Provably Personalized and Robust Federated Learning

Anonymous authors
Paper under double-blind review

Abstract

Clustering clients with similar objectives and learning a model per cluster is an intuitive and interpretable approach to personalization in federated learning. However, doing so with provable and optimal guarantees has remained an open challenge. In this work, we formalize personalized federated learning as a stochastic optimization problem where the stochastic gradients on a client may correspond to one of K distributions. In such a setting, we show that using i) a simple thresholding-based clustering algorithm, and ii) local client gradients obtains optimal convergence guarantees. In fact, our rates asymptotically match those obtained if we knew the true underlying clustering of the clients. Furthermore, our algorithms are provably robust in the Byzantine setting where some fraction of the gradients are corrupted.

1 Introduction

We consider the federated learning setting in which there are N clients with individual loss functions $\{f_i\}_{i\in[N]}$ who seek to jointly train a model or multiple models. The defacto algorithm for problems in this setting is FedAvg (McMahon et al., 2017) which has an objective of the form

$$x^*_{\text{FedAvg}} = \arg\min_{x\in\mathcal{X}} \frac{1}{N} \sum_{i\in[N]} f_i(x). \tag{1}$$

From (1), we see that FedAvg optimizes the average of the client losses. In many real-world cases however, clients' data distributions are heterogeneous, making such an approach unsuitable since the global optimum (1) may be very far from the optima of individual clients. Rather, we want algorithms which identify clusters of the clients that have relevant data for each other and that only perform training within each cluster. However, this is a challenging exercise since 1) it is unclear what it means for data distributions of two clients to be useful for each other, or 2) how to automatically identify such subsets without expensive multiple retraining (Zamir et al., 2018). In this work we propose algorithms which iteratively and simultaneously 1) identify K clusters amongst the clients by clustering their gradients and 2) optimize the clients' losses within each cluster.

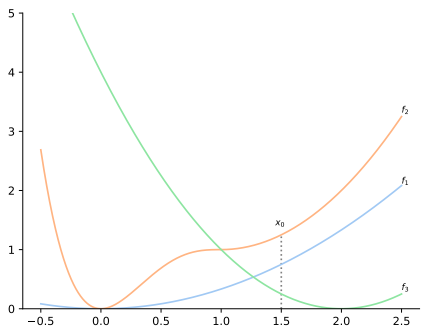
1.1 Clustering Methods for Personalized Federated Learning

As stated, our task at hand in this work is to simultaneously learn the clustering structure amongst clients and minimize their losses. There are two main criteria that must hold in order for this clustering problem to be solvable.

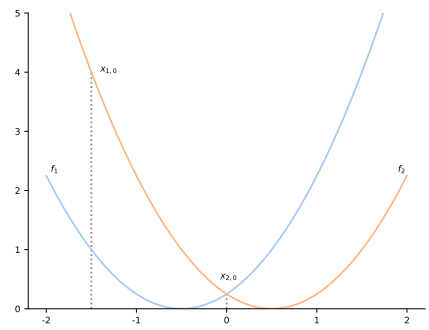
1. **Intra-cluster Similarity**: Points within a cluster must be sufficiently close.

2. **Inter-cluster Separation**: Points from different clusters must be sufficiently far apart.

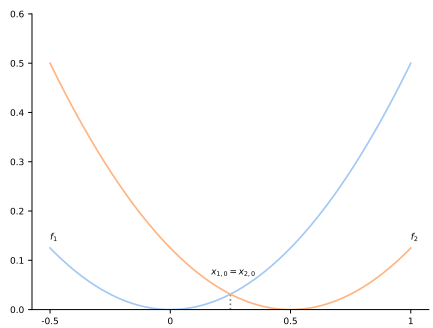
Since our approach in this work is to cluster clients by clustering their gradients, these criteria are difficult to satisfy. We can see why by examining a naive algorithm, Myopic-Clustering (Algorithm 1).



(a) Myopic-Clustering ($\eta = 0.5$)



(b) IFCA/HypCluster



(c) Clustered Federated Learning

Figure 1: Counter-examples for existing clustering-based personalized federated learning algorithms.

Myopic-Clustering (Algorithm 1). This algorithm initially sets K models and iteratively updates them with the gradients of the closest clients. However, Myopic-Clustering has two issues: 1) If it makes a clustering mistake at one step, the models could be updated in directions away from their optima, or the gradients of clients in the same cluster could drift apart, thus violating the intra-cluster similarity criterion. Furthermore, these errors can compound over rounds. 2) Even if Myopic-Clustering clusters clients perfectly at each step, the clients' gradients will approach zero as the models converge to their optima. This means that clients from different clusters will appear to belong to the same cluster as the algorithm converges, the inter-cluster separation criterion will no longer be satisfied, and all clients will collapse into a single cluster. The following example (Figure 1a) demonstrates these failure modes of Myopic-Clustering.

Let $N = 3$ and $K = 2$, with client loss functions

$$
\begin{aligned}
f_1(x) &= \frac{1}{6\eta}x^2 \\
f_2(x) &= \begin{cases} 4(x-1)^3 + 3(x-1)^4 + 1 & x < 1 \\ \frac{1}{2\eta}(x-1)^2 + 1 & x \geq 1, \end{cases} \\
f_3(x) &= \frac{1}{2\eta}(x-2)^2,
\end{aligned}
$$

where η is the learning rate of the algorithm. With this structure, clients $\{1, 2\}$ share the same global minimum and belong to the same cluster, and client $\{3\}$ belongs to its own cluster. Suppose Myopic-Clustering is initialized at $x_0 = 1.5$. At step 1, it will compute the client gradients as $1/2\eta$, $1/2\eta$, and $-1/2\eta$ respectively, correctly clustering clients $\{1, 2\}$ together and client $\{3\}$ alone. After updates, the clients' parameters will next be $x_{1,1} = 1, x_{2,1} = 1, x_{3,1} = 2$ respectively. At this point, clients $\{2, 3\}$ will be

Algorithm 1 Myopic-Clustering

Input Learning rate: η. Initial parameters: $\{x_{1,0} = ... = x_{N,0} = x_0\}$.

for round $t \in [T]$ **do**
  for client i in [N] **do**
    Client i sends $g_i(x_{i,t-1})$ to server.
    Server clusters $\{g_i(x_{i,t-1})\}_{i\in[N]}$, generating cluster centers $\{v_{k,t}\}_{k\in[K]}$.
    Server sends $v_{k_i,t}$ to client i, where k_i denotes the cluster to which client i is assigned.
    Client i computes update: $x_{i,t} = x_{i,t-1} - \eta v_{k_i,t}$.

Output: Personalized parameters: $\{x_{1,T}, ..., x_{N,T}\}$.

incorrectly clustered together since their gradients will both be 0, while client $\{1\}$ will be clustered alone. As the algorithm proceeds, clients $\{2, 3\}$ will always be clustered together and will remain at $x = 1$ and $x = 2$ respectively, while client $\{1\}$ will converge to its optimum at $x = 0$. Consequently, two undesirable things happen: 1) Client $\{2\}$ gets stuck at the saddle point at $x = 1$ which occurred when it was incorrectly clustered with client $\{3\}$ at $t = 1$ and subsequently did not recover. 2) All gradients converge to 0, so at the end of the algorithm all clients are clustered together.

Federated-Clustering (Algorithm 2), our proposed algorithm, checks at every step the gradient values of all N clients at the current parameters of all K clusters. This verification process avoids the type of errors made by Myopic-Clustering. For instance, on the example above, at $t = 1$ when Myopic-Clustering makes its terminal mistake, Federated-Clustering computes the gradients of all clients at client $\{1\}$'s current parameters – $g_1(1) = 1/3\eta$, $g_2(1) = 0$, and $g_3(1) = -1/\eta$. Therefore it correctly clusters $\{1, 2\}$ together at this point, and client $\{2\}$'s parameters update beyond the saddle-point and converge to the global minimum at $x = 0$.

To further motivate the design choice for our algorithms, we now discuss three clustering-based algorithms in the literature on personalized federated learning. In particular, we generate counter-examples on which they fail and show how our algorithm avoids such pitfalls.

The first two algorithms IFCA (Ghosh et al., 2021) and HypCluster (Mansour et al., 2021) are closely related. They both cluster loss function values rather than gradients, and like our algorithm they avoid the myopic nature of Myopic-Clustering by, at each step, computing all client losses at all current cluster parameters to determine the clustering. However, as we show in the next example (Figure 1b), they are brittle and sensitive to initialization.

IFCA (Ghosh et al., 2021). Let $N = 2$, $K = 2$ with loss functions

$$f_1(x) = (x + 0.5)^2$$
$$f_2(x) = (x - 0.5)^2,$$

and initialize clusters 1 and 2 at $x_{1,0} = -1.5$ and $x_{2,0} = 0$ respectively. Given this setup, both clients initially select cluster 2 since their losses at $x_{2,0}$ are smaller than at $x_{1,0}$.

Option I: At $x_{2,0} = 0$, the client gradients will average to 0. Consequently the models will remain stuck at their initializations, and both clients will be incorrectly assigned to cluster 2.

Option II: Both clients individually run τ steps of gradient descent starting at their selected model $x_{2,0}$ (i.e. perform the `LocalUpdate` function in line 18 of IFCA). Since the clients' individually updated models will be symmetric around 0 after this process, the server will compute cluster 2's model update in line 15 of IFCA as: $x_{2,1} \leftarrow 0 = x_{2,0}$. Consequently, the outcome is the same as in Option I: the models never update and both clients are incorrectly assigned to cluster 2.

HypCluster (Mansour et al., 2021). This algorithm is a centralized version of Option II of IFCA. The server alternately clusters clients by loss function value and runs stochastic gradient descent per-cluster using the clients' data. It performs as Option II of IFCA on the example above.

By clustering clients based on gradient instead of loss value, our algorithm Federated-Clustering initially computes the clients' gradients of $+1$ and -1 respectively at $x_{2,0} = 0$, and given the continued separation of their gradients around 0 as the algorithm converges, correctly identifies that they belong to different clusters.

Finally, we discuss Clustered Federated Learning, the algorithm proposed in Sattler et al. (2019), which runs the risk of clustering too finely, as in the next example (Figure 1c).

Clustered Federated Learning (Sattler et al., 2019). Clustered Federated Learning operates by recursively bi-partitioning the set of clients based on the clients' gradient values at the FedAvg optimum. Consider the following example. Let $N = 3$ and $K = 2$ with client gradients

$$\begin{aligned} g_1(x) &= x \\ g_2(x) &= x - 1/2 \\ g_3(x) &= \begin{cases} x & \text{w.p. } 1/2 \\ x - 1 & \text{w.p. } 1/2. \end{cases} \end{aligned}$$

Therefore the correct clustering here is $\{1\}$ and $\{2, 3\}$. The FedAvg optimum is $x^*_{\text{FedAvg}} = 1/4$, at which the clients' gradient values are $g_1(1/4) = 1/4$, $g_2(1/4) = -1/4$ and $g_3 = 1/4$ w.p. $1/2$. Based on this computation, Clustered Federated Learning partitions the client set into $\{1, 2\}$ and $\{3\}$ w.p. $1/2$ and then proceeds to run the algorithm separately on each sub-cluster. Therefore, the algorithm never corrects its initial error in separating clients $\{2\}$ and $\{3\}$. Our algorithm, Federated-Clustering, avoids this type of mistake by considering all clients during every clustering.

1.2 Our Contributions

To address the shortcomings in current approaches, we propose two personalized federated learning algorithms, which simultaneously cluster similar clients and optimize their loss objectives in a personalized manner. In each round of the procedure, we examine the client gradients to identify the cluster structure as well as to update the model parameters. Importantly, ours is the first method with theoretical guarantees for general non-convex loss functions, and not just restrictive toy settings. We show that our method enjoys both nearly optimal convergence, while also being robust to some malicious (Byzantine) client updates. This is again the first theoretical proof of the utility of personalization for Byzantine robustness. Specifically in this work,

- We show that existing or naive clustering methods for personalized learning, with stronger assumptions than ours, can fail in simple settings (Figure 1).
- We design a robust clustering subroutine (Algorithm 3) whose performance improves with the separation between the cluster means and the number of data points being clustered. We prove nearly matching lower bounds showing its near-optimality (Theorem 2), and we show that the error due to malicious clients scales smoothly with the fraction of such clients (Theorem 1).
- We propose two personalized learning algorithms (Algorithm 2 and Algorithm 4) which converge at the optimal $\mathcal{O}(1/\sqrt{n_i T})$ rate in T for stochastic gradient descent for smooth non-convex functions and linearly scale with n_i, the number of clients in client i's cluster.
- We empirically verify our theoretical assumptions and demonstrate experimentally that our learning algorithms benefit from collaboration, scale with the number of collaborators, are competitive with SOTA personalized federated learning algorithms, and are not sensitive to the model's initial weights (Section 4).

2 Related Work

Personalization via Clustering. Personalization in federated learning has recently enjoyed tremendous attention (see Tan et al. (2022); Kulkarni et al. (2020) for surveys). We focus on gradient-based clustering

methods for personalized federated learning. Several recent works propose and analyze clustering methods. Sattler et al. (2019) alternately train a global model with FedAvg and partition clients into smaller clusters based on the global model's performance on their local data. Mansour et al. (2021) and Ghosh et al. (2021) instead train personalized models from the start (as we do) without maintaining a global model. They iteratively update K models and, using empirical risk minimization, assign each of N clients one of the models at every step. Mansour et al. (2021) and Ghosh et al. (2021) both provide generalization guarantees for the assigned models and Ghosh et al. (2021) additionally provide convergence guarantees on training data. The spirit of our work is similar to these works with some important differences: 1) Ghosh et al. (2021) assume strong convexity of the clients' loss objectives (along with strong initializaton conditions), while our guarantees hold for all smooth (convex and non-convex) functions; 2) Ghosh et al. (2021); Mansour et al. (2021) cluster clients based on empirical risk minimization whereas we cluster clients based on similarity of their gradients.

Multitask learning. Our work is closely related to multitask learning, which simultaneously trains separate models for different-but-related tasks. Smith et al. (2017) and Li et al. (2021) both cast personalized federated learning as a multitask learning problem. In the first, the per-task models jointly minimize an objective that encodes relationships between the tasks. In the second, models are trained locally (for personalization) but regularized to be close to an optimal global model (for task-relatedness). These settings are quite similar to our setting. However, we use assumptions on gradient (dis)similarity across the domain space to encode relationships between tasks, and we do not maintain a global model.

Robustness. Our methods are provably robust in the Byzantine (Lamport et al., 2019; Blanchard et al., 2017) setting, where clients can make arbitrary updates to their gradients to corrupt the training process. Karimireddy et al. (2021) create an optimization algorithm for such settings with provably robust convergence guarantees. Our clustering algorithm builds upon their procedure for minimizing the effect of malicious gradients. The personalization algorithm in Li et al. (2021) also has robustness properties, but they are only demonstrated empirically and analyzed on toy examples.

3 Federated-Clustering

In this section, we propose an algorithm, Federated-Clustering (Algorithm 2), which avoids the shortcomings of the algorithms discussed in Section 1.1. At a high level, it works as follows. Each client i maintains a personalized model which, at every step, it broadcasts to the other clients $j \neq i$. Then each client j computes its gradient on clients i's model parameters and sends the gradient to client i. Finally, client i runs a clustering procedure on the received gradients, determines which other clients have gradients closest to its own at its current model, and updates its current model by averaging the gradients of these similar clients. By the end of the algorithm, ideally each client has a model which has been trained only on the data of similar clients.

The core of our algorithm is a clustering procedure, Threshold-Clustering (Algorithm 3), which identifies clients with similar gradients at each step. This clustering procedure, which we discuss in the next section, has two important properties: it is robust and its error rate is near-optimal.

Notation. For an arbitrary integer N, we let $[N] = \{1, ..., N\}$. We take $a \gtrsim b$ to mean there is a sufficiently large constant c such that $a \geq cb$, $a \lesssim b$ to mean there is a sufficiently small constant c such that $a \leq cb$, and $a \approx b$ to mean there is a constant c such that $a = cb$. We write $i \sim j$ if clients i and j belong to the same cluster and $i \not\sim j$ if they belong to different clusters. Finally, n_i denotes the number of clients in client i's cluster, $\delta_i = {}^{n_i}/_{N}$ denotes the fraction of clients in client i's cluster, and β_i denotes the fraction of clients that are malicious from client i's perspective.

