MULTI-LEVEL LOCAL SGD: DISTRIBUTED SGD FOR HETEROGENEOUS HIERARCHICAL NETWORKS

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ABSTRACT

We propose Multi-Level Local SGD, a distributed stochastic gradient method for learning a smooth, non-convex objective in a multi-level communication network with heterogeneous workers. Our network model consists of a set of disjoint sub-networks, with a single hub and multiple workers; further, workers may have different operating rates. The hubs exchange information with one another via a connected, but not necessarily complete communication network. In our algorithm, sub-networks execute a distributed SGD algorithm, using a hub-and-spoke paradigm, and the hubs periodically average their models with neighboring hubs. We first provide a unified mathematical framework that describes the Multi-Level Local SGD algorithm. We then present a theoretical analysis of the algorithm; our analysis shows the dependence of the convergence error on the worker node heterogeneity, hub network topology, and the number of local, sub-network, and global iterations. We back up our theoretical results via simulation-based experiments using both convex and non-convex objectives.

1 INTRODUCTION

Stochastic Gradient Descent (SGD) is a key algorithm in modern Machine Learning and optimization (Amari, 1993). To support distributed data as well as reduce training time, Zinkevich et al. (2010) introduced a distributed form of SGD. Traditionally, distributed SGD is run within a hub-and-spoke network model: a central parameter server (hub) coordinates with worker nodes. At each iteration, the hub sends a model to the workers. The workers each train on their local data, taking a gradient step, then return their locally trained model to the hub to be averaged. Distributed SGD can be an efficient training mechanism when message latency is low between the hub and workers, allowing gradient updates to be transmitted quickly at each iteration. However, as noted in Moritz et al. (2016), message transmission latency is often high in distributed settings, which causes a large increase in overall training time. A practical way to reduce this communication overhead is to allow the workers to take multiple local gradient steps before communicating their local models to the hub. This form of distributed SGD is referred to as Local SGD (Lin et al., 2018; Stich, 2019). There is a large body of work that analyzes the convergence of Local SGD and the benefits of multiple local training rounds (McMahan et al., 2017; Wang & Joshi, 2018; Li et al., 2019).

Local SGD is not applicable to all scenarios. Workers may be heterogeneous in terms of their computing capabilities, and thus the time required for local training is not uniform. The vast majority of previous work uses a synchronous model, where all clients train for the same number of rounds before sending updates to the hub (Dean et al., 2012; Ho et al., 2013; Cipar et al., 2013). Further, most works assume a hub-and-spoke model, but this does not capture many real world settings. For example, devices in an ad-hoc network may not all be able to communicate to a central hub in a single hop due to network or communication range limitations. In such settings, a multi-level communication network model may be beneficial. In flying ad-hoc networks (FANETs), a network architecture has been proposed to improve scalability by partitioning the UAVs into mission areas (Bekmezci et al., 2013). Here, clusters of UAVs have their own clusterheads, or hubs, and these hubs communicate through an upper level network, e.g., via satellite. Multi-level networks have also been utilized in Fog and Edge computing, a paradigm designed to improve data aggregation and analysis in wireless sensor networks, autonomous vehicles, power systems, and more (Bonomi et al., 2012; Laboratory, 2017; Satyanarayanan, 2017).

Motivated by these observations, we propose Multi-Level Local SGD (MLL-SGD), a distributed learning algorithm for heterogeneous multi-level networks. Specifically, we consider a two-level network structure. The lower level consists of a disjoint set of hub-and-spoke *sub-networks*, each with a single hub server and a set of workers. The upper level network consists of a connected, but not necessarily complete, *hub network* by which the hubs communicate. For example, in a Fog Computing application, the sub-network workers may be edge devices connected to their local data center, and the data centers act as hubs communicating over a decentralized network. Each sub-network runs one or more Local SGD rounds, in which its workers train for a local training period, followed by model averaging at the sub-network's hub. Periodically, the hubs average their models with neighbors in the hub network. We model heterogeneous workers using a stochastic approach; each worker executes a local training iteration in each time step with a probability proportional to its computational resources. Thus, different workers may take different numbers of gradient steps within each local training period.

We prove the convergence of MLL-SGD for smooth and potentially non-convex loss functions. We assume data is distributed in an IID manner to all workers. Further, we analyze the relationship between the algorithm convergence error and algorithm parameters, and find that, for a fixed step size, error is quadratic in the number of local training iterations and the number of sub-network training iterations, and linear in the average worker operating rate. Our algorithm and analysis are general enough to encompass several variations of SGD as special cases, including classical SGD (Amari, 1993), SGD with weighted workers (McMahan et al., 2017), and Decentralized Local SGD with an arbitrary hub communication network (Wang & Joshi, 2018). Our work provides novel analysis of a distributed learning algorithm in a multi-level network model with heterogeneous workers.

The specific contributions of this paper are as follows: 1) We formalize the multi-level network model with heterogeneous workers, and we define the MLL-SGD algorithm for training models in such a network. 2) We provide theoretical analysis of the convergence guarantees of MLL-SGD with heterogeneous workers. 3) We provide an experimental evaluation that highlights our theoretical convergence guarantees. The experiments illustrate the convergence error of MLL-SGD in practice, as well as provide insight into how each parameter affects convergence.

The rest of the paper is structured as follows. In Section 2, we discuss related work. Section 3 introduces the system model and problem formulation. We describe MLL-SGD in Section 4, and we present our main theoretical results in Section 5. Proofs of these results are deferred to the appendix. We provide experimental results in Section 6. Finally, we conclude in Section 7.

2 Related Work

Distributed SGD is a well studied subject in Machine Learning. Zinkevich et al. (2010) introduced a parallel SGD in a hub-and-spoke model. Variations on Local SGD in the hub-and-spoke model have been studied in several works (Moritz et al., 2016; Zhang et al., 2016; McMahan et al., 2017). Many works have provided convergence bounds of SGD within this model (Wang et al., 2019b; Li et al., 2019). There is also a large body of work on decentralized approaches for optimization using gradient based methods, dual averaging, and deep learning (Tsitsiklis et al., 1986; Jin et al., 2016; Wang et al., 2019a). These previous works, however, do not address a multi-level network structure.

Several recent works analyze Hierarchical Local SGD (HL-SGD), an algorithm for training a model in a multi-level network. HL-SGD assumes that the sub-networks and hub network topologies are hub-and-spoke and workers are homogeneous. Zhou & Cong (2019) and Liu et al. (2020) analyze the convergence error of HL-SGD, while Abad et al. (2020) analyzes convergence time. Unlike HL-SGD, MLL-SGD accounts for heterogeneous worker operating rates and an arbitrary hub communication graph, which more realistically models many real-world distributed platforms.

In practice, workers may be heterogeneous in nature, which means that they may execute training iterations at different rates. For this reason, it can be either costly or impossible for workers to train in a fully synchronous manner, as stragglers may hold up global computation. Lian et al. (2017) addressed this heterogeneity by defining a gossip-based asynchronous SGD algorithm. In Stich (2019), workers are modeled to take gradient steps at an arbitrary subset of all iterations. However, none of these works address a multi-level network model.

Several other works seek to encapsulate many variations of SGD under a single framework. Koloskova et al. (2020) created a generalized model that considers a gossip-based decentralized SGD algorithm where the communication network is time-varying. However, this work does not account for a multi-level network model nor worker heterogeneity. Wang et al. introduced the Cooperative SGD framework (Wang & Joshi, 2018), a model that includes communication reduction through local SGD steps and decentralized mixing between homogeneous workers. Cooperative SGD also allows for of auxiliary variables. These auxiliary variables can be used to model SGD in a multi-level network, but only when the hubs average across the hub network whenever they average within their own sub-network. Our model is more general; it considers heterogeneous workers and it allows for an arbitrary number of averaging rounds within each sub-network between averaging rounds across sub-networks, which is more practical in multi-level networks where inter-hub communication is slow or costly.

3 System Model and Problem Formulation

In this section, we introduce our system model, the objective function that we seek to minimize, and the assumptions we make about the function.

We consider a set of D sub-networks $\mathcal{D} = \{1, \ldots, D\}$. Each sub-network $d \in \mathcal{D}$ has a single hub and a set of workers $\mathcal{M}^{(d)}$, with $|\mathcal{M}^{(d)}| = N^{(d)}$. Workers in $\mathcal{M}^{(d)}$ only communicate with their own hub and not with any other workers or hubs. We define the set of all workers in the system as $\mathcal{M} = \bigcup_{d=1}^{D} \mathcal{M}^{(d)}$. Let $|\mathcal{M}| = N$. Each worker i holds a set $\mathcal{S}^{(i)}$ of local training data. Let $\mathcal{S} = \bigcup_{i=1}^{N} \mathcal{S}^{(i)}$. The set of all D hubs is denoted $\mathcal{C}$. The hubs communicate with one another via an undirected, connected communication graph $G = (\mathcal{C}, E)$. Let $\mathcal{N}_d = \{j \mid e_{d,j} \in E\}$ denote the set of neighbors of the hub in sub-network d in the hub graph G.

Let the model parameters be denoted by $\boldsymbol{x} \in \mathbb{R}^n$. Our goal is to find an $\boldsymbol{x}$ that minimizes the following objective function over the training set:

$$F(\boldsymbol{x}) = \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} f(\boldsymbol{x}; s) \tag{1}$$

where $f(\cdot)$ is the loss function. The workers collaboratively minimize this loss function, in part by executing local iterations of SGD over their training sets. For each executed local iteration, a worker samples a mini-batch of data uniformly at random from its local data. Let ξ be a randomly sampled mini-batch of data and let $g(\boldsymbol{x}; \xi) = \frac{1}{|\xi|} \sum_{s \in \xi} \nabla f(\boldsymbol{x}; s)$ be the mini-batch gradient. For simplicity, we use $g(\boldsymbol{x})$ instead of $g(\boldsymbol{x}; \xi)$ from here on.

Assumption 1. *The objective function and the mini-batch gradients satisfy the following:*

- *1a The objective function $F : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, and the gradient is Lipschitz with constant $L > 0$, i.e., $\|\nabla F(\boldsymbol{x}) - \nabla F(\boldsymbol{y})\|_2 \leq L\|\boldsymbol{x} - \boldsymbol{y}\|_2$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.*
- *1b The function F is lower bounded, i.e., $F(\boldsymbol{x}) \geq F_{inf} > -\infty$ for all $\boldsymbol{x} \in \mathbb{R}^n$.*
- *1c The mini-batch gradients are unbiased, i.e., $\mathbb{E}_{\xi|\boldsymbol{x}}[g(\boldsymbol{x})] = \nabla F(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.*
- *1d There exist scalars $\beta \geq 0$ and $\sigma \geq 0$ such that $\mathbb{E}_{\xi|\boldsymbol{x}}\|g(\boldsymbol{x}) - \nabla F(\boldsymbol{x})\|_2^2 \leq \beta||\nabla F(\boldsymbol{x})||_2^2 + \sigma^2$ for all $\boldsymbol{x} \in \mathbb{R}^n$.*

Assumption 1a requires that the gradients do not change too rapidly, and Assumption 1b requires that our objective function is lower bounded by some F_{inf}. Assumptions 1c and 1d assume that the local data at each worker can be used as an unbiased estimate for the full dataset with the same bounded variance. These assumptions are common in convergence analysis of SGD algorithms (e.g., Bottou et al. (2018)).

4 Algorithm

We now present our Multi-Level Local SGD (MLL-SGD) algorithm. The pseudocode is shown in Algorithm 1. Each sub-network trains in parallel and, periodically, the hubs average their models

Algorithm 1 Multi-Level Local SGD

1: **Initialize:** $\boldsymbol{y}_1^{(d)}$ for hubs $d = 1, \ldots, D$
2: **for** $k = 1, \ldots, K$ **do**
3: **parallel for** $d \in \mathcal{D}$ **do**
4: **parallel for** $i \in \mathcal{M}^{(d)}$ **do**
5: $\boldsymbol{x}_k^{(i)} \leftarrow \boldsymbol{y}_k^{(d)}$ ▷ Workers receive updated model from hub
6: **for** $j = k, \ldots, k + \tau - 1$ **do**
7: $\boldsymbol{x}_{k+1}^{(i)} \leftarrow \boldsymbol{x}_k^{(i)} - \eta \boldsymbol{g}_k^{(i)}$ ▷ Local iteration (probabilistic)
8: **end for**
9: **end parallel for**
10: $\boldsymbol{z}^{(d)} \leftarrow \sum_{i \in \mathcal{M}^{(d)}} v^{(i)} \boldsymbol{x}_{k+1}^{(i)}$ ▷ Hub d computes average of its workers' models
11: **if** $k \mod q \cdot \tau = 0$ **then**
12: $\boldsymbol{y}_{k+1}^{(d)} \leftarrow \sum_{j \in \mathcal{N}^{(d)}} \mathbf{H}_{j,d} \boldsymbol{z}^{(j)}$ ▷ Hub d averages its model with neighboring hubs
13: **else**
14: $\boldsymbol{y}_{k+1}^{(d)} \leftarrow \boldsymbol{z}^{(d)}$
15: **end if**
16: **end parallel for**
17: **end for**

with neighboring hubs. The steps corresponding to Local SGD are shown in lines 5-10. Each hub and worker stores a copy of the model. For worker $i \in \mathcal{M}^{(d)}$, we denote its copy of the local model by $\boldsymbol{x}^{(i)}$. We denote the model at hub d by $\boldsymbol{y}^{(d)}$. The hub first sends its model to its workers, and the workers update their local models to match their hub's model. Workers then execute multiple local training iterations, shown in line 7, to refine their local models independently. To represent the different rates of computation at each worker, we use a probabilistic approach. We assume that, on average, a worker i can execute $\tau^{(i)}$ local iterations for every τ time steps ($\tau^{(i)} \leq \tau$). We thus define the N-vector $\boldsymbol{p}$ where each entry $\boldsymbol{p}_i = \frac{\tau^{(i)}}{\tau}$ is the probability with which worker i executes a local gradient step in each iteration k. Worker i updates its local model at iteration k as follows:

$$\boldsymbol{x}_{k+1}^{(i)} = \boldsymbol{x}_k^{(i)} - \eta \boldsymbol{g}_k^{(i)} \tag{2}$$

where η is the step size and $\boldsymbol{g}_k^{(i)}$ is a random variable such that

$$\boldsymbol{g}_k^{(i)} = \begin{cases} g(\boldsymbol{x}_k^{(i)}) & \text{w/ probability } \boldsymbol{p}_i \\ \mathbf{0} & \text{w/ probability } 1 - \boldsymbol{p}_i. \end{cases} \tag{3}$$

Note that in expectation, worker i takes $\tau^{(i)}$ gradient steps in τ time steps.

After τ time steps, the hub updates its models based on the models of its workers (line 10). For each worker i, we assign a positive weight $w^{(i)}$. Let $v^{(i)}$ be the weight for worker i normalized within its sub-network: $v^{(i)} = \frac{w^{(i)}}{\sum_{j \in \mathcal{M}^{(d(i))}} w^{(j)}}$, where $d(i)$ denotes the sub-network of worker i. Each hub's updates its model to be a weighted model average over the workers in its sub-network: $\boldsymbol{y}^{(d)} = \sum_{i \in \mathcal{M}^{(d)}} v^{(i)} \boldsymbol{x}^{(i)}$.

Weights may be assigned for different reasons. If all worker gradients are treated equally, then $w^{(i)} = 1$ and $v^{(i)} = \frac{1}{N^{(d(i))}}$. We may also weight a worker's gradient proportional to its local dataset size, in which case $w^{(i)} = |\mathcal{S}^{(i)}|$ and $v^{(i)} = \frac{|\mathcal{S}^{(i)}|}{\sum_{r \in \mathcal{M}^{(d(i))}} |\mathcal{S}^{(r)}|}$. The latter approach is used in Federated Averaging (McMahan et al., 2017).

After q iterations of Local SGD in each sub-network ($q \cdot \tau$ time steps), the hubs average their models with their neighbors in the hub communication network (line 12). The weight assigned to each hub's model is defined by a $D \times D$ matrix $\mathbf{H}$ so that:

$$\boldsymbol{y}^{(d)} = \sum_{j \in \mathcal{N}^{(d)}} \mathbf{H}_{j,d} \boldsymbol{y}^{(j)}. \tag{4}$$

Define the total weight in the network to be $w_{tot} = \sum_{i \in \mathcal{M}} w^{(i)}$. Let $\boldsymbol{b}$ be a D-vector with each component d given by $\boldsymbol{b}_d = (\sum_{i \in \mathcal{M}^{(d)}} w^{(i)})/w_{tot}$. We assume $\mathbf{H}$ meets the following requirements.

Assumption 2. *The matrix* $\mathbf{H}$ *satisfies the following:*

2a If $(i, j) \in E$*, then* $\mathbf{H}_{i,j} > 0$*. Otherwise,* $\mathbf{H}_{i,j} = 0$*.*

2b $\mathbf{H}$ *is column stochastic, i.e.,* $\sum_{i=1}^{D} \mathbf{H}_{i,j} = 1$*.*

2c For all $i, j \in \mathcal{D}$*, we have* $\boldsymbol{b}_i \mathbf{H}_{i,j} = \boldsymbol{b}_j \mathbf{H}_{j,i}$*.*

Assumption 2 implies that $\mathbf{H}$ has one as a simple eigenvalue, with corresponding right eigenvector $\boldsymbol{b}$ and left eigenvector $\mathbf{1}_D$. Further, all of its other eigenvalues have magnitude strictly less than 1 (since G is connected) (Rotaru & Nägeli, 2004). By defining $\mathbf{H}$ in this way, we ensure that the contributions from the workers' gradients in each hub are incorporated in proportion to the workers' weights. This weighted averaging approach allows us to naturally extend Federated Averaging to the multi-level network model.

