some of the analysis later on somewhat cleaner, and our results would still hold if we chose the vectors in the support of $\mathcal{D}$ to have entries summing to 1.

2.2 MULTI-TASK PHASE RETRIEVAL WITH MISSING DATA

In this section we make formal the discussion in Section 1.1 and situate it in the notation above. First consider a synthetic dataset consisting of a single image $y \triangleq y^{\mathbf{X},w}$, where w is arbitrary and $\mathbf{X}$ is a random Gaussian image. From Eq. (1) we know that

$$|\langle w, p_j \rangle| = y_j \ \forall \, j \in [d].$$

If $S = \{1, ..., n\}$, then the problem of recovering selection vector w from synthetic dataset $\{y\}$ is merely that of recovering w from pairs (p_j, y_j), and this is exactly the problem of *phase retrieval* over Gaussians. More precisely, because w is assumed to be sparse, this is the problem of *sparse phase retrieval* over Gaussians.

If $S \subsetneq \{1, ..., n\}$, then it's clearly impossible to recover the private coordinates of w from $y^{\mathbf{X},w}$ alone. But it may still be possible to recover the public coordinates: formally, we can hope to recover $[w]_S$ given pairs $([p_j]_S, y_j)$, where the p_j's are sampled independently from $\mathcal{N}(0, \mathsf{Id}_n)$. This can be thought of as a *missing-data* version of sparse phase retrieval where some known subset of the coordinates of the inputs, those indexed by S^c, are unobserved.

But recall our ultimate goal is to say something about the *private images*. It turns out that because we actually observe multiple synthetic images, corresponding to multiple vectors w, it becomes possible to recover x_s for some $s \in S^c$ (even in the extreme case where $S = \emptyset$!). This corresponds to the following inverse problem which is *formally equivalent* to Problem 1, but phrased in a self-contained way which may be of independent interest.

Problem 2 (Multi-task phase retrieval with missing data). Let $S \subsetneq [n]$ and $S^c = [n] \backslash S$. Let $\mathbf{X} \in \mathbb{R}^{d \times n}$ be a matrix whose entries are i.i.d. draws from $\mathcal{N}(0, 1)$, with rows denoted by $p_1, ..., p_d$ and columns denoted by $x_1, ..., x_n$. Let $w_1, \ldots, w_m \sim \mathcal{D}$.

For every $j \in [d]$, we get a tuple $\left([p_j]_S, y_j^{(1)}, \ldots, y_j^{(m)}\right)$ satisfying

$$|\langle w_i, p_j \rangle| = y_j^{(i)} \ \forall \, i \in [m], j \in [d].$$

Using just these, output $x \in \mathbb{R}^d$ such that for some $s \in S^c$, $|x_i| = |(x_s)_i|$ for all $i \in [d]$.

3 PROOF OVERVIEW

At a high level, our algorithm has three components:

1. Learn the public coordinates of all the selection vectors $w_1, ..., w_m$ used to generate the synthetic dataset.
2. Recover the $m \times m$ rescaled Gram matrix $\mathbf{M}$ whose (i, j)-th entry is $k \cdot \langle w_i, w_j \rangle$.
3. Use $\mathbf{M}$ and the synthetic dataset to recover a private image.

Step 1 draws upon techniques in Gaussian phase retrieval, while Step 2 follows by leveraging the correspondence between the covariance matrix of a Gaussian and the covariance matrix of its corresponding *folded Gaussian* (see Definition 2.1). Step 3 is the trickiest part and calls for leveraging delicate properties of the distribution $\mathcal{D}$ over selection vectors.