3.1 Analysis of Clustering Procedure

Given the task of clustering N points into K clusters, at step l our clustering procedure has current estimates of the K cluster-centers, $v_{1,l}, ..., v_{K,l}$. To update each estimate $v_{k,l+1} \leftarrow v_{k,l}$, it constructs a ball of radius $\tau_{k,l}$ around $v_{k,l}$. If a point falls inside the ball, the point retains its value; if it falls outside the ball, its value is mapped to the current cluster-center estimate. The values of all the points are then averaged to set $v_{k,l+1}$

(update rule (4)). The advantage of this rule is that it is very conservative. If our algorithm is confident that its current cluster-center estimate is close to the true cluster mean (i.e. there are many points nearby), it will confidently improve its estimate by taking a large step in the right direction (where the step size and direction are determined mainly by the nearby points). If our algorithm is not confident about being close to the cluster mean, it will tentatively improve its estimate by taking a small step in the right direction (where the step size and direction are small since the majority of points are far away and thus do not change the current estimate).

To analyze the theoretical properties of this procedure, we look at a natural setting in which clients within the same cluster have i.i.d. data (for analysis of our federated learning algorithm, we will relax this strong notion of intra-cluster similarity). In particular, in our setting there are N points $\{z_1, ..., z_N\}$ which can be partitioned into K clusters within which points are i.i.d. We assume the following.

- **Assumption 1** (Intra-cluster Similarity): For all $i \sim j$,

$$z_i \overset{i.i.d.}{\sim} z_j.$$

- **Assumption 2** (Inter-cluster Separation): For all $i \not\sim j$,

$$\|\mathbb{E}z_i - \mathbb{E}z_j\|^2 \geq \Delta^2.$$

- **Assumption 3** (Bounded Variance): For all z_i,

$$\mathbb{E}\|z_i - \mathbb{E}z_i\|^2 \leq \sigma^2.$$

Theorem 1. *Suppose there N points $\{z_i\}_{i\in[N]}$ for which Assumptions [1-3] hold with inter-cluster separation parameter $\Delta \gtrsim \sigma/\delta_i$. Running Algorithm 3 for*

$$l \gtrsim \max\left\{1, \max_{i\in[N]} \frac{\log(\sigma/\Delta)}{\log(1 - \delta_i/2)}\right\}$$

steps with fraction of malicious clients $\beta_i \lesssim \delta_i$ and thresholding radius $\tau \approx \sqrt{\delta_i \sigma \Delta}$ guarantees that

$$\mathbb{E}\|v_{k_i,l} - \mathbb{E}z_i\|^2 \lesssim \frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \beta_i \sigma \Delta. \tag{2}$$

Proof. See A.1. □

Supposing $\beta_i = 0$, if we knew the identity of all points within z_i's cluster, we would simply take their mean as the cluster-center estimate, incurring estimation error of σ^2/n_i (i.e. the sample-mean's variance). Since we don't know the identity of points within clusters, the additional factor of σ^3/Δ in (2) is the price we pay to learn the clusters. This additional term scales with the difficulty of the clustering problem. If true clusters are well-separated and/or the variance of the points within each cluster is small (i.e. Δ is large, σ^2 is small), then the clustering problem is easier and our bound is tighter. If clusters are less-well-separated and/or the variance of the points within each cluster is large, accurate clustering is more difficult and our bound weakens.

Setting τ. To achieve the rate in (2), we set $\tau \approx \sqrt{\sigma\Delta}$, which is the geometric mean of the standard deviation, σ, of points belonging to the same cluster and the distance, Δ, to a different cluster. The intuition for this choice is that we want the radius for each cluster to be at least as large as the standard deviation of the points belonging to that cluster in order to capture in-cluster points. The radius could be significantly larger than the standard deviation if Δ is large, thus capturing many non-cluster points as well. However, the conservative nature of our update rule (4) offsets this risk. By only updating the center with a step-size proportional to the *fraction* of points inside the ball, it limits the influence of any mistakenly captured points.

Threshold-Clustering has two important properties which we now discuss: it is Byzantine robust and has a near-optimal error rate.

3.1.1 Robustness

We construct the following definition to characterize the robustness of Algorithm 3.

Definition 1 (Robustness)**.** *An algorithm $\mathcal{A}$ is* robust *if the error introduced by bad clients can be bounded i.e. malicious clients do not have an arbitrarily large effect on the convergence. Specifically, for a specific objective, let $\mathcal{E}_1$ be the base error of $\mathcal{A}$ with no bad clients, let β be the fraction of bad clients, and let $\mathcal{E}_2$ be some bounded error added by the bad points. Then $\mathcal{A}$ is* robust *if*

$$Err(\mathcal{A}) \leq \mathcal{E}_1 + \beta\mathcal{E}_2.$$

Threat model. Our clustering procedure first estimates the centers of the K clusters from the N points and constructs a ball of radius τ_k around the estimated center of each cluster k. If a point falls inside the ball, the point retains its value; if it falls outside the ball, its value is mapped to the current cluster-center estimate. Following the update rule (4), the values of all the points are averaged to update the cluster-center estimate. Therefore, a bad point that wants to distort the estimate of the k'th cluster's center has the most influence by placing itself just within the boundary of the ball around that cluster-center, i.e. at τ_k-distance from the cluster-center.

From (2) we see that the base squared-error of Algorithm 3 in estimating z_i's cluster-center is $\lesssim \sigma^2/n_i + \sigma^3/\Delta$, and that the bad points introduce extra squared-error of order $\sigma\Delta$. Given our threat model, this is exactly expected. The radius around z_i's cluster-center is order $\sqrt{\sigma\Delta}$. Therefore, bad points placing themselves at the edge of the ball around z_i's cluster-center estimate will be able to distort the estimate by order $\sigma\Delta$. The scaling of this extra error by β_i satisfies our definition of robustness, and the error smoothly vanishes as $\beta_i \to 0$.

3.1.2 Near-Optimality

The next result shows that the upper bound (2) on the estimation error of Algorithm 3 nearly matches the best-achievable lower bound. In particular, it is tight within a factor of σ/Δ.

Theorem 2 (Near-optimality of **Threshold-Clustering**)**.** *For any algorithm $\mathcal{A}$, there exists a mixture of distributions $\mathcal{D}_1 = (\mu_1, \sigma^2)$ and $\mathcal{D}_2 = (\mu_2, \sigma^2)$ with $\|\mu_1 - \mu_2\| \geq \Delta$ such that the estimator $\hat{\mu}_1$ produced by $\mathcal{A}$ has error*

$$\mathbb{E}\|\hat{\mu}_1 - \mu_1\|^2 \geq \Omega\left(\frac{\sigma^4}{\Delta^2} + \frac{\sigma^2}{n_i}\right).$$

Proof. See A.2. □

3.2 Analysis of Federated-Clustering

We now have the tools to analyze our personalized federated learning algorithm, Federated-Clustering (Algorithm 2). First, we establish the necessary assumptions for clustering to be possible in this setting: intra-cluster similarity, inter-cluster separation, bounded variance of stochastic gradients, and smoothness of loss objectives.

- **Assumption 4** (Intra-cluster Similarity): For all x, $i \sim j$, and some constant $A \geq 0$,

  $$\|\nabla f_i(x) - \nabla \bar{f}_i(x)\|^2 \leq A^2 \|\nabla \bar{f}_i(x)\|^2,$$

  where $\bar{f}_i(x) \triangleq \frac{1}{n_i}\sum_{j\sim i} f_j(x)$.

- **Assumption 5** (Inter-cluster Separation): For all x, $i \not\sim j$, and some constant $D \geq 0$,

  $$\|\nabla f_i(x) - \nabla f_j(x)\|^2 \geq \Delta^2 - D^2\|\nabla f_i(x)\|^2.$$

We note that Assumptions 4 and 5 are quite permissive. Rather than requiring that clients in the same cluster have close gradients at all points, Assumption 4 merely requires that they have the same *optimum*,

Algorithm 2 Federated-Clustering

Input Learning rate: η. Initial parameters for each client: $\{x_{1,0}, ..., x_{N,0}\}$.

for client $i \in [N]$ **do**
  Send $x_{i,0}$ to all clients $j \neq i$.
for round $t \in [T]$ **do**
  for client i in [N] **do**
    Compute $g_i(x_{j,t-1})$ and send to client j for all $j \neq i \in [N]$.
    Compute $v_{i,t} \leftarrow$ `Threshold-Clustering`$(\{g_j(x_{i,t-1})\}_{j\in[N]}; 1 \text{ cluster}; g_i(x_{i,t-1}))$.
    Update parameter: $x_{i,t} = x_{i,t-1} - \eta v_{i,t}$.
    Send $x_{i,t}$ to all clients $j \neq i$.
Output: Personalized parameters: $\{x_{1,T}, ..., x_{N,T}\}$.

but allows their gradients to diverge elsewhere in domain space. This assumption is closely related to the strong growth condition (equation (1) in Vaswani et al. (2019)), which is shown to be a very useful notion in practical deep learning. We also experimentally verify that it holds in our setting (Figure 2). Similarly, rather than requiring that clients in different clusters have separated gradients at all points, Assumption 5 merely requires that their gradients be sufficiently far apart at their respective *optima*.

- **Assumption 6** (Bounded Variance of Stochastic Gradients): For all x,

$$\mathbb{E}\|g_i(x) - \nabla f_i(x)\|^2 \leq \sigma^2,$$

  where $\mathbb{E}[g_i(x)|x] = \nabla f_i(x)$ and each client i's stochastic gradients $g_i(x)$ are independent.

- **Assumption 7** (Smoothness of Loss Functions): For any x, y,

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|.$$

Theorem 3. *Let* Assumptions [4-7] *hold with inter-cluster separation parameter* $\Delta \gtrsim {}^{\max(1,A^4)\max(1,D^2)\sigma}/_{\delta_i}$. *Under these conditions, suppose we run Algorithm 2 for* T *rounds with learning rate* $\eta \leq {}^{1}/_{L}$, *fraction of malicious clients* $\beta_i \lesssim \delta_i$, *and batch size* $|B| \gtrsim \sqrt{\max(1, A^2)({}^{\sigma^2}/_{n_i} + {}^{\sigma^3}/_{\Delta} + \beta_i\sigma\Delta)m_i}$, *where* m_i *is the size of client* i*'s training dataset. If, in each round* $t \in [T]$, *we cluster with radius* $\tau \approx \sqrt{\delta_i\sigma\Delta}$ *for*

$$l \geq \max\left\{1, \max_{i\in N]} \frac{\log({}^{\sigma}/_{\sqrt{|B|}\Delta})}{\log(1 - {}^{\delta_i}/_{2})}\right\}$$

steps, then

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2 \lesssim \sqrt{\frac{\max(1, A^2)({}^{\sigma^2}/_{n_i} + {}^{\sigma^3}/_{\Delta} + \beta_i\sigma\Delta)}{T}}. \tag{3}$$

Proof. See A.3. □

We note a few things. 1) The rate in (3) is the optimal rate in T for stochastic gradient descent on non-convex functions (Arjevani et al., 2019). 2) The dependence on $\sqrt{\sigma^2/n_i}$ is intuitive, since convergence error should increase as the variance of points in the cluster increases and decrease as the number of points in the cluster increases. 3) The dependence on $\sqrt{\beta_i\sigma\Delta}$ is also expected. We choose a radius $\tau \approx \sqrt{\sigma\Delta}$ for clustering. Given our threat model, the most adversarial behavior of the bad clients from client i's perspective is to place themselves at the edge of the ball surrounding the estimated location of i's gradient, thus adding error of order $\sqrt{\beta_i\sigma\Delta}$. When there are no malicious clients, this extra error vanishes.

Algorithm 3 Threshold-Clustering

Input Points to be clustered: $\{z_1, ..., z_N\}$. Number of clusters: K. Cluster-center initializations: $\{v_{1,0}, ..., v_{K,0}\}$.

1: **for** round $l \in [M]$ **do**
2: **for** cluster k in [K] **do**
3: Set radius $\tau_{k,l}$.
4: Update cluster-center estimate:

$$v_{k,l} = \frac{1}{N}\sum_{i=1}^{N}\Big(z_i \mathbb{1}(\|z_i - v_{k,l-1}\| \leq \tau_{k,l}) + v_{k,l-1}\mathbb{1}(\|z_i - v_{k,l-1}\| > \tau_{k,l})\Big). \tag{4}$$

5: **Output:** Cluster-center estimates $\{v_1 = v_{1,M}, ..., v_K = v_{K,M}\}$.

3.3 Improving Communication Efficiency with Momentum

Federated-Clustering is inefficient, requiring N^2 rounds of communication between clients at each step (each client computes their gradient at every other client's parameter). We pay this price to ensure that the intra-cluster similarity and inter-cluster separation criteria are satisfied throughout the algorithm, resulting in higher likelihood of a correct clustering. Recall on the other hand that Myopic-Clustering, while communication efficient (N rounds per step), may not recover from past clustering mistakes. With our next algorithm, Momentum-Clustering (Algorithm 4), we propose a practical middle-ground by clustering momentums instead of gradients. Since momentums change much more slowly from round-to-round than gradients, a past clustering mistake will not have as much of a deleterious impact on future correct clustering and convergence as when clustering gradients.

In Algorithm 4, at each step each client computes their momentum and sends it to the server. The server clusters the N momentums, computes an update per-cluster, and sends the update to the clients in each cluster. Therefore, communication is limited to N rounds per step.

3.3.1 Analysis of Momentum-Clustering

The analysis of the momentum based method requires adapting the intra-cluster similarity and inter-cluster separation assumptions from before.

- **Assumption 8** (Intra-cluster Similarity): For all $i \sim j$ and $t \in [T]$,

$$m_{i,t} \overset{i.i.d.}{\sim} m_{j,t},$$

  where $m_{i,t}$ is defined as in (6).

- **Assumption 9** (Inter-cluster Separation): For all $i \not\sim j$ and $t \in [T]$,

$$\|\mathbb{E}m_{i,t} - \mathbb{E}m_{j,t}\|^2 \geq \Delta^2.$$

Note that the intra-cluster similarity assumption in this momentum setting is stronger than in the gradient setting (Assumption 4): namely we require that the momentum of clients in the same cluster be i.i.d. at all points. This stronger assumption is the price we pay for a simpler and more practical algorithm. Finally, due to the fact that momentums are low-variance counterparts of gradients (Lemma 13), we can eliminate constraints on the batch size and still achieve the same rate.

Theorem 4. *Let* Assumptions [6-9] *hold with inter-cluster separation parameter* $\Delta \gtrsim \sigma/\delta_i$. *Under these conditions, suppose we run Algorithm 4 for* T *rounds with learning rate* $\eta \lesssim \min\left\{\frac{1}{L}, \sqrt{\frac{\mathbb{E}(f_i(x_{i,0}) - f_i^*)}{LT(\sigma^2/n_i + \sigma^3/\Delta)}}\right\}$ *(*f_i^* *is the global minimum of* f_i*), fraction of bad clients* $\beta_i \lesssim \delta_i$*, and momentum parameter* $\alpha \gtrsim L\eta$*. If, in each*

Algorithm 4 Momentum-Clustering

Input Learning rate: η. Initial parameters: $\{x_{1,0} = ... = x_{N,0}\}$.

1: **for** round $t \in [T]$ **do**
2: **for** client i in [N] **do**
3: Client i sends

$$m_{i,t} = \alpha g_i(x_{i,t-1}) + (1-\alpha)m_{i,t-1} \tag{6}$$

to server.

4: Server generates cluster centers

$$\{v_{k,t}\}_{k\in[K]} \leftarrow \texttt{Threshold-Clustering}(\{m_{i,t}\}; K \text{ clusters}; \{v_{k,t-1}\}_{k\in[K]})$$

and sends $v_{k_i,t}$ to client i, where k_i denotes the cluster to which i is assigned in this step.

5: Client i computes update: $x_{i,t} = x_{i,t-1} - \eta v_{k_i,t}$.