5 Analysis

We note that hubs are essentially stateless in our model, as the hub models are copied to all workers after each sub-network or hub averaging. Thus, our analysis focuses on how worker models evolve. We first present an equivalent formulation of the MLL-SGD algorithm in terms of the evolution of the worker models. We then present our main result on the convergence of MLL-SGD.

The system behavior can be summarized by the following update rule for worker models:

$$\mathbf{X}_{k+1} = (\mathbf{X}_k - \eta \, \mathbf{G}_k) \, \mathbf{T}_k \tag{5}$$

where $\mathbf{X}_k = [\boldsymbol{x}_k^{(1)}, \ldots, \boldsymbol{x}_k^{(N)}]$, $\mathbf{G}_k = [\boldsymbol{g}_k^{(1)}, \ldots, \boldsymbol{g}_k^{(N)}]$, and $\mathbf{T}_k$ is a time-varying operator that captures the three stages in MLL-SGD: local iterations, hub-and-spoke averaging within each sub-network, and averaging across the hub network. We define $\mathbf{T}_k$ as follows:

$$\mathbf{T}_k = \begin{cases} \mathbf{Z} & \text{if } k \bmod q\tau = 0 \\ \mathbf{V} & \text{if } k \bmod \tau = 0 \text{ and } k \bmod q\tau \neq 0 \\ \mathbf{I} & \text{otherwise.} \end{cases} \tag{6}$$

For local iterations, $\mathbf{T}_k = \mathbf{I}$, as there are no interactions between workers or hubs. For sub-network averaging, $\mathbf{V}$ is an $N \times N$ block diagonal matrix, with each block $\mathbf{V}^{(d)}$ corresponding to a single sub-network d. The matrix $\mathbf{V}^{(d)}$ is an $N^{(d)} \times N^{(d)}$ matrix where each entry is $\mathbf{V}_{i,j}^{(d)} = v^{(i)}$. Finally, we define an $N \times N$ matrix $\mathbf{Z}$ that captures the sub-network averaging and hub network averaging in one operation that involves all workers. The components of $\mathbf{Z}$ are given by

$$\mathbf{Z}_{i,j} = \mathbf{H}_{d(i),d(j)} v^{(i)}. \tag{7}$$

Let $\boldsymbol{a}$ be an N-vector with each component $\boldsymbol{a}_i = \frac{w^{(i)}}{w_{tot}}$ representing the weight of worker i, normalized over all worker weights. We observe that $\mathbf{Z}$ and $\mathbf{V}$ satisfy the following: each have a right eigenvector of $\boldsymbol{a}$ and left eigenvector of $\mathbf{1}_N^T$ with eigenvalue 1 and all other eigenvalues have magnitude strictly less than 1. The proof of these properties can be found in the appendix. These properties are necessary (but not sufficient) to ensure that the worker models converge to a consensus model, where each worker's updates have been incorporated according to the worker's weight.

As is common, we study an averaged model over all workers in the system (Yuan et al., 2016; Wang & Joshi, 2018). Specifically, we define a weighted average model:

$$\boldsymbol{u}_k = \mathbf{X}_k \, \boldsymbol{a}. \tag{8}$$

We identify the recurrence relation of $\boldsymbol{u}_k$. If we multiply $\boldsymbol{a}$ on both sides of (5):

$$\mathbf{X}_{k+1} \, \boldsymbol{a} = (\mathbf{X}_k - \eta \, \mathbf{G}_k) \, \mathbf{T}_k \, \boldsymbol{a} \tag{9}$$

$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k - \eta \, \mathbf{G}_k \, \boldsymbol{a} \tag{10}$$

$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k - \eta \sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{g}_k^{(i)} \tag{11}$$

where (10) follows from $\boldsymbol{a}$ being a right eigenvector of $\mathbf{V}$ and $\mathbf{Z}$ with eigenvalue 1. We note that $\boldsymbol{u}_k$ is updated via an approximate stochastic gradient descent step since we use a weighted average of several mini-batch gradients. Since $F(\cdot)$ may be non-convex, SGD may converge to a local minimum or saddle point. Thus, we study the gradients of $\boldsymbol{u}_k$ as k increases.

We next provide the main theoretical result of the paper.

Theorem 1. *Under Assumptions 1 and 2, if η satisfies the following for all $i \in \mathcal{M}$:*

$$(4\boldsymbol{p}_i - \boldsymbol{p}_i^2 - 2) \geq \eta L \left(\boldsymbol{a}_i \boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i \boldsymbol{p}_i^2 + \boldsymbol{p}_i^2\right) + 8L^2\eta^2 q^2 \tau^2 \Gamma \tag{12}$$

where $\Gamma = \frac{\zeta}{1-\zeta^2} + \frac{2}{1-\zeta} + \frac{\zeta}{(1-\zeta)^2}$ and $\zeta = \max\{|\lambda_2(\mathbf{H})|, |\lambda_N(\mathbf{H})|\}$, then the expected square norm of the average gradient, averaged over K iterations, is bounded as follows:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\|\nabla F(\boldsymbol{u}_k)\|_2^2\right] &\leq \frac{2\left(F(\boldsymbol{x}_1) - F_{inf}]\right)}{\eta K} + \sigma^2 \eta L \sum_{i=1}^{N} \boldsymbol{a}_i^2 \boldsymbol{p}_i \\
&\quad + 4L^2\eta^2\sigma^2 q^3 \tau^3 \left(\frac{1}{q\tau} - \frac{1}{K}\right)\left(\frac{\zeta^2}{1-\zeta^2} + \frac{2\zeta}{1-\zeta} + \frac{1}{(1-\zeta)^2}\right)\mathrm{P} \\
&\quad + 4L^2\eta^2\sigma^2 \left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2 \frac{(q-1)(2q+1)}{6} + \frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P} \qquad (13) \\
\xrightarrow{K\to\infty} \quad & \sigma^2 \eta L \sum_{i=1}^{N} \boldsymbol{a}_i^2 \boldsymbol{p}_i + 4L^2\eta^2\sigma^2 q^2 \tau^2 \left(\frac{\zeta^2}{1-\zeta^2} + \frac{2\zeta}{1-\zeta} + \frac{1}{(1-\zeta)^2}\right)\mathrm{P} \\
&\quad + 4L^2\eta^2\sigma^2 \left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2 \frac{(q-1)(2q+1)}{6} + \frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P} \qquad (14)
\end{aligned}$$

where $\mathrm{P} = \sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i$.

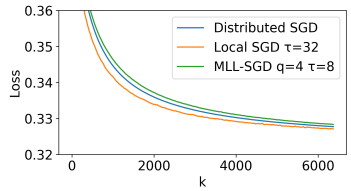
The proof of Theorem 1 is provided in the appendix.

The first term in (13) is the same as in centralized SGD (Bottou et al., 2018). As $K \to \infty$, this term goes to zero. The second term is similar to centralized SGD as well. If the gradient of F has a large Lipschitz constant, the step size in the algorithm is large, and/or the stochastic gradients have high variance, then the convergence error will be larger. This term is also related to the convergence error in distributed SGD (Bottou et al., 2018), which is equivalent to MLL-SGD when there is one sub-network, $q = \tau = 1$, $\boldsymbol{a}_i = 1/N$, and $\boldsymbol{p}_i = 1$ for all i. MLL-SGD has a dependence on the probabilities of gradient steps and worker weights, replacing the $\frac{1}{N}$ in the equivalent term in distributed SGD.

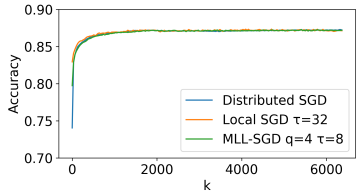
The third and fourth terms in (13) are additive errors that depend on the topology of the hub network. The value of ζ is given by the second largest eigenvalue of $\mathbf{H}$, by magnitude, which is an indication of the sparsity of the hub network. When worker weights are uniform, a fully connected hub graph G will have $\zeta = 0$, while a sparse G will typically have ζ close to 1. It is interesting to note that ζ only depends on $\mathbf{H}$, and not $\mathbf{Z}$ or $\mathbf{V}$, meaning the convergence error does not depend on how worker weights are distributed within sub-networks.

We also note the third and fourth terms depend on P, the weighted average probability of the workers. The convergence error increases as the average worker operating rate increases. This relation is expected as more local iterations will increase convergence error (McMahan et al., 2017). It is interesting to note that the convergence error does not depend on the distribution of $\boldsymbol{p}$, meaning that a skewed and uniform distribution with the same average probability would have the same convergence error. We observe that the condition on η in (12) cannot always be satisfied given certain probabilities. Specifically, when there exists a $\boldsymbol{p}_i \leq 2 - \sqrt{2} \approx 0.59$, then the left-hand side will be non-positive, and the inequality can no longer be satisfied. Although this may be a conservative bound, intuitively, when $\boldsymbol{p}_i$'s are below this threshold, the algorithm may not make sufficient progress in each time step to guarantee convergence.

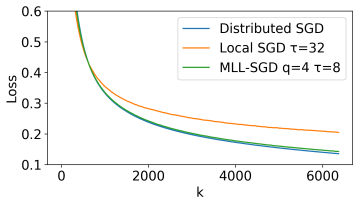
The third and fourth terms also grow with q and τ, the number of local iterations per hub network averaging and sub-network averaging steps, respectively. The longer workers train locally without reconciling their models, the more their models will diverge, leading to larger convergence error. We



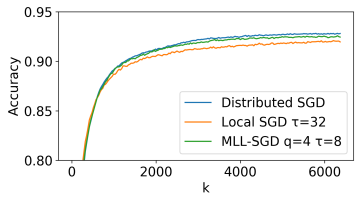
(a) Training loss, as a function of the time step k.



(b) Test accuracy, as a function of the time step k.



(c) Training loss, as a function of the time step k.



(d) Test accuracy, as a function of the time step k.

Figure 1: a) and b) Logistic regression model trained on MNIST dataset. c) and d) CNN model trained on EMNIST dataset.

can see that τ plays a slightly larger role in convergence error than q. For a given $q \cdot \tau$, meaning a given number of time steps between hub averaging steps, a larger τ leads to higher convergence error than a larger q would. Thus, there is a slight penalty to performing more local iterations between sub-network averaging steps. We explore this more in Section 6.

6 Experiments

To visualize the convergence behavior of MLL-SGD in practice, we simulate our system in Python. We train a logistic regression model and a Convolutional Neural Network (CNN) to see the effect on convergence with convex and non-convex loss functions. We train the logistic regression model on the MNIST dataset (Bottou et al., 1994). MNIST is a set of hand-written digits with 60,000 digits in the training set and 10,000 digits in the test set. We train a binary classification model with half the classes being 0 and the other half being 1 and use a step size of 0.2. The CNN is modeled in PyTorch (Paszke et al., 2019) with two convolution layers and two fully connected layers. We use the EMNIST (Cohen et al., 2017) dataset for the CNN, a set of hand-written digits and letters. We use the Letters subset with 124,800 letters in the training set and 20,800 letters in the test set. We train the CNN with a step size of 0.01. For these experiments, we partition the datasets among the workers in an IID fashion. The workers weights are assigned weights based on dataset size. Each sub-network has the same number of workers in every experiment. Unless otherwise specified, the hub network used for MLL-SGD is a manually generated graph with $\zeta = 0.74$.

We compare MLL-SGD with Distributed SGD and Local SGD. Distributed SGD is equivalent to MLL-SGD when there is one hub, $q = \tau = 1$, and $\boldsymbol{a}_i = 1/N$ and $\boldsymbol{p}_i = 1$ for all i. Local SGD is the same as Distributed SGD except τ can be greater than 1. For these experiments, we let $\tau = 32$ for Local SGD. We measure training loss of the averaged model $\boldsymbol{u}_k$ every 32 iterations.

We first study the impact of the multi-level network structure and heterogeneous weights on the convergence of the training loss and test accuracy. For MLL-SGD, we configure a multi-level network with 10 hubs, each with 10 workers. For this experiment, each worker is manually assigned a different sized dataset so that their weights are heterogeneous. Workers were randomly split into five groups of 20 workers each with a different sized dataset: 0.25%, 0.5%, 1%, 1.25%, and 2%. We set $\tau = 8$ and $q = 4$. For Distributed SGD and Local SGD, we use 100 workers.

Figures 1a and 1b give results for the logistic regression model on the MNIST dataset, and Figures 1c and 1d compare the same algorithms on the CNN model with the EMNIST dataset. In both cases, MLL-SGD performs well and achieves similar accuracy to Local SGD in about the same number of global iterations. Looking closely at the regression model, we can see a minor degradation in

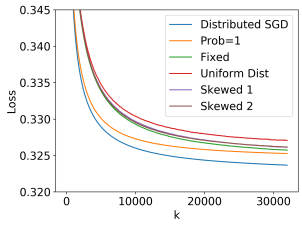


Figure 2: Effect of worker operating rates (logistic regression on MNIST).

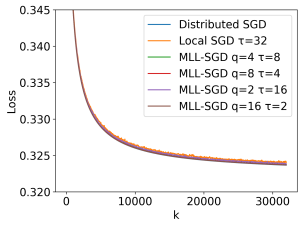


Figure 3: Effect of different q and τ, with $q\tau$ constant (logistic regression on MNIST).

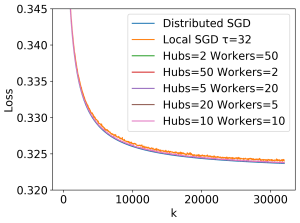


Figure 4: Effect of worker distribution in sparse hub network (logistic regression on MNIST).

the convergence rate when compared to Distributed and Local SGD. However, MLL-SGD actually outperforms Local SGD when training the CNN model, perhaps due to the larger q and smaller τ.

We next study how heterogeneous worker probabilities affect the convergence of the models. According to Theorem 1, the average probability across workers plays a role in the error bound. To see if this holds in practice, we compare four different setups, all of which includes 10 hubs, each with 10 workers, $\boldsymbol{a}_i = 1/N$, and an average probability of 0.55: (i) all workers with a $\boldsymbol{p}_i = 0.55$ (Fixed); (ii) workers in each sub-network with probability ranging from 0.1 to 1 at steps of 0.1 (Uniform Distribution); (iii) 90 workers with $\boldsymbol{p}_i = 0.5$ and 10 workers with $\boldsymbol{p}_i = 1$ (Skewed 1); (iv) 90 workers with $\boldsymbol{p}_i = 0.6$ and 10 workers with $\boldsymbol{p}_i = 0.1$ (Skewed 2).

The results of this experiment are shown in Figure 2. Despite having the same average probability of 0.55, the uniform distribution has the worst convergence error at the end of training, while the fixed probability scenario achieves lower loss. This indicates that, in practice, the distribution of worker operating rates can impact performance.

We also investigate the effect of changing q and τ. According to Theorem 1, τ has a greater effect on convergence error than q. We train the logistic regression model on the MNIST dataset in 10 hubs with 10 workers each. For all i, we let $\boldsymbol{p}_i = 1$ and $\boldsymbol{a}_i = 1/N$. For Distributed SGD and Local SGD, we use 100 workers. The results are shown in Figure 3. Interestingly, when comparing these cases of MLL-SGD where $q\tau$ are equal, the difference in convergence rate and error is negligible. Although Theorem 1 indicates τ should have a larger impact than q on convergence error, in this experiment, changing q or τ while keeping $q\tau$ constant had little effect on convergence.

Finally, we investigate how the number of sub-networks affects the convergence of MLL-SGD. From a pool of 100 workers, we distribute them across 2, 5, 10, 20, and 50 sub-networks. The hub network is a path graph, which yields the largest ζ while keeping the network connected. This hub network topology the worst-case scenario in terms of the convergence bound. Note that as the number of hubs increases, the larger ζ becomes. We let $\boldsymbol{a}_i = 1/N$ and $\boldsymbol{p}_i = 1$ for all workers i. For Distributed SGD and Local SGD, we again use 100 workers.

The results are shown in Figure 4. We can see that the convergence rate is very similar in all cases. This indicates that, in practice, a fixed number of workers, distributed over a variable number of hubs will have little effect on overall performance. A hub may experience network congestion when managing a large number of workers in a hub-and-spoke model. By employing a multi-level network approach, one could mitigate network congestion with little degradation in performance.

7 Conclusion

We have introduced MLL-SGD, a variation of Distributed SGD in a multi-level network model. Our algorithm incorporates the heterogeneity of worker devices using a stochastic approach. We provide theoretical analysis of the algorithm's convergence, and we show how the convergence error depends on the average worker rate, the hub network topology, and the number of local, sub-network averaging, and hub averaging steps. Finally, we provide experimental results that support our theoretical analysis. In future work, we plan to analyze the effects of non-IID data on convergence error.

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A Proof of Theorem 1

For our proof we adopt a similar approach to that in Wang & Joshi (2018). This section is structured as follows. We first define notation and make some observations in Section A.1. Our supporting lemmas are stated in Section A.2. We close with the full proof of Theorem 1 in Section A.3.