Learning the Public Coordinates of Any Selection Vector We begin by describing how to carry out Step 1 above. First consider the case where $S = \{1, \ldots, n\}$, that is, where every image is public. Recall from the discussion in Section 1.1 and 2.2 that in this case, the question of recovering w from synthetic image $y^{\mathbf{X},w}$ is equivalent to Gaussian phase retrieval. One way to get a reasonable approximation to w is to consider the $n \times n$ matrix

$$\mathbf{N} \triangleq \mathop{\mathbf{E}}_{p,y}[y^2 \cdot (pp^\top - \mathsf{Id})], \qquad p \sim \mathcal{N}(0, \mathsf{Id}_n) \text{ and } y = |\langle w, p \rangle|.$$

It is a standard calculation (see Lemma B.3) to show that $\mathbf{N}$ is a rank-one matrix proportional to $ww^\top$. And as every one of $p_1, \ldots, p_d$ is an independent sample from $\mathcal{N}(0, \mathsf{Id})$, and $y_i^{\mathbf{X},w}$ satisfies

$\langle w, p_i \rangle = y_i^{\mathbf{X},w}$ for every pixel $i \in [d]$, one can approximate $\mathbf{N}$ with the matrix

$$\widehat{\mathbf{N}} \triangleq \frac{1}{d} \sum_{i=1}^{d} \left(y_i^{\mathbf{X},w} \right)^2 \cdot (p_i p_i^\top - \mathsf{Id}).$$

This is the basis for the spectral initialization procedure that is present in many works on Gaussian phase retrieval, see e.g. Candes et al. (2015); Netrapalli et al. (2013). $\widehat{\mathbf{N}}$ will not be a sufficiently good spectral approximation to $\mathbf{N}$ when $d \ll n$, so instead we use a standard post-processing step based on the canonical SDP for sparse PCA (see (2)). Instead of taking the top eigenvector of $\widehat{\mathbf{N}}$, we can take the top eigenvector of the SDP solution and argue that as long as $d = \widetilde{\Omega}(\mathrm{poly}(k_{\mathsf{pub}}) \log n)$, this will be sufficiently close to w that we can exactly recover $\mathrm{supp}(w)$.

Now what happens when $S \subsetneq \{1, \ldots, n\}$? Interestingly, if one simply modifies the definition of $\mathbf{N}$ to be $\mathbf{E}_{p,y}[y^2 \cdot ([p]_S [p]_S^\top - \mathsf{Id})]$ and defines the corresponding empirical analogue $\widehat{\mathbf{N}}$ formed from the pairs $\{([p_i]_S, y_i^{\mathbf{X},w})\}_{i \in [d]}$, one can still argue (see Lemma B.3) that the $\mathbf{N}$ is a rank-1 $|S| \times |S|$ matrix proportional to $[w]_S [w]_S^\top$ and that the top eigenvector of the solution to a suitable SDP formed from $\widehat{\mathbf{N}}$ will be close to w (see Lemma B.4).

Recovering the Gram Matrix via Folded Gaussians As we noted earlier, it is information-theoretically impossible to recover $[w_i]_{S^c}$ for any $i \in [m]$ given only $y^{\mathbf{X},w_i}$ and $[w_i]_S$, but we now show it's possible to recover the inner products $\langle [w_i]_{S^c}, [w_j]_{S^c} \rangle$ for any $i, j \in [m]$. For the remainder of the overview, we will work in the extreme case where $S^c = \{1, ..., n\}$, though it is not hard (see Section B.7) to combine the algorithms we are about to discuss with the algorithm for recovering the public coordinates to handle the case of general S. For brevity, let $k \triangleq k_{\mathsf{priv}}$.

First note that the $m \times d$ matrix whose rows consist of $y^{\mathbf{X},w_1}, ..., y^{\mathbf{X},w_m}$ can be written as

$$\mathbf{Y} \triangleq \begin{pmatrix} |\langle p_1, w_1 \rangle| & \cdots & |\langle p_d, w_1 \rangle| \\ \vdots & \ddots & \vdots \\ |\langle p_1, w_m \rangle| & \cdots & |\langle p_d, w_m \rangle| \end{pmatrix}.$$