6: **Output:** Personalized parameters: $\{x_{1,T}, ..., x_{N,T}\}$.

round $t \in [T]$, we cluster with radius $\tau \approx \sqrt{\delta_i \sigma \Delta}$ for

$$l \geq \max\left\{1, \max_{i\in N]} \frac{\log(\sqrt{\alpha}\sigma/\Delta)}{\log(1-\delta_i/2)}\right\}$$

steps, then for all $i \in [N]$

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2 \lesssim \sqrt{\frac{\sigma^2/n_i + \sigma^3/\Delta}{T}} + \frac{\beta_i\sigma\Delta}{T^{\frac{1}{4}}(\sigma^2/n_i + \sigma^3/\Delta)^{\frac{1}{4}}}. \tag{5}$$

Proof. See A.4. □

We see from (5) that when there are no malicious clients ($\beta_i = 0$), Momentum-Clustering achieves the same $\sqrt{\sigma^2/n_i T}$ convergence rate observed in (3), with no restrictions on the batch size.

4 Experiments

In this section, we 1) use a synthetic dataset to verify the assumptions and demonstrate the theoretical performance from analysis in the previous section and 2) use the MNIST dataset (LeCun et al., 2010) to compare the proposed algorithm with state-of-the-art federated learning algorithms. Algorithms are implemented with PyTorch (Paszke et al., 2017) and all experiments are running with Intel Xeon Gold 6132 CPU @ 2.60 GHz.

4.1 Synthetic dataset

Construction of synthetic dataset. We consider a synthetic linear regression task with squared loss and construct $K = 4$ clusters, each with $n_i = 75$ clients. Clients in cluster $k \in [K]$ share the same minimizer $x_k^\star \in \mathbb{R}^d$. For each client i in cluster k, we generate sample matrix $A_i \in \mathbb{R}^{d\times n}$ from gaussian distribution and obtain the associated $y_i = A_i^\top x_k^\star \in \mathbb{R}^n$. We choose the model dimension $d = 10$ to be greater than the number of local samples $n = 9$ such that the local linear system $y_i = A_i^\top x$ is overdetermined and the error $\|x^\star - x\|_2^2$ is large. A desired federated clustering algorithm determines the minimizer by incorporating information from other clients $j \sim i$ in the same cluster.

Estimate constants in Assumption 4 and 5. Considering the above synthetic dataset,

- we estimate the intra-cluster variance ratio A^2 by finding the upper bound of

$$\|\nabla f_i(x) - \nabla \bar{f}_i(x)\|_2^2 / \|\nabla \bar{f}_i(x)\|_2^2;$$

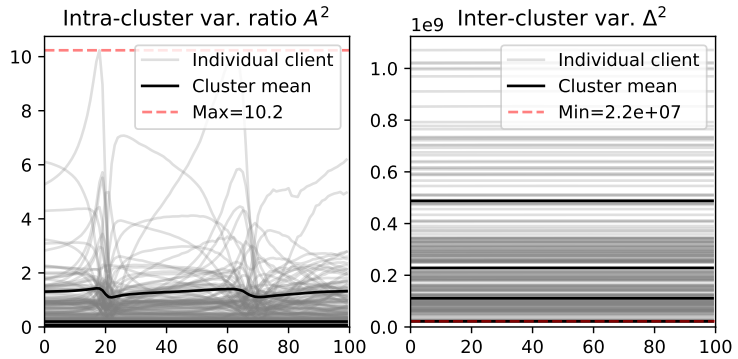


Figure 2: Estimated A^2 and Δ^2 on synthetic dataset ($D = 0$).

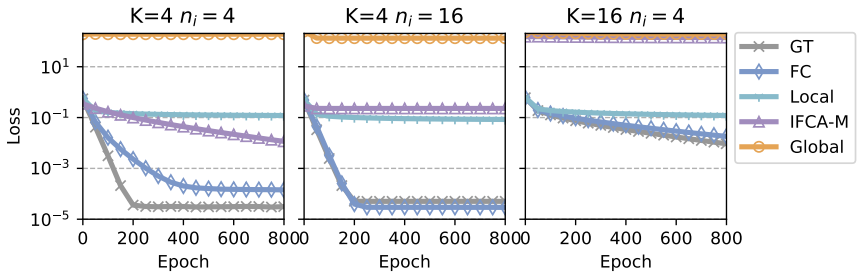


Figure 3: Performances of algorithms on synthetic dataset.

- we estimate the inter-cluster variance Δ^2 by setting $D = 0$ and computing the lower bound of $\|\nabla f_i(x) - \nabla f_j(x)\|_2^2$.

We run Federated-Clustering with perfect clustering assignments and estimate these bounds over time. The result is shown in Figure 2 where grey lines are individual clients and black lines are averaged with a cluster and red dashed lines are the estimated bounds. As we can see, the intra-cluster variance ratio does not grow with time and therefore upper bounded by constant A^2. On the other hand, the inter-cluster variance is also lower bounded by positive constants. Therefore, we empirically showed that Assumptions 4 and 5 are possible in practice.

Performances. We compare our algorithm with standalone training ("local"), IFCA (Ghosh et al., 2020), FedAvg or "global" (McMahan et al., 2017), and distributed training with ground truth (GT) cluster information. We consider the synthetic dataset with cluster $(K, n_i) = (4, 4)$ and compare its performance against increasing $n_i = 16$ and increasing cluster $K = 16$ separately. All other setups remain the same except step sizes are tuned separately. The performances of algorithms are shown in Figure 3. Our algorithm outperforms other non-oracle baselines in all cases. While Federated-Clustering is slightly worse than ground truth in the left subplot, their performances are almost identical in the middle subplot where $n_i = 16$ is larger. This observation is consistent with the linear-in-n_i scaling, $\sigma^2/n_i T$, in (3).

4.2 MNIST experiment

In this section, we compare the baselines and federated learning algorithms on MNIST dataset. The dataset is constructed as follows which is similar to Ghosh et al. (2020). The data samples are randomly shuffled and split into $K = 4$ clusters with $n_i = 75$ clients in each cluster. We consider two different tasks: 1) the *rotation* task transforms images in cluster k for $k * 90$ degrees; 2) the *private label* task transforms labels in cluster k with $T_k(y) : y \mapsto (y + k \mod 10)$ so that same image can have different labels within K clusters.

Table 1: Test losses and accuracies for personalization algorithms.

	Rotation		Private label	
	Acc.(%)	Loss	Acc.(%)	Loss
Local	71.3	0.517	75.2	0.489
Global	46.6	0.631	22.2	0.803
Ditto	62.0	0.576	61.7	0.578
IFCA	54.6	0.588	65.4	0.531
KNN	52.1	2.395	63.2	1.411
FC (ours)	**75.4**	**0.475**	**77.0**	**0.468**
GT (oracle)	84.7	0.432	85.1	0.430

Personalization algorithm hyperparameters. For these two classification tasks, we additionally consider KNN-personalization (Marfoq et al., 2021) and Ditto (Li et al., 2021). The prediction of KNN-personalization is made through a combination of a FedAvg model and a KNN model with tuned linear combination coefficients, i.e. $\lambda = 0.5$ and $\lambda = 0.9$ for rotation and for private label tasks separately. The hyper-parameter of Ditto is fixed to be 1. To reduce the computation cost of our federated clustering algorithm, in each iteration we randomly divide N clients into 16 subgroups and apply federated clustering to each subgroup simultaneously. The clipping radius τ_k^l is adaptively chosen to be the 20%-percentile distances to the cluster center.

Performances. The experiment results are listed in Table 1. As an image can have different labels in the private label task, a model trained over the pool of all datasets only admit inferior performance. Therefore, a naive distributed training algorithm that gives a global model, i.e. FedAvg, Ditto, KNN, performs poorly compared to training alone. On the other hand, our federated clustering algorithm outperforms standalone training and personalization baselines. This experiment suggests that our federated clustering algorithm explores the cluster structure and benefits from collaborative training.

5 Conclusion

We develop gradient-based clustering algorithms to achieve personalization in federated learning. Our algorithms have optimal convergence guarantees, they asymptotically match the achievable rates when the true clustering of clients is known, and our analysis holds under light assumptions (e.g. for all smooth convex and non-convex losses). Furthermore, our algorithms are provably robust in the Byzantine setting where some fraction of the clients can arbitrarily corrupt their gradients. Future directions involve developing bespoke analysis for the convex-loss case and developing more communication-efficient versions of our algorithms. Further, our analysis can be used to show that our algorithms are incentive-compatible and lead to *stable coalitions* as in Donahue & Kleinberg (2021). This would form a strong argument towards encouraging participants in a federated learning system. Investigating such incentives and fairness concerns is another promising future direction.

References

Yossi Arjevani, Yair Carmon, John C. Duchi, Dylan J. Foster, Nathan Srebro, and Blake Woodworth. Lower bounds for non-convex stochastic optimization. 2019.

Peva Blanchard, El Mahdi El Mhamdi, Rachid Guerraoui, and Julien Stainer. Machine learning with adversaries: Byzantine tolerant gradient descent. *Advances in Neural Information Processing Systems*, 30, 2017.

Kate Donahue and Jon Kleinberg. Optimality and stability in federated learning: A game-theoretic approach. *Advances in Neural Information Processing Systems*, 34:1287–1298, 2021.

Avishek Ghosh, Jichan Chung, Dong Yin, and Kannan Ramchandran. An efficient framework for clustered federated learning. *Advances in Neural Information Processing Systems*, 33:19586–19597, 2020.

Avishek Ghosh, Jichan Chung, Dong Yin, and Kannan Ramchandran. An efficient framework for clustered federated learning. In *34th Conference on Neural Information Processing Systems*, 2021.

Sai Praneeth Karimireddy, Lie He, and Martin Jaggi. Byzantine-robust learning on heterogeneous datasets via bucketing. In *International Conference on Learning Representations*, 2021.

Viraj Kulkarni, Milind Kulkarni, and Aniruddha Pant. Survey of personalization techniques for federated learning. In *2020 Fourth World Conference on Smart Trends in Systems, Security and Sustainability (WorldS4)*, pp. 794–797. IEEE, 2020.

L. Lamport, R. Shostak, and M. Pease. The byzantine generals problem. *In Concurrency: the Works of Leslie Lamport.*, 2019.

Yann LeCun, Corinna Cortes, and CJ Burges. Mnist handwritten digit database. *ATT Labs [Online]. Available: http://yann.lecun.com/exdb/mnist*, 2, 2010.

T. Li, Shengyuan Hu, Ahmad Beirami, and Virginia Smith. Fair and robust federated learning through personalization. In *38th International Conference on Machine Learning*, 2021.

Yishay Mansour, Mehryar Mohri, Jae Ro, and Ananda Theertha Suresh. Three approaches for personalization with applications to federated learning. 2021.

Othmane Marfoq, Giovanni Neglia, Laetitia Kameni, and Richard Vidal. Personalized federated learning through local memorization. *arXiv preprint arXiv:2111.09360*, 2021.

Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial intelligence and statistics*, pp. 1273–1282. PMLR, 2017.

H. Brendan McMahon, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Proceedings of the 20th Conference on Artificial Intelligence and Statistics*, volume 54, 2017.

Adam Paszke, Sam Gross, Soumith Chintala, Gregory Chanan, Edward Yang, Zachary DeVito, Zeming Lin, Alban Desmaison, Luca Antiga, and Adam Lerer. Automatic differentiation in pytorch. In *NIPS-W*, 2017.

Felix Sattler, Klaus-Robert Muller, and Wojciech Samek. Clustered federated learning: Model-agnostic distributed multi-task optimization under privacy constraints. 2019.

V. Smith, Chao-Kai Chiang, Maziar Sanjabi, and Ameet Talwalkar. Federated multi-task learning. In *31st Conference on Neural Information Processing Systems*, 2017.

Alysa Ziying Tan, Han Yu, Lizhen Cui, and Qiang Yang. Towards personalized federated learning. *IEEE Transactions on Neural Networks and Learning Systems*, 2022.

Sharan Vaswani, Francis Bach, and Mark Schmidt. Fast and faster convergence of sgd for over-parameterized models and an accelerated perceptron. In *Proceedings of the 22nd International Conference on Artificial Intelligence and Statistics (AISTATS)*, volume 89, 2019.

Amir R Zamir, Alexander Sax, William Shen, Leonidas J Guibas, Jitendra Malik, and Silvio Savarese. Taskonomy: Disentangling task transfer learning. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 3712–3722, 2018.

A Proofs

A.1 Proof of Theorem 1

First we establish some notation.

Notation.

- $\mathcal{G}_i$ are the good points and $\mathcal{B}_i$ the bad points from point z_i's perspective. Therefore $|\mathcal{G}_i| + |\mathcal{B}_i| = N$.
- k_i denotes the cluster to which client i is assigned at the end of Threshold-Clustering.
- To facilitate the proof, we introduce a variable $c_{k_i,l}^2$ that quantifies the distance from the cluster-center-estimates to the true cluster means at each step of thresholding. Specifically, for client i's cluster k_i at round l of Threshold-Clustering, we set

$$c_{k_i,l}^2 = \mathbb{E}\|v_{k_i,l-1} - \mathbb{E}z_i\|^2.$$

- For client i's cluster k_i at round l of Threshold-Clustering, we use thresholding radius

$$\tau_{k_i,l}^2 \approx c_{k_i,l}^2 + \delta_i \sigma \Delta.$$

- We introduce a variable $y_{j,l}$ to denote the points clipped by Threshold-Clustering:

$$v_{k,l} = \frac{1}{N} \sum_{j\in[N]} \underbrace{z_j \mathbb{1}(\|z_j - v_{k,l-1}\| \leq \tau_{k,l}) + v_{k,l-1}\mathbb{1}(\|z_j - v_{k,l-1}\| > \tau_{k,l})}_{y_{j,l}}.$$

Proof of Theorem 1. We prove the main result with the following sequence of inequalities, and then justify the labeled steps afterwards.

$$\begin{aligned}
\mathbb{E}\|v_{k_i,l} - \mathbb{E}z_i\|^2 &= \mathbb{E}\left\|\frac{1}{N}\sum_{j\in[N]} y_{j,l} - \mathbb{E}z_i\right\|^2 \\
&= \mathbb{E}\left\|(1-\beta_i)\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l} - \mathbb{E}z_i\right) + \beta_i\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l} - \mathbb{E}z_i\right)\right\|^2 \\
&\overset{(i)}{\leq} (1+\beta_i)(1-\beta_i)^2\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l}\right) - \mathbb{E}z_i\right\|^2 + \left(1+\frac{1}{\beta_i}\right)\beta_i^2\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l}\right) - \mathbb{E}x_i\right\|^2 \\
&\lesssim \underbrace{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l}\right) - \mathbb{E}z_i\right\|^2}_{\mathcal{E}_1} + \beta_i \underbrace{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l}\right) - \mathbb{E}z_i\right\|^2}_{\mathcal{E}_2} \\
&\overset{(ii)}{\lesssim} \left((1-\delta_i)c_{k_i,l}^2 + \frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta}\right) + \beta_i(c_{k_i,l}^2 + \delta_i\sigma\Delta) \\
&\overset{(iii)}{\lesssim} (1-{}^{\delta_i}/_2)c_{k_i,l}^2 + \left(\frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \beta_i\sigma\Delta\right) \\
&\overset{(iv)}{\lesssim} (1-{}^{\delta_i}/_2)^l c_{k_i,1}^2 + \left(\frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \beta_i\sigma\Delta\right)\sum_{q=0}^{l-1}(1-{}^{\delta_i}/_2)^q \\
&\overset{(v)}{\lesssim} (1-{}^{\delta_i}/_2)^l\sigma^2 + \left(\frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \beta_i\sigma\Delta\right) \\
&\overset{(vi)}{\lesssim} \frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \beta_i\sigma\Delta. \qquad (7)
\end{aligned}$$

Justifications for the labeled steps are:

- (i) Young's inequality: $\|x+y\|^2 \leq (1+\epsilon)x^2 + (1+{}^1/_\epsilon)y^2$ for any $\epsilon > 0$.

- (ii) We prove this bound in Lemmas 1 and 5. Importantly, it shows that the clustering error is composed of two quantities: $\mathcal{E}_1$, the error contributed by good points from the cluster's perspective, and $\mathcal{E}_2$, the error contributed by the bad points from the cluster's perspective.

- (iii) Assumption that $\beta_i \lesssim \delta_i$

- (iv) Since $\mathbb{E}\|v_{k_i,l} - \mathbb{E}z_i\|^2 = c^2_{k_i,l+1}$, the inequality forms a recursion which we unroll over l steps.

- (v) Assumption that $c^2_{k_i,1} = \mathbb{E}\|v_{k_i,0} - \mathbb{E}z_i\|^2 \leq \sigma^2$. Also, the partial sum in the second term can be upper-bounded by a large-enough constant.