A.1 Preliminaries

For simplicity of notation, we let $\|\cdot\|$ denote the l_2 vector norm. Let the weighted Frobenius norm of an $N \times M$ matrix $\boldsymbol{X}$ with an N-vector $\boldsymbol{a}$ be defined as follows:

$$\|\boldsymbol{X}\|_{F_{\boldsymbol{a}}}^2 = \left|\mathrm{Tr}((\mathrm{diag}(\boldsymbol{a}))^{1/2}\,\boldsymbol{X}\boldsymbol{X}^T\,(\mathrm{diag}(\boldsymbol{a}))^{1/2})\right| = \sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{a}_i|\boldsymbol{x}_{i,j}|^2. \tag{15}$$

The matrix operator norm for a square matrix $\mathbf{Q}$ is defined as:

$$\|\mathbf{Q}\|_{op} = \sqrt{\lambda_{max}(\mathbf{Q}^T\,\mathbf{Q})}. \tag{16}$$

We define the set of Bernoulli random variables $\Theta = \{\theta_k^1, \ldots, \theta_k^N\}$, where

$$\theta_k^i = \begin{cases} 1 & \text{with probability } \boldsymbol{p}_i \\ 0 & \text{with probability } (1-\boldsymbol{p}_i). \end{cases}$$

Let $\Xi_k = \{\xi_k^{(1)}, ..., \xi_k^{(N)}\}$ be the set of mini-batches used by the N workers at time step k. Without loss of generality, we assign a mini-batch to each worker, even if it does not execute a gradient step in that iteration. An equivalent definition of $\boldsymbol{g}_k^{(i)}$ is then

$$\boldsymbol{g}_k^{(i)} = \theta_k^i g(\xi_k^{(i)}). \tag{17}$$

For simplicity of notation, let $\mathbb{E}_k$ be equivalent to $\mathbb{E}_{\Theta_k, \Xi_k | X_k}$.

We note that Assumption 1c implies:

$$\mathbb{E}_k[\boldsymbol{g}_k^{(i)}] = \boldsymbol{p}_i \mathbb{E}_k[g(\boldsymbol{x}_k^{(i)})] \tag{18}$$

$$= \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)}). \tag{19}$$

Further, when $i \neq j$:

$$\mathbb{E}_k[(\boldsymbol{g}_k^{(i)})^T \boldsymbol{g}_k^{(j)}] = \boldsymbol{p}_i \boldsymbol{p}_j \mathbb{E}_k[(g(\boldsymbol{x}_k^{(i)}))]^T \mathbb{E}_k[g(\boldsymbol{x}_k^{(j)})] \tag{20}$$

$$= \boldsymbol{p}_i \boldsymbol{p}_j \nabla F(\boldsymbol{x}_k^{(i)})^T \nabla F(\boldsymbol{x}_k^{(j)}). \tag{21}$$

We also note that Assumption 1d implies:

$$\mathbb{E}_k \left\| \boldsymbol{g}_k^{(i)} - \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 = \mathbb{E}_k \left[\left\| \boldsymbol{g}_k^{(i)} \right\|^2 + \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 - 2(\boldsymbol{g}_k^{(i)})^T \nabla F(\boldsymbol{x}_k^{(i)}) \right] \tag{22}$$

$$= \mathbb{E}_k \left\| \boldsymbol{g}_k^{(i)} \right\|^2 + \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 - 2\mathbb{E}_k (\boldsymbol{g}_k^{(i)})^T \nabla F(\boldsymbol{x}_k^{(i)}) \tag{23}$$

$$= \boldsymbol{p}_i \mathbb{E}_k \left\| g(\boldsymbol{x}_k^{(i)}) \right\|^2 + \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 - 2\boldsymbol{p}_i \mathbb{E}_k g(\boldsymbol{x}_k^{(i)})^T \nabla F(\boldsymbol{x}_k^{(i)}) \tag{24}$$

$$= \boldsymbol{p}_i \mathbb{E}_k \left\| g(\boldsymbol{x}_k^{(i)}) \right\|^2 + \boldsymbol{p}_i \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 - 2\boldsymbol{p}_i \mathbb{E}_k g(\boldsymbol{x}_k^{(i)})^T \nabla F(\boldsymbol{x}_k^{(i)}) + (1 - \boldsymbol{p}_i) \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 \tag{25}$$

$$= \boldsymbol{p}_i \mathbb{E}_k \left\| g(\boldsymbol{x}_k^{(i)}) - \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 + (1 - \boldsymbol{p}_i) \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 \tag{26}$$

$$\leq \boldsymbol{p}_i \beta \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 + \boldsymbol{p}_i \sigma^2 + (1 - \boldsymbol{p}_i) \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 \tag{27}$$

$$= (\boldsymbol{p}_i(\beta - 1) + 1) \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 + \boldsymbol{p}_i \sigma^2. \tag{28}$$

Finally, we define the weighted average stochastic gradient and the weighted average batch gradient as:

$$\mathcal{G}_k = \sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{g}_k^{(i)}, \;\; \mathcal{H}_k = \sum_{i=1}^{N} \boldsymbol{a}_i \nabla F(\boldsymbol{x}_k^{(i)}).$$

A.2 Lemmas and Propositions

Next, we state our supporting lemmas and propositions.

Proposition 1. *The matrices* $\mathbf{Z}$ *and* $\mathbf{V}$ *satisfy the following properties:*

1. $\mathbf{Z}$ *and* $\mathbf{V}$ *each have a right eigenvector of* $\boldsymbol{a}$ *with eigenvalue 1.*
2. $\mathbf{Z}$ *and* $\mathbf{V}$ *each have a left eigenvector of* $\boldsymbol{1}_N^T$ *with eigenvalue 1.*
3. *All other eigenvalues of* $\mathbf{Z}$ *and* $\mathbf{V}$ *have magnitude strictly less than 1.*

Proof. Assumption 2 indicates that $\mathbf{H}$ is a *Generalized Diffusion Matrix* as defined in Rotaru & Nägeli (2004). Recall Assumption 2:

Assumption 2. *The matrix* $\mathbf{H}$ *satisfies the following:*

2a If $(i, j) \in E$*, then* $\mathbf{H}_{i,j} > 0$*. Otherwise,* $\mathbf{H}_{i,j} = 0$*.*

2b $\mathbf{H}$ *is column stochastic, i.e.,* $\sum_{i=1}^{D} \mathbf{H}_{i,j} = 1$*.*

2c For all $i, j \in \mathcal{D}$*, we have* $\boldsymbol{b}_i \mathbf{H}_{i,j} = \boldsymbol{b}_j \mathbf{H}_{j,i}$*.*

If we show this implies that $\mathbf{Z}$ and $\mathbf{V}$ are Generalized Diffusion Matrices with the same properties to those in Assumption 2 with vector $\boldsymbol{a}$, then the properties in the proposition are satisfied.

Since $\mathbf{H}$ and $\boldsymbol{b}$ are non-negative, then $\mathbf{Z}$ is also non-negative. It is also clear that $\mathbf{Z}$ is column stochastic by construction. It is left to prove that:

$$\mathbf{Z}_{i,j}\, \boldsymbol{a}_j = \mathbf{Z}_{j,i}\, \boldsymbol{a}_i. \tag{29}$$

Applying the definition of $\mathbf{Z}$ to the left side, we have:

$$\mathbf{Z}_{i,j}\, \boldsymbol{a}_j = \mathbf{H}_{d(i),d(j)} v^{(i)} \boldsymbol{a}_j \tag{30}$$

Since we know that $\mathbf{H}$ is a Generalized Diffusion Matrix with vector $\boldsymbol{b}$, we know that:

$$\mathbf{H}_{i,j} \boldsymbol{b}_j = \mathbf{H}_{j,i} \boldsymbol{b}_i \tag{31}$$

$$\mathbf{H}_{i,j} = \mathbf{H}_{j,i} \frac{\boldsymbol{b}_i}{\boldsymbol{b}_j}. \tag{32}$$

Plugging this in for $\mathbf{H}_{d(i),d(j)}$, we have:

$$\mathbf{Z}_{i,j}\, \boldsymbol{a}_j = \mathbf{H}_{d(j),d(i)} \frac{\boldsymbol{b}_{d(i)}}{\boldsymbol{b}_{d(j)}} v^{(i)} \boldsymbol{a}_j \tag{33}$$

$$= \mathbf{H}_{d(j),d(i)} \frac{\sum_{r \in \mathcal{M}^{(d(i))}} w^{(r)}}{w_{tot}} \frac{w_{tot}}{\sum_{r \in \mathcal{M}^{(d(j))}} w^{(r)}} \frac{w^{(i)}}{\sum_{r \in \mathcal{M}^{(d(i))}} w^{(r)}} \frac{w^{(j)}}{w_{tot}} \tag{34}$$

$$= \mathbf{H}_{d(j),d(i)} \frac{w^{(i)}}{w_{tot}} \frac{w^{(j)}}{\sum_{r \in \mathcal{M}^{(d(j))}} w^{(r)}} \tag{35}$$

$$= \mathbf{H}_{d(j),d(i)} v^{(j)} \boldsymbol{a}_i \tag{36}$$

$$= \mathbf{Z}_{j,i}\, \boldsymbol{a}_i. \tag{37}$$

Therefore, $\mathbf{Z}$ is a Generalized Diffusion Matrix.

We can show that $\mathbf{V}$ is also a Generalized Diffusion Matrix with the vector $\boldsymbol{a}$. $\mathbf{V}$ is constructed to be non-negative and column stochastic. It is left to prove that

$$\mathbf{V}_{i,j}\, \boldsymbol{a}_j = \mathbf{V}_{j,i}\, \boldsymbol{a}_i. \tag{38}$$

When i, j are outside a block $\mathbf{V}^{(d)}$, then $\mathbf{V}_{i,j} = \mathbf{V}_{j,i} = 0$, so the equation is trivially satisfied. When within a block, in terms of w, we have:

$$\mathbf{V}_{i,j}\, \boldsymbol{a}_j = \mathbf{V}_{j,i}\, \boldsymbol{a}_i \tag{39}$$

$$\frac{w^{(i)}}{\sum_{r \in \mathcal{M}^{(d(i))}} w^{(r)}} \frac{w^{(j)}}{w_{tot}} = \frac{w^{(j)}}{\sum_{r \in \mathcal{M}^{(d(j))}} w^{(r)}} \frac{w^{(i)}}{w_{tot}}. \tag{40}$$

Noting that we are within a block, therefore $d(i) = d(j)$, we can see that both sides are equal:

$$w^{(i)} w^{(j)} = w^{(j)} w^{(i)}. \tag{41}$$

Therefore, $\mathbf{V}$ is a Generalized Diffusion Matrix.

□

Proposition 2. *Given a diffusion matrix* $\mathbf{H}$ *with the properties in Assumption 2, if* $\mathbf{Z}$ *constructed as follows,*

$$\mathbf{Z}_{i,j} = \mathbf{H}_{d(i),d(j)} v^{(i)} \tag{42}$$

then the largest eigenvalues of $\mathbf{Z}$ *are the eigenvalues of* $\mathbf{H}$*, and zero otherwise.*

Proof. In order to prove the relationship of the eigenvalues of $\mathbf{Z}$ and $\mathbf{H}$, we prove the following two points separately:

1. The rank of $\mathbf{Z}$ is the same as $\mathbf{H}$.
2. All non-zero eigenvalues of $\mathbf{H}$ are eigenvalues of $\mathbf{Z}$ with the same multiplicity.

For the rank of $\mathbf{Z}$, we take a look at how each column is constructed. Consider column j of $\mathbf{Z}$:

$$\mathbf{Z}_j = [\mathbf{H}_{1,d(j)}v^{(1)}, \ldots, \mathbf{H}_{1,d(j)}v^{(N^{(1)})}, \mathbf{H}_{2,d(j)}v^{(N^{(1)}+1)}, \ldots, \mathbf{H}_{D,d(j)}v^{(N)}]^T. \tag{43}$$

For two columns i and j where $d(i) = d(j)$, these columns are identical. Therefore, the rank of $\mathbf{Z}$ will be, at most, the number of hubs, D. Further, we can see that the elements of a column j in $\mathbf{Z}$ are simply scaled elements of column $d(j)$ in $\mathbf{H}$. So any linearly dependent columns in $\mathbf{H}$ will also be linearly dependent in $\mathbf{Z}$. Therefore, the rank of the two matrices are the same.

For the second point, we show there is a bijective mapping from eigenpairs of $\mathbf{H}$ to eigenpairs of $\mathbf{Z}$. Let $(\lambda, \boldsymbol{y})$ be an eigenpair of $\mathbf{H}$ (with $\lambda \neq 0$), i.e.

$$\mathbf{H}\boldsymbol{y} = \lambda \boldsymbol{y}. \tag{44}$$

Define the N-vector $\boldsymbol{x}$ with components $\boldsymbol{x}_i = v^{(i)}\boldsymbol{y}_{d(i)}$. We will show that $\mathbf{Z}\,\boldsymbol{x} = \lambda\boldsymbol{x}$. Looking at the i-th entry of the vector $\mathbf{Z}\,\boldsymbol{x}$, we have

$$(\mathbf{Z}\,\boldsymbol{x})_i = \sum_{j=1}^{N} \mathbf{Z}_{i,j}\,\boldsymbol{x}_j. \tag{45}$$

Applying the definition of $\mathbf{Z}$ and $\boldsymbol{x}$, we obtain

$$(\mathbf{Z}\,\boldsymbol{x})_i = \sum_{j=1}^{N} \frac{1}{v^{(i)}}\mathbf{H}_{d(i),d(j)}v^{(j)}\boldsymbol{y}_{d(j)} \tag{46}$$

$$= \frac{1}{v^{(i)}} \sum_{l=1}^{D} \mathbf{H}_{d(i),l}\boldsymbol{y}_l \sum_{k\in\mathcal{M}^{(l)}} v^{(k)} \tag{47}$$

$$= \frac{1}{v^{(i)}} \sum_{l=1}^{D} \mathbf{H}_{d(i),l}\boldsymbol{y}_l. \tag{48}$$

Note that the m-th entry of the vector $\mathbf{H}\boldsymbol{y}$ equals $\sum_{l=1}^{D} \mathbf{H}_{m,l}\boldsymbol{y}_l = \lambda\boldsymbol{y}_m$. Applying this equality, we obtain

$$(\mathbf{Z}\,\boldsymbol{x})_i = \frac{1}{v^{(i)}}\lambda\boldsymbol{y}_{d(i)} \tag{49}$$

$$= \lambda\boldsymbol{x}_i. \tag{50}$$

Therefore, for any eigenpair $(\lambda, \boldsymbol{y})$ of $\mathbf{H}$, we can find an eigenpair $(\lambda, \boldsymbol{x})$ of $\mathbf{Z}$. It is left to prove that this mapping is a bijection.

Suppose eigenvalue λ of $\mathbf{H}$ has multiplicity $k > 1$. We consider any two of the k eigenpairs $(\lambda, \boldsymbol{c})$ and $(\lambda, \boldsymbol{d})$. Let the corresponding eigenpairs of $\mathbf{Z}$ be $(\lambda, \boldsymbol{e})$ and $(\lambda, \boldsymbol{f})$. We know that $\boldsymbol{e} \neq \boldsymbol{f}$ because $\boldsymbol{c}$ and $\boldsymbol{d}$ are unique, and there must exist an index i such that $v^{(i)}\boldsymbol{c}_{d(i)} \neq v^{(i)}\boldsymbol{d}_{d(i)}$. Therefore, the mapping of eigenpairs of $\mathbf{H}$ to eigenpairs of $\mathbf{Z}$ is a bijection. □

Proposition 3. *Given definition of* $\mathbf{Z}$ *and* $\mathbf{V}$ *in Proposition 1, it is the case that*

$$\mathbf{Z}\,\mathbf{V} = \mathbf{V}\,\mathbf{Z} = \mathbf{Z}\,. \tag{51}$$

Proof. First, we prove that $\mathbf{V}\,\mathbf{Z} = \mathbf{Z}$. Note that the i-th row of $\mathbf{V}$ contains either v_i or zero. Looking at an arbitrary entry i, j of $\mathbf{V}\,\mathbf{Z}$ we have:

$$(\mathbf{V}\,\mathbf{Z})_{i,j} = v^{(i)} \sum_{r\in M_{d(i)}} \mathbf{Z}_{r,j} \tag{52}$$

$$(\mathbf{V}\,\mathbf{Z})_{i,j} = v^{(i)}\mathbf{H}_{d(i),d(i)} \tag{53}$$

$$(\mathbf{V}\,\mathbf{Z})_{i,j} = \mathbf{Z}_{i,j}\,. \tag{54}$$

Next we prove that $\mathbf{Z}\,\mathbf{V} = \mathbf{Z}$. Note that for any row i in $\mathbf{Z}$, $\mathbf{Z}_{i,j} = \mathbf{Z}_{i,k}$ when $d(j) = d(k)$.

$$(\mathbf{Z}\,\mathbf{V})_{i,j} = \mathbf{Z}_{i,j} \sum_{r=1}^{N} \mathbf{V}_{r,j}\,. \tag{55}$$

Since $\mathbf{V}$ is column stochastic:

$$(\mathbf{Z}\,\mathbf{V})_{i,j} = \mathbf{Z}_{i,j}\,. \tag{56}$$

□

Proposition 4. *Let $\boldsymbol{A} = \boldsymbol{a}\,\boldsymbol{1}^T$. Given our definition of $\boldsymbol{T}_k$ in (6),*

$$\boldsymbol{T}_k\,\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{T}_k = \boldsymbol{A} \tag{57}$$

for all k.

