ROBUST WEIGHT INITIALIZATION FOR TANH NEURAL NETWORKS WITH FIXED POINT ANALYSIS

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ABSTRACT

As a neural network's depth increases, it can achieve strong generalization performance. However, training deep networks is challenging due to gradient and signal propagation issues. To address these challenges, extensive theoretical research and various methods have been introduced. Despite these advances, effective weight initialization methods for tanh neural networks remain insufficiently studied. This paper presents a novel weight initialization method for Neural Networks with tanh activation function. Based on an analysis of the fixed points of the function $\tanh(ax)$, our proposed method aims to determine values of a that mitigate activation saturations. A series of experiments on various classification datasets and Physics-Informed Neural Networks demonstrate that the proposed method outperforms Xavier initialization methods (with or without normalization) in terms of robustness to network size variations, data efficiency, and convergence speed.

1 INTRODUCTION

Deep learning has enabled substantial advancements in state-of-the-art performance across various domains (LeCun et al., 2015; He et al., 2016). In general, the expressivity of neural networks increases exponentially with depth (Poole et al., 2016; Raghu et al., 2017), enabling strong generalization performance. However, deeper networks often face challenges such as vanishing or exploding gradients and poor signal propagation (Bengio et al., 1993). These challenges have driven the development of effective weight initialization methods tailored to various activation functions. Xavier initialization (Glorot & Bengio, 2010) ensures signals stay in the non-saturated region for sigmoid and hyperbolic tangent activations, while He initialization (He et al., 2015) maintains stable variance for ReLU networks. Especially in ReLU neural networks, several weight initialization methods have been proposed to mitigate the dying ReLU problem, which hinders signal propagation in deep networks (Lu et al., 2019; Lee et al., 2024). However, to the best of our knowledge, research on initialization methods that are robust to the size of tanh networks remains insufficient. Tanh networks commonly use Xavier initialization (Raissi et al., 2019; Jagtap et al., 2022; Rathore et al., 2024) and are widely applied in various domains, such as Physics-Informed Neural Networks (PINNs) (Raissi et al., 2019) and Recurrent Neural Networks (RNNs) (Rumelhart et al., 1986), with performance often dependent on model size and initialization randomness (Liu et al., 2022).

The main contribution of this paper is the proposal of a simple weight initialization method for FeedForward Neural Networks (FFNNs) with tanh activation function. The proposed method is data-efficient and demonstrates robustness to variations in network size, reducing the dependency on normalization techniques such as Batch Normalization (Ioffe, 2015) and Layer Normalization (Ba, 2016). As a result, it reduces the requirement for extensive hyperparameter tuning, such as the number of hidden layers and units, and avoids normalization computational overhead. The theoretical foundation for this approach is provided through the fixed point of the function $\tanh(ax)$. We investigate the performance of the proposed method on two tasks: classification and Physics-Informed Neural Networks (PINNs). For classification tasks, we assess the method's performance across various FFNN sizes using datasets including MNIST, Fashion MNIST, CIFAR-10, and CIFAR-100. The results demonstrate improved validation accuracy and loss compared to Xavier initialization with Batch Normalization (BN) or Layer Normalization (LN). For PINNs, the method exhibits robustness across diverse network sizes and demonstrates its effectiveness in solving a wide range of

PDE problems. Notably, for both tasks, the proposed method outperforms Xavier initialization in terms of data efficiency, achieving improved performance even with limited data.

Contributions. Our contributions can be summarised as follows:

- We identify the conditions under which activation values remain non-vanishing as the depth of the neural network increases, using a fixed-point analysis (Section 3.1 and 3.2).
- We propose a novel weight initialization method for tanh neural networks that is robust to variations in network size and demonstrates high data efficiency (Section 3.2 and 3.3).
- We experimentally show that the proposed method is more data-efficient than Xavier initialization, with or without normalization, on image benchmarks and PINNs (Section 4).
- We experimentally show that the proposed method is more robust to network size variations on image benchmarks and PINNs (Section 4).

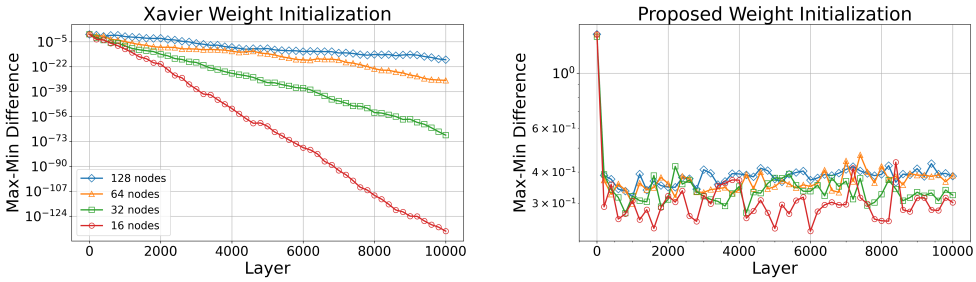
2 Related works

The expressivity of neural networks grows exponentially with depth, resulting in improved generalization performance (Poole et al., 2016; Raghu et al., 2017). Weight initialization is crucial for training deep networks effectively (Saxe et al., 2014; Mishkin & Matas, 2016). Xavier (Glorot & Bengio, 2010) and He He et al. (2015) initialization are common initialization methods typically used with tanh and ReLU activation functions, respectively. Various initialization methods have been proposed to facilitate the training of deeper ReLU neural networks (Lu et al., 2019; Bachlechner et al., 2021; Zhao et al., 2022; Lee et al., 2024). In contrast, research on weight initialization for neural networks using tanh activation remains limited. Despite this, tanh neural networks have gained popularity in recent years, particularly in applications such as physics-informed neural networks (PINNs), where their performance can be sensitive to the randomness of initialization methods.

PINNs have shown promising results in solving forward, inverse, and multiphysics problems arising in science and engineering. (Lu et al., 2021; Karniadakis et al., 2021; Cuomo et al., 2022b;a; Yin et al., 2021; Wu et al., 2023; Hanna et al., 2022; Bararnia & Esmaeilpour, 2022; Shukla et al., 2020; Zhu et al., 2024; Hosseini et al., 2023; Mao et al., 2020). PINNs approximate solutions to partial differential equations (PDEs) using neural networks and are trained by minimizing a loss defined by the sum of least-squares that incorporates the residual of PDE, boundary conditions, and initial conditions. This loss is usually minimized using gradient-based optimizers such as Adam (Kingma, 2014), L-BFGS (Liu & Nocedal, 1989), or a combination of both. Universal approximation theories (Cybenko, 1989; Hornik et al., 1989; Hornik, 1991; Park et al., 2020; Guliyev & Ismailov, 2018b; Shen et al., 2022; Guliyev & Ismailov, 2018a; Maiorov & Pinkus, 1999; Yarotsky, 2017; Gripenberg, 2003) guarantee the capability and performance of neural networks as an approximation of the analytic solution to PDE. However, PINNs still face challenges in accuracy, stability, computational complexity, and tuning optimal hyperparameters of loss terms.

To alleviate these issues, many authors have introduced enhanced versions of PINNs: (1) the self-adaptive loss balanced PINNs (lbPINNs) that automatically adjust the hyperparameters of loss terms during the training process (Xiang et al., 2022), (2) the Bayesian PINNs (B-PINNs) that are specialized to deal with forward and inverse nonlinear problems with noisy data (Yang et al., 2021), (3) Rectified PINNs (RPINNs) that are trained with the gradient information from the numerical solution by the multigrid method and designed for solving stationary PDEs (Peng et al., 2022), (4) Auxiliary Pinns (A-PINNs) that effectively handle integro-differential equations (Yuan et al., 2022), (5) conservative PINNs (cPINNs) and exetended PINNs (XPINNs) that adopt the domain decomposition technique (Jagtap et al., 2020; Jagtap & Karniadakis, 2020), (6) parrel PINNs that reduces the computational cost of cPINNs and XPINNs (Shukla et al., 2021), (7) gradient-enhanced PINNs (gPINNs) that use the gradient of the PDE loss term with respect to the network inputs (Yu et al., 2022).