- (vi) Assumption that $l \geq \max\left\{1, \frac{\log(\sigma/\Delta)}{\log(1-\delta_i/2)}\right\}$

From (7), we see that $c^2_{k_i,l} \lesssim \frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \beta_i\sigma\Delta$. Plugging this into the expression for $\tau^2_{k_i,l}$ gives $\tau^2_{k_i,l} \approx \frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \delta_i\sigma\Delta \approx \delta_i\sigma\Delta$ for large n_i and Δ. □

Lemma 1 (Clustering Error due to Good Points)**.**

$$\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l}\right) - \mathbb{E}z_i\right\|^2 \lesssim (1-\delta_i)c^2_{k_i,l} + \frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta}$$

Proof of Lemma 1. We prove the main result with the following sequence of inequalities and justify the labeled steps afterward.

$$
\begin{aligned}
\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l}\right) - \mathbb{E}z_i\right\|^2 &= \mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-\mathbb{E}z_j)\right) + \left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l}-\mathbb{E}z_i)\right)\right\|^2 \\
&\overset{(i)}{\leq} \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-\mathbb{E}z_j)\right\|^2 + \left(1+\frac{\delta_i}{2}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l}-\mathbb{E}z_i)\right\|^2 \\
&\overset{(ii)}{\lesssim} \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(\mathbb{E}y_{j,l}-\mathbb{E}z_j)\right\|^2 + \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-\mathbb{E}y_{j,l})\right\|^2 \\
&\quad + \left(1+\frac{\delta_i}{2}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l}-\mathbb{E}z_i)\right\|^2 \\
&\leq \left(1+\frac{2}{\delta_i}\right)\delta_i^2\|\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-z_j)\|^2 + \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-\mathbb{E}y_{j,l})\right\|^2 \\
&\quad + \left(1+\frac{\delta_i}{2}\right)(1-\delta_i)^2\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l}-\mathbb{E}z_i\|^2 \\
&\lesssim \delta_i \underbrace{\|\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-z_j)\|^2}_{\mathcal{T}_i} + \left(1+\frac{2}{\delta_i}\right)\underbrace{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-\mathbb{E}y_{j,l})\right\|^2}_{\mathcal{T}_2} \\
&\quad + \left(1+\frac{\delta_i}{2}\right)(1-\delta_i)^2\underbrace{\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l}-\mathbb{E}z_i\|^2}_{\mathcal{T}_3} \\
&\overset{(iii)}{\lesssim} \delta_i\left(c_{k_i,l}^2 + \frac{(c_{k_i,l}^2+\sigma^2)\sigma}{\delta_i\Delta}\right) + \left(1+\frac{2}{\delta_i}\right)\frac{n_i}{|\mathcal{G}_i|^2}\sigma^2 \\
&\quad + (1-\delta_i)^2\left(1+\frac{\delta_i}{2}\right)\left(\left(1+\frac{\delta_i}{2}+\frac{\sigma^2}{\delta_i\Delta^2}\right)c_{k_i,l}^2 + \frac{\sigma^3}{\Delta}\right) \\
&\lesssim \left(1-\delta_i+\frac{\sigma}{\Delta}+\frac{\sigma^2}{\delta_i\Delta^2}\right)c_{k_i,l}^2 + \left(\frac{\sigma^2}{n_i}+\frac{\sigma^3}{\Delta}\right) \\
&\overset{(iv)}{\lesssim} (1-\delta_i)c_{k_i,l}^2 + \left(\frac{\sigma^2}{n_i}+\frac{\sigma^3}{\Delta}\right).
\end{aligned}
$$

- (i), (ii) Young's inequality
- (iii) We prove this bound in Lemmas 2, 3, and 4. Importantly, it shows that, from point i's perspective, the error of its cluster-center-estimate is composed of three quantities: $\mathcal{T}_1$, the error introduced by our thresholding procedure on the good points which belong to i's cluster (and therefore ideally are included within the thresholding radius); $\mathcal{T}_2$, which accounts for the variance of the points in i's cluster; and $\mathcal{T}_3$, the error due to the good points which don't belong to i's cluster (and therefore ideally are forced outside the thresholding radius).
- (iv) Assumption that $\Delta \gtrsim \sigma/\delta_i$.

□

Lemma 2 (Bound $\mathcal{T}_1$: Error due to In-Cluster Good Points)**.**

$$\|\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-z_j)\|^2 \lesssim c_{k_i,l}^2 + \frac{(c_{k_i,l}^2+\sigma^2)\sigma}{\delta_i\Delta}.$$

Proof of Lemma 2. By definition of $y_{j,l}$,

$$\begin{aligned}
\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}\|y_{j,l}-z_j\| &= \mathbb{E}[\|v_{k_i,l-1}-z_j\|\mathbb{1}(\|v_{k_i,l-1}-z_j\|>\tau_{k_i,l})] \\
&\leq \frac{\mathbb{E}[\|v_{k_i,l-1}-z_j\|^2\mathbb{1}(\|v_{k_i,l-1}-z_j\|>\tau_{k_i,l})]}{\tau_{k_i,l}} \\
&\leq \frac{\mathbb{E}\|v_{k_i,l-1}-z_j\|^2}{\tau_{k_i,l}} \\
&\lesssim \frac{\mathbb{E}\|v_{k_i,l-1}-\mathbb{E}z_i\|^2+\mathbb{E}\|\mathbb{E}z_j-z_j\|^2}{\tau_{k_i,l}} \\
&\leq \frac{c_{k_i,l}^2+\sigma^2}{\tau_{k_i,l}}
\end{aligned}$$

Finally, by Jensen's inequality and plugging in the value for $\tau_{k_i,l}$,

$$\begin{aligned}
\|\mathbb{E}(y_{j,l}-z_j)\|^2 &\leq (\mathbb{E}\|y_{j,l}-z_j\|)^2 \\
&\lesssim \frac{(c_{k_i,l}^2+\sigma^2)^2}{\tau_{k_i,l}^2} \\
&\lesssim \frac{c_{k_i,l}^4}{c_{k_i,l}^2}+\frac{c_{k_i,l}^2\sigma^2}{\delta_i\sigma\Delta}+\frac{\sigma^4}{\delta_i\sigma\Delta} \\
&= c_{k_i,l}^2+\frac{(c_{k_i,l}^2+\sigma^2)\sigma}{\delta_i\Delta}.
\end{aligned}$$

□

Lemma 3 (Bound $\mathcal{T}_2$: Variance of Clipped Points)**.**

$$\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-\mathbb{E}y_{j,l})\right\|^2 \leq \frac{n_i}{|\mathcal{G}_i|^2}\sigma^2.$$

Proof of Lemma 3. Note that the elements in the sum $\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-\mathbb{E}y_{j,l})$ are not independent. Therefore, we cannot get rid of the cross terms when expanding the squared-norm. However, if for each round of thresholding we were sample a fresh batch of points to set the new cluster-center estimate, then the terms would be independent. With this resampling strategy, our bounds would only change by a constant factor. Therefore, for ease of analysis, we will assume the terms in the sum are independent. In that case,

$$\begin{aligned}
\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l}-\mathbb{E}y_{j,l})\right\|^2 &\leq \frac{n_i}{|\mathcal{G}_i|^2}\mathbb{E}\|y_{j,l}-\mathbb{E}y_{j,l}\|^2 \\
&\leq \frac{n_i}{|\mathcal{G}_i|^2}\mathbb{E}\|z_j-\mathbb{E}z_j\|^2 \\
&\leq \frac{n_i}{|\mathcal{G}_i|^2}\sigma^2,
\end{aligned}$$

where the second-to-last inequality follows from the contractivity of the thresholding procedure. □

Lemma 4 (Bound $\mathcal{T}_3$: Error due to Out-of-Cluster Good Points)**.**

$$\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l}-\mathbb{E}z_i\|^2 \lesssim \left(1+\frac{\delta_i}{2}+\frac{\sigma^2}{\delta_i\Delta^2}\right)c_{k_i,l}^2+\frac{\sigma^3}{\Delta}.$$

Proof of Lemma 4. By Young's inequality,

$$\begin{aligned}
\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l}-\mathbb{E}z_i\|^2 &\le \left(1+\frac{\delta_i}{2}\right)\mathbb{E}\|v_{k_i,l-1}-\mathbb{E}z_i\|^2+\left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l}-v_{k_i,l-1}\|^2\\
&\le \left(1+\frac{\delta_i}{2}\right)c_{k_i,l}^2+\left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l}-v_{k_i,l-1}\|^2\\
&= \left(1+\frac{\delta_i}{2}\right)c_{k_i,l}^2+\left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}[\|z_j-v_{k_i,l-1}\|^2\mathbb{1}\{\|z_j-v_{k_i,l-1}\|\le\tau_{k_i,l}\}]\\
&\le \left(1+\frac{\delta_i}{2}\right)c_{k_i,l}^2+\left(1+\frac{2}{\delta_i}\right)\tau_{k_i,l}^2\mathbb{P}_{j\in\mathcal{G}_i:j\not\sim i}(\|z_j-v_{k_i,l-1}\|\le\tau_{k_i,l}).
\end{aligned}$$

We now have to bound the probability in the expression above. Note that if $\|v_{k_i,l-1}-z_j\|\le\tau_{k_i,l}$, then

$$\begin{aligned}
\|\mathbb{E}z_j-\mathbb{E}z_i\|^2 &\lesssim \|z_j-\mathbb{E}z_j\|^2+\|z_j-\mathbb{E}v_{k_i,l-1}\|^2+\|\mathbb{E}v_{k_i,l-1}-\mathbb{E}z_i\|^2\\
&\lesssim \|z_j-\mathbb{E}z_j\|^2+\|z_j-\mathbb{E}z_j\|^2+\|\mathbb{E}z_j-\mathbb{E}v_{k_i,l-1}\|^2+\mathbb{E}\|v_{k_i,l-1}-\mathbb{E}z_i\|^2+\mathbb{E}\|z_i-\mathbb{E}z_i\|^2\\
&\lesssim \|z_j-\mathbb{E}z_j\|^2+\tau_{k_i,l}^2+c_{k_i,l}^2+\sigma^2.
\end{aligned}$$

By Assumption 2, this implies that

$$\Delta^2\lesssim\|z_j-\mathbb{E}z_j\|^2+\tau_{k_i,l}^2+c_{k_i,l}^2+\sigma^2$$

which means that

$$\|z_j-\mathbb{E}z_j\|^2\gtrsim\Delta^2-(\tau_{k_i,l}^2+c_{k_i,l}^2+\sigma^2).$$

By Markov's inequality,

$$\mathbb{P}(\|z_j-\mathbb{E}z_j\|^2\gtrsim\Delta^2-(\tau_{k_i,l}^2+c_{k_i,l}^2+\sigma^2))\le\frac{\sigma^2}{\Delta^2-(\tau_{k_i,l}^2+c_{k_i,l}^2+\sigma^2)}\lesssim\frac{\sigma^2}{\Delta^2}$$

as long as

$$\Delta^2\gtrsim\tau_{k_i,l}^2+c_{k_i,l}^2+\sigma^2,$$

which holds due to the constraint on Δ in the theorem statement. Therefore

$$\begin{aligned}
\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l}-\mathbb{E}z_i\|^2 &\lesssim \left(1+\frac{\delta_i}{2}\right)c_{k_i,l}^2+\left(1+\frac{2}{\delta_i}\right)\frac{(c_{k_i,l}^2+\delta_i\sigma\Delta)\sigma^2}{\Delta^2}\\
&\le \left(1+\frac{\delta_i}{2}+\frac{\sigma^2}{\delta_i\Delta^2}\right)c_{k_i,l}^2+\frac{\sigma^3}{\Delta}.
\end{aligned}$$

□

Lemma 5 (Clustering Error due to Bad Points)**.**

$$\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i}y_{j,l}\right)-\mathbb{E}z_i\right\|^2\lesssim c_{k_i,l}^2+\delta_i\sigma\Delta$$

Proof of Lemma 5.

$$\begin{aligned}
\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i}y_{j,l}\right)-\mathbb{E}z_i\right\|^2 &\le \mathbb{E}_{j\in\mathcal{B}_i}\|y_{j,l}-\mathbb{E}z_i\|^2\\
&\lesssim \mathbb{E}_{j\in\mathcal{B}_i}\|y_{j,l}-v_{k_i,l-1}\|^2+\mathbb{E}\|v_{k_i,l-1}-\mathbb{E}z_i\|^2\\
&\lesssim c_{k_i,l}^2+\delta_i\sigma\Delta.
\end{aligned}$$

The last inequality follows from the intuition that bad points will position themselves at the edge of the thresholding ball, a distance $\tau_{k_i,l}$ away from the current center-estimate $v_{k_i,l-1}$. Therefore we cannot do better than upper-bounding $\mathbb{E}_{j\in\mathcal{B}_i}\|y_{j,l}-v_{k_i,l-1}\|^2$ by $\tau_{k_i,l}^2\approx c_{k_i,l}^2+\delta_i\sigma\Delta$, the squared-radius of the ball. □

A.2 Proof of Theorem 2

Proof of Theorem 2. Let

$$\mathcal{D}_1 = \begin{cases} \delta & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

and

$$\mathcal{D}_2 = \begin{cases} \delta & \text{w.p. } 1-p \\ 0 & \text{w.p. } p \end{cases}$$

and define the mixture $\mathcal{M} = \frac{1}{2}\mathcal{D}_1 + \frac{1}{2}\mathcal{D}_2$. Also consider the mixture $\tilde{\mathcal{M}} = \frac{1}{2}\tilde{\mathcal{D}}_1 + \frac{1}{2}\tilde{\mathcal{D}}_2$, where $\tilde{\mathcal{D}}_1 = 0$ and $\tilde{\mathcal{D}}_2 = \delta$. It is impossible to distinguish whether a sample comes from $\mathcal{M}$ or $\tilde{\mathcal{M}}$. Therefore, if you at least know a sample came from either $\mathcal{M}$ or $\tilde{\mathcal{M}}$ but not which one, the best you can do is to estimate μ_1 with $\hat{\mu}_1 = \frac{\delta p}{2}$, half-way between the mean of $\mathcal{D}_1$, which is δp, and the mean of $\tilde{\mathcal{D}}_1$, which is 0. In this case

$$\mathbb{E}\|\hat{\mu}_1 - \mu_1\|^2 = \frac{\delta^2 p^2}{4}.$$

If $p \leq \frac{1}{2}$, then

$$\Delta = (1-p)\delta - p\delta = (1-2p)\delta. \tag{8}$$

Also,

$$\sigma^2 = \delta^2 p(1-p). \tag{9}$$

Equating δ^2 in (8) and (9),

$$\frac{\Delta^2}{(1-2p)^2} = \frac{\sigma^2}{p(1-p)},$$

which can be rearranged to

$$(4\sigma^2 + \Delta^2)p^2 - (4\sigma^2 + \Delta^2)p + \sigma^2 = 0.$$

Solving for p,

$$p = \frac{1}{2} - \frac{\Delta}{2\sqrt{4\sigma^2 + \Delta^2}}. \tag{10}$$

Note that,

$$\begin{aligned} \frac{\delta^2 p^2}{4} &= \frac{\sigma^2 p^2}{4p(1-p)} \\ &= \frac{\sigma^2 p}{4(1-p)}. \end{aligned} \tag{11}$$

Plugging the expression for p from (10) into (11), we can see that

$$\frac{\delta^2 p^2}{4} = \frac{\sigma^2}{4}\left(\frac{\sqrt{4\sigma^2 + \Delta^2} - \Delta}{\Delta}\right) = \frac{\sigma^2}{4}\left(\sqrt{1 + \frac{4\sigma^2}{\Delta^2}} - 1\right) \geq \frac{\sigma^2}{4}\left(\frac{2\sigma^2}{\Delta^2} - \frac{2\sigma^4}{\Delta^4}\right).$$

The last step used an immediately verifiable inequality that $\sqrt{1+x} \geq 1 + \frac{x}{2} - \frac{x^2}{8}$ for all $x \in [0, 8]$. Finally, we can choose $\Delta^2 \geq 2\sigma^2$ to give the result that

$$\mathbb{E}\|\hat{\mu}_1 - \mu_1\|^2 \geq \frac{\delta^2 p^2}{4} \geq \frac{\sigma^4}{4\Delta^2}.$$

Finally, suppose that there is only a single cluster with $K = 1$. Then, given n stochastic samples. standard information theoretic lower bounds show that we will have an error at least

$$\mathbb{E}\|\hat{\mu}_1 - \mu_1\|^2 \geq \frac{\sigma^2}{4n}.$$

Combining these two lower bounds yields the proof of the theorem. □

A.3 Proof of Theorem 3

First we establish some notation.