Proof. We prove each of the three cases of $\mathbf{T}_k$: $\mathbf{I}$, $\mathbf{V}$, and $\mathbf{Z}$. Clearly, $\mathbf{I}\,\mathbf{A} = \mathbf{A}\,\mathbf{I} = \mathbf{A}$. It is left to prove $\mathbf{V}\,\mathbf{A} = \mathbf{A}\,\mathbf{V} = \mathbf{A}$ and $\mathbf{Z}\,\mathbf{A} = \mathbf{A}\,\mathbf{Z} = \mathbf{A}$.

We can see that $\mathbf{Z}\,\mathbf{A} = \mathbf{A}$ since $\boldsymbol{a}$ is a right eigenvector of $\mathbf{Z}$ with eigenvalue 1: $\mathbf{Z}\,\mathbf{A} = \mathbf{Z}\,\boldsymbol{a}\,\mathbf{1}^T = \boldsymbol{a}\,\mathbf{1}^T = \mathbf{A}$. Similarly, we can see that $\mathbf{A}\,\mathbf{Z} = \boldsymbol{a}\,\mathbf{1}^T\,\mathbf{Z} = \boldsymbol{a}\,\mathbf{1}^T = \mathbf{A}$ as $\mathbf{1}^T$ is a left eigenvector of $\mathbf{Z}$. The same holds for $\mathbf{V}$. □

Lemma 1. *Under Assumptions 1c and 1d, the variance of the weighted average stochastic gradient is bounded as follows:*

$$\mathbb{E}_k[\|\mathcal{G}_k - \mathcal{H}_k\|^2] \leq \sum_{i=1}^{N} \boldsymbol{a}_i^2 \left[(\boldsymbol{p}_i(\beta - 1) + 1) \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 + \boldsymbol{p}_i \sigma^2 \right] + \sum_{l=1}^{N} \sum_{j \neq l}^{N} \boldsymbol{a}_l \boldsymbol{a}_j (1 - \boldsymbol{p}_j)(1 - \boldsymbol{p}_l) \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}). \tag{58}$$

Proof.

$$\mathbb{E}_k[\|\mathcal{G}_k - \mathcal{H}_k\|^2] = \mathbb{E}_k \left[\left\| \sum_{i=1}^{N} \boldsymbol{a}_i (\boldsymbol{g}_k^{(i)} - \nabla F(\boldsymbol{x}_k)) \right\|^2 \right] \tag{59}$$

$$= \mathbb{E}_k \left[\sum_{i=1}^{N} \boldsymbol{a}_i^2 \|\boldsymbol{g}_k^{(i)} - \nabla F(\boldsymbol{x}_k)\|^2 + \sum_{l=1}^{N} \sum_{j \neq l}^{N} \boldsymbol{a}_l \boldsymbol{a}_j \left\langle \boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}), \boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}) \right\rangle \right] \tag{60}$$

$$= \sum_{i=1}^{N} \boldsymbol{a}_i^2 \mathbb{E}_k \|\boldsymbol{g}_k^{(i)} - \nabla F(\boldsymbol{x}_k)\|^2 + \sum_{l=1}^{N} \sum_{j \neq l}^{N} \boldsymbol{a}_l \boldsymbol{a}_j \mathbb{E}_k \left[\left\langle \boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}), \boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}) \right\rangle \right]. \tag{61}$$

Looking at the cross-terms in (61):

$$\begin{aligned}
&\mathbb{E}_k \left[\left\langle \boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}), \boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}) \right\rangle \right] \\
&\qquad = \mathbb{E}_k \left[(\boldsymbol{g}_k^{(j)})^T \boldsymbol{g}_k^{(l)} \right] - \mathbb{E}_k \left[(\boldsymbol{g}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}) \right] \\
&\qquad\quad - \mathbb{E}_k \left[\nabla F(\boldsymbol{x}_k^{(j)})^T \boldsymbol{g}_k^{(l)} \right] + \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(j)}) && (62) \\
&\qquad = \boldsymbol{p}_j \boldsymbol{p}_l \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}) - \boldsymbol{p}_j \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}) \\
&\qquad\quad - \boldsymbol{p}_l \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}) + \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(j)}) && (63) \\
&\qquad = (1 - \boldsymbol{p}_j)(1 - \boldsymbol{p}_l) \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}). && (64)
\end{aligned}$$

Plugging (64) into (61) we have:

$$\mathbb{E}_k[\|\mathcal{G}_k - \mathcal{H}_k\|^2] = \sum_{i=1}^{N} \boldsymbol{a}_i^2 \mathbb{E}_k \|\boldsymbol{g}_k^{(i)} - \nabla F(\boldsymbol{x}_k)\|^2 + \sum_{l=1}^{N}\sum_{j\neq l}^{N} \boldsymbol{a}_l \boldsymbol{a}_j (1-\boldsymbol{p}_j)(1-\boldsymbol{p}_l) \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}) \quad (65)$$

$$\leq \sum_{i=1}^{N} \boldsymbol{a}_i^2 \left[(\boldsymbol{p}_i(\beta-1)+1) \left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \boldsymbol{p}_i \sigma^2 \right] + \sum_{l=1}^{N}\sum_{j\neq l}^{N} \boldsymbol{a}_l \boldsymbol{a}_j (1-\boldsymbol{p}_j)(1-\boldsymbol{p}_l) \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}) \quad (66)$$

where (66) follows from Assumption 1d and (28). □

Lemma 2. *Under Assumptions 1c and 1d, the squared norm of the stochastic gradients is bounded by:*

$$\mathbb{E}_k\left[\|\mathcal{G}_k\|^2\right] \leq \sum_{i=1}^{N} \left[\boldsymbol{a}_i^2 \left(\boldsymbol{p}_i(\beta+1) - \boldsymbol{p}_i^2\right) + \boldsymbol{a}_i \boldsymbol{p}_i^2\right] \left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \sum_{i=1}^{N} \boldsymbol{a}_i^2 \boldsymbol{p}_i \sigma^2 \quad (67)$$

Proof.

$$\mathbb{E}_k\left[\|\mathcal{G}_k\|^2\right] = \mathbb{E}_k\left[\|\mathcal{G}_k - \mathbb{E}_k[\mathcal{G}_k]\|^2\right] + \|\mathbb{E}_k[\mathcal{G}_k]\|^2 \quad (68)$$

$$= \mathbb{E}_k\left[\|\mathcal{G}_k - \sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\|^2\right] + \left\|\sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \quad (69)$$

$$= \mathbb{E}_k\left[\left\|\mathcal{G}_k - \sum_{i=1}^{N} \boldsymbol{a}_i \nabla F(\boldsymbol{x}_k^{(i)}) + \sum_{i=1}^{N} \boldsymbol{a}_i (1-\boldsymbol{p}_i) \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2\right] + \left\|\sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \quad (70)$$

Applying the definition of $\mathcal{G}_k$ to (70) we get:

$$\mathbb{E}_k\left[\|\mathcal{G}_k\|^2\right]$$

$$= \mathbb{E}_k\left[\left\|\sum_{i=1}^{N}\left[\boldsymbol{a}_i \boldsymbol{g}_k^{(i)} - \boldsymbol{a}_i \nabla F(\boldsymbol{x}_k^{(i)}) + \boldsymbol{a}_i (1-\boldsymbol{p}_i) \nabla F(\boldsymbol{x}_k^{(i)})\right]\right\|^2\right] + \left\|\sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \quad (71)$$

$$= \mathbb{E}_k\left[\sum_{i=1}^{N}\left\|\boldsymbol{a}_i \boldsymbol{g}_k^{(i)} - \boldsymbol{a}_i \nabla F(\boldsymbol{x}_k^{(i)}) + \boldsymbol{a}_i (1-\boldsymbol{p}_i) \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2\right] + \left\|\sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2$$

$$+ \mathbb{E}_k\left[\sum_{j=1}^{N}\sum_{l=1, l\neq j}^{N} \boldsymbol{a}_l \boldsymbol{a}_j \Big\langle (\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)})) + (1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)}), (\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)})) + (1-\boldsymbol{p}_l)\nabla F(\boldsymbol{x}_k^{(l)}) \Big\rangle\right]. \quad (72)$$

Let the cross-terms in (72) be

$$CR = \mathbb{E}_k\left[\sum_{j=1}^{N}\sum_{l=1, l\neq j}^{N} \boldsymbol{a}_l \boldsymbol{a}_j \Big\langle (\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)})) + (1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)}), (\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)})) + (1-\boldsymbol{p}_l)\nabla F(\boldsymbol{x}_k^{(l)}) \Big\rangle\right]. \quad (73)$$

We can simplify CR as follows:

$$
\begin{aligned}
CR = &\sum_{j=1}^{N}\sum_{l=1,l\neq j}^{N} \boldsymbol{a}_l\boldsymbol{a}_j\mathbb{E}_k\Big[(\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)})) \\
&+ (\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(1-\boldsymbol{p}_l)\nabla F(\boldsymbol{x}_k^{(l)}) + (1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)})^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)})) \\
&\qquad + (1-\boldsymbol{p}_j)(1-\boldsymbol{p}_l)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)})^T\Big] \qquad (74)\\
= &\sum_{j=1}^{N}\sum_{l=1,l\neq j}^{N} \boldsymbol{a}_l\boldsymbol{a}_j\Big[\mathbb{E}_k\left[(\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}))\right] \\
&\qquad + (\mathbb{E}_k[\boldsymbol{g}_k^{(j)}] - \nabla F(\boldsymbol{x}_k^{(j)}))^T(1-\boldsymbol{p}_l)\nabla F(\boldsymbol{x}_k^{(l)}) \\
&\qquad + (1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)})^T(\mathbb{E}_k[\boldsymbol{g}_k^{(l)}] - \nabla F(\boldsymbol{x}_k^{(l)})) \\
&\qquad + (1-\boldsymbol{p}_j)(1-\boldsymbol{p}_l)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)})^T\Big]. \qquad (75)
\end{aligned}
$$

Applying Assumption 1c to (75), we get:

$$
\begin{aligned}
CR = &\sum_{j=1}^{N}\sum_{l=1,l\neq j}^{N} \boldsymbol{a}_l\boldsymbol{a}_j\Big[\mathbb{E}_k\left[(\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}))\right] \\
&+ (\boldsymbol{p}_j - 1)(1-\boldsymbol{p}_l)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)}) + (\boldsymbol{p}_l - 1)(1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)}) \\
&\qquad + (1-\boldsymbol{p}_j)(1-\boldsymbol{p}_l)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)})\Big] \qquad (76)\\
= &\sum_{j=1}^{N}\sum_{l=1,l\neq j}^{N} \boldsymbol{a}_l\boldsymbol{a}_j\Big[\mathbb{E}_k\left[(\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}))\right] \\
&\qquad - (1-\boldsymbol{p}_l)(1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)})\Big]. \qquad (77)
\end{aligned}
$$

Applying (77) to (72) we have:

$$
\begin{aligned}
\mathbb{E}_k\left[\|\mathcal{G}_k\|^2\right] = &\mathbb{E}_k\left[\sum_{i=1}^{N}\left\|\boldsymbol{a}_i\boldsymbol{g}_k^{(i)} - \boldsymbol{a}_i\nabla F(\boldsymbol{x}_k^{(i)}) + \boldsymbol{a}_i(1-\boldsymbol{p}_i)\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2\right] \\
&+ \sum_{j=1}^{N}\sum_{l=1,l\neq j}^{N} \boldsymbol{a}_l\boldsymbol{a}_j\Big[\mathbb{E}_k\left[(\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}))\right] \\
&\qquad - (1-\boldsymbol{p}_l)(1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)})\Big] \\
&\qquad + \left\|\sum_{i=1}^{N}\boldsymbol{a}_i\boldsymbol{p}_i\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \qquad (78)
\end{aligned}
$$

Expanding the first term in (78) we have:

$$\begin{aligned}
\mathbb{E}_k\left[\|\mathcal{G}_k\|^2\right] = \mathbb{E}_k\left[\sum_{i=1}^N \left\|\boldsymbol{a}_i\boldsymbol{g}_k^{(i)} - \boldsymbol{a}_i\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2\right] + \sum_{i=1}^N \boldsymbol{a}_i^2(1-\boldsymbol{p}_i)^2\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \\
+ \sum_{i=1}^N \underbrace{\mathbb{E}_k\left[\left\langle \boldsymbol{a}_i\boldsymbol{g}_k^{(i)} - \boldsymbol{a}_i\nabla F(\boldsymbol{x}_k^{(i)}), \boldsymbol{a}_i(1-\boldsymbol{p}_i)\nabla F(\boldsymbol{x}_k^{(i)})\right\rangle\right]}_{CR1} \\
+ \sum_{i=1}^N \underbrace{\mathbb{E}_k\left[\left\langle \boldsymbol{a}_i(1-\boldsymbol{p}_i)\nabla F(\boldsymbol{x}_k^{(i)}), \boldsymbol{a}_i\boldsymbol{g}_k^{(i)} - \boldsymbol{a}_i\nabla F(\boldsymbol{x}_k^{(i)})\right\rangle\right]}_{CR2} \\
+ \sum_{j=1}^N \sum_{l=1,l\neq j}^N \boldsymbol{a}_l\boldsymbol{a}_j\Big[\mathbb{E}_k\left[(\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}))\right] \\
- (1-\boldsymbol{p}_l)(1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)})\Big] \\
+ \left\|\sum_{i=1}^N \boldsymbol{a}_i\boldsymbol{p}_i\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \qquad (79)
\end{aligned}$$

We simplify $CR1$:

$$\begin{aligned}
CR1 &= \left[\left\langle \boldsymbol{a}_i\boldsymbol{p}_i\mathbb{E}_k(g(\boldsymbol{x}_k^{(i)}))^T - \boldsymbol{a}_i\nabla F(\boldsymbol{x}_k^{(i)}), \boldsymbol{a}_i(1-\boldsymbol{p}_i)\nabla F(\boldsymbol{x}_k^{(i)})\right\rangle\right] && (80)\\
&= \left[\left\langle \boldsymbol{a}_i(\boldsymbol{p}_i-1)\nabla F(\boldsymbol{x}_k^{(i)}), \boldsymbol{a}_i(1-\boldsymbol{p}_i)\nabla F(\boldsymbol{x}_k^{(i)})\right\rangle\right] && (81)\\
&= -\boldsymbol{a}_i^2(1-\boldsymbol{p}_i)^2\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. && (82)
\end{aligned}$$

Similarly, for $CR2$:

$$CR2 = -\boldsymbol{a}_i^2(1-\boldsymbol{p}_i)^2\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \qquad (83)$$

Plugging (82) and (83) back into (79):

$$\begin{aligned}
\mathbb{E}_k\left[\|\mathcal{G}_k\|^2\right] = \mathbb{E}_k\left[\sum_{i=1}^N \left\|\boldsymbol{a}_i\boldsymbol{g}_k^{(i)} - \boldsymbol{a}_i\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2\right] - \sum_{i=1}^N \boldsymbol{a}_i^2(1-\boldsymbol{p}_i)^2\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \\
+ \sum_{j=1}^N \sum_{l=1,l\neq j}^N \boldsymbol{a}_l\boldsymbol{a}_j\Big[\mathbb{E}_k\left[(\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}))\right] \\
- (1-\boldsymbol{p}_l)(1-\boldsymbol{p}_j)\nabla F(\boldsymbol{x}_k^{(j)})^T\nabla F(\boldsymbol{x}_k^{(l)})\Big] \\
+ \left\|\sum_{i=1}^N \boldsymbol{a}_i\boldsymbol{p}_i\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \qquad (84)
\end{aligned}$$

We can simplify by observing that:

$$\begin{aligned}
\mathbb{E}_k\left[\|\mathcal{G}_k - \mathcal{H}_k\|^2\right] = \mathbb{E}_k\left[\sum_{i=1}^N \left\|\boldsymbol{a}_i\boldsymbol{g}_k^{(i)} - \boldsymbol{a}_i\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2\right] \\
+ \sum_{j=1}^N \sum_{l=1,l\neq j}^N \boldsymbol{a}_l\boldsymbol{a}_j\mathbb{E}_k\Big[(\boldsymbol{g}_k^{(j)} - \nabla F(\boldsymbol{x}_k^{(j)}))^T(\boldsymbol{g}_k^{(l)} - \nabla F(\boldsymbol{x}_k^{(l)}))\Big] \qquad (85)
\end{aligned}$$

which gives us:

$$\mathbb{E}_k\left[\|\mathcal{G}_k\|^2\right] = \mathbb{E}_k\left[\|\mathcal{G}_k - \mathcal{H}_k\|^2\right] - \sum_{i=1}^{N} \boldsymbol{a}_i^2 (1-\boldsymbol{p}_i)^2 \left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \left\|\sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 - \sum_{j=1}^{N} \sum_{l=1, l\neq j}^{N} \boldsymbol{a}_l \boldsymbol{a}_j (1-\boldsymbol{p}_l)(1-\boldsymbol{p}_j) \nabla F(\boldsymbol{x}_k^{(j)})^T \nabla F(\boldsymbol{x}_k^{(l)}) \quad (86)$$

