Gradient Properties of Hard Thresholding Operator

Saeed Damadi
Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, MD 21250
sdamadi1@umbc.edu

Jinglai Shen
Department of Mathematics and Statistics
University of Maryland, Baltimore County
Baltimore, MD 21250
shenj@umbc.edu

Abstract

Sparse optimization receives increasing attention in many applications such as compressed sensing, variable selection in regression problems, and recently neural network compression in machine learning. For example, the problem of compressing a neural network is a bi-level, stochastic, and nonconvex problem that can be cast into a sparse optimization problem. Hence, developing efficient methods for sparse optimization plays a critical role in applications. The goal of this paper is to develop analytical techniques for general, large size sparse optimization problems using the hard thresholding operator. To this end, we study the iterative hard thresholding (IHT) algorithm, which has been extensively studied in the literature because it is scalable, fast, and easily implementable. In spite of extensive research on the IHT scheme, we develop several new techniques that not only recover many known results but also lead to new results. Specifically, we first establish a new and critical gradient descent property of the hard thresholding (HT) operator. This property allows one to study the IHT when the stepsize is less than or equal to 1/L, where L>0 is the Lipschitz constant of the gradient of an objective function. Note that the existing techniques in the literature can only handle the case when the stepsize is strictly less than 1/L. By exploiting this we introduce and study HT-stable and HT-unstable stationary points and show no matter how close an initialization is to a HT-unstable stationary point, the IHT sequence leaves it. Finally, we show the IHT sequence converges globally if the function values of HT-stable stationary points are distinct, which is a new assumption that has not been found in the literature.

1 Introduction

Solving sparse problems has gained increasing attention in the fields of statistics, finance, and engineering. These problems emerge in statistics as variable selection in linear regression problems [22, 63, 15, 19], mixed-integer programs [8, 34, 18], portfolio optimization in finance [9, 12], compressed sensing in signal processing [25, 21], and compressing deep neural networks in machine learning [39, 26], just to name a few. Due to the use of ℓ_0-(pseudo) norm[1], these problems are discontinuous and nonconvex. The ℓ_0-norm case have been addressed by the hard thresholding (HT) techniques specially the iterative HT (IHT) scheme [6, 3, 36, 61]. The Lasso-type, Basic Pursuit(BP)-type, and BP denoising(BPDN)-type problems consider ℓ_1-norm as a convex approximation of ℓ_0-norm [51, 40]. Nonconvex approximation of ℓ_0-norm as ℓ_p-(pseudo) norm ($0 < p < 1$) has also been studied well [13, 24, 30, 53, 58, 55]. Sparse optimization problems can also be formulated as mixed-integer programs [11]. Intrinsic combinatorics involved in sparse optimization problems makes it an NP-hard problem (even for a quadratic loss [17, 41]) so it is difficult to find a global minimizer. However, greedy algorithms have developed to find local minimizers. To this end, following the

[1] ℓ_0 is not mathematically a norm because for any norm $\|\cdot\|$ and $\alpha \in \mathbb{R}$, $\|\alpha\boldsymbol{\theta}\| = |\alpha|\|\boldsymbol{\theta}\|$, while $\|\alpha\boldsymbol{\theta}\|_0 = |\alpha|\|\boldsymbol{\theta}\|_0$ if and only if $|\alpha| = 1$

Preprint. Under review.

Algorithm 1 The iterative hard thresholding (IHT)

Require: $\mathbf{x}^0 \in \mathbb{R}^n$ such that $\|\mathbf{x}^0\|_0 \leq s$ and stepsize $\gamma > 0$.
1: $\mathbf{x}^{k+1} \in H_s(\mathbf{x}^k - \gamma \nabla f(\mathbf{x}^k))$ for $k = 0, 1, \ldots$

ideas of matching pursuit (MP) and orthogonal MP (OMP) [38, 50] as greedy algorithms, numerous other greedy algorithms have been developed such as stagewise OMP (StOMP) [20], regularized OMP (ROMP) [43, 44], Compressive Sampling MP (CoSaMP) [42], and Gradient Support Pursuit (GraSP) [1]. It should be noted that sparse optimization is not restricted to finding a sparse vector. For example [23, 27, 16], finding a low-rank matrix is considered. The problem of finding a low-rank matrix is a counterpart to finding a sparse vector when it comes to applications dealing with matrices. In addition to devising algorithms for solving sparse optimization problems, developing first and second order optimality conditions have also been addressed well [47, 2, 4, 32, 37, 46, 10].

The general sparse optimization problem is the following:

$$\begin{array}{l} \min f(\mathbf{x}) \\ \text{s.t. } C_s \cap \mathcal{X} \end{array} \tag{1}$$

where $C_s = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_0 \leq s\}$ (sparsity constraint) is the union of finitely many subspaces of dimension s such that $1 \leq s < n$, $\mathcal{X}$ is a constraint set in $\mathbb{R}^n$, and the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is lower bounded and continuously differentiable, i.e., C^1. In this paper we address a special case of Problem (1) where $\mathcal{X} = \mathbb{R}^n$ as follows:

$$(\text{P}) : \quad \begin{array}{l} \min f(\mathbf{x}) \\ \text{s.t. } \mathbf{x} \in C_s \end{array} \tag{2}$$

To address Problem (2) the following fundamental questions arise:

(Q1) What are the necessary/sufficient conditions for a local/global minimizer of Problem (2)?

(Q2) What are the characteristics of accumulation points of algorithms solving Problem (2)?

(Q3) Under what condition(s) does an accumulation point become a local/global minimizer?

(Q4) If an accumulation point is a local/global minimizer, what is the rate of convergence?

By considering the IHT algorithm, we will answer the above questions. This algorithm has been extensively studied in the literature. It was originally devised for solving compressed sensing problems in 2008 [6, 7]. Since then, there has been a large body of literature studying the IHT-type algorithms from different standpoints. For example, [3, 36, 37, 48, 61] consider convergence of iterations, [28, 35] study the limit of the objective function value sequence, [34, 62] address duality, [60, 57] extend it to Newton's-type IHT, [14, 31, 33, 59] consider the stochastic version, [5, 29, 52, 56] address accelerated IHT, and [54, 1] solve logistic regression problem using the IHT.

Summary of Contributions

By considering the IHT Algorithm 1 for Problem (2), we develop the following results:

- We establish a new critical gradient descent property of the hard thresholding (HT) operator that has not been found in the literature. This property allows one to study the IHT when the stepsize is less than or equal to $1/L$, where $L > 0$ is the Lipschitz constant of the gradient of an objective function. Note that the existing techniques in the literature can only handle the case when the stepsize is strictly less than $1/L$. As an example, one can refer to [35] that needs the stepsize to be greater than or equal to $1/L$.
- We introduce the notion of *HT-stable/unstable stationary* points. Using them we establish the *escapability* property of *HT-unstable stationary* points and local *reachability* property of strictly *HT-stable stationary* points.
- We also show that the IHT sequence converges globally under a new assumption that has not been found in the literature. In addition, Q-linearly convergence of the IHT algorithm towards a local minimum when the objective function is both RSS and restricted strictly convex is shown.

According to our results, we address (Q1) and (Q2) by establishing a new gradient descent property of the hard thresholding (HT) operator and introducing the notion of *HT-stable/unstable stationary* points. By considering RSS, restricted strictly convex, and RSC properties we address (Q3) and (Q4). Table 1 is provided to compare our results with those in the literature. It shows what has been done chronologically and demonstrates our results.