Notation.

- $\mathcal{G}_i$ are the good clients and $\mathcal{B}_i$ the bad clients from client i's perspective. Therefore $|\mathcal{G}_i| + |\mathcal{B}_i| = N$.
- $\mathbb{E}_x$ denotes conditional expectation given the parameter, e.g. $\mathbb{E}_x g(x) = \mathbb{E}[g(x)|x]$. $\mathbb{E}$ denotes expectation over all randomness.
- $\overline{X_t} \triangleq \frac{1}{T}\sum_{t=1}^{T} X_t$ for a general variable X_t indexed by t.
- We use $f_i(x)$ to denote average loss on a general batch B of samples. That is, if $f_i(x;b)$ is the loss on a single sample b, we define $f_i(x) = \frac{1}{|B|}\sum_{b\in B} f_i(x;b)$.
- $\bar{f}_i(x) \triangleq \frac{1}{n_i}\sum_{j\in\mathcal{G}_i : j\sim i} f_j(x)$
- We introduce a variable ρ^2 to bound the variance of the gradients

  $$\mathbb{E}\|g_i(x) - \mathbb{E}_x g_i(x)\|^2 \leq \rho^2,$$

  and show in Lemma 6 how this can be written in terms of the variance of the gradients computed over a batch size of 1.
- l_t is the number of rounds that Threshold-Clustering is run in round t of Federated-Clustering.
- k_i denotes the cluster to which client i is assigned.
- $v_{i,l,t}$ denotes the gradient update for client i in round t of Federated-Clustering and round l of Threshold-Clustering. That is, $v_{i,l_t,t}$ corresponds to the quantity returned in Step 6 of Algorithm 2.
- To facilitate the proof, we introduce a variable $c_{k_i,l,t}$ that quantifies the distance from the cluster-center-estimates to the true cluster means. Specifically, for client i's cluster k_i at round t of Federated-Clustering and round l of Threshold-Clustering we set

  $$c^2_{k_i,l,t} = \mathbb{E}\|v_{i,l-1,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2.$$

- For client i's cluster k_i at round t of Federated-Clustering and round l of Threshold-Clustering, we use thresholding radius

  $$\tau^2_{k_i,l,t} \approx c^2_{k_i,l,t} + A^4\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2 + \delta_i\rho\Delta.$$

- Finally, we introduce a variable $y_{j,l,t}$ to denote the points clipped by Threshold-Clustering:

  $$v_{i,l,t} = \frac{1}{N}\sum_{j\in[N]} \underbrace{\mathbb{1}(\|g_j(x_{i,t-1}) - v_{i,l-1,t}\| \leq \tau_{k_i^t,l}) + v_{i,l-1,t}\mathbb{1}(\|g_j(x_{i,t-1}) - v_{i,l-1,t}\| > \tau_{k_i^t,l})}_{y_{j,l,t}}.$$

Proof of Theorem 3. In this proof, our goal is to bound $\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2}$ for each client i, thus showing convergence. Recall that our thresholding procedure clusters the gradients of clients at each round and estimates the center of each cluster. These estimates are then used to update the parameters of the clusters. Therefore, we expect $\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2}$ to be bounded in terms of the error of this estimation procedure. The following sequence of inequalities shows this.

By L-smoothness of f_i and setting $\eta \leq 1/L$,

$$\begin{aligned}
f_i(x_{i,t}) &\leq f_i(x_{i,t-1}) + \langle \nabla f_i(x_{i,t-1}), x_{i,t} - x_{i,t-1} \rangle + \frac{L}{2}\|x_{i,t} - x_{i,t-1}\|^2 \\
&= f_i(x_{i,t-1}) - \eta \langle \nabla f_i(x_{i,t-1}), v_{i,l_t,t} \rangle + \frac{L\eta^2}{2}\|v_{i,l_t,t}\|^2 \\
&= f_i(x_{i,t-1}) + \frac{\eta}{2}\|v_{i,l_t,t} - \nabla f_i(x_{i,t-1})\|^2 - \frac{\eta}{2}\|\nabla f_i(x_{i,t-1})\|^2 - \frac{\eta}{2}(1 - L\eta)\|v_{i,l_t,t}\|^2 \\
&\leq f_i(x_{i,t-1}) + \eta\|v_{i,l_t,t} - \nabla \bar{f}_i(x_{i,t-1})\|^2 + \eta A^2 \|\nabla \bar{f}_i(x_{i,t-1})\|^2 - \frac{\eta}{2}\|\nabla f_i(x_{i,t-1})\|^2 - \frac{\eta}{2}(1 - L\eta)\|v_{i,l_t,t}\|^2.
\end{aligned} \tag{12}$$

Recall that $v_{i,l_t,t}$ is client i's cluster-center estimate at round t of optimization, so the second term on the right side of (12) is the error due to the clustering procedure. In Lemma 7, we show that, in expectation, this error is bounded by

$$\delta_i \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta\right).$$

Therefore, subtracting f_i^* from both sides, summing (12) over t, dividing by T, taking expectations, applying Lemma 7 to (12), and applying the constraint on Δ from the theorem statement, we have

$$\eta \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} \lesssim \frac{\mathbb{E}(f_i(x_{i,0}) - f_i^*)}{T} + \eta A^2 \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \eta\left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta\right). \tag{13}$$

The third term on the right side of (12) reflects the fact that clients in the same cluster may have different loss objectives. Far from their optima, these loss objectives may look very different and therefore be hard to cluster together.

In order to bound this term, we use a similar argument as above. By L-smoothness of f_i's and setting $\eta \leq 1/L$,

$$\begin{aligned}
\bar{f}_i(x_{i,t}) &\leq \bar{f}_i(x_{i,t-1}) + \langle \nabla \bar{f}_i(x_{i,t-1}), x_{i,t} - x_{i,t-1} \rangle + \frac{L}{2}\|x_{i,t} - x_{i,t-1}\|^2 \\
&= \bar{f}_i(x_{i,t-1}) - \eta \langle \nabla \bar{f}_i(x_{i,t-1}), v_{i,l_t,t} \rangle + \frac{L\eta^2}{2}\|v_{i,l_t,t}\|^2 \\
&= \bar{f}_i(x_{i,t-1}) + \frac{\eta}{2}\|v_{i,l_t,t} - \nabla \bar{f}_i(x_{i,t-1})\|^2 - \frac{\eta}{2}\|\nabla \bar{f}_i(x_{i,t-1})\|^2 - \frac{\eta}{2}(1 - L\eta)\|v_{i,l_t,t}\|^2.
\end{aligned}$$

Subtracting $\bar{f}_i^*$ from both sides, summing over t, dividing by T, taking expectations, and applying Lemma 7,

$$\eta \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} \lesssim \frac{\mathbb{E}(\bar{f}_i(x_{i,0}) - \bar{f}_i^*)}{T} + \frac{\eta\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \eta\left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta\right). \tag{14}$$

Combining (13) and (14), and applying the constraint on Δ from the theorem statement, we have that

$$\eta \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} \lesssim \frac{\mathbb{E}(f_i(x_{i,0}) - f_i^*) + A^2 \mathbb{E}(\bar{f}_i(x_0) - \bar{f}_i^*)}{T} + \eta \max(1, A^2)\left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta\right). \tag{15}$$

Dividing both sides of (15) by $\eta = 1/L$ and noting that $\rho^2 = \sigma^2/|B|$ from Lemma 6,

$$\begin{aligned}
\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} &\lesssim \frac{\mathbb{E}(f_i(x_{i,0}) - f_i^*) + A^2 \mathbb{E}(\bar{f}_i(x_{i,0}) - \bar{f}_i^*)}{\eta T} + \max(1, A^2)\left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta\right) \\
&\leq \frac{L(\mathbb{E}(f_i(x_{i,0}) - f_i^*) + A^2 \mathbb{E}(\bar{f}_i(x_{i,0}) - \bar{f}_i^*))}{T} + \frac{\max(1, A^2)(\sigma^2/n_i + \sigma^3/\Delta + \beta_i \sigma \Delta)}{\sqrt{|B|}} \\
&\lesssim \sqrt{\frac{\max(1, A^2)(\sigma^2/n_i + \sigma^3/\Delta + \beta_i \sigma \Delta)}{T}},
\end{aligned}$$

where the last inequality follows from setting

$$|B| \gtrsim \max(1, A^2)\left(\frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \beta_i \sigma \Delta\right) T.$$

Since $T = {}^{m_i}/_{|B|}$, this is equivalent to setting

$$|B| \gtrsim \sqrt{\max(1, A^2)\left(\frac{\sigma^2}{n_i} + \frac{\sigma^3}{\Delta} + \beta_i \sigma \Delta\right) m_i}.$$

□

Lemma 6 (Variance reduction using batches)**.** *If, for a single sample b,*

$$\mathbb{E}_x \|g_i(x; b) - \mathbb{E}_x g_i(x; b)\|^2 \leq \sigma^2,$$

then for a batch B of samples,

$$\mathbb{E}_x \|g_i(x) - \mathbb{E}_x g_i(x)\|^2 \leq \frac{\sigma^2}{|B|}.$$

Proof of Lemma 6.

$$\begin{aligned}\mathbb{E}_x \|g_i(x) - \mathbb{E}_x g_i(x)\|^2 &= \frac{1}{|B|^2} \sum_{b \in B} \mathbb{E}_x \|g_i(x; b) - \mathbb{E}_x g_i(x; b)\|^2 \\ &\leq \frac{\sigma^2}{|B|}.\end{aligned}$$

□

Lemma 7 (Bound on Clustering Error)**.**

$$\overline{\mathbb{E}\|v_{i,l_t,t} - \nabla \bar{f}_i(x_{i,t-1})\|^2} \lesssim \delta_i \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta\right)$$

Proof of Lemma 7. We prove the main result with the following sequence of inequalities and justify the labeled steps afterwards.

$$
\begin{aligned}
\overline{\mathbb{E}\|v_{i,l_t,t} - \nabla \bar{f}_i(x_{i,t-1})\|^2} &= \overline{\mathbb{E}\|v_{i,l_t,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2} \\
&= \overline{\mathbb{E}\left\| \frac{1}{N} \sum_{j\in[N]} y_{j,l_t,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1}) \right\|^2} \\
&= \overline{\mathbb{E}\left\| (1-\beta_i)\left(\frac{1}{|\mathcal{G}_i|} \sum_{j\in\mathcal{G}_i} y_{j,l_t,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1}) \right) + \beta_i \left(\frac{1}{|\mathcal{B}_i|} \sum_{j\in\mathcal{B}_i} y_{j,t,l} - \mathbb{E}_x \bar{g}_i(x_{i,t-1}) \right) \right\|^2} \\
&\overset{(i)}{\leq} (1+\beta_i)(1-\beta_i)^2 \overline{\mathbb{E}\left\| \left(\frac{1}{|\mathcal{G}_i|} \sum_{j\in\mathcal{G}_i} y_{j,l_t,t} \right) - \mathbb{E}_x \bar{g}_i(x_{i,t-1}) \right\|^2} \\
&\quad + \left(1 + \frac{1}{\beta_i}\right) \beta_i^2 \overline{\mathbb{E}\left\| \left(\frac{1}{|\mathcal{B}_i|} \sum_{j\in\mathcal{B}_i} y_{j,l_t,t} \right) - \mathbb{E}_x \bar{g}_i(x_{i,t-1}) \right\|^2} \\
&\leq \underbrace{\overline{\mathbb{E}\left\| \left(\frac{1}{|\mathcal{G}_i|} \sum_{j\in\mathcal{G}_i} y_{j,l_t,t} \right) - \mathbb{E}_x \bar{g}_i(x_{i,t-1}) \right\|^2}}_{\mathcal{E}_1} + \beta_i \underbrace{\overline{\mathbb{E}\left\| \left(\frac{1}{|\mathcal{B}_i|} \sum_{j\in\mathcal{B}_i} y_{j,l_t,t} \right) - \mathbb{E}_x \bar{g}_i(x_{i,t-1}) \right\|^2}}_{\mathcal{E}_2} \\
&\overset{(ii)}{\lesssim} (1-\delta_i+\beta_i)\overline{c^2_{k_i,l_t,t}} + (\delta_i + \beta_i A^4)\overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} \\
&\quad + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta \right) \\
&\overset{(iii)}{\lesssim} (1 - {}^{\delta_i}/_{2})\overline{c^2_{k_i,l_t,t}} + \delta_i \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta \right) \\
&\overset{(iv)}{\lesssim} \overline{(1 - {}^{\delta_i}/_{2})^{l_t} c^2_{k_i,1,t}} + \delta_i \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta \right) \\
&\overset{(v)}{\leq} \frac{\rho}{\Delta} \overline{c^2_{k_i,1,t}} + \delta_i \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta \right). \quad (16)
\end{aligned}
$$

Justifications for the labeled steps are:

- (i) Young's inequality: $\|x+y\|^2 \leq (1+\epsilon)x^2 + (1 + {}^{1}/_{\epsilon})y^2$ for any $\epsilon > 0$.
- (ii) We prove this bound in Lemmas 8 and 12. Importantly, it shows that the clustering error is composed of two quantities: $\mathcal{E}_1$, the error contributed by good points from the cluster's perspective, and $\mathcal{E}_2$, the error contributed by the bad points from the cluster's perspective.
- (iii) Assumption that $\beta_i \lesssim \min(\delta_i, {}^{\delta_i}/_{A^4})$
- (iv) Since $\mathbb{E}\|v_{i,l_t,t} - \nabla \bar{f}_i(x_{i,t-1})\|^2 = c^2_{k_i,l_t+1,t}$, the inequality forms a recursion which we unroll over l_t steps.
- (v) Assumption that $l_t \geq \max(1, \frac{\log(\rho/\Delta)}{\log(1-\delta_i/2)})$

Finally, we note that

$$
\begin{aligned}
c_{k_i,1,t}^2 &= \mathbb{E}\|g_i(x_{i,t-1}) - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2 \\
&\lesssim \mathbb{E}\|g_i(x_{i,t-1}) - \mathbb{E}_x g_i(x_{i,t-1})\|^2 + \mathbb{E}\|\mathbb{E}_x g_i(x_{i,t-1}) - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2 \\
&\leq \rho^2 + A^2 \mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2.
\end{aligned}
$$

Applying this bound to (16), and applying the bound on Δ from the theorem statement, we have

$$
\begin{aligned}
&\overline{\mathbb{E}\|v_{i,l_t,t} - \nabla \bar{f}_i(x_{i,t-1})\|^2} \\
&\lesssim \left(\delta_i + \frac{\rho A^2}{\Delta}\right) \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta\right) \\
&\lesssim \delta_i \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i \rho \Delta\right).
\end{aligned}
$$

□

Lemma 8 (Clustering Error due to Good Points)**.**

$$
\begin{aligned}
&\overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t}\right) - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\right\|^2} \\
&\lesssim (1-\delta_i)\overline{c_{k_i,l_t,t}^2} + \delta_i \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta} D^2 \overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta}\right)
\end{aligned}
$$

Proof of Lemma 8. We prove the main result in the sequence of inequalities below and then justify the labeled steps.