Applying Lemma 1 to (86):

$$\mathbb{E}_k\left[\|\mathcal{G}_k\|^2\right] \leq \sum_{i=1}^{N} \boldsymbol{a}_i^2 \left[(\boldsymbol{p}_i(\beta-1)+1)\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \boldsymbol{p}_i \sigma^2\right] - \sum_{i=1}^{N} \boldsymbol{a}_i^2 (1-\boldsymbol{p}_i)^2 \left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \left\|\sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \quad (87)$$

$$\leq \sum_{i=1}^{N} \boldsymbol{a}_i^2 \left[(\boldsymbol{p}_i(\beta-1)+1)\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \boldsymbol{p}_i \sigma^2\right] - \sum_{i=1}^{N} \boldsymbol{a}_i^2 (1-\boldsymbol{p}_i)^2 \left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{p}_i^2 \left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \quad (88)$$

$$= \sum_{i=1}^{N} \left[\boldsymbol{a}_i^2 \left(\boldsymbol{p}_i(\beta+1) - \boldsymbol{p}_i^2\right) + \boldsymbol{a}_i \boldsymbol{p}_i^2\right] \left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \sum_{i=1}^{N} \boldsymbol{a}_i^2 \boldsymbol{p}_i \sigma^2 \quad (89)$$

where equation (88) follows from Jensen's inequality. □

Lemma 3. *Under Assumption 1c, the expected inner product of the batch gradient and the weighted average stochastic gradient is equal to:*

$$\mathbb{E}_k[\langle \nabla F(\boldsymbol{u}_k), \mathcal{G}_k\rangle] = \frac{1}{2}\|\nabla F(\boldsymbol{u}_k)\|^2 + \sum_{i=1}^{N} \frac{\boldsymbol{a}_i}{2} \left\|\boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 - \sum_{i=1}^{N} \frac{\boldsymbol{a}_i}{2} \left\|\nabla F(\boldsymbol{u}_k) - \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \quad (90)$$

Proof.

$$\mathbb{E}_k[\langle \nabla F(\boldsymbol{u}_k), \mathcal{G}_k\rangle] = \mathbb{E}_k\left[\left\langle \nabla F(\boldsymbol{u}_k), \sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{g}_k^{(i)}\right\rangle\right] \quad (91)$$

$$= \left\langle \nabla F(\boldsymbol{u}_k), \sum_{i=1}^{N} \boldsymbol{p}_i \boldsymbol{a}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\rangle \quad (92)$$

$$= \sum_{i=1}^{N} \boldsymbol{a}_i \left\langle \nabla F(\boldsymbol{u}_k), \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\rangle \quad (93)$$

$$= \sum_{i=1}^{N} \frac{\boldsymbol{a}_i}{2} \left[\|\nabla F(\boldsymbol{u}_k)\|^2 + \|\boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\|^2 - \|\nabla F(\boldsymbol{u}_k) - \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\|^2\right] \quad (94)$$

$$= \frac{1}{2}\|\nabla F(\boldsymbol{u}_k)\|^2 + \sum_{i=1}^{N} \frac{\boldsymbol{a}_i}{2} \left\|\boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 - \sum_{i=1}^{N} \frac{\boldsymbol{a}_i}{2} \left\|\nabla F(\boldsymbol{u}_k) - \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \quad (95)$$

where (92) follows from (19), and (94) follows from the fact that, for arbitrary vectors $\boldsymbol{y}$ and $\boldsymbol{z}$, $2\boldsymbol{y}^T\boldsymbol{z} = ||\boldsymbol{y}||^2 + ||\boldsymbol{z}||^2 - ||\boldsymbol{y}-\boldsymbol{z}||^2$. □

Lemma 4. *Under Assumption 1, following the update rule given in (5), if all model parameters are initialized at the same $\boldsymbol{x}_1$, the expected weighted average gradient is bounded as follows:*

$$\mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\|\nabla F(\boldsymbol{u}_k)\|^2\right] \leq \frac{2\left(F(\boldsymbol{x}_1) - F_{inf}]\right)}{\eta K} + \sigma^2\eta L\sum_{i=1}^{N}\boldsymbol{a}_i^2\boldsymbol{p}_i + \frac{2L^2}{K}\sum_{k=1}^{K}\mathbb{E}\|\boldsymbol{X}_k(\boldsymbol{I}-\boldsymbol{A})\|_{F_{\boldsymbol{a}}}^2$$
$$-\frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{N}\boldsymbol{a}_i\left((4\boldsymbol{p}_i - \boldsymbol{p}_i^2 - 2) - \eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i\boldsymbol{p}_i^2 + \boldsymbol{p}_i^2\right)\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \quad (96)$$

where $\boldsymbol{A} = \boldsymbol{a}\,\boldsymbol{1}^T$.

Proof. According to Assumption 1a,

$$\mathbb{E}_k[F(\boldsymbol{u}_{k+1})] - F(\boldsymbol{u}_k) \leq \mathbb{E}_k\left[\langle\nabla F(\boldsymbol{u}_k)), \boldsymbol{u}_{k+1} - \boldsymbol{u}_k\rangle + \frac{L}{2}\|\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\|_2^2\right] \quad (97)$$
$$= -\eta\mathbb{E}_k[\langle\nabla F(\boldsymbol{u}_k), \mathcal{G}_k\rangle] + \frac{\eta^2 L}{2}\mathbb{E}_k\left[\|\mathcal{G}_k\|_2^2\right]. \quad (98)$$

Plugging in Lemmas 2 and 3, we get:

$$\mathbb{E}_k[F(\boldsymbol{u}_{k+1})] - F(\boldsymbol{u}_k)$$
$$\leq -\eta\left[\frac{1}{2}\|\nabla F(\boldsymbol{u}_k)\|^2 + \sum_{i=1}^{N}\frac{\boldsymbol{a}_i}{2}\boldsymbol{p}_i^2\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 - \sum_{i=1}^{N}\frac{\boldsymbol{a}_i}{2}\|\nabla F(\boldsymbol{u}_k) - \boldsymbol{p}_i\nabla F(\boldsymbol{x}_k^{(i)})\|^2\right]$$
$$+\frac{\eta^2 L}{2}\sum_{i=1}^{N}\left(\boldsymbol{a}_i^2\boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i^2\boldsymbol{p}_i^2 + \boldsymbol{a}_i\boldsymbol{p}_i^2\right)\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 + \frac{\sigma^2\eta^2 L}{2}\sum_{i=1}^{N}\boldsymbol{a}_i^2\boldsymbol{p}_i \quad (99)$$
$$= -\frac{\eta}{2}\|\nabla F(\boldsymbol{u}_k)\|^2 + \frac{\eta}{2}\sum_{i=1}^{N}\boldsymbol{a}_i\|\nabla F(\boldsymbol{u}_k) - \boldsymbol{p}_i\nabla F(\boldsymbol{x}_k^{(i)})\|^2 + \frac{\sigma^2\eta^2 L}{2}\sum_{i=1}^{N}\boldsymbol{a}_i^2\boldsymbol{p}_i$$
$$-\frac{\eta}{2}\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i^2 - \eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i\boldsymbol{p}_i^2 + \boldsymbol{p}_i^2\right)\right)\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \quad (100)$$

After some rearranging, we obtain:

$$\|\nabla F(\boldsymbol{u}_k)\|^2 \leq \frac{2\left(F(\boldsymbol{u}_k) - \mathbb{E}_k[F(\boldsymbol{u}_{k+1})]\right)}{\eta} + \sigma^2\eta L\sum_{i=1}^{N}\boldsymbol{a}_i^2\boldsymbol{p}_i + \sum_{i=1}^{N}\boldsymbol{a}_i\|\nabla F(\boldsymbol{u}_k) - \boldsymbol{p}_i\nabla F(\boldsymbol{x}_k^{(i)})\|^2$$
$$-\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i^2 - \eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i\boldsymbol{p}_i^2 + \boldsymbol{p}_i^2\right)\right)\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \quad (101)$$

Taking the total expectation over all iterations:

$$\mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\|\nabla F(\boldsymbol{u}_k)\|^2\right] \leq \frac{2\left(F(\boldsymbol{x}_1) - F_{inf}]\right)}{\eta K} + \sigma^2\eta L\sum_{i=1}^{N}\boldsymbol{a}_i^2\boldsymbol{p}_i$$
$$+\frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{N}\boldsymbol{a}_i\mathbb{E}\|\nabla F(\boldsymbol{u}_k) - \boldsymbol{p}_i\nabla F(\boldsymbol{x}_k^{(i)})\|^2$$
$$-\frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i^2 - \eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i\boldsymbol{p}_i^2 + \boldsymbol{p}_i^2\right)\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \quad (102)$$

The third term in (102) can be bounded as:

$$\sum_{i=1}^{N} \boldsymbol{a}_i \mathbb{E}\|\nabla F(\boldsymbol{u}_k) - \boldsymbol{p}_i \nabla F(\boldsymbol{x}_k^{(i)})\|^2$$

$$= \sum_{i=1}^{N} \boldsymbol{a}_i \mathbb{E}\|\nabla F(\boldsymbol{u}_k) - \nabla F(\boldsymbol{x}_k^{(i)}) + (1-\boldsymbol{p}_i)\nabla F(\boldsymbol{x}_k^{(i)})\|^2 \tag{103}$$

$$\leq \sum_{i=1}^{N} \left[2\boldsymbol{a}_i \mathbb{E}\|\nabla F(\boldsymbol{u}_k) - \nabla F(\boldsymbol{x}_k^{(i)})\|^2 + 2\boldsymbol{a}_i (1-\boldsymbol{p}_i)^2 \mathbb{E}\|\nabla F(\boldsymbol{x}_k^{(i)})\|^2 \right] \tag{104}$$

$$\leq \sum_{i=1}^{N} 2\boldsymbol{a}_i L^2 \mathbb{E}\|\boldsymbol{u}_k - \boldsymbol{x}_k^{(i)}\|^2 + \sum_{i=1}^{N} 2\boldsymbol{a}_i (1-\boldsymbol{p}_i)^2 \mathbb{E}\|\nabla F(\boldsymbol{x}_k^{(i)})\|^2 \tag{105}$$

where (104) follows from the fact that $\|\boldsymbol{y}+\boldsymbol{z}\|^2 \leq 2\|\boldsymbol{y}\|^2 + 2\|\boldsymbol{z}\|^2$, and (105) follows from (104) by Assumption 1a.

Recalling the definition of the weighted Frobenius norm and the definition of u, we can simplify the first term in (105):

$$\sum_{i=1}^{N} 2\boldsymbol{a}_i L^2 \mathbb{E}\|\boldsymbol{u}_k - \boldsymbol{x}_k^{(i)}\|^2 = 2L^2 \mathbb{E}\|\boldsymbol{u}_k \mathbf{1}^T - \mathbf{X}_k\|_{F_{\boldsymbol{a}}}^2 \tag{106}$$

$$= 2L^2 \mathbb{E}\left\|\mathbf{X}_k \boldsymbol{a}\mathbf{1}^T - \mathbf{X}_k\right\|_{F_{\boldsymbol{a}}}^2 \tag{107}$$

$$= 2L^2 \mathbb{E}\|\mathbf{X}_k(\mathbf{I}-\mathbf{A})\|_{F_{\boldsymbol{a}}}^2. \tag{108}$$

Plugging (105) and (108) back into (102), we obtain:

$$\mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\|\nabla F(\boldsymbol{u}_k)\|^2\right] \leq \frac{2\left(F(\boldsymbol{x}_1) - F_{inf}]\right)}{\eta K} + \sigma^2 \eta L \sum_{i=1}^{N} \boldsymbol{a}_i^2 \boldsymbol{p}_i$$
$$+ \frac{2L^2}{K}\sum_{k=1}^{K}\mathbb{E}\|\mathbf{X}_k(\mathbf{I}-\mathbf{A})\|_{F_{\boldsymbol{a}}}^2 + \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{N} 2\boldsymbol{a}_i(1-\boldsymbol{p}_i)^2 \mathbb{E}\|\nabla F(\boldsymbol{x}_k^{(i)})\|^2$$
$$- \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{N} \boldsymbol{a}_i\left(\boldsymbol{p}_i^2 - \eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i\boldsymbol{p}_i^2 + \boldsymbol{p}_i^2\right)\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \tag{109}$$

$$= \frac{2\left(F(\boldsymbol{x}_1) - F_{inf}]\right)}{\eta K} + \sigma^2 \eta L \sum_{i=1}^{N} \boldsymbol{a}_i^2 \boldsymbol{p}_i + \frac{2L^2}{K}\sum_{k=1}^{K}\mathbb{E}\|\mathbf{X}_k(\mathbf{I}-\mathbf{A})\|_{F_{\boldsymbol{a}}}^2$$
$$- \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{N} \boldsymbol{a}_i\left((4\boldsymbol{p}_i - \boldsymbol{p}_i^2 - 2) - \eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i\boldsymbol{p}_i^2 + \boldsymbol{p}_i^2\right)\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2. \tag{110}$$

□

Lemma 5. *Given the properties of* $\boldsymbol{Z}$ *and* $\boldsymbol{V}$ *given in Propositions 1 and 2, it is the case that:*

$$\left\|\boldsymbol{Z}^j - \boldsymbol{A}\right\|_{op} = \zeta^j, \quad \|\boldsymbol{V} - \boldsymbol{A}\|_{op} = 1, \quad \|\boldsymbol{I} - \boldsymbol{A}\|_{op} = 1 \tag{111}$$

where $\boldsymbol{A} = \boldsymbol{a}\,\boldsymbol{1}^T$ *and* $\zeta = \max\{|\lambda_2(\mathbf{H})|, |\lambda(\mathbf{H})|\}$.

Proof. According to the definition of the matrix operator norm,

$$\left\|\mathbf{Z}^j - \mathbf{A}\right\|_{op} = \sqrt{\lambda_{max}((\mathbf{Z}^j - \mathbf{A})^T(\mathbf{Z}^j - \mathbf{A}))} \tag{112}$$

$$= \sqrt{\lambda_{max}(\mathbf{Z}^{2j} - \mathbf{A}\,\mathbf{Z}^j - \mathbf{Z}^j\,\mathbf{A} + \mathbf{A})} \tag{113}$$

$$= \sqrt{\lambda_{max}(\mathbf{Z}^{2j} - \mathbf{A})} \tag{114}$$

where (113) follows from $\mathbf{A}^j = \mathbf{A}$, and (114) follows from $\mathbf{A}\,\mathbf{Z} = \mathbf{Z}\,\mathbf{A} = \mathbf{A}$.

We can simplify (114) further:

$$= \sqrt{\lambda_{max}(\mathbf{Z}^{2j} - \mathbf{A}^{2j})} \tag{115}$$

$$= \sqrt{\lambda_{max}(\mathbf{Z} - \mathbf{A})^{2j}} \tag{116}$$

where (116) follows from the commutability of $\mathbf{Z}$ and $\mathbf{A}$.

Based on Proposition 2, the non-zero eigenvalues of $\mathbf{Z}$ are the same as $\mathbf{H}$. As shown in Lemma 6 of Rotaru & Nägeli (2004), for a matrix $\mathbf{Z}$ with the properties in Proposition 1, the spectral norm of $\mathbf{Z} - \mathbf{A}$ is equal to ζ.

Therefore:

$$\left\|\mathbf{Z}^j - \mathbf{A}\right\|_{op} = \sqrt{\zeta^{2j}} \tag{117}$$

$$= \zeta^j. \tag{118}$$

Similarly for $\mathbf{V}$:

$$\|\mathbf{V} - \mathbf{A}\|_{op} = \sqrt{\lambda_{max}((\mathbf{V} - \mathbf{A})^T(\mathbf{V} - \mathbf{A}))} \tag{119}$$

$$= \sqrt{\lambda_{max}(\mathbf{V} - \mathbf{A}\,\mathbf{V} - \mathbf{V}\,\mathbf{A} + \mathbf{A})} \tag{120}$$

$$= \sqrt{\lambda_{max}(\mathbf{V} - \mathbf{A})} \tag{121}$$

$$\tag{122}$$

where (120) follows from $\mathbf{A}^j = \mathbf{A}$ and $\mathbf{V}^j = \mathbf{V}$, and (121) follows from $\mathbf{A}\,\mathbf{V} = \mathbf{V}\,\mathbf{A} = \mathbf{A}$.