Observe that without absolute values, each column would be an independent draw from the m-variate Gaussian $\mathcal{N}(0, \mathbf{M})$, where $\mathbf{M}$ is the Gram matrix defined above. Instead, with the absolute values, each column of $\mathbf{Y}$ is actually an independent draw from the *folded Gaussian* $\mathcal{N}^{\mathsf{fold}}(0, \mathbf{M})$ (Definition 2.1). The key point is that the covariance of $\mathcal{N}^{\mathsf{fold}}(0, \mathbf{M})$ can be directly related to $\mathbf{M}$ (see Corollary B.6), so by estimating the covariance of the folded Gaussian $\mathcal{N}^{\mathsf{fold}}(0, \mathbf{M})$ using the columns of $\mathbf{Y}$, we can obtain a good enough approximation $\widetilde{\mathbf{M}}$ to $\mathbf{M}$ that we can simply round every entry of $\widetilde{\mathbf{M}}$ so that the rounded matrix exactly equals $\mathbf{M}$. Furthermore, we only need to *entrywise* approximate the covariance of $\mathcal{N}^{\mathsf{fold}}(0, \mathbf{M})$ for all this to work, which is why it suffices for d to grow *logarithmically* in m.

Discerning Structure From the Gram Matrix By this point, we have access to the Gram matrix $\mathbf{M}$. Equivalently, we now know for any $i, j \in [m]$ whether $\mathrm{supp}(w_i) \cap \mathrm{supp}(w_j) \neq \emptyset$, that is, for any pair of synthetic images, we can tell whether the set of private images generating one of them overlaps with the set generating the other. Note that $\mathbf{M} = k \cdot \mathbf{W}\mathbf{W}^\top$, where $\mathbf{W}$ is the matrix whose i-th row is w_i, so if we could factorize $\mathbf{M}$ in this way, we would be able to recover which private vectors generated each synthetic vector. Of course, this kind of factorization problem, even if we constrain the factors to be row-sparse like $\mathbf{W}$, has multiple solutions. One reason is that any permutation of the columns of $\mathbf{W}$ would also be a viable solution, but this is not really an issue because the ordering of the private images is not identifiable to begin with.

A more serious issue is that if m is too small, there might be row-sparse factorizations of $\mathbf{M}$ which appear to be valid, but which we could definitively rule out upon sampling more synthetic vectors. For instance, suppose the first $k+1$ selections vectors all satisfied $|\mathrm{supp}(w_j) \cap \mathrm{supp}(w_{j'})| = k-1$. Ignoring the fact that this is highly unlikely, in such a scenario it is impossible to distinguish between the case where the corresponding synthetic images all have the same $k-1$ private images in common, and the case where there is a group $T \subseteq [n]$ of $k+1$ private images such that each of these synthetic images is comprised of a subset of T of size k. But if we then sampled a new selection vector w_{k+2} for which $|\mathrm{supp}(w_j) \cap \mathrm{supp}(w_{k+2})| = 1$ for all $j \in [k+1]$, we could rule out the latter.

This is indicative of a more general issue, namely that one cannot always recover the identity of a collection of subsets (even up to relabeling) if one only knows the sizes of their pairwise intersections!

This leads to the following natural combinatorial question. What families of sets are *uniquely identified* (up to trivial ambiguities) by the sizes of the pairwise intersections? One answer to this question, as we show, is the family of all subsets of $\{1, ..., k+2\}$ of size k (see Lemma B.11). This leads us to the following definition:

Definition 3.1 (Floral Submatrices)**.** A $\binom{k+2}{k} \times \binom{k+2}{k}$ matrix $\mathbf{H}$ is *floral* if the following holds. Fix some lexicographic ordering on $\mathcal{C}^k_{[k+2]}$ and index $\mathbf{H}$ according to this ordering. There is some permutation matrix $\mathbf{\Pi}$ for which the matrix $\mathbf{H}' \triangleq \mathbf{\Pi}^\top \mathbf{H} \mathbf{\Pi}$ satisfies that for every pair of $S, S' \in \mathcal{C}^k_{[k+2]}$, $\mathbf{H}'_{S,S'} = |S \cap S'|$. See Example B.18 in the supplement.

The upshot is that if we can identify a floral submatrix of $\mathbf{M}$, then we know for certain that the subsets of private images picked by those selection vectors comprise all size-k subsets of some subset of $[n]$ of size $k+2$. In summary, using the pairwise intersection size information provided by the Gram matrix $\mathbf{M}$, we can pinpoint collections of selection vectors which share a nontrivial amount of common structure.