2 Related work

To answer (Q1) [3] introduces L-*stationarity* property as a necessary condition for an optimal solution of Problem (2). The L-*stationarity* property is defined when the gradient of the objective function is Lipschitz. Also, [3] addresses (Q2) by showing any accumulation point of the IHT algorithm is L-stationary. Lu in [36] restricts the objective function to be convex and shows that the IHT sequence converges to a local minimum when the objective function is regularized by ℓ_0-norm and $\mathcal{X}$ is a box constraint. Jain et al. [28] put more restriction on the objective value function and show that the objective value function sequence generated by the IHT algorithm converges to a value attained under a more restricted sparsity constraint. The restrictions used in [28] are Restricted Strong Smoothness (RSS) and Restricted Strong Convexity (RSC). The RSS and RSC properties are introduced by [45] and first used by [1] for sparsity optimization problems. Currently, they have become standard restrictions for analyzing sparsity optimization problems. Under RSS and RSC properties for the objective function, one is able to address (Q3) and (Q4).

Finding a closed-form expression for $P_{C_s \cap \mathcal{X}}$ when $\mathcal{X}$ is an arbitrary set is difficult. However, [4] shows orthogonal projection of a point onto $C_s \cap \mathcal{X}$ can be efficiently computed when $\mathcal{X}$ is a symmetric closed convex set. In this context, two types of sets are of interest: nonnegative symmetric sets and sign free sets. To address (Q1) in a more generalized setting, Beck and Hallak [4] characterize L-*stationary* points of Problem (1) when $\mathcal{X}$ is either nonnegative symmetric set or sign free. Also, Lu in [37] considers the same setting as [4] and introduces a new optimality condition that is stronger than L-*stationary*. He devises a Nonmonotone Projected Gradient (NPG) algorithm and shows an accumulation of the NPG sequence is the global optimal of Problem (1). Pan et al. [47] consider Problem (1) when $\mathcal{X} = \mathbb{R}^n_+$. They develop an Improved IHT algorithm (IIHT) that employs the Armijo-type stepsize rule. They show when the objective function is RSS and RSC, the IIHT sequence converges to a local minimum. A recent work by Zhou et al. [61] develops Newton Hard-Thresholding Pursuit (NHTP) for solving problem (2). They show that when accumulation points of the NHTP sequence are L-*stationary* and are isolated, the sequence converges with a locally Q-quadratic rate. Table 1 compares current results in the literature.

3 Definitions

We provide some definitions that will be used throughout the paper. These definitions are the HT operator (HTO) and HTO inequality, RSS and RSC functions.

Definition 1 (Restricted Strong Smoothness (RSS))**.** *A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be restricted strongly smooth with modulus $L_s > 0$ or is L_s-RSS if*

$$f(\mathbf{y}) \leq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{L_s}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ such that } \|\mathbf{x}\|_0 \leq s, \|\mathbf{y}\|_0 \leq s. \tag{3}$$

Definition 2 (Restricted Strong Convexity (RSC))**.** *A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be restricted strongly convex with modulus $\beta_s > 0$ or is β_s-RSC if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\beta_s}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ such that } ||\mathbf{x}||_0 \leq s, ||\mathbf{y}||_0 \leq s. \tag{4}$$

Definition 3 (The HT operator)**.** *The HT operator $H_s(\cdot)$ denotes the orthogonal projection onto multiple subspaces of $\mathbb{R}^n$ with dimension $1 \leq s < n$, that is,*

$$H_s(\mathbf{x}) \in \arg\min_{\|\mathbf{z}\|_0 \leq s} \|\mathbf{z} - \mathbf{x}\|_2. \tag{5}$$

Claim 1. *The HT operator keeps the s largest entries of its input in absolute values.*

For a vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{I}_s^{\mathbf{x}} \subset \{1, \ldots, n\}$ denotes the set of indices corresponding to the first s largest elements of $\mathbf{x}$ in absolute values. For example $H_2([1, -3, 1]^\top)$ is either $[0, -3, 1]^\top$ or $[1, -3, 0]^\top$

Table 1: Comparison of results for the deterministic IHT-type algorithms.

Paper	Objective function	Constraints	Stepsize	Optimality conditions	Method	Convergence
[3]	$f(\mathbf{x})$: L-LG	$\mathbf{x} \in C_s$	$0 < \gamma < 1/L$	$\mathbf{x}^* \in H_s(\mathbf{x}^* - \frac{1}{L}\nabla f(\mathbf{x}^*))$	IHT	Any accumulation point of the IHT sequence satisfies $\mathbf{x}^* \in H_s(\mathbf{x}^* - \frac{1}{L}\nabla f(\mathbf{x}^*))$(Theorem 3.1).
[36]	$f(\mathbf{x}) + \lambda\|\mathbf{x}\|_0$ $f(\mathbf{x})$ is convex	$\mathbf{x} \in C_s$ $l \leq \mathbf{x} \leq u$	$0 < \gamma < 1/L$	Local minimizer		The IHT sequence converges to a local minimizer. (Theorem 3.3)
[4]	$f(\mathbf{x})$: L-LG	$\mathbf{x} \in C_s \cap \mathcal{X}$, $\mathcal{X}$ is nonnegative symmetric or sign free	$0 < \gamma < 1/L$	Basic feasibility	BFS	The sequence of BFS converges to a basic feasible point (Lemma 7.1).
[37]	$f(\mathbf{x})$: L-LG	$\mathbf{x} \in C_s \cap \mathcal{X}$, $\mathcal{X}$ is nonnegative symmetric or sign free	$0 < \gamma < 1/L$	$\mathbf{x}^* = P_{C_s \cap \mathcal{X}}(\mathbf{x}^* - \frac{1}{L}\nabla f(\mathbf{x}^*))$	NPG	An accumulation point of NPG sequence satisfies the optimality condition (Theorem 4.3).
[48]	$f(\mathbf{x})$ L_s-RSS & s-RC	$\mathbf{x} \in C_s \cap \mathbb{R}^n_+$	$0 < \gamma < 1/L_s$	$\mathbf{x}^* \in P_{C_s \cap \mathbb{R}^n_+}(\mathbf{x}^* - \frac{1}{L}\nabla f(\mathbf{x}^*))$ or L_s-stationary	IIHT	Any accumulation point of IIHT sequence converges to a L_s-stationary point (Theorem 3.1).
[48]	$f(\mathbf{x})$ L_s-RSS & β_s-RSC	$\mathbf{x} \in C_s \cap \mathbb{R}^n_+$	$0 < \gamma < 1/L_s$	$\mathbf{x}^* \in P_{C_s \cap \mathbb{R}^n_+}(\mathbf{x}^* - \frac{1}{L}\nabla f(\mathbf{x}^*))$ or L_s-stationary	IIHT	The IIHT sequence converges to a local minimizer (Theorem 3.2). If $\|\mathbf{x}^*\|_0 < s$, then $\mathbf{x}^*$ is a global minimizer. When $\|\mathbf{x}^*\|_0 = s$, the IIHT sequence converges Q-linearly (Theorem 3.4).
[61]	$f(\mathbf{x})$ L_s-RSS	$\mathbf{x} \in C_s$	$0 < \gamma < 1/L_s$	$\mathbf{x}^* \in H_s(\mathbf{x}^* - \frac{1}{L_s}\nabla f(\mathbf{x}^*))$ or L_s-stationary	NHTP	Any accumulation point $\mathbf{x}^*$ of the NHTP sequence is an L_s-stationary point (Theorem 9). If $\mathbf{x}^*$ is isolated, the entire sequence converges.
[61]	$f(\mathbf{x})$ L_s-RSS & β_s-RSC & Restricted Hessian is Lipschitz	$\mathbf{x} \in C_s$	$0 < \gamma < 1/L_s$	$\mathbf{x}^* \in H_s(\mathbf{x}^* - \frac{1}{L_s}\nabla f(\mathbf{x}^*))$ or L_s-stationary	NHTP	The NHTP sequence converges to a L_s-stationary point (Theorem 10). Locally, it converges quadratically.
Ours	$f(\mathbf{x})$ L_s-RSS	$\mathbf{x} \in C_s$	$0 < \gamma \leq 1/L_s$	$\mathbf{x}^* \in H_s(\mathbf{x}^* - \frac{1}{L_s}\nabla f(\mathbf{x}^*))$ or HT-stable	IHT	If $f(\mathbf{x}^*) \neq f(\mathbf{y}^*)$ for all $\mathbf{x}^*, \mathbf{y}^*$ HT-stable points, by starting from a $\mathbf{x}^0 \in C_s$ the IHT sequence converges to some HT-stable points. Corollary 5: global convergence
Ours	$f(\mathbf{x})$ L_s-RSS	$\mathbf{x} \in C_s$	$0 < \gamma < 1/L_s$	$\mathbf{x}^* \in H_s(\mathbf{x}^* - \frac{1}{L_s}\nabla f(\mathbf{x}^*))$ or HT-stable	IHT	If all HT-stable points are isolated, by starting from a $\mathbf{x}^0 \in C_s$ the IHT sequence converges to some HT-stable points. Corollary 5: global convergence
Ours	$f(\mathbf{x})$ L_s-RSS Strictly convex on $\mathcal{S}_\mathcal{J}$ with $\|\mathcal{J}\| = s$	$\mathbf{x} \in C_s$	$0 < \gamma \leq 1/L_s$	$\mathbf{x}^* = H_s(\mathbf{x}^* - \frac{1}{L_s}\nabla f(\mathbf{x}^*))$ or HT-strictly stable	IHT	By starting from $\mathbf{x}^0 \in \mathcal{B}(\mathbf{x}^*, \delta) \cap C_s$ the IHT sequence converges to HT-strictly stable $\mathbf{x}^*$ point Q-linearly (Proposition 1).