Note that the eigenvalues of each block $\mathbf{V}^{(d)}$ are $N^{(d)} - 1$ zeros and a one. The set of eigenvalues of $\mathbf{V}$ will include D ones. If $D > 1$, then based on Lemma 6 of Rotaru & Nägeli (2004) and Proposition 1, the spectral norm of $\mathbf{V} - \mathbf{A}$ is 1, so

$$\|\mathbf{V} - \mathbf{A}\|_{op} = \sqrt{1} \tag{123}$$

$$= 1. \tag{124}$$

Since the eigenvalues of $\mathbf{I}$ are all 1, and $\mathbf{I}$ is commutable with $\mathbf{A}$, we can similarly say:

$$\|\mathbf{I} - \mathbf{A}\|_{op} = 1. \tag{125}$$

□

Lemma 6. *Given two matrices $\boldsymbol{C} \in \mathbb{R}^{N \times M}$ and $\boldsymbol{D} \in \mathbb{R}^{M \times N}$, and an N-vector $\boldsymbol{a}$,*

$$\left|\mathrm{Tr}((\mathrm{diag}(\boldsymbol{a}))^{1/2}\,\boldsymbol{C}\,\boldsymbol{D}\,(\mathrm{diag}(\boldsymbol{a}))^{1/2})\right| \leq \|\,\boldsymbol{C}\,\|_{\mathrm{F_a}} \|\,\boldsymbol{D}\,\|_{\mathrm{F_a}}. \tag{126}$$

Proof. We define the i-th row of $\mathbf{C}$ as $\boldsymbol{c}_i^T$ and the i-th column of $\mathbf{D}$ as $\boldsymbol{d}_i$. We can rewrite the trace as:

$$\mathrm{Tr}((\mathrm{diag}(\boldsymbol{a}))^{1/2}\,\mathbf{C}\,\mathbf{D}\,(\mathrm{diag}(\boldsymbol{a}))^{1/2}) = \sum_{i=1}^{N}\sum_{j=1}^{M} \boldsymbol{a}_i\,\mathbf{C}_{i,j}\,\mathbf{D}_{j,i} \tag{127}$$

$$= \sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{c}_i^T \boldsymbol{d}_i. \tag{128}$$

Placing a squared norm around (128), we can apply the Cauchy-Schwartz inequality:

$$\left|\sum_{i=1}^{N} \boldsymbol{a}_i \boldsymbol{c}_i^T \boldsymbol{d}_i\right|^2 \leq \left(\sum_{i=1}^{N} \boldsymbol{a}_i \|\boldsymbol{c}_i^T\|^2\right)\left(\sum_{i=1}^{N} \boldsymbol{a}_i \|\boldsymbol{d}_i\|^2\right) \tag{129}$$

$$= \left(\sum_{i=1}^{N}\sum_{j=1}^{M} \boldsymbol{a}_i\,\mathbf{C}_{i,j}^2\right)\left(\sum_{i=1}^{N}\sum_{j=1}^{M} \boldsymbol{a}_i\,\mathbf{D}_{i,j}^2\right) \tag{130}$$

$$= \|\,\mathbf{C}\,\|_{\mathrm{F_a}}^2 \|\,\mathbf{D}\,\|_{\mathrm{F_a}}^2. \tag{131}$$

□

Lemma 7. *Given two matrices $\boldsymbol{C} \in \mathbb{R}^{M\times N}$ and $\boldsymbol{D} \in \mathbb{R}^{N\times N}$, and an N-vector $\boldsymbol{a}$, then*

$$\| \boldsymbol{C}\boldsymbol{D} \|_{F_{\boldsymbol{a}}} \leq \| \boldsymbol{C} \|_{F_{\boldsymbol{a}}} \| \boldsymbol{D} \|_{op}. \tag{132}$$

Proof. We define the i-th row of $\mathbf{C}$ as $\boldsymbol{c}_i^T$ and the set $\mathcal{I} = \{i \in [1, M] : \|\boldsymbol{c}_i^T\| \neq 0\}$. We can rewrite the squared Frobenius norm as:

$$\begin{aligned}
\| \mathbf{C}\,\mathbf{D} \|_{F_{\boldsymbol{a}}}^2 &= \sum_{i=1}^{M} \|\boldsymbol{c}_i^T\, \mathbf{D}\, (\mathrm{diag}(\boldsymbol{a}))^{1/2} \|^2 && (133)\\
&= \sum_{i\in\mathcal{I}}^{M} \|\boldsymbol{c}_i^T\, \mathbf{D}\, (\mathrm{diag}(\boldsymbol{a}))^{1/2} \|^2 && (134)\\
&= \sum_{i\in\mathcal{I}}^{M} \|\boldsymbol{c}_i^T\, (\mathrm{diag}(\boldsymbol{a}))^{1/2} \|^2 \frac{\|\boldsymbol{c}_i^T\, \mathbf{D}\, (\mathrm{diag}(\boldsymbol{a}))^{1/2} \|^2}{\|\boldsymbol{c}_i^T\, (\mathrm{diag}(\boldsymbol{a}))^{1/2} \|^2} && (135)\\
&\leq \sum_{i\in\mathcal{I}}^{M} \|\boldsymbol{c}_i^T\, (\mathrm{diag}(\boldsymbol{a}))^{1/2} \|^2 \| \mathbf{D} \|_{op}^2 && (136)\\
&= \| \mathbf{C} \|_{F_{\boldsymbol{a}}}^2 \| \mathbf{D} \|_{op}^2. && (137)
\end{aligned}$$

□

A.3 Proof of Theorem 1

We recall Theorem 1.

Theorem 1. *Under Assumptions 1 and 2, if η satisfies the following for all $i \in \mathcal{M}$:*

$$(4\boldsymbol{p}_i - \boldsymbol{p}_i^2 - 2) \geq \eta L \left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1) - \boldsymbol{a}_i\boldsymbol{p}_i^2 + \boldsymbol{p}_i^2\right) + 8L^2\eta^2 q^2\tau^2\Gamma \tag{12}$$

where $\Gamma = \frac{\zeta}{1-\zeta^2} + \frac{2}{1-\zeta} + \frac{\zeta}{(1-\zeta)^2}$ and $\zeta = \max\{|\lambda_2(\mathbf{H})|, |\lambda_N(\mathbf{H})|\}$, then the expected square norm of the average gradient, averaged over K iterations, is bounded as follows:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\|\nabla F(\boldsymbol{u}_k)\|_2^2\right] &\leq \frac{2\left(F(\boldsymbol{x}_1) - F_{inf}]\right)}{\eta K} + \sigma^2\eta L \sum_{i=1}^{N} \boldsymbol{a}_i^2\boldsymbol{p}_i \\
&\quad + 4L^2\eta^2\sigma^2 q^3\tau^3 \left(\frac{1}{q\tau} - \frac{1}{K}\right)\left(\frac{\zeta^2}{1-\zeta^2} + \frac{2\zeta}{1-\zeta} + \frac{1}{(1-\zeta)^2}\right)\mathrm{P} \\
&\quad + 4L^2\eta^2\sigma^2 \left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2\frac{(q-1)(2q+1)}{6} + \frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P} && (13)\\
\xrightarrow{K\to\infty} \quad & \sigma^2\eta L \sum_{i=1}^{N} \boldsymbol{a}_i^2\boldsymbol{p}_i + 4L^2\eta^2\sigma^2 q^2\tau^2 \left(\frac{\zeta^2}{1-\zeta^2} + \frac{2\zeta}{1-\zeta} + \frac{1}{(1-\zeta)^2}\right)\mathrm{P} \\
&\quad + 4L^2\eta^2\sigma^2 \left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2\frac{(q-1)(2q+1)}{6} + \frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P} && (14)
\end{aligned}$$

where $\mathrm{P} = \sum_{i=1}^{N} \boldsymbol{a}_i\boldsymbol{p}_i$.

We now give the proof of Theorem 1 using Lemmas 2-7.

Proof. Using our intermediate result from Lemma 4, we decompose $\mathbf{X}_k(\mathbf{I}-\mathbf{A})$ using our recursive definition of $\mathbf{X}_k$:

$$\begin{aligned}
\mathbf{X}_k(\mathbf{I}-\mathbf{A}) &= (\mathbf{X}_{k-1} - \eta\,\mathbf{G}_{k-1})\,\mathbf{T}_{k-1}(\mathbf{I}-\mathbf{A}) && (138)\\
&= \mathbf{X}_{k-1}(\mathbf{I}-\mathbf{A})\,\mathbf{T}_{k-1} - \eta\,\mathbf{G}_{k-1}(\mathbf{T}_{k-1}-\mathbf{A}) && (139)\\
&= [(\mathbf{X}_{k-2} - \eta\,\mathbf{G}_{k-2})\,\mathbf{T}_{k-2}(\mathbf{I}-\mathbf{A})]\,\mathbf{T}_{k-1} - \eta\,\mathbf{G}_{k-1}(\mathbf{T}_{k-1}-\mathbf{A}) && (140)\\
&= [\mathbf{X}_{k-2}(\mathbf{I}-\mathbf{A})\,\mathbf{T}_{k-2} - \eta\,\mathbf{G}_{k-2}(\mathbf{T}_{k-2}-\mathbf{A})]\,\mathbf{T}_{k-1} - \eta\,\mathbf{G}_{k-1}(\mathbf{T}_{k-1}-\mathbf{A}) && (141)\\
&= \mathbf{X}_{k-2}(\mathbf{I}-\mathbf{A})\,\mathbf{T}_{k-2}\,\mathbf{T}_{k-1} - \eta\,\mathbf{G}_{k-2}(\mathbf{T}_{k-2}\,\mathbf{T}_{k-1}-\mathbf{A}) - \eta\,\mathbf{G}_{k-1}(\mathbf{T}_{k-1}-\mathbf{A}). && (142)
\end{aligned}$$

where (139) follows from the commutability of $\mathbf{T}_k$ and $\mathbf{A}$ by Proposition 4.

Continuing this, we end up with:

$$\mathbf{X}_k(\mathbf{I}-\mathbf{A}) = \mathbf{X}_1(\mathbf{I}-\mathbf{A})\prod_{l=1}^{k-1}\mathbf{T}_l - \eta\sum_{s=1}^{k-1}\mathbf{G}_s\left(\prod_{l=s}^{k-1}\mathbf{T}_l - \mathbf{A}\right). \tag{143}$$

Since all workers initialize their models to the same vector, $\mathbf{X}_1(\mathbf{I}-\mathbf{A})\prod_{l=1}^{k-1}\mathbf{T}_k = \mathbf{0}$, and thus we have:

$$\mathbb{E}\left\|\mathbf{X}_k(\mathbf{I}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2 = \eta^2\mathbb{E}\left\|\sum_{s=1}^{k-1}\mathbf{G}_s\left(\prod_{l=s}^{k-1}\mathbf{T}_l - \mathbf{A}\right)\right\|_{F_{\boldsymbol{a}}}^2. \tag{144}$$

Let $k = jq\tau + l\tau + f$, where j is the number of hub network averaging rounds, l is the number of sub-network averaging rounds since the last hub network averaging round, and f is the number of local iterations since the last sub-network averaging round. Define:

$$\Phi_{s,k-1} = \prod_{l=s}^{k-1}\mathbf{T}_l\,.$$

Noting that $\mathbf{V}^j = \mathbf{V}$, and $\mathbf{V}\,\mathbf{Z} = \mathbf{Z}\,\mathbf{V} = \mathbf{Z}$ by Proposition 3, $\Phi_{s,k-1}$ can be expressed as:

$$\Phi_{s,k-1} = \begin{cases} \mathbf{I} & jq\tau + l\tau < s < jq\tau + l\tau + f \\ \mathbf{V} & jq\tau < s \leq jq\tau + l\tau \\ \mathbf{Z} & (j-1)q\tau < s \leq jq\tau \\ \mathbf{Z}^2 & (j-2)q\tau < s \leq (j-1)q\tau \\ \vdots & \\ \mathbf{Z}^j & 1 \leq s \leq q\tau. \end{cases} \tag{145}$$

For $r < j$, let

$$\mathbf{Y}_r = \sum_{s=rq\tau+1}^{(r+1)q\tau}\mathbf{G}_s\,, \quad \mathbf{Q}_r = \sum_{s=rq\tau+1}^{(r+1)q\tau}\nabla F(\mathbf{X}_s)$$

We also let $\mathbf{Y}_{j_1} = \sum_{s=jq\tau+1}^{jq\tau+l\tau}\mathbf{G}_s$, $\mathbf{Y}_{j_2} = \sum_{s=jq\tau+l\tau+1}^{jq\tau+l\tau+f}\mathbf{G}_s$, $\mathbf{Q}_{j_1} = \sum_{s=jq\tau+1}^{jq\tau+l\tau}\nabla F(\mathbf{X}_s)$, and $\mathbf{Q}_{j_2} = \sum_{s=jq\tau+l\tau+1}^{jq\tau+l\tau+f}\nabla F(\mathbf{X}_s)$. With this in mind, we can split the sum in (144) into batches for each hub network averaging period:

$$\sum_{s=1}^{q\tau}\mathbf{G}_s\left(\Phi_{s,k-1} - \mathbf{A}\right) = \mathbf{Y}_0(\mathbf{Z}^j - \mathbf{A}) \tag{146}$$

$$\sum_{s=q\tau+1}^{2q\tau}\mathbf{G}_s\left(\Phi_{s,k-1} - \mathbf{A}\right) = \mathbf{Y}_1(\mathbf{Z}^{j-1} - \mathbf{A}) \tag{147}$$

$$\ldots$$

$$\sum_{s=(j-1)q\tau+1}^{jq\tau}\mathbf{G}_s\left(\Phi_{s,k-1} - \mathbf{A}\right) = \mathbf{Y}_{j-1}(\mathbf{Z} - \mathbf{A}) \tag{148}$$

$$\sum_{s=jq\tau+1}^{jq\tau+l\tau+f}\mathbf{G}_s\left(\Phi_{s,k-1} - \mathbf{A}\right) = \mathbf{Y}_{j_1}(\mathbf{V} - \mathbf{A}) + \mathbf{Y}_{j_2}(\mathbf{I} - \mathbf{A}). \tag{149}$$

Summing this all together, we get:

$$\sum_{s=1}^{k-1}\mathbf{G}_s\left(\Phi_{s,k-1} - \mathbf{A}\right) = \sum_{r=0}^{j-1}\mathbf{Y}_r(\mathbf{Z}^{j-r} - \mathbf{A}) + \mathbf{Y}_{j_1}(\mathbf{V} - \mathbf{A}) + \mathbf{Y}_{j_2}(\mathbf{I} - \mathbf{A}). \tag{150}$$

Plugging (150) into (144):

$$\mathbb{E}\left\|\mathbf{X}_k(\mathbf{I}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2 = \eta^2\mathbb{E}\left\|\sum_{r=0}^{j-1}\mathbf{Y}_r(\mathbf{Z}^{j-r}-\mathbf{A}) + \mathbf{Y}_{j_1}(\mathbf{V}-\mathbf{A}) + \mathbf{Y}_{j_2}(\mathbf{I}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2 \tag{151}$$

$$= \eta^2\mathbb{E}\Bigg\|\sum_{r=0}^{j-1}(\mathbf{Y}_r-\mathbf{Q}_r)(\mathbf{Z}^{j-r}-\mathbf{A}) + (\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1})(\mathbf{V}-\mathbf{A})$$

$$+ (\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2})(\mathbf{I}-\mathbf{A}) + \sum_{r=0}^{j-1}\mathbf{Q}_r(\mathbf{Z}^{j-r}-\mathbf{A}) + \mathbf{Q}_{j_1}(\mathbf{V}-\mathbf{A}) + \mathbf{Q}_{j_2}(\mathbf{I}-\mathbf{A})\Bigg\|_{F_{\boldsymbol{a}}}^2 \tag{152}$$

$$\leq \underbrace{2\eta^2\mathbb{E}\left\|\sum_{r=0}^{j-1}(\mathbf{Y}_r-\mathbf{Q}_r)(\mathbf{Z}^{j-r}-\mathbf{A}) + (\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1})(\mathbf{V}-\mathbf{A}) + (\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2})(\mathbf{I}-\mathbf{A}))\right\|_{F_{\boldsymbol{a}}}^2}_{T_1}$$

$$+ \underbrace{2\eta^2\mathbb{E}\left\|\sum_{r=0}^{j-1}\mathbf{Q}_r(\mathbf{Z}^{j-r}-\mathbf{A}) + \mathbf{Q}_{j_1}(\mathbf{V}-\mathbf{A}) + \mathbf{Q}_{j_2}(\mathbf{I}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2}_{T_2} \tag{153}$$

where (153) follows from the fact that $\|\boldsymbol{y}+\boldsymbol{z}\|^2 \leq 2\|\boldsymbol{y}\|^2 + 2\|\boldsymbol{z}\|^2$.