$$
\begin{aligned}
&\overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t}\right) - \mathbb{E}_x\bar{g}_i(x_{i,t-1})\right\|^2} \\
&= \overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t}\right) - \frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}\mathbb{E}_x g_j(x_{i,t-1}) - \left(\frac{1}{n_i}-\frac{1}{|\mathcal{G}_i|}\right)\sum_{j\in\mathcal{G}_i:j\sim i}\mathbb{E}_x g_j(x_{i,t-1})\right\|^2} \\
&= \overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}_x g_j(x_{i,t-1}))\right) + \left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l_t,t}-\mathbb{E}_x\bar{g}_i(x_{i,t-1}))\right)\right\|^2} \\
&\overset{(i)}{\leq} \left(1+\frac{2}{\delta_i}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}_x g_j(x_{i,t-1}))\right\|^2} + \left(1+\frac{\delta_i}{2}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l_t,t}-\mathbb{E}_x\bar{g}_i(x_{i,t-1}))\right\|^2} \\
&\overset{(ii)}{\lesssim} \left(1+\frac{2}{\delta_i}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(\mathbb{E}y_{j,l_t,t}-\mathbb{E}_x g_j(x_{i,t-1}))\right\|^2} + \left(1+\frac{2}{\delta_i}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})\right\|^2} \\
&\quad + \left(1+\frac{\delta_i}{2}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l_t,t}-\mathbb{E}_x\bar{g}_i(x_{i,t-1}))\right\|^2} \\
&\overset{(iii)}{\lesssim} \left(1+\frac{2}{\delta_i}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}\mathbb{E}(y_{j,l_t,t}-g_j(x_{i,t-1}))\right\|^2} + \left(1+\frac{2}{\delta_i}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(\mathbb{E}_x g_j(x_{i,t-1})-\mathbb{E}g_j(x_{i,t-1}))\right\|^2} \\
&\quad + \left(1+\frac{2}{\delta_i}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})\right\|^2} + \left(1+\frac{\delta_i}{2}\right)\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l_t,t}-\mathbb{E}_x\bar{g}_i(x_{i,t-1}))\right\|^2} \\
&\overset{(iv)}{\lesssim} \left(1+\frac{2}{\delta_i}\right)\delta_i^2\underbrace{\overline{\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}\|\mathbb{E}(y_{j,l_t,t}-g_j(x_{i,t-1}))\|^2}}_{\mathcal{T}_1} + \left(1+\frac{2}{\delta_i}\right)\frac{n_i}{|\mathcal{G}_i|^2}\rho^2 \\
&\quad + \left(1+\frac{2}{\delta_i}\right)\underbrace{\overline{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})\right\|^2}}_{\mathcal{T}_2} + \left(1+\frac{\delta_i}{2}\right)(1-\delta_i)^2\underbrace{\overline{\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-\mathbb{E}_x\bar{g}_i(x_{i,t-1})\|^2}}_{\mathcal{T}_3} \\
&\overset{(v)}{\lesssim} \delta_i\left(\overline{c^2_{k_i,l_t,t}} + \overline{\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\delta_i\Delta}\right) + \left(1+\frac{2}{\delta_i}\right)\frac{n_i}{|\mathcal{G}_i|^2}\rho^2 \\
&\quad + \left(1+\frac{\delta_i}{2}\right)(1-\delta_i)^2\left(\left(1+\frac{\delta_i}{2}\right)\overline{c^2_{k_i,l_t,t}} + \delta_i\overline{\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta}D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\Delta}\right) \\
&\overset{(vi)}{\lesssim} \delta_i\left(\overline{c^2_{k_i,l_t,t}} + \overline{\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\delta_i\Delta}\right) + \frac{\rho^2}{n_i} \\
&\quad + (1-\delta_i)\left(\overline{c^2_{k_i,l_t,t}} + \delta_i\overline{\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta}D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\Delta}\right) \\
&\lesssim (1-\delta_i)\overline{c^2_{k_i,l_t,t}} + \delta_i\overline{\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta}D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \left(\frac{\rho^2}{n_i}+\frac{\rho^3}{\Delta}\right).
\end{aligned}
$$

Justifications for the labeled steps are:

- (i),(ii),(iii) Young's inequality
- (iv) First, we can can interchange the sum and the norm due to independent stochasticity of the gradients. Then by the Tower Property and Law of Total Variance for the 1st and 3rd steps

respectively,

$$\begin{aligned}\mathbb{E}\|\mathbb{E}_x g_j(x_{i,t-1}) - \mathbb{E}g_j(x_{i,t-1}))\|^2 &= \mathbb{E}\|\mathbb{E}_x g_j(x_{i,t-1}) - \mathbb{E}[\mathbb{E}_x g_j(x_{i,t-1})]\|^2 \\ &= \mathrm{Var}(\mathbb{E}_x(g_j(x_{i,t-1}))) \\ &= \mathrm{Var}(g_j(x_{i,t-1})) - \mathbb{E}(\mathrm{Var}_x(g_j(x_{i,t-1}))) \\ &\leq \mathrm{Var}(g_j(x_{i,t-1})) - \mathbb{E}\|g_j(x_{i,t-1}) - \mathbb{E}_x g_j(x_{i,t-1})\|^2 \\ &\lesssim \rho^2,\end{aligned}$$

where the last inequality follows since the two terms above it are both bounded by ρ^2.

- (v) We prove this bound in Lemmas 9, 10, and 11. It shows that, from point i's perspective, the error of its cluster-center-estimate is composed of three quantities: $\mathcal{T}_1$, the error introduced by our thresholding procedure on the good points which belong to i's cluster (and therefore ideally are included within the thresholding radius); $\mathcal{T}_2$, which accounts for the variance of the clipped points in i's cluster; and $\mathcal{T}_3$, the error due to the good points which don't belong to i's cluster (and therefore ideally are forced outside the thresholding radius).
- (vi) $(1 + {}^{x}\!/_{2})^2(1-x)^2 \leq 1 - x$ for all $x \in [0, 1]$

□

Lemma 9 (Bound $\mathcal{T}_1$: Error due to In-Cluster Good Points)**.**

$$\overline{\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}\|\mathbb{E}(y_{j,l_t,t} - g_j(x_{i,t-1}))\|^2} \lesssim \overline{c^2_{k_i,l_t,t}} + \overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\delta_i\Delta}.$$

Proof of Lemma 9. In this sequence of steps, we bound the clustering error due to good points from client i's cluster. By definition of $y_{j,l_t,t}$,

$$\begin{aligned}\mathbb{E}\|y_{j,l_t,t} - g_j(x_{i,t-1})\| &= \mathbb{E}[\|v_{i,l_t-1,t} - g_j(x_{i,t-1})\|\mathbb{1}(\|v_{i,l_t-1,t} - g_j(x_{i,t-1})\| > \tau_{k_i,l_t,t})] \\ &\leq \frac{\mathbb{E}[\|v_{i,l_t-1,t} - g_j(x_{i,t-1})\|^2\mathbb{1}(\|v_{i,l_t-1,t} - g_j(x_{i,t-1})\| > \tau_{k_i,l_t,t})]}{\tau_{k_i^t,l_t}} \\ &\leq \frac{\mathbb{E}\|v_{i,l_t-1,t} - g_j(x_{i,t-1})\|^2}{\tau_{k_i,l_t,t}}.\end{aligned}$$

Therefore, by Jensen's inequality and plugging in the value for $\tau_{k_i,l_t,t}$,

$$\begin{aligned}&\|\mathbb{E}(y_{j,l_t,t} - g_j(x_{i,t-1}))\|^2 \\ &\leq (\mathbb{E}\|y_{j,l_t,t} - g_j(x_{i,t-1})\|)^2 \\ &\leq \frac{(\mathbb{E}\|v_{i,l_t-1,t} - g_j(x_{i,t-1})\|^2)^2}{\tau^2_{k_i,l_t,t}} \\ &\lesssim \frac{(\mathbb{E}\|v_{i,l_t-1,t} - \mathbb{E}_x\bar{g}_i(x_{i,t-1})\|^2 + \mathbb{E}\|\mathbb{E}_x\bar{g}_i(x_{i,t-1}) - \mathbb{E}_x g_j(x_{i,t-1})\|^2 + \mathbb{E}\|g_j(x_{i,t-1}) - \mathbb{E}_x g_j(x_{i,t-1})\|^2)^2}{\tau^2_{k_i,l_t,t}} \\ &\leq \frac{(c^2_{k_i,l_t,t} + A^2\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2 + \rho^2)^2}{\tau^2_{k_i,l_t,t}} \\ &\lesssim \frac{(c^2_{k_i,l_t,t} + A^2\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2 + \rho^2)^2}{c^2_{k_i,l_t,t} + A^4\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2 + \delta_i\rho\Delta} \\ &\lesssim \left(1 + \frac{\rho}{\delta_i\Delta}\right)c^2_{k_i,l_t,t} + \left(1 + \frac{\rho A^2}{\delta_i\Delta}\right)\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2 + \frac{\rho^3}{\delta_i\Delta} \\ &\lesssim c^2_{k_i,l_t,t} + \mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2 + \frac{\rho^3}{\delta_i\Delta},\end{aligned} \tag{17}$$

where the last inequality follows from constraints on Δ. The second inequality follows from Young's inequality. The second-to-last inequality follows by separating the fraction into a sum of fractions and selecting terms from the denominator for each fraction that cancel with terms in the numerator to achieve the desired rate.

Summing (17) over t and dividing by T, we have

$$\begin{aligned}\overline{\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}\|\mathbb{E}_x(y_{j,l_t,t}-g_j(x_{i,t-1}))\|^2} &= \overline{\mathbb{E}\|y_{j,l_t,t}-g_j(x_{i,t-1})\|^2} \\ &\lesssim \overline{c^2_{k_i,l_t,t}} + \overline{\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\delta_i\Delta}.\end{aligned}$$

□

Lemma 10 (Bound $\mathcal{T}_2$: Variance of Clipped Points)**.**

$$\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})\right\|^2 \leq \frac{n_i}{|\mathcal{G}_i|^2}\rho^2.$$

Proof of Lemma 10. The first thing to note is that the elements in the sum $\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})$ are not independent. Therefore, we cannot get rid of the cross terms when expanding the squared-norm. However, if for each round of thresholding we sampled a fresh batch of points to set the new cluster-center estimate, then the terms would be independent. With such a resampling strategy, our bounds in these proofs only change by a constant factor. Therefore, for ease of analysis, we will assume the terms in the sum are independent. In that case,

$$\begin{aligned}\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})\right\|^2 &\leq \frac{n_i}{|\mathcal{G}_i|^2}\mathbb{E}\|y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t}\|^2 \\ &\leq \frac{n_i}{|\mathcal{G}_i|^2}\mathbb{E}\|g_j(x_{i,t-1})-\mathbb{E}g_j(x_{i,t-1})\|^2 \\ &\leq \frac{n_i}{|\mathcal{G}_i|^2}\rho^2,\end{aligned}$$

where the second-to-last inequality follows from the contractivity of the thresholding procedure. □

Lemma 11 (Bound $\mathcal{T}_3$: Error due to Out-of-Cluster Good Points)**.**

$$\overline{\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-\mathbb{E}_x\bar{g}_i(x_{i,t-1})\|^2} \lesssim \left(1+\frac{\delta_i}{2}\right)\overline{c^2_{k_i,l_t,t}} + \delta_i\overline{\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta}D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\Delta}.$$

Proof of Lemma 11. This sequence of steps bounds the clustering error due to points not from client i's cluster. Using Young's inequality for the first step,

$$\begin{aligned}&\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-\mathbb{E}_x\bar{g}_i(x_{i,t-1})\|^2 \\ &\leq \left(1+\frac{\delta_i}{2}\right)\mathbb{E}\|v_{i,l_t-1,t}-\mathbb{E}_x\bar{g}_i(x_{i,t-1})\|^2 + \left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-v_{i,l_t-1,t}\|^2 \\ &\leq \left(1+\frac{\delta_i}{2}\right)c^2_{k_i,l_t,t} + \left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-v_{i,l_t-1,t}\|^2 \\ &= \left(1+\frac{\delta_i}{2}\right)c^2_{k_i,l_t,t} + \left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}[\|g_j(x_{i,t-1})-v_{i,l_t-1,t}\|^2\mathbb{1}\{\|g_j(x_{i,t-1})-v_{i,l_t-1,t}\|\leq\tau_{k_i,l_t,t}\}] \\ &\leq \left(1+\frac{\delta_i}{2}\right)c^2_{k_i,l_t,t} + \left(1+\frac{2}{\delta_i}\right)\tau^2_{k_i,l_t,t}\mathbb{P}_{j\in\mathcal{G}_i:j\not\sim i}(\|g_j(x_{i,t-1})-v_{i,l_t-1,t}\|\leq\tau_{k_i,l_t,t}) \\ &\lesssim \left(1+\frac{\delta_i}{2}\right)c^2_{k_i,l_t,t} + \left(\frac{1}{\delta_i}\right)(c^2_{k_i,l_t,t}+A^4\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2+\delta_i\rho\Delta)\mathbb{P}_{j\in\mathcal{G}_i:j\not\sim i}(\|g_j(x_{i,t-1})-v_{i,l_t-1,t}\|\leq\tau_{k_i,l_t,t}).\end{aligned}$$

The next step is to bound the probability term in the inequality above. Note that if $\|v_{i,l_t-1,t} - g_j(x_{i,t-1})\| \leq \tau_{k_i^t,l_t}$, then

$$\|\mathbb{E}_x g_j(x_{i,t-1}) - \mathbb{E}_x g_i(x_{i,t-1})\|^2 \lesssim \|g_j(x_{i,t-1}) - \mathbb{E}_x g_j(x_{i,t-1})\|^2 + \|g_j(x_{i,t-1}) - v_{i,l_t-1,t}\|^2 + \|v_{i,l_t-1,t} - \mathbb{E}_x g_i(x_{i,t-1})\|^2$$

$$\begin{aligned}
&\lesssim \|g_j(x_{i,t-1}) - \mathbb{E}_x g_j(x_{i,t-1})\|^2 + \tau^2_{k_i,l_t,t} \\
&\quad + \|v_{i,l_t-1,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2 + \|\mathbb{E}_x g_i(x_{i,t-1}) - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2 \\
&\lesssim \|g_j(x_{i,t-1}) - \mathbb{E}_x g_j(x_{i,t-1})\|^2 + \tau^2_{k_i,l_t,t} \\
&\quad + \|v_{i,l_t-1,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2 + A^2\|\nabla \bar{f}_i(x_{i,t-1})\|^2.
\end{aligned}$$

By Assumption 4, the previous inequality implies

$$\Delta^2 - D^2\|\nabla f_i(x_{i,t-1})\|^2 \lesssim \|g_j(x_{i,t-1}) - \mathbb{E}_x g_j(x_{i,t-1})\|^2 + \tau^2_{k_i,l_t,t} + A^2\|\nabla \bar{f}_i(x_{i,t-1})\|^2 + \|v_{i,l_t-1,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2$$

which, summing over t and dividing by T, implies

$$\overline{\|g_j(x_{i,t-1}) - \mathbb{E}_x g_j(x_{i,t-1})\|^2} + \overline{\|v_{i,l_t-1,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2} + A^2\overline{\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + D^2\overline{\|\nabla f_i(x_{i,t-1})\|^2} \gtrsim \Delta^2 - \overline{\tau^2_{k_i,l_t,t}}.$$

By Markov's inequality, the probability of this event is upper-bounded by

$$\begin{aligned}
&\frac{\rho^2 + \overline{\mathbb{E}\|v_{i,l_t-1,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2} + A^2\overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2}}{\Delta^2 - \overline{\tau^2_{k_i,l_t,t}}} \\
&\lesssim \frac{\rho^2 + \overline{c^2_{k_i,l_t,t}} + A^2\overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2}}{\Delta^2},
\end{aligned}$$

where the second inequality holds due to the constraint on Δ from the theorem statement. Therefore,

$$\begin{aligned}
&\overline{\mathbb{E}_{j\in\mathcal{G}_i: j\not\sim i}\|y_{j,l_t,t} - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\|^2} \\
&\lesssim \left(1 + \frac{\delta_i}{2} + \frac{\rho}{\Delta} + \frac{\rho^2 + \overline{c^2_{k_i,l_t,t}} + A^2\overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2}}{\delta_i\Delta^2}\right)\overline{c^2_{k_i,l_t,t}} \\
&\quad + \left(\frac{\rho A^2}{\Delta} + \frac{A^4(\rho^2 + \overline{c^2_{k_i,l_t,t}} + A^2\overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2})}{\delta_i\Delta^2}\right)\overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} \\
&\quad + \frac{\rho}{\Delta}D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\Delta} \\
&\lesssim \left(1 + \frac{\delta_i}{2}\right)\overline{c^2_{k_i,l_t,t}} + \delta_i\overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \frac{\rho}{\Delta}D^2\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2} + \frac{\rho^3}{\Delta},
\end{aligned}$$

where for the last inequality we again apply the constraint on Δ. □

Lemma 12 (Clustering Error due to Bad Points)**.**

$$\overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l_t,t}\right) - \mathbb{E}_x \bar{g}_i(x_{i,t-1})\right\|^2} \lesssim \overline{c^2_{k_i,l_t,t}} + A^4\overline{\mathbb{E}\|\nabla \bar{f}_i(x_{i,t-1})\|^2} + \delta_i\rho\Delta$$

Proof of Lemma 12. This lemma bounds the clustering error due to the bad clients from client i's perspective. The goal of such clients would be to corrupt the cluster-center estimate of client i's cluster as much as possible at each round. They can have the maximum negative effect by setting their gradients to be just inside the thresholding radius around client i's cluster-center estimate. This way, the gradients will keep their value (rather than be assigned the value of the current cluster-center estimate per our update rule), but they will

have maximal effect in moving the cluster-center estimate from its current position. Therefore, in step 3 of the inequalities below, we can not do better than bounding the distance between these bad points and the current cluster center estimate (i.e. $\|y_{j,l_t,t} - v_{i,l_t-1,t}\|^2$) by the thresholding radius ($\tau^2_{k_i,l_t,t}$).

$$\begin{aligned}
\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l_t,t}\right) - \mathbb{E}_x\bar{g}_i(x_{i,t-1})\right\|^2 &\leq \mathbb{E}_{j\in\mathcal{B}_i}\|y_{j,l_t,t} - \mathbb{E}_x\bar{g}_i(x_{i,t-1})\|^2 \\
&\lesssim \mathbb{E}_{j\in\mathcal{B}_i}\|y_{j,l_t,t} - v_{i,l_t-1,t}\|^2 + \mathbb{E}\|v_{i,l_t-1,t} - \mathbb{E}_x\bar{g}_i(x_{i,t-1})\|^2 \\
&\leq \tau^2_{k_i,l_t,t} + c^2_{k_i,l_t,t} \\
&\lesssim c^2_{k_i,l_t,t} + A^4\mathbb{E}\|\nabla\bar{f}_i(x_{i,t-1})\|^2 + \delta_i\rho\Delta.
\end{aligned}$$

The last inequality applies the definition of $\tau_{k_i,l_t,t}$, and the result of the lemma follows by summing this inequality over t and dividing by T. □

A.4 Proof of Theorem 4

First we establish some notation.