While these advancements address various challenges in PINNs, activation functions, and their initialization strategies remain crucial for achieving optimal performance. The tanh activation function is known to perform well in PINNs (Raissi et al., 2019). Detailed experimental results supporting this can be found in Appendix C.2. Xavier initialization is commonly used as the standard choice for tanh networks in existing studies (Jin et al., 2021; Son et al., 2023; Yao et al., 2023; Gnanasam-



(a) Xavier Initialization

(b) Proposed Initialization

Figure 1: Difference between maximum and minimum activation values at each layer when propagating $3,000$ input data through a $10,000$-layer tanh FFNN, using Xavier initialization **(left)** and the proposed initialization **(right)**. Experiments were conducted on networks with $10,000$ hidden layers, each having the same number of nodes: 16, 32, 64, or 128.

bandam et al., 2023; Song et al., 2024). However, our experimental results indicate that this method is dependent on network size and that performance improvements achieved through Batch Normalization or Layer Normalization are limited. Additionally, the method demonstrates sensitivity to the amount of training data. Although there has been a recent result on an initialization method for PINNs, the method relies on transfer learning (Tarbiyati & Nemati Saray, 2023). Thus, we propose a weight initialization method that is robust to variations in network size, achieves high data efficiency, and eliminates the need for both transfer learning and normalization techniques.

3 Proposed Weight Initialization Method

In this section, we discuss the proposed weight initialization method. Section 3.1 introduces the theoretical motivation behind the method. Section 3.2 presents how to derive the initial weight matrix that satisfies the conditions outlined in Section 3.1. Finally, in Section 3.3, we suggest the optimal hyperparameter σ_z in the proposed method.

3.1 Theoretical Motivation

Experimental results in Figure 1 reveal that when Xavier initialization is employed in FFNNs with tanh activation, the distribution of activation values tends to cluster around zero in deeper layers. This vanishing of activation values can hinder the training process due to a discrepancy between the activation values and the desired output. However, theoretically preventing this phenomenon is not straightforward. In this section, we provide a theoretical analysis based on a fixed point of $\tanh(ax)$ to bypass the phenomena. Before giving the theoretical foundations, consider the basic results for a tanh activation function. Recall that x^* is a fixed point of a function f if x^* belongs to both the domain and the codomain of f, and $f(x^*) = x^*$.

Lemma 1. *For a fixed $a > 0$ define the function $\phi_a : \mathbb{R} \to \mathbb{R}$ given as*

$$\phi_a(x) := \tanh(ax).$$

Then, there exists a fixed point x^*. Furthermore,

(1) *if $0 < a \leq 1$, then ϕ has a unique fixed point $x^* = 0$.*

(2) *if $a > 1$, then ϕ has three distinct fixed points: $x^* = -\xi_a$, 0, ξ_a such that $\xi_a > 0$.*

Proof. The proof is detailed in in Appendix A.1. □

The function $\tanh(x)$ satisfies $\tanh(x) < x$ for all $x > 0$. However, according to Lemma 1, the behavior of $\tanh(ax)$ changes when $a > 1$. When $x > \xi_a$, the inequality $\tanh(ax) < x$ holds. When $x < \xi_a$, the inequality $\tanh(ax) > x$ is satisfied. At $x = \xi_a$, the equality $\tanh(ax) = x$

holds. In Lemma 2, we address the convergence properties of iteratively applying $\tanh(ax)$ for any $x > 0$.

Lemma 2. *For a given initial value $x_0 > 0$ define*

$$x_{n+1} = \phi_a(x_n), \quad n = 0, 1, 2, \ldots.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges regardless of the positive initial value $x_0 > 0$. Moreover,

(1) *if $0 < a \leq 1$, then $x_n \to 0$ as $n \to \infty$.*

(2) *if $a > 1$, then $x_n \to \xi_a$ as $n \to \infty$.*

Proof. The proof is detailed in Appendix A.2. □

According to Lemma 2, for $a > 1$ and $x_0 > \xi_a > 0$, the sequence $\{x_n\}$ satisfies $\tanh(ax_n) > \tanh(ax_{n+1}) > \xi_a$ for all $n \in \mathbb{N}$. Similarly, when $0 < x_0 < \xi_a$, the sequence satisfies $\tanh(ax_n) < \tanh(ax_{n+1}) < \xi_a$ for all $n \in \mathbb{N}$. Given $a > 1$ and $x_0 < 0$, the sequence converges to $-\xi_a$ as $n \to \infty$ due to the odd symmetry of $\tanh(ax)$. Therefore, when $a > 1$, for an arbitrary initial value $x_0 > 0$ or $x_0 < 0$, the sequence $\{x_n\}$ converges to ξ_a or $-\xi_a$, respectively, as $n \to \infty$.

Note that the parameter a in Lemma 2 does not change across all iterations. In Propositions 3 and Corollary 4, we address cases where the value of a varies with each iteration.

Proposition 3. *Let $\{a_n\}_{n=1}^{\infty}$ be a positive real sequence, i.e., $a_n > 0$ for all $n \in \mathbb{N}$, such that only finitely many elements are greater than 1. Suppose that $\{\Phi_m\}_{m=1}^{\infty}$ is a sequence of functions defined as for each $m \in \mathbb{N}$*

$$\Phi_m = \phi_{a_m} \circ \phi_{a_{m-1}} \circ \cdots \circ \phi_{a_1}.$$

Then for any $x \in \mathbb{R}$

$$\lim_{m \to \infty} \Phi_m(x) = 0.$$

Proof. The proof is detailed in Appendix A.3. □

Corollary 4. *Let $\epsilon > 0$ be given. Suppose that $\{a_n\}_{n=1}^{\infty}$ be a positive real sequence such that only finitely many elements are lower than $1 + \epsilon$. Then for any $x \in \mathbb{R} \setminus \{0\}$*

$$\left| \lim_{m \to \infty} \Phi_m(x) \right| \geq \xi_{1+\epsilon}$$

Proof. The proof is detailed in Appendix A.4. □

Based on Proposition 3 and Corollary 4, if there exists a sufficiently large N such that all elements a_n for $n \geq N$ are either less than 1 or greater than $1 + \epsilon$, then for any $x_0 \in \mathbb{R} \setminus \{0\}$, the sequence either converges to 0 or satisfies $|\Phi_m(x_0)| \geq \xi_{1+\epsilon}$ as $m \to \infty$, respectively. This result implies that if the sequence $\{\Phi_m\}_{m=1}^{M}$ is finite, a_n for $N \leq n \leq M$, where N is an arbitrarily chosen index close to M, significantly influence the values of $\Phi_M(x_0)$.

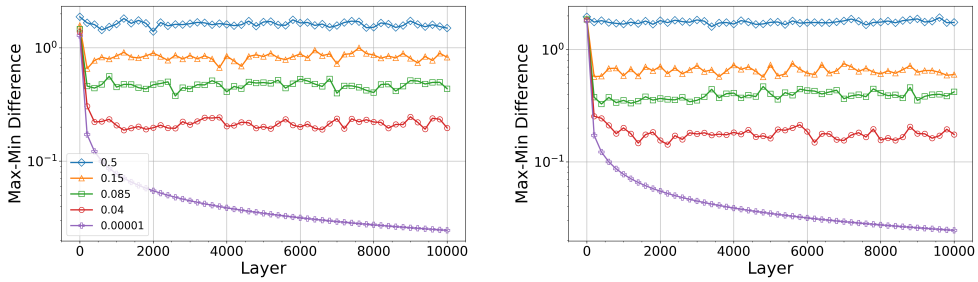
3.2 The derivation of the proposed weight initialization method

Remark Based on the theoretical motivations discussed in the previous section, we propose a weight initialization method that satisfies the following conditions during the initial forward pass: (i) It avoids the phenomenon where activation values cluster around zero in deeper layers. (ii) It ensures that the distribution of activation values in deeper layers is approximately normal.