We first put a bound on T_1:

$$T_1 = 2\eta^2\mathbb{E}\left\|\sum_{r=0}^{j-1}(\mathbf{Y}_r-\mathbf{Q}_r)(\mathbf{Z}^{j-r}-\mathbf{A}) + (\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1})(\mathbf{V}-\mathbf{A}) + (\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2})(\mathbf{I}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2 \tag{154}$$

$$= 2\eta^2\Bigg(\sum_{r=0}^{j-1}\mathbb{E}\left\|(\mathbf{Y}_r-\mathbf{Q}_r)(\mathbf{Z}^{j-r}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E}\left\|(\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1})(\mathbf{V}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2$$

$$+ \mathbb{E}\left\|(\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2})(\mathbf{I}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2\Bigg)$$

$$+ \underbrace{2\eta^2\sum_{n=0}^{j-1}\sum_{l=0,l\neq n}^{j-1}\mathbb{E}\left|\underbrace{\mathrm{Tr}\left((\mathrm{diag}(\boldsymbol{a}))^{1/2}(\mathbf{Z}^{j-n}-\mathbf{A})(\mathbf{Y}_n-\mathbf{Q}_n)^T(\mathbf{Y}_l-\mathbf{Q}_l)(\mathbf{Z}^{j-l}-\mathbf{A})(\mathrm{diag}(\boldsymbol{a}))^{1/2}\right)}_{TR}\right|}_{TR_0}$$

$$+ \underbrace{4\eta^2\sum_{l=0}^{j-1}\mathbb{E}\left|\mathrm{Tr}\left((\mathrm{diag}(\boldsymbol{a}))^{1/2}(\mathbf{V}-\mathbf{A})(\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1})^T(\mathbf{Y}_l-\mathbf{Q}_l)(\mathbf{Z}^{j-l}-\mathbf{A})(\mathrm{diag}(\boldsymbol{a}))^{1/2}\right)\right|}_{TR_1}$$

$$+ \underbrace{4\eta^2\sum_{l=0}^{j-1}\mathbb{E}\left|\mathrm{Tr}\left((\mathrm{diag}(\boldsymbol{a}))^{1/2}(\mathbf{I}-\mathbf{A})(\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2})^T(\mathbf{Y}_l-\mathbf{Q}_l)(\mathbf{Z}^{j-l}-\mathbf{A})(\mathrm{diag}(\boldsymbol{a}))^{1/2}\right)\right|}_{TR_2}$$

$$+ \underbrace{4\eta^2\mathbb{E}\left|\mathrm{Tr}\left((\mathrm{diag}(\boldsymbol{a}))^{1/2}(\mathbf{V}-\mathbf{A})(\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1})^T(\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2})(\mathbf{I}-\mathbf{A})(\mathrm{diag}(\boldsymbol{a}))^{1/2}\right)\right|}_{TR_3}. \tag{155}$$

TR can be bounded as:

$$
\begin{aligned}
TR &\leq \left\|(\mathbf{Z}^{j-n}-\mathbf{A})(\mathbf{Y}_n-\mathbf{Q}_n)^T\right\|_{F_{\boldsymbol{a}}} \left\|(\mathbf{Y}_l-\mathbf{Q}_l)(\mathbf{Z}^{j-l}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}} && (156)\\
&\leq \left\|(\mathbf{Z}^{j-n}-\mathbf{A})\right\|_{op} \|\mathbf{Y}_n-\mathbf{Q}_n\|_{F_{\boldsymbol{a}}} \|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}} \left\|(\mathbf{Z}^{j-l}-\mathbf{A})\right\|_{op} && (157)\\
&\leq \zeta^{2j-n-l} \|\mathbf{Y}_n-\mathbf{Q}_n\|_{F_{\boldsymbol{a}}} \|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}} && (158)\\
&\leq \frac{1}{2}\zeta^{2j-n-l}\left[\|\mathbf{Y}_n-\mathbf{Q}_n\|_{F_{\boldsymbol{a}}}^2 + \|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2\right] && (159)
\end{aligned}
$$

where (156) follows from Lemma 6, (157) follows from Lemma 7, and (158) follows from Lemma 5. We can similarly bound TR_1 and TR_3:

$$
\begin{aligned}
TR_1 &\leq 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l}\left[\mathbb{E}\left\|\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1}\right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E}\|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2\right] && (160)\\
TR_2 &\leq 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l}\left[\mathbb{E}\left\|\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2}\right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E}\|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2\right] && (161)\\
TR_3 &\leq 2\eta^2 \left[\mathbb{E}\left\|\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1}\right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E}\left\|\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2}\right\|_{F_{\boldsymbol{a}}}\right]. && (162)
\end{aligned}
$$

Summing TR_0 through TR_3, we get:

$$
\begin{aligned}
\sum_{t=0}^{3} TR_t \leq\ & \eta^2 \sum_{n=0}^{j-1}\sum_{l=0,l\neq n}^{j-1} \zeta^{2j-n-l}\left[\mathbb{E}\|\mathbf{Y}_n-\mathbf{Q}_n\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E}\|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2\right]\\
&+ 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l}\left[\mathbb{E}\left\|\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1}\right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E}\|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2\right]\\
&+ 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l}\left[\mathbb{E}\left\|\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2}\right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E}\|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2\right]\\
&+ 2\eta^2\mathbb{E}\left\|\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1}\right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2\mathbb{E}\left\|\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2}\right\|_{F_{\boldsymbol{a}}}^2 && (163)\\
\leq\ & 2\eta^2 \sum_{n=0}^{j-1}\sum_{l=0,l\neq n}^{j-1} \zeta^{2j-n-l}\mathbb{E}\|\mathbf{Y}_n-\mathbf{Q}_n\|_{F_{\boldsymbol{a}}}^2\\
&+ 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l}\mathbb{E}\|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l}\mathbb{E}\|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2\\
&+ 2\eta^2 \sum_{l=0}^{j} \zeta^{j-l}\mathbb{E}\left\|\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1}\right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \sum_{l=0}^{j} \zeta^{j-l}\mathbb{E}\left\|\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2}\right\|_{F_{\boldsymbol{a}}}^2 && (164)\\
=\ & 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n}\mathbb{E}\|\mathbf{Y}_n-\mathbf{Q}_n\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0,l\neq n}^{j-1} \zeta^{j-l} + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l}\mathbb{E}\|\mathbf{Y}_l-\mathbf{Q}_l\|_{F_{\boldsymbol{a}}}^2\\
&+ 2\eta^2\mathbb{E}\left\|\mathbf{Y}_{j_1}-\mathbf{Q}_{j_1}\right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} + 2\eta^2\mathbb{E}\left\|\mathbf{Y}_{j_2}-\mathbf{Q}_{j_2}\right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} && (165)
\end{aligned}
$$

where (164) follows from the symmetry of the n and l indices.

Plugging (165) back into (155):

$$
\begin{aligned}
T_1 \leq\ & 2\eta^2 \sum_{r=0}^{j-1} \mathbb{E} \left\| (\mathbf{Y}_r - \mathbf{Q}_r) \right\|_{F_{\boldsymbol{a}}}^2 \left\| (\mathbf{Z}^{j-r} - \mathbf{A}) \right\|_{op}^2 + 2\eta^2 \mathbb{E} \left\| (\mathbf{Y}_{j_1} - \mathbf{Q}_{j_1}) \right\|_{F_{\boldsymbol{a}}}^2 \left\| \mathbf{V} - \mathbf{A} \right\|_{op}^2 \\
& + 2\eta^2 \mathbb{E} \left\| (\mathbf{Y}_{j_2} - \mathbf{Q}_{j_2}) \right\|_{F_{\boldsymbol{a}}}^2 \left\| \mathbf{I} - \mathbf{A} \right\|_{op}^2 + 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n} \mathbb{E} \left\| \mathbf{Y}_n - \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0, l \neq n}^{j-1} \zeta^{j-l} \\
& + 2\eta^2 \mathbb{E} \left\| \mathbf{Y}_{j_1} - \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} + 2\eta^2 \mathbb{E} \left\| \mathbf{Y}_{j_2} - \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} \\
& + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Y}_l - \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \qquad (166) \\
\leq\ & 2\eta^2 \sum_{r=0}^{j-1} \mathbb{E} \left\| (\mathbf{Y}_r - \mathbf{Q}_r) \right\|_{F_{\boldsymbol{a}}}^2 \zeta^{2(j-r)} + 2\eta^2 \mathbb{E} \left\| (\mathbf{Y}_{j_1} - \mathbf{Q}_{j_1}) \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \mathbb{E} \left\| (\mathbf{Y}_{j_2} - \mathbf{Q}_{j_2}) \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n} \mathbb{E} \left\| \mathbf{Y}_n - \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0, l \neq n}^{j-1} \zeta^{j-l} \\
& + 2\eta^2 \mathbb{E} \left\| \mathbf{Y}_{j_1} - \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} + 2\eta^2 \mathbb{E} \left\| \mathbf{Y}_{j_2} - \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} \\
& + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Y}_l - \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \qquad (167)
\end{aligned}
$$

where (166) follows from Lemma 7, and (167) follows from Lemma 5.

We further bound T_1:

$$
\begin{aligned}
T_1 \leq\ & 2\eta^2 \sum_{r=0}^{j-1} \mathbb{E} \left\| (\mathbf{Y}_r - \mathbf{Q}_r) \right\|_{F_{\boldsymbol{a}}}^2 \zeta^{2(j-r)} + 2\eta^2 \mathbb{E} \left\| (\mathbf{Y}_{j_1} - \mathbf{Q}_{j_1}) \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \mathbb{E} \left\| (\mathbf{Y}_{j_2} - \mathbf{Q}_{j_2}) \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n} \mathbb{E} \left\| \mathbf{Y}_n - \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 \frac{\zeta}{1-\zeta} \\
& + 2\eta^2 \mathbb{E} \left\| \mathbf{Y}_{j_1} - \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \frac{1}{1-\zeta} + 2\eta^2 \mathbb{E} \left\| \mathbf{Y}_{j_2} - \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \frac{1}{1-\zeta} \\
& + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Y}_l - \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \qquad (168) \\
=\ & 2\eta^2 \sum_{r=0}^{j-1} \left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta} \right) \mathbb{E} \left\| (\mathbf{Y}_r - \mathbf{Q}_r) \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \left(\frac{2-\zeta}{1-\zeta} \right) \mathbb{E} \left\| \mathbf{Y}_{j_1} - \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \left(\frac{2-\zeta}{1-\zeta} \right) \mathbb{E} \left\| \mathbf{Y}_{j_2} - \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \qquad (169)
\end{aligned}
$$

where (168) follows from the summation formulae of a power series:

$$
\sum_{l=0}^{j} \zeta^{j-l} \leq \sum_{l=-\infty}^{j} \zeta^{j-l} \leq \frac{1}{1-\zeta}, \qquad \sum_{l=0}^{j-1} \zeta^{j-l} \leq \sum_{l=-\infty}^{j-1} \zeta^{j-l} \leq \frac{\zeta}{1-\zeta}. \qquad (170)
$$

Taking a closer look at $\mathbb{E}\left\|(\mathbf{Y}_r - \mathbf{Q}_r)\right\|_{F_{\boldsymbol{a}}}^2$ for $0 \leq r < j$:

$$\mathbb{E}\left\|(\mathbf{Y}_r - \mathbf{Q}_r)\right\|_{F_{\boldsymbol{a}}}^2 = \mathbb{E}\left\|\sum_{s=rq\tau+1}^{(r+1)q\tau}(\mathbf{G}_s - \nabla F(\mathbf{X}_s))\right\|_{F_{\boldsymbol{a}}}^2 \tag{171}$$

$$= \sum_{i=1}^{N}\boldsymbol{a}_i\mathbb{E}\left\|\sum_{s=rq\tau+1}^{(r+1)q\tau}(g_s^i - \nabla F(\boldsymbol{x}_s^{(i)}))\right\|^2 \tag{172}$$

$$\leq \sum_{i=1}^{N}\boldsymbol{a}_i q\tau\sum_{s=rq\tau+1}^{(r+1)q\tau}\mathbb{E}\left\|(g_s^i - \nabla F(\boldsymbol{x}_s^{(i)}))\right\|^2 \tag{173}$$

$$\leq q\tau\left(\sum_{i=1}^{N}\boldsymbol{a}_i\sum_{s=rq\tau+1}^{(r+1)q\tau}(\boldsymbol{p}_i(\beta-1)+1)\,\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2\right) + q^2\tau^2\sigma^2\sum_{i=1}^{N}\boldsymbol{a}_i\boldsymbol{p}_i \tag{174}$$

$$= q\tau\left(\sum_{i=1}^{N}\boldsymbol{a}_i\sum_{s=rq\tau+1}^{(r+1)q\tau}(\boldsymbol{p}_i(\beta-1)+1)\,\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2\right) + q^2\tau^2\sigma^2\,\mathrm{P}\,. \tag{175}$$

where (174) follows from Assumption 1d and (28).

Similarly, for $r = j_1$ and $r = j_2$:

$$\mathbb{E}\left\|(\mathbf{Y}_{j_1} - \mathbf{Q}_{j_1})\right\|_{F_{\boldsymbol{a}}}^2 \leq l\tau\left(\sum_{i=1}^{N}\boldsymbol{a}_i\sum_{s=jq\tau+1}^{jq\tau+l\tau}(\boldsymbol{p}_i(\beta-1)+1)\,\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2\right) + l^2\tau^2\sigma^2\,\mathrm{P} \tag{176}$$

$$\mathbb{E}\left\|(\mathbf{Y}_{j_2} - \mathbf{Q}_{j_2})\right\|_{F_{\boldsymbol{a}}} \leq (f-1)\left(\sum_{i=1}^{N}\boldsymbol{a}_i\sum_{s=jq\tau+l\tau+1}^{jq\tau+l\tau+f-1}(\boldsymbol{p}_i(\beta-1)+1)\,\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2\right) + (f-1)^2\sigma^2\,\mathrm{P}\,. \tag{177}$$

Plugging (175), (176), and (177) into (169), we can bound T_1 as follows:

$$\begin{aligned}
T_1 \leq\ & 2\eta^2\sigma^2\left(\left(q^2\tau^2\sum_{r=0}^{j-1}\left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta}\right)\right) + \left(\frac{2-\zeta}{1-\zeta}\right)\left(l^2\tau^2 + (f-1)^2\right)\right)\mathrm{P} \\
& + 2\eta^2 q\tau\sum_{r=0}^{j-1}\left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta}\right)\sum_{s=rq\tau+1}^{(r+1)q\tau}\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i(\beta-1)+1\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2 \\
& + 2\eta^2\left(\frac{2-\zeta}{1-\zeta}\right)l\tau\left(\sum_{i=1}^{N}\boldsymbol{a}_i\sum_{s=jq\tau+1}^{jq\tau+l\tau}(\boldsymbol{p}_i(\beta-1)+1)\,\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2\right) \\
& + 2\eta^2\left(\frac{2-\zeta}{1-\zeta}\right)(f-1)\left(\sum_{i=1}^{N}\boldsymbol{a}_i\sum_{s=jq\tau+l\tau+1}^{jq\tau+l\tau+f-1}(\boldsymbol{p}_i(\beta-1)+1)\,\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2\right).
\end{aligned} \tag{178}$$

Referring back to Lemma 4, our goal is to sum T_1 over $k = 1, \ldots, K$ iterations. First, we sum over the j-th sub-network update period up to the j-th hub network averaging, for $l = 0, \ldots, q-1$ and

$f = 1, \ldots, \tau$:

$$\sum_{l=0}^{q-1}\sum_{f=1}^{\tau} T_1 \leq 2\eta^2\sigma^2 q^3\tau^3 \sum_{r=0}^{j-1}\left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta}\right)\mathrm{P}$$
$$+ 2\eta^2\sigma^2\left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^3\frac{q(q-1)(2q+1)}{6} + q\frac{\tau(\tau-1)(2\tau+1)}{6}\right)\mathrm{P}$$
$$+ 2\eta^2 q^2\tau^2 \sum_{r=0}^{j-1}\left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta}\right)\sum_{s=rq\tau+1}^{(r+1)q\tau}\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i(\beta-1)+1\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2$$
$$+ \eta^2\left(\frac{2-\zeta}{1-\zeta}\right)q(q-1)\tau^2\sum_{i=1}^{N}\boldsymbol{a}_i\sum_{s=jq\tau+1}^{j(q\tau+1)}\left(\boldsymbol{p}_i(\beta-1)+1\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2$$
$$+ \eta^2\left(\frac{2-\zeta}{1-\zeta}\right)q^2\tau(\tau-1)\sum_{i=1}^{N}\boldsymbol{a}_i\sum_{s=j(q\tau+1)+1}^{j(q\tau+1)+\tau-1}\left(\boldsymbol{p}_i(\beta-1)+1\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2. \tag{179}$$

Let:

$$\Gamma_r = \left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta}\right). \tag{180}$$

Note that $\Gamma_j = \frac{3-2\zeta}{1-\zeta} > \frac{2-\zeta}{1-\zeta}$. Using this inequality, we can bound the sum of the last three terms of (179) to get $2q^2\tau^2\sum_{r=0}^{j}\Gamma_r$:

$$\sum_{l=0}^{q-1}\sum_{f=1}^{\tau} T_1 \leq 2\eta^2\sigma^2 q^3\tau^3 \sum_{r=0}^{j-1}\left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta}\right)\mathrm{P}$$
$$+ 2\eta^2\sigma^2\left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^3\frac{q(q-1)(2q+1)}{6} + q\frac{\tau(\tau-1)(2\tau+1)}{6}\right)\mathrm{P}$$
$$+ 2\eta^2 q^2\tau^2 \sum_{r=0}^{j}\Gamma_r\sum_{s=rq\tau+1}^{(r+1)q\tau}\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i(\beta-1)+1\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2. \tag{181}$$

Summing (181) over the hub network averaging periods $j = 0, \ldots, K/(q\tau) - 1$, we obtain:

$$\sum_{j=0}^{K/(q\tau)-1}\sum_{l=0}^{q-1}\sum_{f=1}^{\tau} T_1 \leq 2\eta^2\sigma^2 q^3\tau^3 \sum_{j=0}^{K/(q\tau)-1}\sum_{r=0}^{j-1}\left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta}\right)\mathrm{P}$$
$$+ 2\eta^2\sigma^2 K\left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2\frac{(q-1)(2q+1)}{6} + \frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P}$$
$$+ 2\eta^2 q^2\tau^2 \sum_{j=0}^{K/(q\tau)-1}\sum_{r=0}^{j}\Gamma_r\sum_{s=rq\tau+1}^{(r+1)q\tau}\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i(\beta-1)+1\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2 \tag{182}$$
$$= 2\eta^2\sigma^2 q^3\tau^3 \sum_{r=0}^{K/(q\tau)-2}\sum_{j=r+1}^{K/(q\tau)-1}\left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta}\right)\mathrm{P}$$
$$+ 2\eta^2\sigma^2 K\left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2\frac{(q-1)(2q+1)}{6} + \frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P}$$
$$+ 2\eta^2 q^2\tau^2 \sum_{r=0}^{K/(q\tau)-1}\left(\sum_{j=r}^{K/(q\tau)-1}\Gamma_j\right)\left(\sum_{s=rq\tau+1}^{(r+1)q\tau}\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i(\beta-1)+1\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_s^{(i)})\right\|^2\right). \tag{183}$$

Applying the following summation formula to sum over Γ_j, we obtain

$$\sum_{j=r}^{K/(q\tau)-1}\left(\zeta^{2(j-r)}+2\zeta^{j-r}+\frac{\zeta^{j-r+1}}{1-\zeta}\right)\leq\sum_{j=r}^{\infty}\left(\zeta^{2(j-r)}+2\zeta^{j-r}+\frac{\zeta^{j-r+1}}{1-\zeta}\right) \tag{184}$$

$$\leq\frac{1}{1-\zeta^2}+\frac{2}{1-\zeta}+\frac{\zeta}{(1-\zeta)^2}. \tag{185}$$