where $\mathcal{I}_2^{\mathbf{y}} = \{2, 3\}$ and $\mathcal{I}_2^{\mathbf{y}} = \{1, 2\}$, respectively. Therefore, the output of it may not be unique. This clearly shows why HTO is not a convex operator and why there is an inclusion in (5) not an inequality.

Definition 4 (HTO inequality)**.** *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \in H_s(x)$. Then, according to Definition 3, the following for any $\mathcal{I}_s^{\mathbf{y}}$ is called HT operator inequality:*

$$|x_i| \leq |x_j| \quad \forall \quad i \in (\mathcal{I}_s^{\mathbf{y}})^c \text{ and } j \in \mathcal{I}_s^{\mathbf{y}}. \tag{6}$$

Definition 5 (The IHT algorithm)**.** *The IHT scheme in Algorithm 1 is an iterative process that applies the HT operator on the updated vector found by the gradient descent algorithm.*

Definition 6 (Basic stationary point)**.** *When $\mathbf{x}^*$ is a local minimizer of Problem (2), then $\nabla_{\mathcal{I}_s^*} f(\mathbf{x}^*) = 0$ when $\|\mathbf{x}^*\|_0 = s$ or $\nabla f(\mathbf{x}^*) = 0$ when $\|\mathbf{x}^*\|_0 < s$, [3]. Every point satisfying these conditions is called basic stationary point of Problem (2). Points with the aforementioned property are called basic feasible points in [3].*

4 Results

We consider solving Problem (2) using the IHT Algorithm 1 and develop results on the HT operator. Using them, the behavior of the IHT sequence generated by Algorithm 1 is characterized. Towards this end, statements of the main results are provided and all the technical proofs are postponed to the Appendix.

First, we establish a new and critical gradient descent property of the hard thresholding (HT) operator.

Theorem 1 (Gradient descent property)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function that is L_s-RSS, $\mathbf{y} \in H_s(\mathbf{x} - \gamma\nabla f(\mathbf{x}))$ with any $\mathcal{I}_s^{\mathbf{y}}$ and $0 < \gamma \leq \frac{1}{L_s}$, and $\mathbf{x}$ be a sparse vector such that $\|\mathbf{x}\|_0 \leq s$ with any $\mathcal{I}_s^{\mathbf{x}}$. Then,*

$$\frac{\gamma}{2}(1 - L_s\gamma)\|\nabla_{\mathcal{I}_s^{\mathbf{x}} \cup \mathcal{I}_s^{\mathbf{y}}} f(\mathbf{x})\|_2^2 \leq f(\mathbf{x}) - f(\mathbf{y}) \tag{7}$$

where $\nabla_{\mathcal{I}_s^{\mathbf{x}} \cup \mathcal{I}_s^{\mathbf{y}}} f(\mathbf{x})$ is the restriction of the gradient vector to the union of the index sets $\mathcal{I}_s^{\mathbf{x}}$ and $\mathcal{I}_s^{\mathbf{y}}$.

Theorem 1 provides a lower bound on the difference between the current function value evaluated at $\mathbf{x}$ and the one evaluated at the updated point provided by the HTO, i.e., $\mathbf{y}$. Note that, $\mathbf{y}$ may not be a unique vector that has s nonzero elements. Nonetheless, as stated in Theorem 1, Inequality (1) holds for any $\mathbf{y}$ that might be the output of the HTO. As one clearly see, the descent can only be characterized by looking at the entries of the gradient that are restricted to the union of the s largest elements in both $\mathbf{x}$ and $\mathbf{y}$. The rest of the gradient can be ignored. Since one may be interested in characterizing the descent using the distance between $\mathbf{x}$ and $\mathbf{y}$, we provide the following corollary.

Corollary 1 (Neighboring error)**.** *Assume all the assumptions in Theorem 1 hold, then,*

$$\frac{1-L_s\gamma}{6\gamma}\|\mathbf{y}-\mathbf{x}\|_2^2 \leq \frac{\gamma}{2}(1-L_s\gamma)\|\nabla_{\mathcal{I}_s^{\mathbf{x}}\cup\mathcal{I}_s^{\mathbf{y}}} f(\mathbf{x})\|_2^2 \leq f(\mathbf{x})-f(\mathbf{y}) \tag{8}$$

Note that it is shown in [3, Lemma 2.4] that $\frac{1-L_s\gamma}{2\gamma}\|\mathbf{y}-\mathbf{x}\|_2^2 \leq f(\mathbf{x})-f(\mathbf{y})$ with better bound 1/2 instead of 1/6.

Algebraically speaking, characterizing a descent of the function value solely with the information of the current iterate, i.e., $\mathbf{x}$, is of more interest. To this end, we provide another corollary to Theorem 1 that ties the descent to $\mathbf{x}$ only.