Notation Consider a feedforward neural network with L layers. The network processes K training samples, denoted as pairs $\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i=1}^{K}$, where $\boldsymbol{x}_i \in \mathbb{R}^{N_x}$ is training input and $\boldsymbol{y}_i \in \mathbb{R}^{N_y}$ is its corresponding output. The iterative computation at each layer ℓ is defined as follows:

$$\boldsymbol{x}^{\ell} = \tanh(\mathbf{W}^{\ell}\boldsymbol{x}^{\ell-1} + \mathbf{b}^{\ell}) \in \mathbb{R}^{N_\ell} \quad \text{for all } \ell = 1, \ldots, L,$$

where $\mathbf{W}^{\ell} \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$ is the weight matrix, $\mathbf{b}^{\ell} \in \mathbb{R}^{N_\ell}$ is the bias, and $\tanh(\cdot)$ is an element-wise activation hyperbolic tangent function.



(a) Same dimension

(b) Varying dimensions

Figure 2: Difference between maximum and minimum activation values at each layer when propagating $3,000$ input data through a $10,000$-layer tanh FFNN, using the proposed initialization with α set to $0.04, 0.085, 0.15$, and 0.5. Network with $10,000$ hidden layers, each with 32 nodes **(left)**, and a network with alternating hidden layers of 64 and 32 nodes **(right)**.

Signal Propagation Analysis We present a simplified analysis of signal propagation in FFNNs with the tanh activation function. For notational convenience, it is assumed that all hidden layers, as well as the input and output layers, have a dimension of n, i.e., $N_\ell = n$ for all ℓ. Given an arbitrary input vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, the first layer activation $\boldsymbol{x}^1 = \tanh(\mathbf{W}^1 \boldsymbol{x})$ can be expressed component-wise as:

$$x_i^1 = \tanh\left(w_{i1}^1 x_1 + \cdots + w_{in}^1 x_n\right) = \tanh\left(\left(w_{ii}^1 + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{w_{ij}^1 x_j}{x_i}\right) x_i\right), \text{ for } i = 1, \ldots, n.$$

For the $k+1$-th layer, $i = 1, \ldots, n$, this expression can be generalized as:

$$x_i^{k+1} = \tanh\left(a_i^{k+1} x_i^k\right), \text{ where } \quad a_i^{k+1} = w_{ii}^{k+1} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{w_{ij}^{k+1} x_j^k}{x_i^k}. \tag{1}$$

The equation 1 follows the form of $\tanh(ax)$, as discussed in Section 3.2. According to Lemma 2, when $a > 1$, for an arbitrary initial value $x_0 > 0$ or $x_0 < 0$, the sequence $\{x_k\}$ defined by $x_{k+1} = \tanh(ax_k)$ converges to ξ_a or $-\xi_a$, respectively, as $k \to \infty$. This result indicates that the sequence converges to the fixed point ξ_a regardless of the initial value x_0. From the perspective of signal propagation in tanh-based FFNNs, this ensures that the activation values do not vanish as the network depth increases. Furthermore, by Proposition 3, if $a_i^k \leq 1$ for all $N \leq k \leq L$, where N is an arbitrarily chosen index close to L, the value of x_i^L approaches zero. Therefore, to satisfy condition (i), a_i^k remains close to 1, and the inequality $a_i^k \leq 1$ does not hold for all $N \leq k \leq L$.

Proposed Weight Initialization The proposed initial weight matrix is defined as $\mathbf{W}^\ell = \mathbf{D}^\ell + \mathbf{Z}^\ell \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$, where $\mathbf{D}_{i,j}^\ell = 1$ if $i \equiv j \pmod{N_{\ell-1}}$, 0 otherwise, and $\mathbf{Z}^\ell$ is a noise matrix drawn from $\mathcal{N}(0, \sigma_z^2)$, where σ_z is set to $\alpha/\sqrt{N^{\ell-1}}$ and $\alpha = 0.085$. Then a_i^{k+1} follows the distribution:

$$a_i^{k+1} \sim \mathcal{N}\left(1,\, \sigma_z^2 + \sigma_z^2 \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{x_j^k}{x_i^k}\right)^2\right). \tag{2}$$

According to Equation 2, a_i^{k+1} follows a Gaussian distribution with a mean of 1. Additionally, if x_i^k becomes small relative to other elements in $\boldsymbol{x}^k$, the variance of the distribution increases. Consequently, the probability that the absolute value of x_i^{k+1} exceeds that of x_i^k becomes higher. Figure 1 (b) shows that activation values maintain consistent scales in deeper layers.

3.3 Preventing activation saturation via appropriate σ_z tuning

In this section, we determine the appropriate value of α in $\sigma_z = \alpha/\sqrt{N_{\ell-1}}$ that satisfies condition (ii). Condition (ii) is motivated by normalization methods (Ioffe, 2015; Ba, 2016). Firstly,

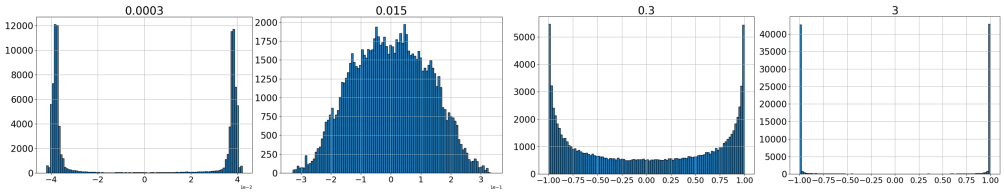


Figure 3: Activation values in the 1000th layer with 32 nodes per hidden layer, using the proposed initial weight matrix for $\sigma_z = 0.0003$, 0.015, 0.3, and 3. The 3000 input data uniformly distributed in $[-1, 1]$.

we experimentally investigated the impact of σ_z on the scale of the activation values. As shown in Figure 2, increasing $\sigma_z = \alpha/\sqrt{N_{\ell-1}}$ broadens the activation range in each layer, while decreasing σ_z narrows it.

When σ_z is Large Setting σ_z to a large value can lead to saturation. If σ_z is too large, Equation (2) implies that the likelihood of a_i^k deviating significantly from 1 increases. This increases the likelihood of activation values being bounded by $\xi_{1+\epsilon}$ in sufficiently deep layers, as stated in Corollary 4. Consequently, in deeper layers, activation values are less likely to approach zero and tend to saturate toward specific values. Please refer to the Figure 3 for the cases where $\sigma_z = 0.3$ and 3.

When σ_z is Small If σ_z is too small, Equation 2 implies that the distribution of a_i^k has a standard deviation close to zero. Consequently, x_i^{k+1} can be approximated as the result of applying $\tanh(x)$ to $x_i > 0$ repeatedly for a finite number of iterations, k. Since $\tanh'(x)$ decreases for $x \geq 0$, the values resulting from finite iterations eventually saturate. Plese refer to the Figure 3 when $\sigma_z = 0.0003$.

For these reasons, we experimentally determined an optimal σ_z that balances between being too large or too small. As shown in Figure 3, $\sigma_z = 0.015$ maintains an approximately normal activation distribution without collapse (See Appendix B.1). Considering the number of hidden layer nodes, we set $\sigma_z = \alpha/\sqrt{N^{\ell-1}}$ with $\alpha = 0.085$. Experimental results for solving the Burgers' equation using PINNs with varying σ_z are provided in Appendix C.3.

4 Experiments

In this section, we conduct a series of experiments to validate the proposed weight initialization method. In Section 4.1, we evaluate the performance of FFNNs with tanh activation function on benchmark datasets. In Section 4.2, we solve PDEs using Physics-Informed Neural Networks (PINNs). Both experiments evaluated the proposed method's robustness to network size and dataset efficiency.

Table 1: Validation accuracy and loss are presented for FFNNs with varying numbers of nodes $(2, 8, 32, 128)$, each with 20 hidden layers using tanh activation function. All models were trained for 20 epochs, and the highest average accuracy and lowest average loss, computed from 10 runs, are presented. The better-performing method is highlighted in bold when comparing different initialization methods under the same experimental settings.