Notation.

- $\mathcal{G}_i$ are the good clients and $\mathcal{B}_i$ the bad clients from client i's perspective.
- $\mathbb{E}_x$ denotes conditional expectation given the parameter, e.g. $\mathbb{E}_x g(x) = \mathbb{E}[g(x)|x]$. $\mathbb{E}$ denotes expectation over all randomness.
- k_i^t is the cluster to which client i is assigned at round t of the algorithm.
- $\overline{X_t} \triangleq \frac{1}{T}\sum_{t=1}^T X_t$ for a general variable X_t indexed by t.
- $\bar{f}_i(x) \triangleq \frac{1}{n_i}\sum_{j\in\mathcal{G}_i:j\sim i} f_j(x)$
- $\bar{m}_{i,t} \triangleq \frac{1}{n_i}\sum_{j\in\mathcal{G}_i:j\sim i} m_{j,t}$
- We introduce a variable ρ^2 to bound the variance of the momentums

  $$\mathbb{E}_x\|m_{i,t} - \mathbb{E}_x m_{i,t}\|^2 \leq \rho^2,$$

  and show in Lemma 13 how this can be written in terms of the variance of the gradients, σ^2.
- l_t is the number of rounds that Threshold-Clustering is run in round t of Federated-Clustering.
- k_i denotes the cluster to which client i is assigned.
- $v_{k_i,l,t}$ denotes the gradient update for client i in round t of Momentum-Clustering and round l of Threshold-Clustering. That is, $v_{k_i,l,t}$ corresponds to the quantity returned in Step 4 of Algorithm 4.
- To facilitate the proof, we introduce a variable $c_{k_i,l,t}$ that quantifies the distance from the cluster-center-estimates to the true cluster means. Specifically, for client i's cluster k_i at round t of Federated-Clustering and round l of Threshold-Clustering we set

  $$c^2_{k_i,l,t} = \mathbb{E}\|v_{k_i,l-1,t} - \mathbb{E}_x\bar{m}_{i,t}\|^2.$$
- For client i's cluster k_i at round t of Federated-Clustering and round l of Threshold-Clustering, we use thresholding radius

  $$\tau^2_{k_i,l,t} \approx c^2_{k_i,l,t} + \delta_i\rho\Delta.$$

- Finally, we introduce a variable, $y_{j,l,t}$, to denote the points clipped by Threshold-Clustering:

$$v_{k_i,l,t} = \frac{1}{N}\sum_{j\in[N]} \underbrace{\mathbb{1}(\|m_{j,t} - v_{k_i,l-1,t}\| \le \tau_{k_i,l,t}) + v_{k_i,l-1,t}\mathbb{1}(\|m_{j,t} - v_{k_i,l-1,t}\| > \tau_{k_i,l,t})}_{y_{j,l,t}}.$$

Proof of Theorem 4. In this proof, our goal is to bound $\overline{\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2}$. We use L-smoothness of the loss objectives to get started, and justify the non-trivial steps afterwards.

$$\begin{aligned}
\mathbb{E}f_i(x_{i,t}) &\overset{(i)}{\le} \mathbb{E}f_i(x_{i,t-1}) + \mathbb{E}\langle \nabla f_i(x_{i,t-1}), x_{i,t} - x_{i,t-1}\rangle + \frac{L}{2}\mathbb{E}\|x_{i,t} - x_{i,t-1}\|^2 \\
&= \mathbb{E}f_i(x_{i,t-1}) - \eta\mathbb{E}\langle \nabla f_i(x_{i,t-1}), v_{k_i,l_t,t}\rangle + \frac{L\eta^2}{2}\mathbb{E}\|v_{k_i,l_t,t}\|^2 \\
&= \mathbb{E}f_i(x_{i,t-1}) + \frac{\eta}{2}\mathbb{E}\|v_{k_i,l_t,t} - \nabla f_i(x_{i,t-1})\|^2 - \frac{\eta}{2}\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2 - \frac{\eta}{2}(1 - L\eta)\mathbb{E}\|v_{k_i,l_t,t}\|^2 \\
&\overset{(ii)}{\lesssim} \mathbb{E}f_i(x_{i,t-1}) + \eta\mathbb{E}\|v_{k_i,l_t,t} - \mathbb{E}_x\bar{m}_{i,t}\|^2 + \eta\mathbb{E}\|\mathbb{E}_x\bar{m}_{i,t} - \nabla f_i(x_{i,t-1})\|^2 \\
&\qquad - \frac{\eta}{2}\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2 - \frac{\eta}{2}(1 - L\eta)\mathbb{E}\|v_{k_i,l_t,t}\|^2 \\
&\overset{(iii)}{\lesssim} \mathbb{E}f_i(x_{i,t-1}) + \eta\left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i\rho\Delta\right) + \eta\mathbb{E}\|\mathbb{E}_x\bar{m}_{i,t} - \nabla f_i(x_{i,t-1})\|^2 \\
&\qquad - \frac{\eta}{2}\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2 - \frac{\eta}{2}(1 - L\eta)\mathbb{E}\|v_{k_i,l_t,t}\|^2. \qquad (18)
\end{aligned}$$

Justifications for the labeled steps are:

- (i) L-smoothness of f_i and $\eta \le 1/L$
- (ii) Young's inequality
- (iii) Lemma 14

Now it remains to bound the $\mathbb{E}\|\mathbb{E}_x\bar{m}_{i,t} - \nabla f_i(x_{i,t-1})\|^2$ term.

$$\begin{aligned}
\mathbb{E}\|\mathbb{E}_x\bar{m}_{i,t} - \nabla f_i(x_{i,t-1})\|^2 &\le \mathbb{E}\|\mathbb{E}_x m_{i,t} - \nabla f_i(x_{i,t-1})\|^2 \\
&= (1-\alpha)^2\mathbb{E}\|\mathbb{E}_x m_{i,t-1} - \nabla f_i(x_{i,t-2}) + \nabla f_i(x_{i,t-2}) - \nabla f_i(x_{i,t-1})\|^2 \\
&\overset{(i)}{\lesssim} (1-\alpha)^2(1+\alpha)\mathbb{E}\|\mathbb{E}_x m_{i,t-1} - \nabla f_i(x_{i,t-2})\|^2 + (1-\alpha)^2\left(1 + \frac{1}{\alpha}\right)L^2\eta^2\mathbb{E}\|v_{k_i,l_{t-1},t-1}\|^2 \\
&\overset{(ii)}{\le} \frac{1}{2}(1 - L\eta)\sum_{t=2}^{T}\mathbb{E}\|v_{k_i,l_{t-1},t-1}\|^2,
\end{aligned}$$

where justifications for the labeled steps are:

- (i) Young's inequality
- (ii) Assumption that $\alpha \gtrsim L\eta$

Plugging this bound back into (18), summing over $t = 1 : T$, and dividing by T gives

$$\frac{\eta}{2T}\sum_{t=1}^{T}\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2 \lesssim \frac{\mathbb{E}(f_i(x_{i,0}) - f_i^*)}{T} + \eta\left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i\rho\Delta\right). \qquad (19)$$

By the variance reduction from momentum (Lemma 13) it follows from (19) that

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2 &\lesssim \frac{\mathbb{E}(f_i(x_{i,0})-f_i^*)}{\eta T}+\left(\frac{\alpha\sigma^2}{n_i}+\frac{\alpha^{3/2}\sigma^3}{\Delta}+\beta_i\sqrt{\alpha}\sigma\Delta\right)\\ &\lesssim \frac{\mathbb{E}(f_i(x_{i,0})-f_i^*)}{\eta T}+\left(\frac{L\eta\sigma^2}{n_i}+\frac{L\eta\sigma^3}{\Delta}+\beta_i\sqrt{L\eta}\sigma\Delta\right).\end{aligned}$$

Finally, setting $\eta \lesssim \min\left\{\frac{1}{L}, \sqrt{\frac{\mathbb{E}(f_i(x_{i,0})-f_i^*)}{LT(\sigma^2/n_i+\sigma^3/\Delta)}}\right\}$,

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\|\nabla f_i(x_{i,t-1})\|^2 \lesssim \sqrt{\frac{\sigma^2/n_i+\sigma^3/\Delta}{T}}+\frac{\beta_i\sigma\Delta}{T^{\frac{1}{4}}(\sigma^2/n_i+\sigma^3/\Delta)^{\frac{1}{4}}}.$$

□

Lemma 13 (Variance reduction using Momentum)**.** *Suppose that for all $i \in [N]$ and x,*

$$\mathbb{E}\|g_i(x)-\mathbb{E}_x g_i(x)\|^2 \le \sigma^2.$$

Then

$$\mathbb{E}\|m_{i,t}-\mathbb{E}_x m_{i,t}\|^2 \le \alpha\sigma^2.$$

Proof of Lemma 13.

$$\begin{aligned}\mathbb{E}\|m_{i,t}-\mathbb{E}m_{i,t}\|^2 &= \mathbb{E}\|\alpha(g_i(x_{i,t-1})-\nabla f_i(x_{i,t-1}))+(1-\alpha)(m_{i,t-1}-\mathbb{E}m_{i,t-1})\|^2\\ &\le \alpha^2\mathbb{E}\|g_i(x_{i,t-1})-\nabla f_i(x_{i,t-1})\|^2+(1-\alpha)^2\mathbb{E}\|m_{i,t-1}-\mathbb{E}m_{i,t-1}\|^2\\ &\le \alpha^2\mathbb{E}\|g_i(x_{i,t-1})-\nabla f_i(x_{i,t-1})\|^2+(1-\alpha)\mathbb{E}\|m_{i,t-1}-\mathbb{E}m_{i,t-1}\|^2\\ &\le \alpha^2\sigma^2\sum_{q=0}^{t-1}(1-\alpha)^q\\ &\le \alpha^2\sigma^2\frac{(1-\alpha)^t-1}{(1-\alpha)-1}\\ &\le \alpha^2\sigma^2\frac{1}{\alpha}\\ &= \alpha\sigma^2.\end{aligned}$$

□

Lemma 14 (Bound on Clustering Error)**.**

$$\overline{\mathbb{E}\|v_{k_i^t,l_t}-\mathbb{E}_x\bar{m}_{i,t}\|^2} \lesssim \frac{\rho^2}{n_i}+\frac{\rho^3}{\Delta}+\beta_i\rho\Delta$$

Proof of Lemma 14. We prove the main result, and then justify each step afterwards.

$$
\begin{aligned}
\overline{\mathbb{E}\|v_{k_i,l_t,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2} &= \overline{\mathbb{E}\left\|\frac{1}{N}\sum_{j\in[N]} y_{j,l_t,t} - \mathbb{E}_x \bar{m}_{i,t}\right\|^2} \\
&= \overline{\mathbb{E}\left\|(1-\beta_i)\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t} - \mathbb{E}_x \bar{m}_{i,t}\right) + \beta_i\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,t,l} - \mathbb{E}_x \bar{m}_{i,t}\right)\right\|^2} \\
&\overset{(i)}{\leq} (1+\beta_i)(1-\beta_i)^2 \overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t}\right) - \mathbb{E}_x \bar{m}_{i,t}\right\|^2} \\
&\quad + \left(1+\frac{1}{\beta_i}\right)\beta_i^2 \overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l_t,t}\right) - \mathbb{E}_x \bar{m}_{i,t}\right\|^2} \\
&\lesssim \overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t}\right) - \mathbb{E}_x \bar{m}_{i,t}\right\|^2} + \beta_i \overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l_t,t}\right) - \mathbb{E}_x \bar{m}_{i,t}\right\|^2} \\
&\overset{(ii)}{\lesssim} (1-\delta_i+\beta_i)\overline{c^2_{k_i,l_t,t}} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i\rho\Delta\right) \\
&\overset{(iii)}{\lesssim} (1-{}^{\delta_i}/_2)\overline{c^2_{k_i,l_t,t}} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i\rho\Delta\right) \\
&\overset{(iv)}{\lesssim} \overline{(1-{}^{\delta_i}/_2)^{l_t} c^2_{k_i,1,t}} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i\rho\Delta\right) \\
&\overset{(v)}{\leq} \frac{\rho}{\Delta}\overline{c^2_{k_i,1,t}} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i\rho\Delta\right). \qquad (20)
\end{aligned}
$$

We justify each step.

- (i) Young's inequality
- (ii) We prove this bound in Lemmas 15 and 19. Importantly, it shows that the clustering error is composed of two quantities: $\mathcal{E}_1$, the error contributed by good points from the cluster's perspective, and $\mathcal{E}_2$, the error contributed by the bad points from the cluster's perspective.
- (iii) Assumption that $\beta_i \lesssim \min(\delta_i, {}^{\delta_i}/_{A^4})$
- (iv) Since $\mathbb{E}\|v_{i,l_t,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2 = c^2_{k_i,l_t+1,t}$, the inequality forms a recursion which we unroll over l_t steps.
- (v) Assumption that $l_t \geq \max(1, \frac{\log(\rho/\Delta)}{\log(1-\delta_i/2)})$

Finally, we note that

$$
\begin{aligned}
c^2_{k_i^t,1,t} &= \mathbb{E}\|\bar{m}_{i,t} - \mathbb{E}_x m_{i,t}\|^2 \\
&\lesssim \mathbb{E}\|m_{j,t} - \mathbb{E}_x m_{j,t}\|^2 + \mathbb{E}\|\mathbb{E}_x m_{j,t} - \mathbb{E}_x m_{i,t}\|^2 \\
&\leq \rho^2.
\end{aligned}
$$

Applying this bound to (20) gives

$$
\overline{\mathbb{E}\|v_{k_i,l_t,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2} \lesssim \frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta} + \beta_i\rho\Delta
$$

□

Lemma 15 (Clustering Error due to Good Points)**.**

$$\overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t}\right) - \mathbb{E}_x\bar{m}_{i,t}\right\|^2} \lesssim (1-\delta_i)\overline{c^2_{k_i,l_t,t}} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta}\right).$$

Proof of Lemma 15. We state the main sequence of steps and then justify them afterwards.