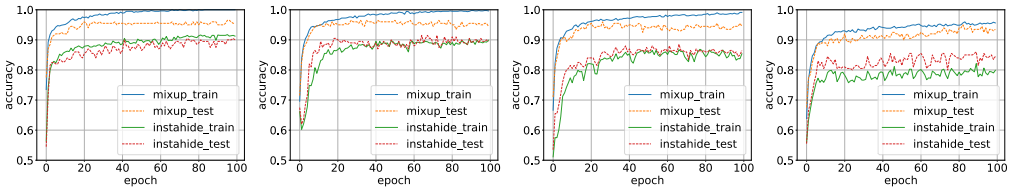
Learning a Private Image With a Floral Submatrix What can we do with this common structure in a floral submatrix? Let $t = \binom{k+2}{k}$. Given that the selection vectors $w_{i_1}, ..., w_{i_t}$ corresponding to the rows of the floral submatrix only involve $k+2$ different private images altogether, and there are $t > k+2$ constraints of the form $|\langle w_{i_j}, p_\ell \rangle| = y_\ell^{\mathbf{X}, w_{i_j}}$ for any pixel $\ell \in [d]$, we can hope that for each pixel, we can uniquely recover the $k+2$ private images from solving this system of equalities, where the unknowns are the values of the $k+2$ private images at that particular pixel. *A priori*, the fact that the number of constraints in this system exceeds the number of unknowns does not immediately guarantee that this system has a unique solution up to multiplying the solution uniformly by -1. Here we exploit the fact that $\mathbf{X}$ is Gaussian to show however that this is the case almost surely (Lemma B.10). Finally, note that this system can be solved in time $\exp(O(k^2))$ by simply enumerating over 2^t sign patterns. We conclude that if we could find a floral submatrix, then we would find not just one, but in fact $k+2$ private images!

Existence of Floral Submatrix, and How to Find It It remains to understand how big m has to be before we can guarantee the existence of a floral submatrix inside the Gram matrix $\mathbf{M}$ with high probability. Obviously if m were big enough that with high probability we see *every possible* synthetic image that could arise from a selection vector w in the support of $\mathcal{D}$, then $\mathbf{M}$ will contain many floral submatrices. One surprising part of our result is that we can ensure the existence of a floral submatrix when m is much smaller. Our proof of this is quite technical, but at a high level it is based on the second moment method (see Lemma B.12).

The final question is: provided a floral submatrix of $\mathbf{M}$ exists, *how do we find it*? Note that naively, we could always brute-force over all $\binom{m}{O(k^2)} \leq n^{O(k^3)}$ principal submatrices with exactly $\binom{k+2}{k}$ rows/columns, and for each such principal submatrix we can check in $\exp(\widetilde{O}(k))$ time whether it is floral.

Surprisingly, we give an algorithm that can identify a floral submatrix of $\mathbf{M}$ in time dominated by the time it takes to write down the entries of the Gram matrix $\mathbf{M}$. Note that an off-the-shelf algorithm for subgraph isomorphism would not suffice as the size of the submatrix in question is $O(k^2)$, and furthermore such an algorithm would need to work for *weighted graphs*. Instead, our approach is to use the constructive nature of the proof in Lemma B.11, that the family of all subsets of $\{1, ..., k+2\}$ of size k is uniquely identified by the sizes of the pairwise intersections. By algorithmizing this proof, we give an efficient procedure for finding a floral submatrix, see Algorithm 3 and Lemma B.17. An important fact we use is that if we restrict our attention to the entries of $\mathbf{M}$ equal to $k-1$ or $k-2$, this corresponds to a graph over the selection vectors which is sparse with high probability.

We defer the formal specification and analysis of our algorithm to the supplement.



(a) $k = 1$ (b) $k = 2$ (c) $k = 3$ (d) $k = 6$

Figure 1: Comparing MixUp and Instahide training on Gaussian dataset with different k.

4 Experiments

We describe an experiment demonstrating the *utility* of InstaHide for Gaussian images and comparing to the utility of another data augmentation scheme, MixUp Zhang et al. (2018). We also informally report on our implementation of LearnPublic and its empirical efficacy.