Corollary 2. *Assume all the assumptions in Theorem 1 hold. Then, the norm of the gradient restricted to any $\mathcal{I}_s^{\mathbf{x}}$ can be bounded as follows:*

$$\frac{\gamma}{2}\|\nabla_{\mathcal{I}_s^{\mathbf{x}}} f(\mathbf{x})\|_2^2 \leq f(\mathbf{x})-f(\mathbf{y}) \tag{9}$$

By this point, we have shown that applying the HTO once, can result in smaller function value provided the gradient over the s largest entries of $\mathbf{x}$ are nonzero. This can be utilized to show the sequence generated by the IHT algorithm is nonincreasing. Specially, if the generated sequence has an accumulation point, the objective value function sequence converges to the objective value of the accumulation point. [2]

Corollary 3 (Function value convergence)**.** *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a bounded below differential function that is L_s-RSS and $\left(\mathbf{x}^k\right)_{k\geq 0}$ be the IHT sequence $\left(\mathbf{x}^k\right)_{k\geq 0}$ with $0<\gamma\leq\frac{1}{L_s}$. Then, $\left(f(\mathbf{x}^k)\right)_{k\geq 0}$ is nonincreasing and converges. Also, if $\mathbf{x}^*$ is an accumulation point of $\left(\mathbf{x}^k\right)_{k\geq 0}$ then $\left(f(\mathbf{x}^k)\right)_{k\geq 0}\to f(\mathbf{x}^*)$.*

Next, we look at *basic stationary points* of Problem (2) and show their properties.

4.1 Optimality condition based on the HT properties

In this subsection, we will show that not all *basic stationary* points of Problem (2) are reachable when the IHT algorithm is run. To do so, the notion of *HT-stable stationary* point is introduced as follows.

Definition 7 (HT-stable stationary point)**.** *For a given constant $\gamma>0$, we say that a sparse vector $\mathbf{x}^*\in C_s$ is HT-stable stationary point of Problem (2) associated with γ if $\nabla_{supp(\mathbf{x}^*)} f(\mathbf{x}^*)=0$, and*

$$\min\left(|x_i^*| : i\in\mathcal{I}_s^{\mathbf{x}^*}\right) \geq \gamma\max\left(|\nabla_j f(\mathbf{x}^*)| : j\notin supp(\mathbf{x}^*)\right) = \gamma\|\nabla_{(supp(\mathbf{x}^*))^c} f(\mathbf{x}^*)\|_\infty. \tag{10}$$

(Note that $\min\left(|x_i^*| : i\in\mathcal{I}_s^{\mathbf{x}^*}\right)$ is unique and does not depend on the choice $\mathcal{I}_s^{\mathbf{x}^*}$.) If $\nabla_{supp(\mathbf{x}^*)} f(\mathbf{x}^*)=0$ but (10) fails, we say that $\mathbf{x}^*$ is HT-unstable stationary point with γ. Moreover, if $\nabla_{supp(\mathbf{x}^*)} f(\mathbf{x}^*)=0$ and (10) holds strictly, namely,

$$\min\left(|x_i^*| : i\in\mathcal{I}_s^{\mathbf{x}^*}\right) > \gamma\max\left(|\nabla_j f(\mathbf{x}^*)| : j\notin supp(\mathbf{x}^*)\right) \tag{11}$$

we say that $\mathbf{x}^*$ is a strictly HT-stable stationary point associated with γ.

[2] A sequence may not converge but it may have an accumulation point. For example $1,-1,1,-1,\ldots$ is not a convergent sequence but it has two accumulation points.

Note that when $\|\mathbf{x}^*\|_0 = s$, $\mathcal{I}_s^{\mathbf{x}^*}$ is unique and equals $\text{supp}(\mathbf{x}^*)$ such that $\text{supp}(\mathbf{x}^*)$ in the above definition can be replaced by $\mathcal{I}_s^{\mathbf{x}^*}$. Moreover, if $\mathbf{x}^*$ is a strictly *HT-stable* stationary point, then we must have $\mathcal{I}_s^{\mathbf{x}^*} = \text{supp}(\mathbf{x}^*)$ (or equivalently $\|\mathbf{x}^*\|_0 = s$) because otherwise, $0 = \min\left(|x_i^*| : i \in \mathcal{I}_s^{\mathbf{x}^*}\right) > \gamma\|\nabla_{(\text{supp}(\mathbf{x}^*))^c} f(\mathbf{x}^*)\|_\infty$ which is impossible.

Remark 1. *The main message of Definition 7 is the following: "only by looking at the gradient restricted to the nonzero entries of a basic feasible point, one cannot say whether it is a local minimizer of Problem (2) or not".*

An *HT-stable* stationary point associated with γ is equivalent to the $\frac{1}{\gamma}$-stationary point of Problem (2) defined in [3, Definition 2.3]. Thus, by [3, Lemma 2.2], $\mathbf{x}^*$ is a *HT-stable* point if and only if $\mathbf{x}^* \in H_s(\mathbf{x}^* - \gamma\nabla f(\mathbf{x}^*))$.

In the following, we present a result that characterizes a *HT-unstable* stationary point. In essence, the following result shows that there always exists a neighborhood around a sparse *HT-unstable* stationary point whose gradient is zero over the nonzero elements and one can decrease the function value by going towards the direction of any nonzero coordinates.

Theorem 2 (Characterization of HT-unstable point)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 and L_s-RSS. Given any $0 < \gamma \leq \frac{1}{L_s}$, if a vector $\tilde{\mathbf{x}} \in C_s$ is such that $\nabla_{supp(\tilde{\mathbf{x}})} f(\tilde{\mathbf{x}}) = 0$ and $\min\left(|\tilde{x}_i| : i \in \mathcal{I}_s^{\tilde{\mathbf{x}}}\right) < \gamma\|\nabla_{(supp(\tilde{\mathbf{x}}))^c} f(\tilde{\mathbf{x}})\|_\infty$ for some $\mathcal{I}_s^{\tilde{\mathbf{x}}}$, then there exist a constant $\nu > 0$ and a neighborhood $\mathcal{N}$ of $\tilde{\mathbf{x}}$ such that $f(\mathbf{y}) \leq f(\mathbf{x}) - \nu$ for any $\mathbf{x} \in \mathcal{N} \cap C_s$ and any $\mathbf{y} \in H_s(\mathbf{x} - \gamma\nabla f(\mathbf{x}))$.*

The above result leads to the following necessary optimality conditions for a global minimizer of Problem (2) in terms of hard thresholding operator H_s. Note that the condition for $0 < \gamma < \frac{1}{L_s}$ has been obtained in [3, Theorem 2.2] without using gradient properties of the HT operator.

Theorem 3 (Characterization of global minimizer)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is L_s-RSS and $\mathbf{x}^*$ is a global minimizer. Then, $\mathbf{x}^*$ is a HT-stable (or $\frac{1}{\gamma}$-) stationary point for any $0 < \gamma \leq \frac{1}{L_s}$. Particularly, the following hold:*

a) For any $0 < \gamma < \frac{1}{L_s}$, $\mathbf{x}^* = H_s(\mathbf{x}^* - \gamma\nabla f(\mathbf{x}^*))$.

b) For $\gamma = \frac{1}{L_s}$, $\mathbf{x}^* \in H_s(\mathbf{x}^* - \gamma\nabla f(\mathbf{x}^*))$.

The following result shows that any accumulation point of an IHT sequence must be a *HT-stable* stationary point.