Dataset	Method	2 Nodes		8 Nodes		32 Nodes		128 Nodes	
		Accuracy	Loss	Accuracy	Loss	Accuracy	Loss	Accuracy	Loss
MNIST	Xavier	49.78	1.632	68	0.958	91.67	0.277	95.45	0.154
	Proposed	**62.82**	**1.185**	**77.95**	**0.706**	**92.51**	**0.255**	**96.12**	**0.134**
FMNIST	Xavier	42.89	1.559	68.55	0.890	81.03	0.533	86.2	0.389
	Proposed	**51.65**	**1.324**	**71.31**	**0.777**	**83.06**	**0.475**	**87.12**	**0.359**
CIFAR-10	Xavier	32.82	1.921	43.51	1.608	48.62	1.473	47.58	1.510
	Proposed	**38.16**	**1.780**	**47.04**	**1.505**	**48.80**	**1.463**	**48.51**	**1.471**
CIFAR-100	Xavier	10.87	4.065	18.53	3.619	23.71	3.301	23.83	3.324
	Proposed	**15.22**	**3.818**	**23.07**	**3.350**	**24.93**	**3.237**	**24.91**	**3.240**

Table 2: Validation accuracy and loss are presented for FFNNs with varying numbers of layers $(10, 50, 100)$, each with 64 number of nodes using the tanh activation function. All models were trained for 40 epochs, and the highest average accuracy and lowest average loss, computed from 10 runs, are presented.

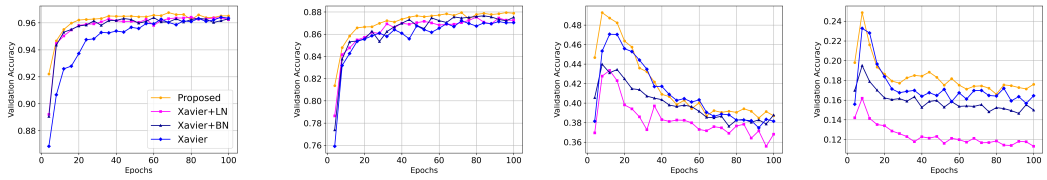
Dataset	Method	3 Layers		10 Layers		50 Layers		100 Layers	
		Accuracy	Loss	Accuracy	Loss	Accuracy	Loss	Accuracy	Loss
MNIST	Xavier	95.98	0.130	96.55	0.112	96.57	0.123	94.08	0.194
	Proposed	**96.32**	**0.123**	**97.04**	**0.102**	**96.72**	**0.109**	**96.06**	**0.132**
FMNIST	Xavier	85.91	0.401	88.73	0.319	87.72	0.344	83.41	0.463
	Proposed	**86.51**	**0.379**	**89.42**	**0.305**	**88.51**	**0.324**	**86.01**	**0.382**
CIFAR-10	Xavier	42.91	1.643	48.39	1.468	47.87	1.474	46.71	1.503
	Proposed	**45.05**	**1.588**	**48.41**	**1.458**	**48.71**	**1.461**	**48.96**	**1.437**
CIFAR-100	Xavier	19.10	3.628	22.73	3.400	24.27	3.283	20.32	3.515
	Proposed	**19.30**	**3.609**	**23.83**	**3.309**	**25.07**	**3.190**	**24.41**	**3.234**

4.1 Classification Task

Experimental Setting To evaluate the effectiveness of the proposed weight initialization method, we conduct experiments on the MNIST, Fashion MNIST (FMNIST), CIFAR-10, and CIFAR-100 (Krizhevsky & Hinton, 2009) datasets with the Adam optimizer. All experiments are conducted with a batch size of 64 and a learning rate of 0.0001. Fifteen percent of the total dataset is allocated for validation. The experiments were conducted in TensorFlow without skip connections and learning rate decay in any of the experiments.

Width Independence in Classification Task We evaluate the proposed weight initialization method in training tanh FFNNs, focusing on its robustness to variations in network width. Four tanh FFNNs are designed, each with 20 hidden layers, and with $2, 8, 32$, and 128 nodes per hidden layer, respectively. In Table 1, for both the MNIST and Fashion MNIST datasets, the network with 128 nodes achieves the highest accuracy and lowest loss when our proposed method is employed. However, for the CIFAR-10 and CIFAR-100 datasets, the network with 32 nodes yields the highest accuracy and lowest loss when employing the proposed method. In summary, Proposed method demonstrates robustness to variations in the number of nodes in tanh FFNNs. Detailed experimental results are provided in Appendix B.2.

Depth Independence in Classification Task The expressivity of neural networks is known to increase exponentially with depth, enabling strong generalization performance (Poole et al., 2016; Raghu et al., 2017). To evaluate the robustness of the proposed weight initialization method to variations in network depth, we conduct experiments on deep FFNNs with tanh activation functions. Specifically, we construct four tanh FFNNs, each with 64 nodes per hidden layer and 3, 10, 50, or 100 hidden layers. respectively. In Table 2, for both the MNIST and Fashion MNIST datasets, the network with 10 hidden layers achieves the highest accuracy and lowest loss when our proposed method is employed. Both initialization methods showed lower performance in networks with 3 layers compared to those with more layers. Moreover, for complex datasets like CIFAR-10 and CIFAR-100, the proposed method demonstrated improved performance when training deeper networks.



(a) MNIST (b) Fashion MNIST (c) CIFAR10 (d) CIFAR100

Figure 4: Validation accuracy for a tanh FFNN with 50 hidden layers (32 nodes each). Xavier + BN and Xavier + LN represent Xavier initialization with Batch Normalization or Layer Normalization applied every 5 layers, respectively.

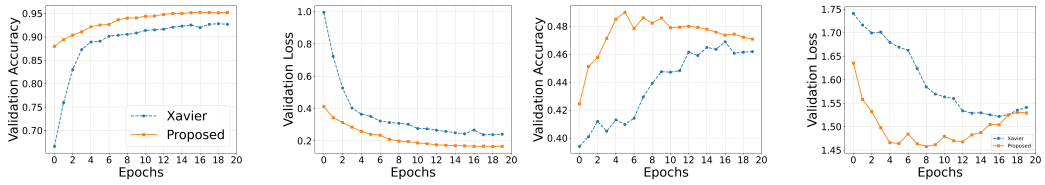
Table 3: Validation accuracy and loss for a 10-layer FFNN (64 nodes per layer) trained on datasets of sizes 10, 20, 30, 50, and 100. Results show the highest average accuracy and lowest average loss over 5 runs after 100 epochs.

Dataset	Method	10		20		30		50		100	
		Accuracy	Loss	Accuracy	Loss	Accuracy	Loss	Accuracy	Loss	Accuracy	Loss
MNIST	Xavier	31.13	2.281	35.03	2.078	45.05	1.771	58.45	1.227	64.02	1.139
	Xavier + BN	22.46	2.267	33.73	2.053	37.13	2.042	39.78	1.944	57.51	1.464
	Xavier + LN	28.52	2.411	41.54	1.796	41.94	1.886	54.97	1.362	65.11	1.093
	Proposed	**37.32**	**2.204**	**46.79**	**1.656**	**48.60**	**1.645**	**61.54**	**1.131**	**68.44**	**1.043**
FMNIST	Xavier	36.16	2.320	41.69	1.814	53.86	1.459	64.53	1.140	63.58	1.048
	Xavier + BN	35.44	**2.136**	38.58	1.925	40.16	1.819	53.93	1.728	59.78	1.237
	Xavier + LN	34.94	2.362	37.90	1.793	53.27	1.470	59.50	1.198	62.01	1.073
	Proposed	**37.31**	2.217	**49.25**	**1.651**	**55.19**	**1.372**	**66.14**	**1.057**	**67.58**	**0.914**

Normalization Methods Xavier initialization is known to cause vanishing gradients and activation problems in deeper networks. These issues are known to be mitigated by employing Batch Normalization (BN) or Layer Normalization (LN) in the network. Therefore, we compared the proposed method with Xavier, Xavier with BN, and Xavier with LN. To validate the effectiveness of normalization, we conducted experiments using a sufficiently deep neural network with 50 hidden layers. As shown in Figure 4, for datasets with relatively fewer features, such as MNIST and FMNIST, Xavier with normalization converges faster than Xavier. However, for feature-rich datasets like CIFAR-10 and CIFAR-100, the accuracy of Xavier with normalization is lower than that of Xavier. Normalization typically incurs a 30% computational overhead, and additional hyperparameter tuning is required to determine which layers should apply normalization. In contrast, the proposed method achieves the best performance among all four approaches across all datasets, without the need for normalization.