$$\begin{aligned}
&\mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t}\right) - \mathbb{E}_x\bar{m}_{i,t})\right\|^2 \\
&= \mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i} y_{j,l_t,t}\right) - \frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}\mathbb{E}_x m_{j,t} - \left(\frac{1}{n_i} - \frac{1}{|\mathcal{G}_i|}\right)\sum_{j\in\mathcal{G}_i:j\sim i}\mathbb{E}_x m_{j,t}\right\|^2 \\
&= \mathbb{E}\left\|\left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t} - \mathbb{E}_x m_{j,t})\right) + \left(\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l_t,t} - \mathbb{E}_x\bar{m}_{i,t})\right)\right\|^2 \\
&\overset{(i)}{\leq} \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t} - \mathbb{E}_x m_{j,t})\right\|^2 + \left(1+\frac{\delta_i}{2}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l_t,t} - \mathbb{E}_x\bar{m}_{i,t})\right\|^2 \\
&\overset{(ii)}{\lesssim} \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}\mathbb{E}y_{j,l_t,t} - \mathbb{E}_x m_{j,t}\right\|^2 + \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t} - \mathbb{E}_x y_{j,l_t,t})\right\|^2 \\
&\quad + \left(1+\frac{\delta_i}{2}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l_t,t} - \mathbb{E}_x\bar{m}_{i,t})\right\|^2 \\
&\overset{(iii)}{\lesssim} \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}\mathbb{E}(y_{j,l_t,t} - m_{j,t})\right\|^2 + \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(\mathbb{E}m_{j,t} - \mathbb{E}_x m_{j,t})\right\|^2 \\
&\quad + \left(1+\frac{2}{\delta_i}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t} - \mathbb{E}y_{j,l_t,t})\right\|^2 + \left(1+\frac{\delta_i}{2}\right)\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\not\sim i}(y_{j,l_t,t} - \mathbb{E}_x\bar{m}_{i,t})\right\|^2 \\
&\overset{(iv)}{\lesssim} \left(1+\frac{2}{\delta_i}\right)\delta_i^2\underbrace{\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}\|\mathbb{E}_x(y_{j,l_t,t} - m_{j,t})\|^2}_{\mathcal{T}_1} + \left(1+\frac{2}{\delta_i}\right)\frac{n_i}{|\mathcal{G}_i|^2}\rho^2 \\
&\quad + \left(1+\frac{2}{\delta_i}\right)\frac{n_i}{|\mathcal{G}_i|^2}\underbrace{\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t} - \mathbb{E}y_{j,l_t,t})\right\|^2}_{\mathcal{T}_2} + \left(1+\frac{\delta_i}{2}\right)(1-\delta_i)^2\underbrace{\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t} - \mathbb{E}_x\bar{m}_{i,t}\|^2}_{\mathcal{T}_3} \\
&\overset{(v)}{\lesssim} \delta_i\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}\|\mathbb{E}_x(y_{j,l_t,t} - m_{j,t})\|^2 + \frac{\rho^2}{n_i} + \left(1+\frac{\delta_i}{2}\right)(1-\delta_i)^2\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t} - \mathbb{E}_x\bar{m}_{i,t}\|^2 \\
&\overset{(vi)}{\lesssim} \delta_i\left(\overline{c^2_{k_i,l_t,t}} + \frac{\rho^3}{\delta_i\Delta}\right) + \frac{\rho^2}{n_i} + \left(1+\frac{\delta_i}{2}\right)(1-\delta_i)^2\left(\left(1+\frac{\delta_i}{2}\right)\overline{c^2_{k_i,l_t,t}} + \frac{\rho^3}{\Delta}\right) \\
&\lesssim \delta_i\left(\overline{c^2_{k_i,l_t,t}} + \frac{\rho^3}{\delta_i\Delta}\right) + \frac{\rho^2}{n_i} + (1-\delta_i)\left(\overline{c^2_{k_i,l_t,t}} + \frac{\rho^3}{\Delta}\right) \\
&\lesssim (1-\delta_i)\overline{c^2_{k_i,l_t,t}} + \left(\frac{\rho^2}{n_i} + \frac{\rho^3}{\Delta}\right).
\end{aligned}$$

Justifications for the labeled steps are:

- (i), (ii), (iii) Young's inequality
- (iv) First, we can can interchange the sum and the norm due to independent stochasticity of the momentums. Then, by the Tower Property and Law of Total Variance for the 1st and 3rd steps

respectively

$$\begin{aligned}\mathbb{E}\|\mathbb{E}_x m_{j,t} - \mathbb{E}m_{j,t}\|^2 &= \mathbb{E}\|\mathbb{E}_x m_{j,t} - \mathbb{E}[\mathbb{E}_x m_{j,t}]\|^2 \\ &= \mathrm{Var}(\mathbb{E}_x m_{j,t}) \\ &= \mathrm{Var}(m_{j,t}) - \mathbb{E}(\mathrm{Var}_x(m_{j,t})) \\ &= \mathrm{Var}(m_{j,t}) - \mathbb{E}\|m_{j,t} - \mathbb{E}_x m_{j,t}\|^2 \\ &\lesssim \rho^2,\end{aligned}$$

where the last inequality follows since the two terms above it are both bounded by ρ^2.

- (v) We prove this bound in Lemmas 16, 17, and 18. It shows that, from point i's perspective, the error of its cluster-center-estimate is composed of three quantities: $\mathcal{T}_1$, the error introduced by our thresholding procedure on the good points which belong to i's cluster (and therefore ideally are included within the thresholding radius); $\mathcal{T}_2$, which accounts for the variance of the clipped points in i's cluster; and $\mathcal{T}_3$, the error due to the good points which don't belong to i's cluster (and therefore ideally are forced outside the thresholding radius).
- (vi) $(1 + x/2)^2(1-x)^2 \leq 1 - x$ for all $x \in [0, 1]$

□

Lemma 16 (Bound $\mathcal{T}_1$: Error due to In-Cluster Good Points)**.**

$$\overline{\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}\|\mathbb{E}(y_{j,l_t,t} - m_{j,t})\|^2} \lesssim \overline{c^2_{k_i,l_t,t}} + \frac{\rho^3}{\delta_i\Delta}.$$

Proof of Lemma 16. By definition of $y_{j,l_t,t}$,

$$\begin{aligned}\mathbb{E}\|y_{j,l_t,t} - m_{j,t}\| &= \mathbb{E}[\|v_{k_i,l_t-1,t} - m_{j,t}\|\mathbb{1}(\|v_{k_i,l_t-1,t} - m_{j,t}\| > \tau_{k_i,l_t,t})] \\ &\leq \frac{\mathbb{E}[\|v_{k_i,l_t-1,t} - m_{j,t}\|^2\mathbb{1}(\|v_{k_i,l_t-1,t} - m_{j,t}\| > \tau_{k_i,l_t,t})]}{\tau_{k_i,l_t,t}}.\end{aligned}$$

Therefore by Jensen's inequality,

$$\begin{aligned}\|\mathbb{E}y_{j,l_t,t} - m_{j,t}\|^2 &\leq (\mathbb{E}\|y_{j,l_t,t} - m_{j,t}\|)^2 \\ &\leq \frac{(\mathbb{E}\|v_{k_i,l_t-1,t} - m_{j,t}\|^2)^2}{\tau^2_{k_i,l_t,t}} \\ &\lesssim \frac{(\mathbb{E}\|v_{k_i,l_t-1,t} - \mathbb{E}_x\bar{m}_{i,t}\|^2 + \mathbb{E}\|\mathbb{E}_x\bar{m}_{i,t} - m_{j,t}\|^2)^2}{\tau^2_{k_i,l_t,t}} \\ &= \frac{(\mathbb{E}\|v_{k_i,l_t-1,t} - \mathbb{E}_x\bar{m}_{i,t}\|^2 + \mathbb{E}\|\mathbb{E}_x m_{j,t} - m_{j,t}\|^2)^2}{\tau^2_{k_i,l_t,t}} \\ &\leq \frac{(c^2_{k_i,l_t,t} + \rho^2)^2}{\tau^2_{k_i,l_t,t}} \\ &\lesssim \frac{(c^2_{k_i,l_t,t} + \rho^2)^2}{c^2_{k_i,l_t,t} + \delta_i\rho\Delta} \\ &\lesssim \left(1 + \frac{\rho}{\delta_i\Delta}\right)c^2_{k_i,l_t,t} + \frac{\rho^3}{\delta_i\Delta} \\ &\lesssim c^2_{k_i,l_t,t} + \frac{\rho^3}{\delta_i\Delta}.\end{aligned}$$

Summing this inequality over t and dividing by T, we have

$$\begin{aligned}\overline{\mathbb{E}_{j\in\mathcal{G}_i:j\sim i}\|\mathbb{E}_x(y_{j,l_t,t}-m_{j,t})\|^2} &\leq \overline{\mathbb{E}\|y_{j,l_t,t}-m_{j,t}\|^2}\\ &\lesssim \overline{c^2_{k_i,l_t,t}}+\frac{\rho^3}{\delta_i\Delta}.\end{aligned}$$

□

Lemma 17 (Bound $\mathcal{T}_2$: Variance of Clipped Points)**.**

$$\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})\right\|^2 \leq \frac{n_i}{|\mathcal{G}_i|^2}\rho^2.$$

Proof of Lemma 17. Note that the elements in the sum $\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})$ are not independent. Therefore, we cannot get rid of the cross terms when expanding the squared-norm. However, if for each round of thresholding we sampled a fresh batch of points to set the new cluster-center estimate, then the terms would be independent. With this resampling strategy, our bounds would only change by a constant factor. Therefore, for ease of analysis, we will assume the terms in the sum are independent. In that case,

$$\begin{aligned}\mathbb{E}\left\|\frac{1}{|\mathcal{G}_i|}\sum_{j\in\mathcal{G}_i:j\sim i}(y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t})\right\|^2 &\leq \frac{n_i}{|\mathcal{G}_i|^2}\mathbb{E}\|y_{j,l_t,t}-\mathbb{E}y_{j,l_t,t}\|^2\\ &\leq \frac{n_i}{|\mathcal{G}_i|^2}\mathbb{E}\|m_{j,t}-\mathbb{E}m_{j,t}\|^2\\ &\leq \frac{n_i}{|\mathcal{G}_i|^2}\rho^2,\end{aligned}$$

where the second-to-last inequality follows from the contractivity of the thresholding procedure. □

Lemma 18 (Bound $\mathcal{T}_3$: Error due to Out-of-Cluster Good Points)**.**

$$\overline{\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-\mathbb{E}_x\bar{m}_{i,t}\|^2} \lesssim \left(1+\frac{\delta_i}{2}\right)\overline{c^2_{k_i,l_t,t}}+\frac{\rho^3}{\Delta}.$$

Proof of Lemma 18. This sequence of steps bounds the clustering error due to points not from client i's cluster. Using Young's inequality for the first step,

$$\begin{aligned}&\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-\mathbb{E}_x\bar{m}_{i,t}\|^2\\ &\leq \left(1+\frac{\delta_i}{2}\right)\mathbb{E}\|v_{k_i^t,l_t-1}-\mathbb{E}_x\bar{m}_{i,t}\|^2+\left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-v_{k_i,l_t-1,t}\|^2\\ &\leq \left(1+\frac{\delta_i}{2}\right)c^2_{k_i,l_t,t}+\left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}\|y_{j,l_t,t}-v_{k_i,l_t-1,t}\|^2\\ &= \left(1+\frac{\delta_i}{2}\right)c^2_{k_i,l_t,t}+\left(1+\frac{2}{\delta_i}\right)\mathbb{E}_{j\in\mathcal{G}_i:j\not\sim i}[\|m_{j,t}-v_{k_i,l_t-1,t}\|^2\mathbb{1}\{\|m_{j,t}-v_{k_i,l_t-1,t}\|\leq\tau_{k_i,l_t,t}\}]\\ &\leq \left(1+\frac{\delta_i}{2}\right)c^2_{k_i,l_t,t}+\left(1+\frac{2}{\delta_i}\right)\tau^2_{k_i,l_t,t}\mathbb{P}_{j\in\mathcal{G}_i:j\not\sim i}(\|m_{j,t}-v_{k_i,l_t-1,t}\|\leq\tau_{k_i,l_t,t})\\ &\lesssim \left(1+\frac{\delta_i}{2}\right)c^2_{k_i,l_t,t}+\left(\frac{1}{\delta_i}\right)(c^2_{k_i,l_t,t}+\delta_i\rho\Delta)\mathbb{P}_{j\in\mathcal{G}_i:j\not\sim i}(\|m_{j,t}-v_{k_i^t,l_t-1}\|\leq\tau_{k_i,l_t,t}).\end{aligned}$$

If $\|v_{k_i^t,l_t-1}-m_{j,t}\|\leq\tau_{k_i,l_t,t}$, then

$$\begin{aligned}\|\mathbb{E}_x m_{j,t}-\mathbb{E}_x m_{i,t}\|^2 &\lesssim \|m_{j,t}-\mathbb{E}_x m_{j,t}\|^2+\|m_{j,t}-v_{k_i,l_t-1,t}\|^2+\|v_{k_i^t,l_t-1}-\mathbb{E}_x\bar{m}_{i,t}\|^2\\ &\lesssim \|m_{j,t}-\mathbb{E}_x m_{j,t}\|^2+\tau^2_{k_i,l_t,t}+\|v_{k_i,l_t-1,t}-\mathbb{E}_x\bar{m}_{i,t}\|^2\end{aligned}$$

By Assumption 9,

$$\Delta^2 \lesssim \|m_{j,t} - \mathbb{E}_x m_{j,t}\|^2 + \tau^2_{k_i,l_t,t} + \|v_{k_i,l_t-1,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2$$

which, summing over t and dividing by T, implies

$$\overline{\|m_{j,t} - \mathbb{E}_x m_{j,t}\|^2} + \overline{\|v_{k_i,l_t-1,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2} \gtrsim \Delta^2 - \overline{\tau^2_{k_i,l_t,t}}.$$

By Markov's inequality, the probability of this event is upper-bounded by

$$\frac{\rho^2 + \overline{\mathbb{E}\|v_{k_i,l_t-1,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2}}{\Delta^2 - \overline{\tau^2_{k_i,l_t,t}}} \lesssim \frac{\rho^2 + \overline{c^2_{k_i,l_t,t}}}{\Delta^2},$$

where the inequality holds due to the constraint on Δ from the theorem statement. Therefore,

$$\begin{aligned}\overline{\mathbb{E}_{j\in\mathcal{G}_i: j\not\sim i}\|y_{j,l_t,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2} &\lesssim \left(1 + \frac{\delta_i}{2} + \frac{\rho}{\Delta} + \frac{\rho^2 + \overline{c^2_{k_i,l_t,t}}}{\delta_i \Delta^2}\right)\overline{c^2_{k_i,l_t,t}} + \frac{\rho^3}{\Delta} \\ &\lesssim \left(1 + \frac{\delta_i}{2}\right)\overline{c^2_{k_i,l_t,t}} + \frac{\rho^3}{\Delta},\end{aligned}$$

where again we apply the constraint on Δ for the second inequality. □

Lemma 19 (Clustering Error due to Bad Points)**.**

$$\overline{\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l_t,t}\right) - \mathbb{E}_x \bar{m}_{i,t}\right\|^2} \lesssim \overline{c^2_{k_i,l_t,t}} + \delta_i \rho \Delta$$

Proof of Lemma 19. This lemma gives a bound on the clustering error due to the bad clients from client i's perspective. The goal of such clients would be to corrupt the cluster-center estimate of client i's cluster as much as possible at each round. They can have the maximum negative effect by setting their gradients to be just inside the thresholding radius around client i's cluster-center estimate. This way, the gradients will keep their value (rather than be assigned the value of the current cluster-center estimate per our update rule), but they will have maximal effect in moving the cluster-center estimate from its current position. Therefore, in step 3 of the inequalities below, we can not do better than bounding the distance between these bad points and the current cluster center estimate (i.e. $\|y_{j,l_t,t} - v_{k_i,l_t-1,t}\|^2$) by the thresholding radius $\tau_{k_i,l_t,t}$.

$$\begin{aligned}\mathbb{E}\left\|\left(\frac{1}{|\mathcal{B}_i|}\sum_{j\in\mathcal{B}_i} y_{j,l_t,t}\right) - \mathbb{E}_x \bar{m}_{i,t}\right\|^2 &\leq \mathbb{E}_{j\in\mathcal{B}_i}\|y_{j,l_t,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2 \\ &\lesssim \mathbb{E}_{j\in\mathcal{B}_i}\|y_{j,l_t,t} - v_{k_i,l_t-1,t}\|^2 + \mathbb{E}\|v_{k_i,l_t-1,t} - \mathbb{E}_x \bar{m}_{i,t}\|^2 \\ &\leq \tau^2_{k_i,l_t,t} + c^2_{k_i,l_t,t} \\ &\lesssim c^2_{k_i,l_t,t} + \delta_i \rho \Delta.\end{aligned}$$

The last inequality applies the definition of $\tau_{k_i,l_t,t}$, and the result of the lemma follows by summing this inequality over t and dividing by T. □