Theorem 4 (Accumulation point of IHT sequence)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be L_s-RSS and C^1 function. Suppose f is bounded below on C_s. Consider an IHT sequence $\left(\mathbf{x}^k\right)_{k\geq 0}$ associated with an arbitrary $\gamma \in (0, \frac{1}{L_s}]$, and let $\mathbf{x}^*$ be an accumulation point of $\left(\mathbf{x}^k\right)_{k\geq 0}$. Then, $\mathbf{x}^*$ is a HT-stable stationary point of Problem (2).*

Remark 2. The above theorem shows that any accumulation point of an IHT sequence is a *HT-stable* stationary point of Problem (2). Since each *HT-stable* stationary point must be a basic stationary point, one can observe that any accumulating point $\mathbf{x}^*$ of an IHT sequence must satisfy $\nabla_{\text{supp}(\mathbf{x}^*)} f(\mathbf{x}^*) = 0$ when $\|\mathbf{x}^*\|_0 = s$, or $\nabla f(\mathbf{x}^*) = 0$ when $\|\mathbf{x}^*\|_0 < s$.

The following result pertains to the objective function values of *HT-stable* and *HT-unstable stationary* points.

Corollary 4 (HT-stable vs HT-unstable stationary points)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be L_s-RSS and C^1 function. Suppose that every (nonempty) sub-level set of f contained in C_s is bounded, i.e., for any $\alpha \in \mathbb{R}$, $\{x \in C_s | f(\mathbf{x}) \leq \alpha\}$ is bounded (and closed). Consider an arbitrary $\gamma \in (0, \frac{1}{L_s}]$. For any HT-unstable stationary point $\mathbf{x}^*$ associated with γ, there exists a HT-stable stationary point $\tilde{\mathbf{x}}^*$ associated with γ such that $f(\mathbf{x}^*) > f(\tilde{\mathbf{x}}^*)$.*

Based on the above corollary, it is easy to see that if there are finitely many *HT-unstable* stationary points (happens when the function is RSC), then there is a *HT-stable* stationary point $\tilde{\mathbf{x}}^*$ such that $f(\mathbf{x}^*) > f(\tilde{\mathbf{x}}^*)$ for any *HT-unstable* stationary point $\mathbf{x}^*$.

The following result provides sufficient conditions for the global convergence of an IHT sequence. We say that a *HT-stable/unstable* stationary point $\mathbf{x}^*$ associated with $\gamma \in (0, \frac{1}{L_s}]$ is isolated if there exists a neighborhood $\mathcal{N}$ of $\mathbf{x}^*$ such that $\mathcal{N}$ does not contain any HT stationary point other than $\mathbf{x}^*$.

Corollary 5 (Global convergence of IHT sequence)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be L_s-RSS and C^1 function. Suppose that every (nonempty) sub-level set of f contained in C_s is bounded. Consider an arbitrary $\gamma \in (0, \frac{1}{L_s}]$. Assume that*

A.1 : For any two distinct HT-stable stationary points $\mathbf{x}^*$ and $\mathbf{y}^*$ associated with γ, $f(\mathbf{x}^*) \neq f(\mathbf{y}^*)$. Then, for any $\mathbf{x}^0 \in C_s$, the IHT sequence $\left(\mathbf{x}^k\right)_{k\geq 0}$ converges to a HT-stable stationary point associated with γ. This convergence results also hold under the following assumption:

A.2 : when $0 < \gamma < \frac{1}{L_s}$, each HT-stable stationary point associated with γ is isolated.

The following corollary shows that any IHT sequence always "escape" from a *HT-unstable* stationary point.

Corollary 6 (Escaping from HT-unstable stationary point)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be L_s-RSS and C^1 function. Suppose f is bounded below on C_s. For any given $\gamma \in (0, \frac{1}{L_s}]$ and any HT-unstable stationary point $\mathbf{x}^*$ associated with γ, there exists a neighborhood $\mathcal{N}$ of $\mathbf{x}^*$ such that for any IHT sequence starting from any $\mathbf{x}^0 \in C_s$, there exists $N \in \mathbb{N}$ such that $\mathbf{x}^k \notin \mathcal{N} \cap C_s$ for all $k \geq N$.*

The next result shows the attraction towards a strictly HT-stable stationary point in a neighborhood of such a stationary point. In what follows, for each index subset $\mathcal{J}$ with $|\mathcal{J}| = s$, a subspace $\mathcal{S}_{\mathcal{J}} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}_{\mathcal{J}^c}\}$ associated with $\mathcal{J}$ is defined. Clearly, C_s is the union of $\mathcal{S}_{\mathcal{J}}$'s for all $\mathcal{J}$'s with $|\mathcal{J}| = s$.

Proposition 1 (Reachability to strictly HT-stable stationary points)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be L_s-RSS and C^1 function. Suppose f is bounded below on C_s and f is strictly convex on $\mathcal{S}_{\mathcal{J}}$ for any index subset $\mathcal{J}$ with $|\mathcal{J}| = s$. Let $\mathbf{x}^*$ be a strictly HT-stable stationary point associated with any given $\gamma \in (0, \frac{1}{L_s}]$. Then there exists a neighborhood $\mathcal{B}$ of $\mathbf{x}^*$ such that for every $\mathbf{x}^0 \in \mathcal{B} \cap C_s$, the IHT sequence $\left(\mathbf{x}^k\right)_{k\geq 0}$ converges to $\mathbf{x}^*$. Moreover, if f is strongly convex on $\mathcal{J}$ for every index subset $\mathcal{J}$ with $|\mathcal{J}| = s$, then for $\mathbf{x}^0 \in \mathcal{B} \cap C_s$, the IHT sequence $\left(\mathbf{x}^k\right)_{k\geq 0}$ Q-linearly converges to $\mathbf{x}^*$.*

In the next subsection, we provide an example to show the *escapability* property of *HT-unstable* points.

5 Simulation

To elaborate on theoretical results including Corollary 3, Theorem 2, the notion of *HT-stationary* points, Corollary 6 which shows *escapability* property of *HT-unstable stationary* points, and Proposition 1 which shows *Reachability* to *HT-stable stationary* points, we use a quadratic function $f(\mathbf{x}) = \frac{1}{m}\sum_{i=1}^{m}(A_{i\bullet}\mathbf{x} - y_i)^2 = \frac{1}{m}\|A\mathbf{x} - \mathbf{b}\|^2$ where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $A_{i\bullet}$ is the i-th row of A, $\mathbf{x} \in \mathbb{R}^n$ is the optimization variable, and $\mathbf{y} \in \mathbb{R}^m$ is the target. This function is both RSS and RSC so both Corollary 6 and Proposition 1 follow. To better visualize the process, we let $m = n = 4$ and $s = 2$. Therefore, there are six *HT-stationary* points where the gradient over the nonzero elements is zero. We use Pytorch [49] to select the matrix A and $\mathbf{y}$. By setting the random seed to be 45966 we draw a 4×4 matrix A whose elements are standard normal. Keeping the same seed, we generate $\mathbf{y}$. The following would be A and $\mathbf{y}$:

$$A = \begin{bmatrix} -1.0655 & 0.2249 & -0.0897 & 0.1876 \\ 1.1627 & -1.1229 & -0.0823 & -0.3059 \\ -0.2011 & 0.5342 & -0.0551 & -1.3459 \\ 0.2308 & -0.6404 & -0.7468 & 0.0378 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -1.7861 \\ -0.3556 \\ -0.1881 \\ 0.3896 \end{bmatrix}$$