Dataset Efficiency in Classification Task Based on the results in Table 2, we evaluated data efficiency on a network with 50 hidden layers, each containing 64 nodes, where Xavier showed strong performance. As shown in Table 3, the highest average accuracy and lowest average loss over 5 runs after 100 epochs are presented for dataset sizes of $10, 20, 30, 50$, and 100. The proposed method achieved the best performance across all dataset sizes.

Non-uniform Hidden Layer Dimensions We validate the performance of the proposed initialization in networks where hidden layer dimensions are not uniform. As shown in Figure 5, the network consists of 60 hidden layers, where the number of nodes alternates between 32 and 16 in each layer. We demonstrate improved performance in terms of both loss and accuracy across all epochs on the MNIST and CIFAR-10 datasets. Additionally, Appendix B.3 presents experiment on network with larger differences in the number of nodes. Motivated by these results, Appendix B.3 explores autoencoders with significant variations in hidden layer dimensions.



(a) MNIST Accuracy (b) MNIST Loss (c) CIFAR10 Accuracy (d) CIFAR10 Loss

Figure 5: Validation accuracy and loss for a tanh FFNN with 60 hidden layers, where the number of nodes alternates between 32 and 16 across layers, repeated 30 times. The model was trained for 20 epochs on the MNIST and CIFAR-10 datasets.

4.2 Physics-Informed Neural Networks

Xavier initialization is the most commonly employed method for training PINNs (Jin et al., 2021; Son et al., 2023; Yao et al., 2023; Gnanasambandam et al., 2023). In this section, we experimentally

show that the proposed method is more robust to variations in network size and achieves better data efficiency compared to Xavier initialization with or without normalization methods.

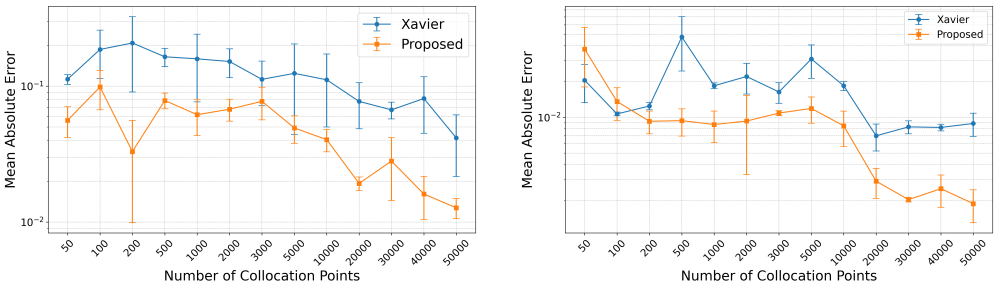
Experimental Setting All experiments on Physics-Informed Neural Networks (PINNs) use full-batch training with a learning rate of 0.001. In this section, we solve the Allen-Cahn, Burgers, Diffusion, and Poisson equations using a tanh FFNN-based PINN with 20,000 collocation points. Details on the PDEs are provided in Appendix C.1.

Network Size Independence in PINNs We construct eight tanh FFNNs, each with 16 nodes per hidden layer and $5, 10, 20, 30, 40, 50, 60,$ or 80 hidden layers. As shown in Table 4, for the Allen-Cahn and Burgers' equations, Xavier+BN and Xavier+LN achieve the lowest loss at a network depth of 30. However, their loss gradually increases as the depth grows. In contrast, the proposed method achieves the lowest loss at depths of 50 and 60, respectively, maintaining strong learning performance even in deeper networks. For the Diffusion and Poisson equations, Xavier+LN achieves the lowest loss at depths of 5 and 10, respectively. While all methods show increasing loss as network depth increases, the proposed method consistently maintains lower loss in deeper networks. Similar results are observed with 32 nodes, double the previous size. Across all tested network sizes and PDEs, the proposed method consistently achieves the lowest loss.

Table 4: A PINN loss is presented for FFNNs with varying numbers of layers $(5, 10, 20, 30, 40, 50, 60, 80)$ using the tanh activation function. The top table shows results with 16 nodes per layer, and the bottom table shows results with 32 nodes per layer. All models were trained for 300 iterations using Adam and 300 iterations using L-BFGS. The median PINN loss from the final iteration for the Burgers and Allen–Cahn equations, computed over 5 runs, is presented.

Allen-Cahn (16 Nodes)	5	10	20	30	40	50	60	80
Xavier	9.58e-04	8.16e-04	7.61e-04	1.06e-03	1.1e-03	1.24e-03	3.55e-03	1.81e-03
Xavier + BN	1.42e-03	8.17e-04	8.56e-04	7.07e-04	7.77e-04	8.87e-04	9.11e-04	2.15e-03
Xavier + LN	6.29e-01	1.77e-03	6.98e-04	1.27e-03	1.82e-03	6.65e-01	3.29e-01	5.86e-01
Proposed	**9.21e-04**	**7.29e-04**	**5.76e-04**	**5.29e-04**	**5.37e-04**	**4.03e-04**	**4.73e-04**	**5.77e-04**
Burgers (16 Nodes)	5	10	20	30	40	50	60	80
Xavier	6.97e-03	1.11e-02	7.9e-03	9.71e-03	2.45e-02	2.65e-02	6.5e-02	5.71e-02
Xavier + BN	8.07e-03	7.72e-03	6.24e-03	1.70e-02	1.50e-02	1.85e-02	2.91e-02	6.84e-02
Xavier + LN	3.89e-02	1.88e-02	9.48e-03	9.28e-03	2.46e-02	3.30e-02	6.91e-02	4.42e-02
Proposed	**6.19e-03**	**5.08e-03**	**5.28e-03**	**9.31e-04**	**3.56e-03**	**8.27e-04**	**3.43e-04**	**2.05e-03**
Diffusion (16 Nodes)	5	10	20	30	40	50	60	80
Xavier	2.52e-03	4.82e-03	9.69e-03	1.33e-02	2.08e-02	1.50e-02	2.92e-02	7.24e-02
Xavier + BN	2.89e-03	5.77e-03	1.05e-02	9.65e-03	2.76e-02	1.07e-02	9.07e-03	1.43e-02
Xavier + LN	1.72e-03	6.10e-03	8.04e-03	9.48e-03	2.14e-02	7.59e-03	2.05e-02	2.21e-02
Proposed	**9.14e-04**	**2.59e-03**	**2.40e-03**	**1.01e-03**	**1.97e-03**	**1.21e-03**	**1.12e-03**	**1.91e-03**
Poisson (16 Nodes)	5	10	20	30	40	50	60	80
Xavier	1.52e-02	2.87e-02	1.28e-01	9.82e-02	1.15e-01	1.37e-01	1.82e-01	2.55e-01
Xavier + BN	1.62e-02	2.02e-02	8.72e-02	1.12e-01	2.45e-01	9.85e-02	1.00e-01	1.34e-01
Xavier + LN	5.39e-01	4.40e-02	1.34e-01	3.91	2.52e+02	2.58	9.79e+02	nan
Proposed	**1.37e-02**	**1.70e-02**	**4.62e-02**	**2.43e-02**	**3.75e-02**	**4.03e-02**	**6.07e-02**	**6.01e-02**