We let $\Gamma=\frac{1}{1-\zeta^2}+\frac{2}{1-\zeta}+\frac{\zeta}{(1-\zeta)^2}$. We can also apply this following summation formula to the first term in (183):

$$\sum_{j=r+1}^{K/(q\tau)-1}\left(\zeta^{2(j-r)}+2\zeta^{j-r}+\frac{\zeta^{j-r+1}}{1-\zeta}\right)\leq\sum_{j=r+1}^{\infty}\left(\zeta^{2(j-r)}+2\zeta^{j-r}+\frac{\zeta^{j-r+1}}{1-\zeta}\right) \tag{186}$$

$$\leq\frac{\zeta^2}{1-\zeta^2}+\frac{2\zeta}{1-\zeta}+\frac{1}{(1-\zeta)^2}. \tag{187}$$

Applying the summation formula in (187), plugging Γ in, and indexing the iterations in terms of k, we bound (183) as:

$$\begin{aligned}\sum_{k=1}^{K}T_1\leq{}&2\eta^2\sigma^2q^3\tau^3\left(\frac{K}{q\tau}-1\right)\left(\frac{\zeta^2}{1-\zeta^2}+\frac{2\zeta}{1-\zeta}+\frac{1}{(1-\zeta)^2}\right)\mathrm{P}\\&+2\eta^2\sigma^2K\left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2\frac{(q-1)(2q+1)}{6}+\frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P}\\&+2\eta^2q^2\tau^2\Gamma\sum_{k=1}^{K}\sum_{i=1}^{N}\boldsymbol{a}_i\left(\boldsymbol{p}_i(\beta-1)+1\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2.\end{aligned} \tag{188}$$

Now we bound T_2:

$$T_2=2\eta^2\mathbb{E}\left\|\sum_{r=0}^{j-1}\mathbf{Q}_r(\mathbf{Z}^{j-r}-\mathbf{A})+\mathbf{Q}_{j_1}(\mathbf{V}-\mathbf{A})+\mathbf{Q}_{j_2}(\mathbf{I}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2 \tag{189}$$

$$\begin{aligned}&=2\eta^2\sum_{r=0}^{j-1}\mathbb{E}\left\|\mathbf{Q}_r(\mathbf{Z}^{j-r}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2+2\eta^2\mathbb{E}\left\|\mathbf{Q}_{j_1}(\mathbf{V}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2+2\eta^2\mathbb{E}\left\|\mathbf{Q}_{j_2}(\mathbf{I}-\mathbf{A})\right\|_{F_{\boldsymbol{a}}}^2\\&\quad+\underbrace{2\eta^2\sum_{n=0}^{j-1}\sum_{l=0,l\neq n}^{j-1}\mathbb{E}\left[\underbrace{\mathrm{Tr}\left((\mathrm{diag}(\boldsymbol{a}))^{1/2}(\mathbf{Z}^{j-n}-\mathbf{A})\,\mathbf{Q}_n^T\,\mathbf{Q}_l(\mathbf{Z}^{j-l}-\mathbf{A})\,(\mathrm{diag}(\boldsymbol{a}))^{1/2}\right)}_{TR'}\right]}_{TR'_0}\\&\qquad+\underbrace{4\eta^2\sum_{l=0}^{j-1}\mathbb{E}\left[\mathrm{Tr}\left((\mathrm{diag}(\boldsymbol{a}))^{1/2}(\mathbf{V}-\mathbf{A})\,\mathbf{Q}_{j_1}^T\,\mathbf{Q}_l(\mathbf{Z}^{j-l}-\mathbf{A})\,(\mathrm{diag}(\boldsymbol{a}))^{1/2}\right)\right]}_{TR'_1}\\&\qquad+\underbrace{4\eta^2\sum_{l=0}^{j-1}\mathbb{E}\left[\mathrm{Tr}\left((\mathrm{diag}(\boldsymbol{a}))^{1/2}(\mathbf{I}-\mathbf{A})\,\mathbf{Q}_{j_2}^T\,\mathbf{Q}_l(\mathbf{Z}^{j-l}-\mathbf{A})\,(\mathrm{diag}(\boldsymbol{a}))^{1/2}\right)\right]}_{TR'_2}\\&\qquad+\underbrace{4\eta^2\mathbb{E}\left[\mathrm{Tr}\left((\mathrm{diag}(\boldsymbol{a}))^{1/2}(\mathbf{V}-\mathbf{A})\,\mathbf{Q}_{j_1}^T\,\mathbf{Q}_{j_2}(\mathbf{I}-\mathbf{A})\,(\mathrm{diag}(\boldsymbol{a}))^{1/2}\right)\right]}_{TR'_3}.\end{aligned} \tag{190}$$

TR' can be bounded as:

$$
\begin{aligned}
TR' &\leq \left\| \left(\mathbf{Z}^{j-n} - \mathbf{A}\right) \mathbf{Q}_n^T \right\|_{F_{\boldsymbol{a}}} \left\| \mathbf{Q}_l (\mathbf{Z}^{j-l} - \mathbf{A}) \right\|_{F_{\boldsymbol{a}}} && (191)\\
&\leq \left\| (\mathbf{Z}^{j-n} - \mathbf{A}) \right\|_{op} \left\| \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}} \left\| (\mathbf{Z}^{j-l} - \mathbf{A}) \right\|_{op} && (192)\\
&\leq \frac{1}{2} \zeta^{2j-n-l} \left[\left\| \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 + \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \right] && (193)
\end{aligned}
$$

where (191) follows from Lemma 6. We can similarly bound TR'_1 through TR'_3:

$$
\begin{aligned}
TR'_1 &\leq 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \left[\mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \right] && (194)\\
TR'_2 &\leq 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \left[\mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \right] && (195)\\
TR'_3 &\leq 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 . && (196)
\end{aligned}
$$

Summing TR'_0 through TR'_3, we get:

$$
\begin{aligned}
\sum_{t=0}^{3} TR'_t \leq\ & \eta^2 \sum_{n=0}^{j-1} \sum_{l=0, l\neq n}^{j-1} \zeta^{2j-n-l} \mathbb{E} \left[\mathbb{E} \left\| \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 + \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \right] \\
& + 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \sum_{l=0}^{j} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \sum_{l=0}^{j} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 && (197)\\
\leq\ & 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n} \mathbb{E} \left\| \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0, l\neq n}^{j-1} \zeta^{j-l} + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} && (198)
\end{aligned}
$$

where (198) follows from the symmetry of the indices n and l.

Plugging (198) back into (190):

$$
\begin{aligned}
T_2 \leq\ & 2\eta^2 \sum_{r=0}^{j-1} \mathbb{E} \left\| \mathbf{Q}_r (\mathbf{Z}^{j-r} - \mathbf{A}) \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} (\mathbf{V} - \mathbf{A}) \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} (\mathbf{I} - \mathbf{A}) \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n} \mathbb{E} \left\| \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0, l \neq n}^{j-1} \zeta^{j-l} + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} \qquad (199) \\
\leq\ & 2\eta^2 \sum_{r=0}^{j-1} \mathbb{E} \left\| \mathbf{Q}_r \right\|_{F_{\boldsymbol{a}}}^2 \left\| (\mathbf{Z}^{j-r} - \mathbf{A}) \right\|_{op}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \left\| \mathbf{V} - \mathbf{A} \right\|_{op}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \left\| \mathbf{I} - \mathbf{A} \right\|_{op}^2 \\
& + 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n} \mathbb{E} \left\| \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0, l \neq n}^{j-1} \zeta^{j-l} + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} \qquad (200) \\
\leq\ & 2\eta^2 \sum_{r=0}^{j-1} \zeta^{j-r} \mathbb{E} \left\| \mathbf{Q}_r \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n} \mathbb{E} \left\| \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0, l \neq n}^{j-1} \zeta^{j-l} + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \sum_{l=0}^{j} \zeta^{j-l} \qquad (201)
\end{aligned}
$$

where (200) follows from Lemma 7, and (201) follows from Lemma 5.

We further bound T_2:

$$
\begin{aligned}
T_2 \leq\ & 2\eta^2 \sum_{r=0}^{j-1} \zeta^{j-r} \mathbb{E} \left\| \mathbf{Q}_r \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \sum_{n=0}^{j-1} \zeta^{j-n} \mathbb{E} \left\| \mathbf{Q}_n \right\|_{F_{\boldsymbol{a}}}^2 \frac{\zeta}{1-\zeta} + 4\eta^2 \sum_{l=0}^{j-1} \zeta^{j-l} \mathbb{E} \left\| \mathbf{Q}_l \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 \frac{1}{1-\zeta} + 2\eta^2 \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \frac{1}{1-\zeta} \qquad (202) \\
\leq\ & 2\eta^2 \sum_{r=0}^{j-1} \left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta} \right) \mathbb{E} \left\| \mathbf{Q}_r \right\|_{F_{\boldsymbol{a}}}^2 \\
& + 2\eta^2 \left(\frac{2-\zeta}{1-\zeta} \right) \mathbb{E} \left\| \mathbf{Q}_{j_1} \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \left(\frac{2-\zeta}{1-\zeta} \right) \mathbb{E} \left\| \mathbf{Q}_{j_2} \right\|_{F_{\boldsymbol{a}}}^2 \qquad (203)
\end{aligned}
$$

where (202) follows from the summation formulae of a power series in (170).

After applying the definition of $\mathbf{Q}$ to (202), we obtain:

$$
\begin{aligned}
T_2 = 2\eta^2 \sum_{r=0}^{j-1} \left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta} \right) \mathbb{E} \left\| \sum_{s=1}^{q\tau} \nabla F(\mathbf{X}_{rq\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 \\
+ 2\eta^2 \left(\frac{2-\zeta}{1-\zeta} \right) \mathbb{E} \left\| \sum_{s=1}^{l\tau} \nabla F(\mathbf{X}_{jq\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 + 2\eta^2 \left(\frac{2-\zeta}{1-\zeta} \right) \mathbb{E} \left\| \sum_{s=1}^{f-1} \nabla F(\mathbf{X}_{jq\tau+l\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 \quad (204) \\
\leq 2\eta^2 q\tau \sum_{r=0}^{j-1} \left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta} \right) \sum_{s=1}^{q\tau} \mathbb{E} \left\| \nabla F(\mathbf{X}_{rq\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 \\
+ 2\eta^2 l\tau \left(\frac{2-\zeta}{1-\zeta} \right) \sum_{s=1}^{l\tau} \mathbb{E} \left\| \nabla F(\mathbf{X}_{jq\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 \\
+ 2\eta^2 (f-1) \left(\frac{2-\zeta}{1-\zeta} \right) \sum_{s=1}^{f-1} \mathbb{E} \left\| \nabla F(\mathbf{X}_{jq\tau+l\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 \quad (205)
\end{aligned}
$$

where (205) follows from (204) by Jensen's inequality.

Summing over all iterates in the j-th sub-network update period, we obtain:

$$
\begin{aligned}
\sum_{l=0}^{q-1} \sum_{f=1}^{\tau} T_2 \leq 2\eta^2 q^2 \tau^2 \sum_{r=0}^{j-1} \left(\left(\zeta^{2(j-r)} + 2\zeta^{j-r} + \frac{\zeta^{j-r+1}}{1-\zeta} \right) \sum_{s=1}^{q\tau} \mathbb{E} \left\| \nabla F(\mathbf{X}_{rq\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 \right) \\
+ \eta^2 q\tau (q-1) \left(\frac{2-\zeta}{1-\zeta} \right) \sum_{s=1}^{q\tau} \mathbb{E} \left\| \nabla F(\mathbf{X}_{jq\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 \\
+ \eta^2 q\tau (\tau-1) \left(\frac{2-\zeta}{1-\zeta} \right) \sum_{s=1}^{\tau-1} \mathbb{E} \left\| \nabla F(\mathbf{X}_{jq\tau+q\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 \quad (206) \\
\leq 2\eta^2 q^2 \tau^2 \sum_{r=0}^{j} \Gamma_r \sum_{s=1}^{q\tau} \mathbb{E} \left\| \nabla F(\mathbf{X}_{rq\tau+s}) \right\|_{F_{\boldsymbol{a}}}^2 . \quad (207)
\end{aligned}
$$

Summing over all iterations and applying the summation bound in (185) to (207):

$$
\sum_{j=0}^{K/(q\tau)-1} \sum_{l=0}^{q-1} \sum_{f=1}^{\tau} T_2 \leq 2\eta^2 q^2 \tau^2 \Gamma \sum_{k=1}^{K} \mathbb{E} \left\| \nabla F(\mathbf{X}_k) \right\|_{F_{\boldsymbol{a}}}^2 . \quad (208)
$$

Summing T_1 and T_2, we obtain

$$
\begin{aligned}
\frac{2L^2}{K} \sum_{k=1}^{K} \mathbb{E} \| \mathbf{X}_k (\mathbf{I} - \mathbf{A}) \|_{F_{\boldsymbol{a}}}^2 \leq \frac{2L^2}{K} \sum_{k=1}^{K} T_1 + \frac{2L^2}{K} \sum_{k=1}^{K} T_2 \quad (209) \\
\leq 4L^2 \eta^2 \sigma^2 q^3 \tau^3 \left(\frac{1}{q\tau} - \frac{1}{K} \right) \left(\frac{\zeta^2}{1-\zeta^2} + \frac{2\zeta}{1-\zeta} + \frac{1}{(1-\zeta)^2} \right) \mathrm{P} \\
+ 4L^2 \eta^2 \sigma^2 \left(\frac{2-\zeta}{1-\zeta} \right) \left(\tau^2 \frac{(q-1)(2q+1)}{6} + \frac{(\tau-1)(2\tau+1)}{6} \right) \mathrm{P} \\
+ 8L^2 \eta^2 q^2 \tau^2 \Gamma \frac{1}{K} \sum_{k=1}^{K} \sum_{i=1}^{N} \boldsymbol{a}_i \left(\boldsymbol{p}_i (\beta - 1) + 1 \right) \mathbb{E} \left\| \nabla F(\boldsymbol{x}_k^{(i)}) \right\|^2 . \quad (210)
\end{aligned}
$$

Plugging T_1 and T_2 back into Lemma 4, we arrive at

$$\mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\|\nabla F(\boldsymbol{u}_k)\|^2\right] \leq \frac{2\left(F(\boldsymbol{x}_1)-F_{inf}]\right)}{\eta K} + \sigma^2\eta L\sum_{i=1}^{N}\boldsymbol{a}_i^2\boldsymbol{p}_i + \frac{2L^2}{K}\sum_{k=1}^{K}T_1 + \frac{2L^2}{K}\sum_{k=1}^{K}T_2$$
$$-\frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{N}\boldsymbol{a}_i\left((4\boldsymbol{p}_i-\boldsymbol{p}_i^2-2)-\eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1)-\boldsymbol{a}_i\boldsymbol{p}_i^2+\boldsymbol{p}_i^2\right)\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \tag{211}$$

$$= \frac{2\left(F(\boldsymbol{x}_1)-F_{inf}]\right)}{\eta K} + \sigma^2\eta L\sum_{i=1}^{N}\boldsymbol{a}_i^2\boldsymbol{p}_i$$
$$+4L^2\eta^2\sigma^2q^3\tau^3\left(\frac{1}{q\tau}-\frac{1}{K}\right)\left(\frac{\zeta^2}{1-\zeta^2}+\frac{2\zeta}{1-\zeta}+\frac{1}{(1-\zeta)^2}\right)\mathrm{P}$$
$$+4L^2\eta^2\sigma^2\left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2\frac{(q-1)(2q+1)}{6}+\frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P}$$
$$-\frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{N}\boldsymbol{a}_i\left((4\boldsymbol{p}_i-\boldsymbol{p}_i^2-2)-\eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1)-\boldsymbol{a}_i\boldsymbol{p}_i^2+\boldsymbol{p}_i^2\right)-8L^2\eta^2q^2\tau^2\Gamma\right)\mathbb{E}\left\|\nabla F(\boldsymbol{x}_k^{(i)})\right\|^2 \tag{212}$$

If η satisfies the following for $i=1,\ldots,N$,

$$(4\boldsymbol{p}_i-\boldsymbol{p}_i^2-2) \geq \eta L\left(\boldsymbol{a}_i\boldsymbol{p}_i(\beta+1)-\boldsymbol{a}_i\boldsymbol{p}_i^2+\boldsymbol{p}_i^2\right)+8L^2\eta^2q^2\tau^2\Gamma \tag{213}$$

then we can simplify (212):

$$\mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\|\nabla F(\boldsymbol{u}_k)\|^2\right] \leq \frac{2\left(F(\boldsymbol{x}_1)-F_{inf}]\right)}{\eta K} + \sigma^2\eta L\sum_{i=1}^{N}\boldsymbol{a}_i^2\boldsymbol{p}_i$$
$$+4L^2\eta^2\sigma^2q^3\tau^3\left(\frac{1}{q\tau}-\frac{1}{K}\right)\left(\frac{\zeta^2}{1-\zeta^2}+\frac{2\zeta}{1-\zeta}+\frac{1}{(1-\zeta)^2}\right)\mathrm{P}$$
$$+4L^2\eta^2\sigma^2\left(\frac{2-\zeta}{1-\zeta}\right)\left(\tau^2\frac{(q-1)(2q+1)}{6}+\frac{(\tau-1)(2\tau+1)}{6}\right)\mathrm{P}\,. \tag{214}$$

□