The restricted Lipschitz constant, i.e., L_s, for the above quadratic function is $\frac{2}{m} \times \lambda_{max}(A^\top A)$ where λ_{max} is the maximum eigenvalue of $A^\top A$. Thus, for the above choice of $A, \mathbf{y}$, the maximum allowable stepsize is $\gamma = \frac{1}{L_s} = 0.06$. Once, γ is fixed, one can determine stability of each stationary

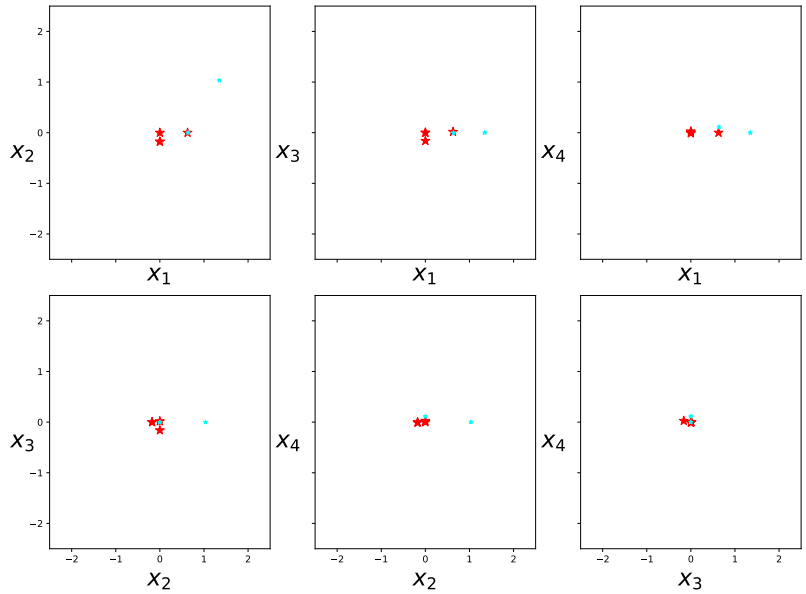

Figure 1: Illustration of *HT-stationary* points on 2-d plains.

point. The following are *HT-stationary* points along with their stability status:

$$\begin{bmatrix} No. & x_1 & x_2 & x_3 & x_4 & HT-stability \\ 1 & 1.3474 & 1.0331 & 0 & 0 & \text{strictly HT-stable} \\ 2 & 0.6278 & 0 & 0.0177 & 0 & \text{HT-unstable} \\ 3 & 0.6387 & 0 & 0 & 0.1123 & \text{strictly HT-stable} \\ 4 & 0 & -0.1758 & 0.0008 & 0 & \text{HT-unstable} \\ 5 & 0 & -0.1776 & 0 & -0.0113 & \text{HT-unstable} \\ 6 & 0 & 0 & -0.1608 & 0.0259 & \text{HT-unstable} \end{bmatrix}$$

where x_1, x_2, x_3, x_4 are four coordinates of each *HT-stationary* point. Since *HT-stationary* points are vectors in $\mathbb{R}^4$, there is no way to show all of them on one 2-d plane. Thus, we use six 2-d plains where each plane shows only two coordinates of *HT-stationary* points. On each 2-d plain we have 6 different points, each one associated with one of the *HT-stationary* points shown in a particular 2-d plain with specified coordinates. In Figure 1 the points with red stars are *HT-unstable* ones, and the blue ones are the *HT-stable* ones. For example, the first 2-d plain (first row-first column) including coordinates $x_1 - x_2$ shows the x_1, x_2 coordinates of all of the six *HT-stationary* points. On the first row-first column 2-d plain, the first *HT-stationary* point is more distinct because it is the only one that has two nonzeros elements associated with $x_1 - x_2$ coordinates. We also can see three points with $x_2 = 0$, two of which are *HT-unstable* points and one is *HT-unstable* one. This is more clear, if one looks at the column x_2 in *HT-stationary* points matrix above. Also, it is clear that we have three *HT-unstable* points with $x_1 = 0$ on the first 2-d plane.

The following also shows the gradient status of each *HT-stationary* point. As you can see, the gradient corresponding to nonzero elements in *HT-stationary* point are zero as follows:

$$\begin{bmatrix} No. & g_1 & g_2 & g_3 & g_4 & HT-stability \\ 1 & 0 & 0 & 0.2060 & -0.3916 & \text{strictly HT-stable} \\ 2 & 0 & -0.3843 & 0 & -0.1070 & \text{HT-unstable} \\ 3 & 0 & -0.4189 & -0.0029 & 0 & \text{strictly HT-stable} \\ 4 & -0.6506 & 0 & 0 & 0.0106 & \text{HT-unstable} \\ 5 & -0.6473 & 0 & -0.0010 & 0 & \text{HT-unstable} \\ 6 & -0.7994 & 0.1297 & 0 & 0 & \text{HT-unstable} \end{bmatrix}$$

where g_1, g_2, g_3, g_4 are the four gradient entries corresponding to each *HT-stationary* point.

We perturb nonzero coordinates of all *HT-unstable* points with a normal random noise with mean zero and standard deviation of $\sigma = 0.5$ to create 4,000 different initialization points. These points

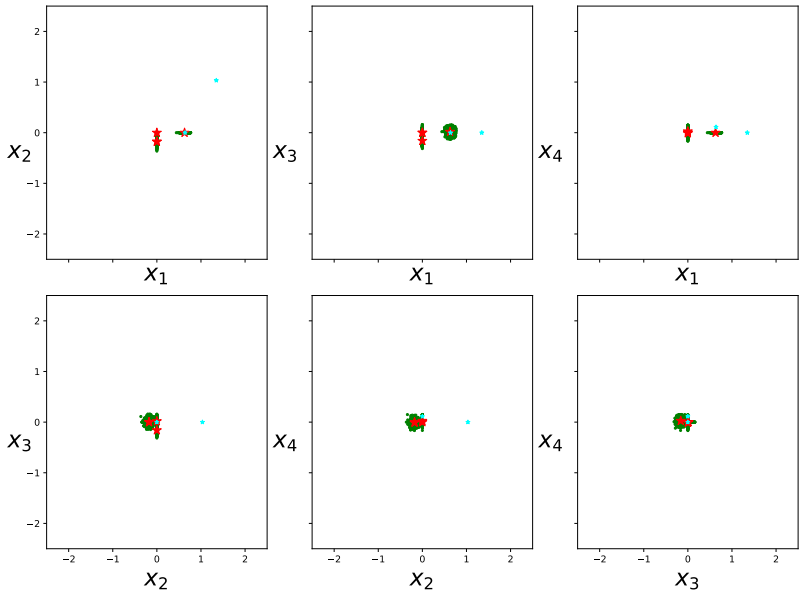

Figure 2: Illustration of 4,000 initialization close to four *HT-unstable stationary* points.

create four clouds around *HT-unstable* points which are shown in Figure 2. Then we run the IHT algorithm for 400 steps. After 300 steps, all of these initializations escape from those *HT-unstable* points and converge to either of the *HT-stable* stationary points on $x_1 - x_2$ or $x_1 - x_4$ 2-d planes. In fact, these two *HT-stable* stationary points are sparse local minimizers. Figure 3 shows the 300-th step of IHT algorithm. The video showing 400 steps of applying IHT algorithm for 4,000 different runs starting close to *HT-unstable* point is on YouTube. These numerical results corroborate our