Allen-Cahn (32 Nodes)	5	10	20	30	40	50	60	80
Xavier	3.13e-01	5.03e-02	3.64e-03	2.37e-03	4.03e-03	5.27e-03	1.73e-02	6.94e-01
Xavier + BN	4.05e-01	8.85e-04	8.41e-04	7.82e-04	9.97e-04	6.80e-04	9.34e-04	6.94e-01
Xavier + LN	3.31e-01	2.10e-03	5.99e-04	6.71e-04	1.49e-03	1.29e-03	3.31e-02	6.93e-01
Proposed	**1.04e-03**	**6.92e-04**	**5.34e-04**	**4.26e-04**	**3.31e-04**	**3.52e-04**	**3.85e-04**	**5.96e-04**
Burgers (32 Nodes)	5	10	20	30	40	50	60	80
Xavier	1.12e-02	3.53e-03	2.72e-03	1.81e-03	7.60e-03	8.56e-03	9.86e-03	1.66e-01
Xavier + BN	5.88e-03	**1.04e-03**	1.79e-03	2.80e-03	5.95e-03	3.66e-02	6.60e-02	1.66e-01
Xavier + LN	4.31e-02	1.21e-02	1.88e-03	7.22e-03	5.54e-03	8.46e-03	9.04e-03	4.86e-02
Proposed	**4.14e-03**	4.11e-03	**1.58e-03**	**1.29e-03**	**7.96e-04**	**5.85e-04**	**9.80e-04**	**1.47e-03**
Diffusion (32 Nodes)	5	10	20	30	40	50	60	80
Xavier	1.69e-03	6.85e-03	7.63e-03	4.50e-03	8.98e-03	5.67e-03	6.33e-01	1.59
Xavier + BN	1.68e-03	2.66e-03	1.08e-02	6.00e-03	8.58e-03	6.60e-03	5.66e-02	1.69e+02
Xavier + LN	8.16e-04	2.85e-03	8.46e-03	4.57e-03	9.40e-03	1.04e-02	2.42e-01	1.67e+02
Proposed	**2.89e-04**	**8.03e-04**	**5.25e-04**	**5.07e-04**	**5.33e-04**	**6.17e-04**	**9.80e-04**	**1.53e-03**
Poisson (32 Nodes)	5	10	20	30	40	50	60	80
Xavier	1.09e-02	1.33e-02	3.13e-02	7.69e-02	6.72e-02	8.90e-02	9.68e+02	1.46e+02
Xavier + BN	1.14e-02	1.47e-02	2.68e-02	3.55e-02	8.25e-02	8.97e-02	4.50e-02	7.75e-01
Xavier + LN	2.36e-02	2.18e-02	3.07e-02	3.85e-01	1.40	4.69	2.60	6.14
Proposed	**9.63e-03**	**8.29e-03**	**1.41e-02**	**1.88e-02**	**1.65e-02**	**1.85e-02**	**1.73e-02**	**3.59e-02**



(a) Burgers Equation

(b) Diffusion Equation

Figure 6: Mean absolute error between the exact solution and PINN-predicted solution with varying numbers of collocation points. The FFNN has 30 hidden layers (32 nodes each) and is trained for 300 iterations using Adam followed by 300 iterations using L-BFGS. The results are averaged over 5 experiments.

Data Efficiency Based on the results in Table 4, we evaluated data efficiency on a network with 30 hidden layers, each containing 32 nodes, where Xavier achieved the lowest PINN loss. As shown in Figure 6, for the Burgers equation, the Mean Absolute Error (MAE) of the proposed initialization shows a clear difference compared to Xavier initialization across varying numbers of collocation points. In contrast, for the Diffusion equation, the difference in MAE between the two methods becomes more pronounced when the number of collocation points exceeds $20,000$. Additionally, Figure 16 illustrates that increasing the number of collocation points enables PINNs with the proposed initialization to predict solutions with lower absolute error. For detailed experiments on the Burgers equation, please refer to Appendix C.4.

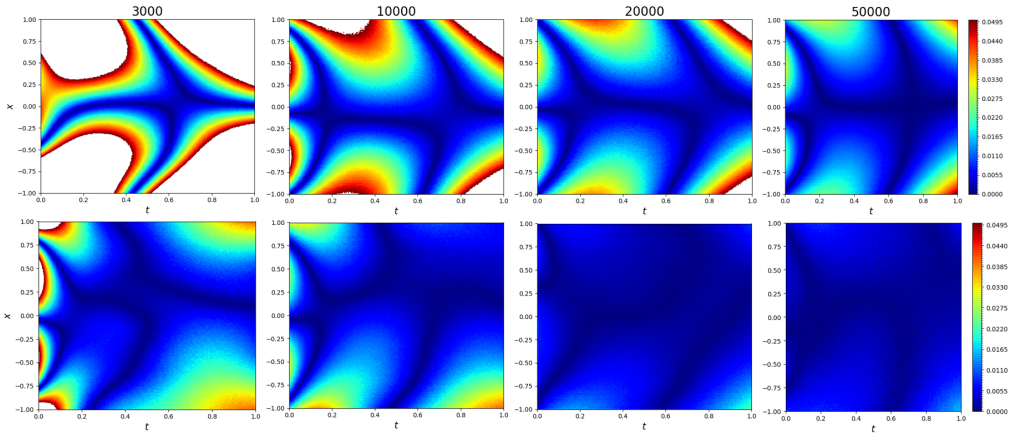


Figure 7: Absolute error between the exact solution and the PINN-predicted solution for the Diffusion equation with varying numbers of collocation points $(3000, 10000, 20000, 50000)$ using **(upper row)** Xavier and **(lower row)** the proposed initialization. The FFNN has 30 hidden layers (32 nodes each) and is trained for 300 iterations using Adam followed by 300 iterations using L-BFGS. The color bar ranges from 0 to 0.05, with values outside this range shown in white.

5 Conclusion

In this study, we proposed a novel weight initialization method for tanh FFNNs, grounded in the theoretical analysis of fixed points of the $\tanh(ax)$ function. The proposed method is experimentally demonstrated to achieve robustness to variations in network size without normalization methods and to exhibit improved data efficiency. With improved data efficiency and stable performance across varying network configurations without normalization, our method reduces the time and effort required for training large datasets and searching for optimal network architectures.

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A PROOFS OF THE THEORETICAL RESULTS

A.1 PROOF OF LEMMA 1

Proof. We define $g(x) = \tanh(ax) - x$. Since $g(x)$ is continuous, and $g(-M) > 0$, $g(M) < 0$ for a large real number $M \in \mathbb{R}^+$, the Intermediate Value Theorem guarantees the existence of a point x such that $g(x) = 0$.

First, consider the case $0 < a \leq 1$. Since $0 < a \leq 1$, the derivative $g'(x) = a \cdot \text{sech}^2(ax) - 1$ satisfies $-1 \leq g'(x) \leq a - 1 < 0$ for all x. Hence, $g(x)$ is strictly decreasing and therefore $g(x)$ has the unique root. At $x = 0$, $\phi(0) = \tanh(a \cdot 0) = 0$. Hence, $x = 0$ is the unique fixed point.

Let us consider the case $a > 1$. For $0 < x \ll 1$, $\tanh(ax) - x \approx (a-1)x$. Since $a > 1$, $\tanh(ax) - x > 0$. On the other hand, since $|\tanh(ax)| < 1$ for all x,

$$\lim_{x\to\infty}[-1 - x] \leq \lim_{x\to\infty}[\tanh(ax) - x] \leq \lim_{x\to\infty}[1 - x].$$

By the squeeze theorem, $\lim_{x\to\infty}[\tanh(ax) - x] = -\infty$. By the intermediate value theorem, therefore, there exists at least one $x > 0$ such that $\tanh(ax) = x$. To establish the uniqueness of the positive fixed point, we investigate the derivative $g'(x) = a\,\text{sech}^2(ax) - 1$. We find the critical points to be $x = \pm\frac{1}{a}\sec^{-1}(\frac{1}{\sqrt{a}})$. It is straightforward to see that $g'(x) > 0$ in $\left(-\frac{1}{a}\sec^{-1}(\frac{1}{\sqrt{a}}), \frac{1}{a}\sec^{-1}(\frac{1}{\sqrt{a}})\right)$ and $g'(x) < 0$ in $\mathbb{R}\backslash\left(-\frac{1}{a}\sec^{-1}(\frac{1}{\sqrt{a}}), \frac{1}{a}\sec^{-1}(\frac{1}{\sqrt{a}})\right)$. i.e. $g(x) = 0$ has exactly two fixed points. Because $g(x)$ is an odd function, if x^* is a solution, then $-x^*$ is also a solution. Thus, for $a > 1$, there exists a unique positive fixed point if $x > 0$ and a unique negative fixed point if $x < 0$. □