4.1 Choice of Architecture and Parameters

As our empirical results are purely for proof-of-concept, we work with a fairly basic neural network architecture. We use a 4-layer neural network as a binary classifier,

$$y = \arg\max(\text{softmax}(W_4\sigma(W_3\sigma(W_2(\sigma(W_1x + b_1)) + b2) + b3) + b4)),$$

where $x \in \mathbb{R}^{10}$, $W_1 \in \mathbb{R}^{100\times 10}$, $W_2 \in \mathbb{R}^{100\times 100}$, $W_3 \in \mathbb{R}^{100\times 100}$, $W_4 \in \mathbb{R}^{2\times 100}$, $b_1 \in \mathbb{R}^{100}$, $b_2 \in \mathbb{R}^{100}$, $b_3 \in \mathbb{R}^{100}$, $b_4 \in \mathbb{R}^{100}$. We initialize the entries of each W_l and b_l to be i.i.d. draws from $\mathcal{N}(u_l, 1)$, where u_l is sampled from $\mathcal{N}(0, \alpha)$ at the outset. We train the neural network for 100 epochs with cross-entropy loss and SGD optimizer with a learning rate of 0.01. We do not need to distinguish between public and private images in our experiments, so let $k_{\mathsf{priv}} = 0$ and $k_{\mathsf{pub}} = k$ for $k \in \{1, 2, 3, 6\}$, and for each choice of k, we use random k-sparse selection vectors whose nonzero entries equal $1/k$. In all of our experiments, we separate our original image data (before generating synthetic data) into two categories: 80% training data and 20% test data. We train on synthetic images generated by MixUp or InstaHide using the training data, and measure the "test accuracy" on the training data and the test data separately. We provide more choices of k in Appendix C.

4.2 Gaussian Data

Settings We considered binary classification on Gaussian data. We generated random images $x_1, \ldots, x_{1000} \in \mathbb{R}^{10}$ from $\mathcal{N}(0, \mathsf{Id})$ and a random vector $v \in \mathbb{R}^{10} \sim \mathcal{N}(0, \mathsf{Id})$. We then ranked all the Gaussian images based on $\sum_i |x_i v_i|$ and labeled the largest half as '1' and the rest as '0'. The point of choosing this labeling function is that it would assign the same label to any x, x' which agree entrywise in magnitudes. Given a synthetic image generated via MixUp or InstaHide using selection vector w, we assigned it the label which is the convex combination of the one-hot encodings of the labels of the original images indexed by w.

Results We compare training and test loss over epochs when training on a synthetic dataset generated by either MixUp or Instahide, as shown in Figure 1. We use the convention in this paper of defining synthetic images under InstaHide to have all nonnegative entries (rather than imposing random sign flips), though we explore in the supplement how random sign flips can affect learnability. Compared to training on MixUp, InstaHide results on lower model performance (accuracy). As we expected, when we increase the k, both MixUp and InstaHide suffer from accuracy loss. Instahide dropped by $\sim 10\%$ accuracy when $k = 6$ compared to classical training $k = 1$, while MixUp dropped by $\sim 5\%$.

4.3 IMPLEMENTATION OF LEARNPUBLIC

We implemented LEARNPUBLIC for $k_{\mathsf{priv}} = 2$ and $n \in \{2000, 5000, 7500, 10000\}$. For $k_{\mathsf{pub}} = 2, 4, 6$, we respectively chose $d = 1000, 1800, 2400$. In particular, our choice of d is meant to work essentially for any choice of n (modulo the logarithmic dependence which does not noticeably manifest in this regime). One heuristic modification that we made to LEARNPUBLIC was to use a *diagonal thresholding* approach from Cai et al. (2016) in place of solving the SDP in (2): namely for every $j \in [n]$ we computed the quantity $\frac{1}{d}\sum_{i=1}^{d} y_i^2 \cdot (x_i)_j^2$, zeroed out all but the principal submatrix of $\widetilde{\mathbf{M}}$ indexed by the top 25 such j, and computed the top eigenvector of the resulting matrix. For each parameter setting we found that, as expected, we were able to recover an average of at least 90% of the support. As this experiment was primarily to demonstrate that d can be much less than n, we did not explore further optimizations to the algorithm.

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