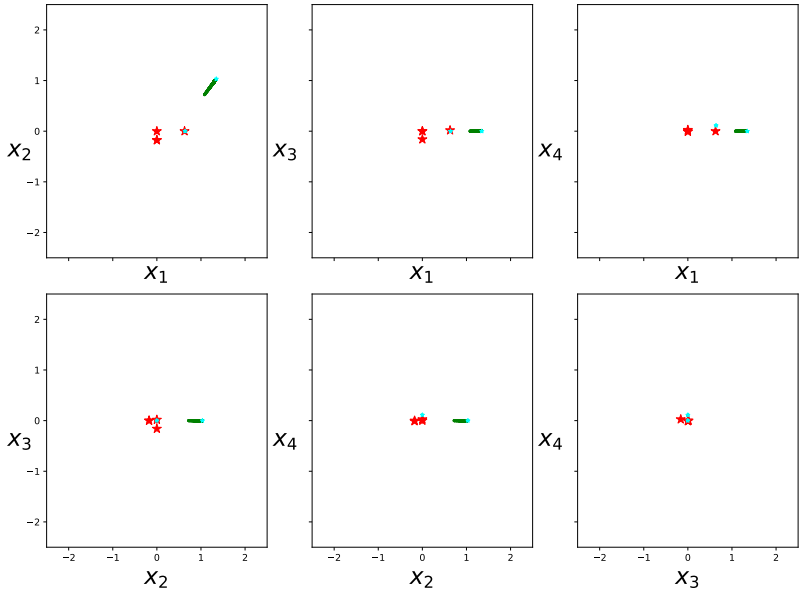

Figure 3: Illustration of 4,000 IHT sequences initialized close to four *HT-unstable stationary* points after 400 steps (please refer to the video of all 400 steps).

theoretical results as expected. By looking at the video, one can easily see *escapability* property of *HT-unstable stationary* points, and *Reachability* to *HT-stable stationary* points.

6 Conclusion

In this paper, we established a new and critical gradient descent property of the hard thresholding operator. Based on this property we introduced the notion of *HT-stationary* points and developed convergence properties of the IHT scheme. Based on the notion of *HT-stationary* points, we showed *escapability* property of *HT-unstable* stationary points and local *reachability* property of *strictly HT-stable* stationary points. We also show that the IHT sequence converges globally if the function values of *HT-stable* stationary points of Problem (2) are distinct. These convergence results hold for any sparse initial conditions if the latter is satisfied.

References

[1] Sohail Bahmani, Bhiksha Raj, and Petros T Boufounos. Greedy sparsity-constrained optimization. *Journal of Machine Learning Research*, 14(Mar):807–841, 2013.

[2] Heinz H Bauschke, D Russell Luke, Hung M Phan, and Xianfu Wang. Restricted normal cones and sparsity optimization with affine constraints. *Foundations of Computational Mathematics*, 14(1):63–83, 2014.

[3] Amir Beck and Yonina C Eldar. Sparsity constrained nonlinear optimization: Optimality conditions and algorithms. *SIAM Journal on Optimization*, 23(3):1480–1509, 2013.

[4] Amir Beck and Nadav Hallak. On the minimization over sparse symmetric sets: projections, optimality conditions, and algorithms. *Mathematics of Operations Research*, 41(1):196–223, 2016.

[5] Thomas Blumensath. Accelerated iterative hard thresholding. *Signal Processing*, 92(3):752–756, 2012.

[6] Thomas Blumensath and Mike E Davies. Iterative thresholding for sparse approximations. *Journal of Fourier analysis and Applications*, 14(5):629–654, 2008.

[7] Thomas Blumensath and Mike E Davies. Iterative hard thresholding for compressed sensing. *Applied and computational harmonic analysis*, 27(3):265–274, 2009.

[8] Sébastien Bourguignon, Jordan Ninin, Hervé Carfantan, and Marcel Mongeau. Exact sparse approximation problems via mixed-integer programming: Formulations and computational performance. *IEEE Transactions on Signal Processing*, 64(6):1405–1419, 2015.

[9] Joshua Brodie, Ingrid Daubechies, Christine De Mol, Domenico Giannone, and Ignace Loris. Sparse and stable markowitz portfolios. *Proceedings of the National Academy of Sciences*, 106(30):12267–12272, 2009.

[10] Max Bucher and Alexandra Schwartz. Second-order optimality conditions and improved convergence results for regularization methods for cardinality-constrained optimization problems. *Journal of Optimization Theory and Applications*, 178(2):383–410, 2018.

[11] Oleg P Burdakov, Christian Kanzow, and Alexandra Schwartz. Mathematical programs with cardinality constraints: reformulation by complementarity-type conditions and a regularization method. *SIAM Journal on Optimization*, 26(1):397–425, 2016.

[12] T-J Chang, Nigel Meade, John E Beasley, and Yazid M Sharaiha. Heuristics for cardinality constrained portfolio optimisation. *Computers & Operations Research*, 27(13):1271–1302, 2000.

[13] Rick Chartrand. Exact reconstruction of sparse signals via nonconvex minimization. *IEEE Signal Processing Letters*, 14(10):707–710, 2007.

[14] Jinghui Chen and Quanquan Gu. Accelerated stochastic block coordinate gradient descent for sparsity constrained nonconvex optimization. In *UAI*, 2016.

[15] Hyonho Chun and Sündüz Keleş. Sparse partial least squares regression for simultaneous dimension reduction and variable selection. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 72(1):3–25, 2010.

[16] Mark A Davenport and Justin Romberg. An overview of low-rank matrix recovery from incomplete observations. *IEEE Journal of Selected Topics in Signal Processing*, 10(4):608–622, 2016.

[17] Geoffrey Mark Davis. *Adaptive nonlinear approximations*. PhD thesis, New York University, 1994.

[18] Antoine Dedieu, Hussein Hazimeh, and Rahul Mazumder. Learning sparse classifiers: Continuous and mixed integer optimization perspectives. *Journal of Machine Learning Research*, 22(135):1–47, 2021.

[19] Loann David Denis Desboulets. A review on variable selection in regression analysis. *Econometrics*, 6(4):45, 2018.

[20] David L Donoho, Yaakov Tsaig, Iddo Drori, and Jean-Luc Starck. Sparse solution of underdetermined systems of linear equations by stagewise orthogonal matching pursuit. *IEEE transactions on Information Theory*, 58(2):1094–1121, 2012.

[21] Yonina C Eldar and Gitta Kutyniok. *Compressed sensing: theory and applications*. Cambridge university press, 2012.

[22] Jianqing Fan and Runze Li. Variable selection via nonconcave penalized likelihood and its oracle properties. *Journal of the American statistical Association*, 96(456):1348–1360, 2001.

[23] Massimo Fornasier, Holger Rauhut, and Rachel Ward. Low-rank matrix recovery via iteratively reweighted least squares minimization. *SIAM Journal on Optimization*, 21(4):1614–1640, 2011.

[24] Simon Foucart and Ming-Jun Lai. Sparsest solutions of underdetermined linear systems via ℓ_q-minimization for $0 < q \leq 1$. *Applied and Computational Harmonic Analysis*, 26(3):395–407, 2009.

[25] Simon Foucart and Holger Rauhut. A mathematical introduction to compressive sensing. pages 1–39. Springer, 2013.

[26] Trevor Gale, Erich Elsen, and Sara Hooker. The state of sparsity in deep neural networks. *arXiv preprint arXiv:1902.09574*, 2019.

[27] Benjamin Haeffele, Eric Young, and Rene Vidal. Structured low-rank matrix factorization: Optimality, algorithm, and applications to image processing. In *International conference on machine learning*, pages 2007–2015. PMLR, 2014.