A.2 PROOF OF LEMMA 2

Proof. (1) Since $(\tanh(ax))' = a\,\text{sech}^2(ax) < 1$ for all $x > 0$, it holds that $x_{n+1} = \phi_a(x_n) < x_n$ for all $n \in \mathbb{N}$. Thus the sequence $\{x_n\}_{n=1}^{\infty}$ is decreasing. Since $x_n > 0$ for all $n \in \mathbb{N}$, by the monotone convergence theorem, it converges to the fixed point $x^* = 0$.
(2) Let $x_0 < \xi_a$. Since $\phi'(x)$ decreasing for $x \geq 0$, with $\phi'(0) > 1$ and ξ_a is the unique fixed point for $x > 0$, it holds that $x_n < x_{n+1} < \xi_a$ for all $n \in \mathbb{N}$. Thus, by the monotone convergence theorem, the sequence converges to the fixed point ξ_a. The proof is similar when $x_0 > \xi_a$. □

A.3 PROOF OF PROPOSITION 3

Proof. Set $N = \max\{n | a_n > 1\}$. Define the sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ such that $b_n = c_n = a_n$ for $n \leq N$, with $b_n = 0$ and $c_n = 1$ for $n > N$. Suppose that $\{\hat{\Phi}_m\}_{m=1}^{\infty}$ and $\{\tilde{\Phi}_m\}_{m=1}^{\infty}$ are sequences of functions defined as for each $m \in \mathbb{N}$

$$\hat{\Phi}_m = \phi_{b_m} \circ \phi_{b_{m-1}} \circ \cdots \circ \phi_{b_1}, \quad \tilde{\Phi}_m = \phi_{c_m} \circ \phi_{c_{m-1}} \circ \cdots \circ \phi_{c_1}.$$

Then, the inequality $\hat{\Phi}_m \leq \Phi_m \leq \tilde{\Phi}_m$ holds for all m. By Lemma 1, for any $x \geq 0$, we have $\lim_{m\to\infty}\hat{\Phi}_m = 0$ and $\lim_{m\to\infty}\tilde{\Phi}_m = 0$. Therefore, the Squeeze Theorem guarantees that $\lim_{m\to\infty}\Phi_m(x) = 0$. □

A.4 PROOF OF COROLLARY 4

Proof. Set $N = \max\{n \mid a_n < 1 + \epsilon\}$. Define the sequence $\{b_n\}_{n=1}^{\infty}$ such that $b_n = a_n$ for $n \leq N$, and $b_n = 1 + \epsilon$ for $n > N$. The remainder of the proof is analogous to the proof of Proposition 3. □

B CLASSIFIACATION TASKS

B.1 ACTIVATION DISTRIBUTION FOR NORMALLY DISTRIBUTED INPUT DATA.

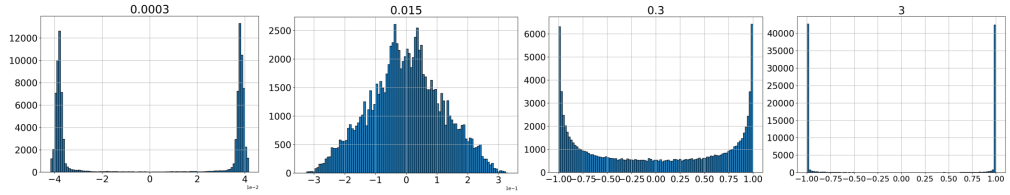


Figure 8: Activation values in the 1000th layer with 32 nodes per hidden layer, using the proposed initial weight matrix for $\sigma_z = 0.0003$, 0.015, 0.3, and 3. The 3000 input data follow a standard normal distribution ($\mathcal{N}(0, 1)$).

B.2 WIDTH INDEPENDENCE IN CLASSIFICATION TASKS

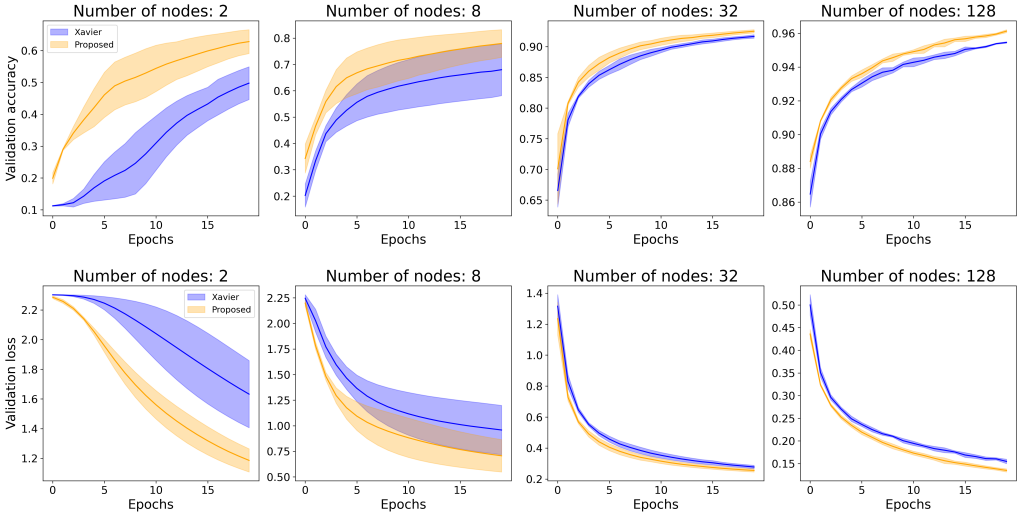


Figure 9: Validation accuracy and loss are presented for tanh FFNNs with varying numbers of nodes $(2, 8, 32, 128)$, each with 20 hidden layers. All models were trained for 20 epochs on the MNIST dataset, with 10 different random seeds.

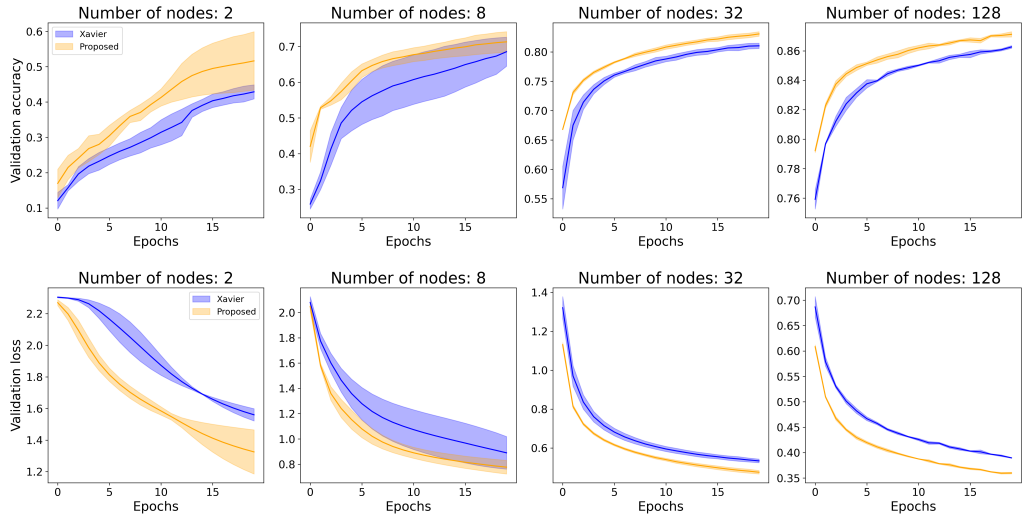


Figure 10: Validation accuracy and loss are presented for tanh FFNNs with varying numbers of nodes $(2, 8, 32, 128)$, each with 20 hidden layers. All models were trained for 20 epochs on the Fashion MNIST dataset, with 10 different random seeds.