[28] Prateek Jain, Ambuj Tewari, and Purushottam Kar. On iterative hard thresholding methods for high-dimensional m-estimation. *Advances in neural information processing systems*, 27, 2014.

[29] Rajiv Khanna and Anastasios Kyrillidis. Iht dies hard: Provable accelerated iterative hard thresholding. In *International Conference on Artificial Intelligence and Statistics*, pages 188–198. PMLR, 2018.

[30] Ming-Jun Lai and Jingyue Wang. An unconstrained ℓ_q minimization with $0 < q \leq< 1$ for sparse solution of underdetermined linear systems. *SIAM Journal on Optimization*, 21(1):82–101, 2011.

[31] Xingguo Li, Raman Arora, Han Liu, Jarvis Haupt, and Tuo Zhao. Nonconvex sparse learning via stochastic optimization with progressive variance reduction. *arXiv preprint arXiv:1605.02711*, 2016.

[32] Xue Li and Wen Song. The first-order necessary conditions for sparsity constrained optimization. *Journal of the Operations Research Society of China*, 3(4):521–535, 2015.

[33] Guannan Liang, Qianqian Tong, Chunjiang Zhu, and Jinbo Bi. An effective hard thresholding method based on stochastic variance reduction for nonconvex sparse learning. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 1585–1592, 2020.

[34] Bo Liu, Xiao-Tong Yuan, Lezi Wang, Qingshan Liu, and Dimitris N Metaxas. Dual iterative hard thresholding: From non-convex sparse minimization to non-smooth concave maximization. In *International Conference on Machine Learning*, pages 2179–2187. PMLR, 2017.

[35] Haoyang Liu and Rina Foygel Barber. Between hard and soft thresholding: optimal iterative thresholding algorithms. *Information and Inference: A Journal of the IMA*, 9(4):899–933, 2020.

[36] Zhaosong Lu. Iterative hard thresholding methods for l_0 regularized convex cone programming. *Mathematical Programming*, 147(1):125–154, 2014.

[37] Zhaosong Lu. Optimization over sparse symmetric sets via a nonmonotone projected gradient method. *arXiv: Optimization and Control*, 2015.

[38] Stéphane G Mallat and Zhifeng Zhang. Matching pursuits with time-frequency dictionaries. *IEEE Transactions on signal processing*, 41(12):3397–3415, 1993.

[39] Dmitry Molchanov, Arsenii Ashukha, and Dmitry Vetrov. Variational dropout sparsifies deep neural networks. In *International Conference on Machine Learning*, pages 2498–2507. PMLR, 2017.

[40] Seyedahmad Mousavi and Jinglai Shen. Solution uniqueness of convex piecewise affine functions based optimization with applications to constrained ℓ_1 minimization. *ESAIM: Control, Optimisation and Calculus of Variations*, 25:56, 2019.

[41] Balas Kausik Natarajan. Sparse approximate solutions to linear systems. *SIAM journal on computing*, 24(2):227–234, 1995.

[42] Deanna Needell and Joel A Tropp. Cosamp: Iterative signal recovery from incomplete and inaccurate samples. *Applied and computational harmonic analysis*, 26(3):301–321, 2009.

[43] Deanna Needell and Roman Vershynin. Uniform uncertainty principle and signal recovery via regularized orthogonal matching pursuit. *Foundations of computational mathematics*, 9(3):317–334, 2009.

[44] Deanna Needell and Roman Vershynin. Signal recovery from incomplete and inaccurate measurements via regularized orthogonal matching pursuit. *IEEE Journal of selected topics in signal processing*, 4(2):310–316, 2010.

[45] Sahand N Negahban, Pradeep Ravikumar, Martin J Wainwright, and Bin Yu. A unified framework for high-dimensional analysis of m-estimators with decomposable regularizers. *Statistical science*, 27(4):538–557, 2012.

[46] Li-Li Pan, Nai-Hua Xiu, and Sheng-Long Zhou. On solutions of sparsity constrained optimization. *Journal of the Operations Research Society of China*, 3(4):421–439, 2015.

[47] LiLi Pan, NaiHua Xiu, and Jun Fan. Optimality conditions for sparse nonlinear programming. *Science China Mathematics*, 60(5):759–776, 2017.

[48] Lili Pan, Shenglong Zhou, Naihua Xiu, and Houduo Qi. A convergent iterative hard thresholding for sparsity and nonnegativity constrained optimization. *Pacific Journal of Optimization*, 13(2):325–353, 2017.

[49] Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, et al. Pytorch: An imperative style, high-performance deep learning library. *arXiv preprint arXiv:1912.01703*, 2019.

[50] Yagyensh Chandra Pati, Ramin Rezaiifar, and Perinkulam Sambamurthy Krishnaprasad. Orthogonal matching pursuit: Recursive function approximation with applications to wavelet decomposition. In *Proceedings of 27th Asilomar conference on signals, systems and computers*, pages 40–44. IEEE, 1993.

[51] Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B (Methodological)*, 58(1):267–288, 1996.

[52] Trung Vu and Raviv Raich. Accelerating iterative hard thresholding for low-rank matrix completion via adaptive restart. In *ICASSP 2019-2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 2917–2921. IEEE, 2019.

[53] Meng Wang, Weiyu Xu, and Ao Tang. On the performance of sparse recovery via ℓ_p-minimization $0 < p \leq 1$. *IEEE Transactions on Information Theory*, 57(11):7255–7278, 2011.

[54] Rui Wang, Naihua Xiu, and Shenglong Zhou. Fast newton method for sparse logistic regression. *arXiv*, (1901.02768), 2019.

[55] Joong-Ho Won, Kenneth Lange, and Jason Xu. A unified analysis of convex and non-convex lp-ball projection problems. *arXiv preprint arXiv:2203.00564*, 2022.

[56] Fan Wu and Wei Bian. Accelerated iterative hard thresholding algorithm for l_0 regularized regression problem. *Journal of Global Optimization*, 76(4):819–840, 2020.

[57] Chen Zhao, Naihua Xiu, Houduo Qi, and Ziyan Luo. A lagrange–newton algorithm for sparse nonlinear programming. *Mathematical Programming*, pages 1–26, 2021.

[58] Le Zheng, Arian Maleki, Haolei Weng, Xiaodong Wang, and Teng Long. Does ℓ_p-minimization outperform ℓ_1-minimization? *IEEE Transactions on Information Theory*, 63(11):6896–6935, 2017.

[59] Pan Zhou, Xiaotong Yuan, and Jiashi Feng. Efficient stochastic gradient hard thresholding. *Advances in Neural Information Processing Systems*, 31, 2018.

[60] Shenglong Zhou, Lili Pan, and Naihua Xiu. Subspace newton method for the l_0-regularized optimization. *arXiv*, (2004.05132), 2020.

[61] Shenglong Zhou, Naihua Xiu, and Hou-Duo Qi. Global and quadratic convergence of newton hard-thresholding pursuit. *J. Mach. Learn. Res.*, 22(12):1–45, 2021.

[62] Wenxing Zhu, Zhengshan Dong, Yuanlong Yu, and Jianli Chen. Lagrange dual method for sparsity constrained optimization. *IEEE Access*, 6:28404–28416, 2018.

[63] Hui Zou and Trevor Hastie. Regularization and variable selection via the elastic net. *Journal of the royal statistical society: series B (statistical methodology)*, 67(2):301–320, 2005.