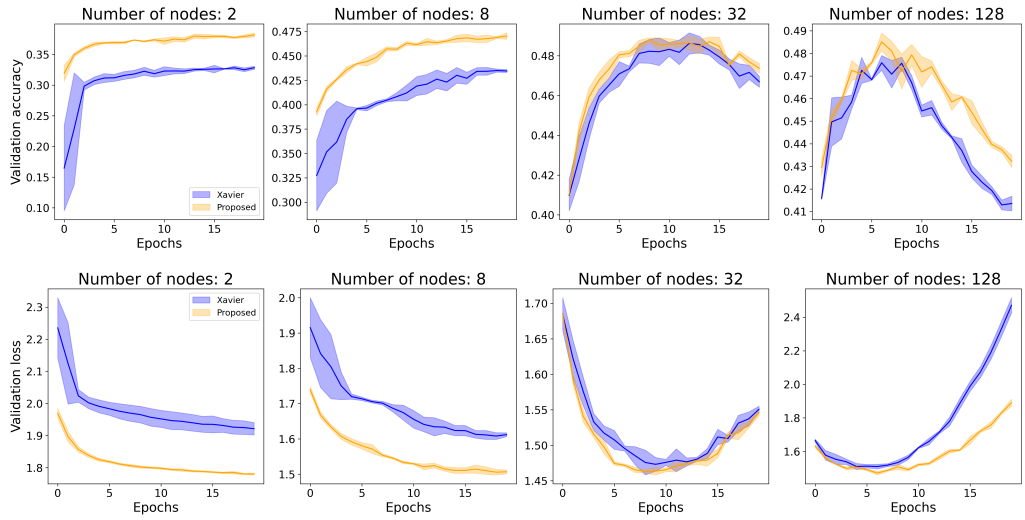


Figure 11: Validation accuracy and loss are presented for tanh FFNNs with varying numbers of nodes $(2, 8, 32, 128)$, each with 20 hidden layers. All models were trained for 20 epochs on the CIFAR-10 dataset, with 10 different random seeds.

B.3 Non-uniform Hidden Layer Dimensions

Tanh neural networks have been less commonly used compared to ReLU networks due to higher computational complexity, the vanishing gradient problem, and ReLU's superior empirical performance in many deep learning tasks. However, recent success of PINNs with tanh neural networks has led to a resurgence in their usage. In this section, we compare the performance of four methods on architectures that are generally challenging to train: (1) tanh activation with Xavier initialization, (2) tanh activation with the proposed initialization, (3) ReLU activation with He initialization + BN, and (4) ReLU activation with orthogonal initialization.

FFNN The experiments were conducted on an FFNN with alternating hidden layers of 16 and 4 nodes, repeated 50 times, over 100 epochs. The results are shown in Figure 13 (a). Both Xavier and

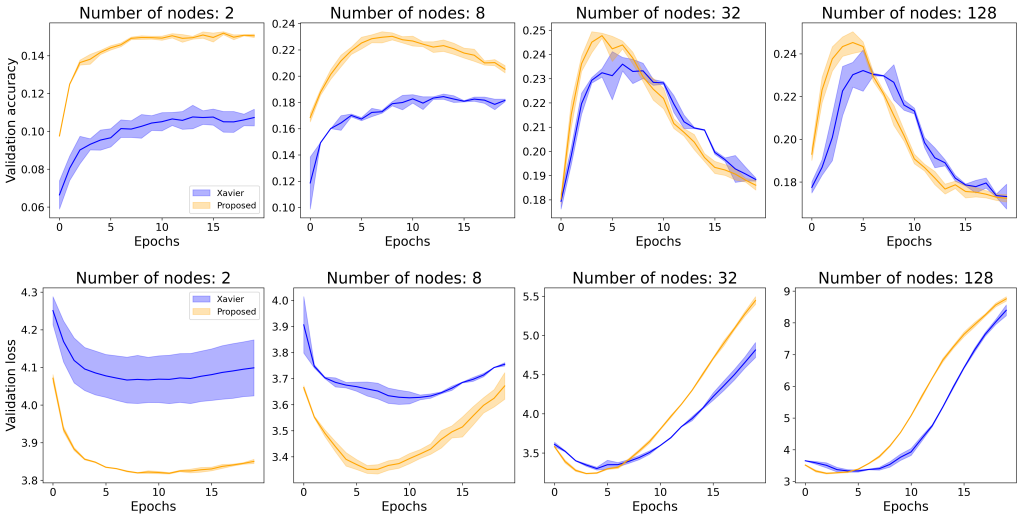
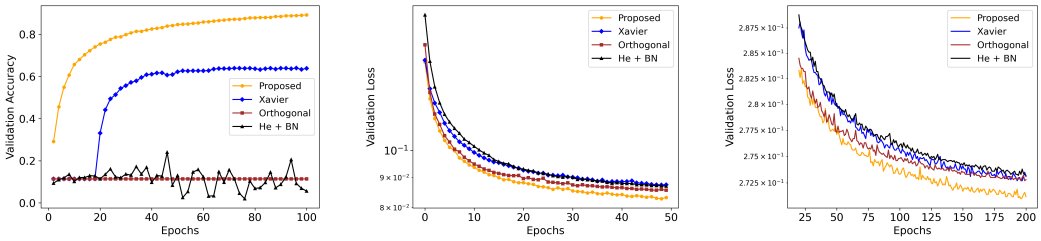


Figure 12: Validation accuracy and loss are presented for tanh FFNNs with varying numbers of nodes $(2, 8, 32, 128)$, each with 20 hidden layers. All models were trained for 20 epochs on the CIFAR-100 dataset, with 10 different random seeds.

the proposed method successfully trained the network, with the proposed method showing overall better performance. Given its strong performance despite significant differences in the number of nodes between hidden layers, we further tested the proposed method on autoencoders with large variations in layer sizes, as shown in Figure 13 (b).

Autoencoder The autoencoder architecture consists of an encoder and a decoder, both employing batch normalization and dropout (0.2) for regularization. The encoder compresses the input through layers of sizes 512, 256, 128, and finally maps to a latent space of 64 units. The decoder reconstructs the input by symmetrically expanding the latent space through layers of sizes 128, 256, and 512, followed by a final output layer with sigmoid activation. In Figure 13 (b), the model is trained on the MNIST dataset with a batch size of 256, while in (c), it is trained on the FMNIST dataset with a batch size of 512.



(a) FFNN (MNIST) (b) Autoencoder (MNIST) (c) Autoencoder (FMNIST)

Figure 13: **(a)** Validation loss for an FFNN with alternating hidden layers of 16 and 4 nodes, repeated 50 times, comparing four methods: Tanh with Xavier initialization, Tanh with the proposed initialization, ReLU with He initialization + BN, and ReLU with orthogonal initialization. **(b)** Validation loss for an autoencoder with encoder-decoder layers of 512, 256, 128, and 64 units, comparing the same four methods. **(c)** Same as (b), but on the FMNIST dataset.

C PHYSICS INFORMED NEURAL NETWORKS

C.1 PDE DETAILS

Allen-Cahn Equation The diffusion coefficient is set to $d = 0.01$. The initial condition is defined as $u(x, 0) = x^2 \cos(\pi x)$ for $x \in [-1, 1]$, with boundary conditions $u(-1, t) = -1$ and $u(1, t) = -1$, applied over the time interval $t \in [0, 1]$. The Allen-Cahn equation is expressed as:

$$\frac{\partial u}{\partial t} - d\frac{\partial^2 u}{\partial x^2} = -\frac{u^3 + u}{d}$$

where $u(x, t)$ represents the solution, d is the diffusion coefficient, and the nonlinear term $u^3 - u$ models the phase separation dynamics.

Burgers' Equation The Burgers' equation, a viscosity coefficient of $\nu = 0.01$ is employed. The initial condition is given by $u(x, 0) = -\sin(\pi x)$ for $x \in [-1, 1]$, with boundary conditions $u(-1, t) = 0$ and $u(1, t) = 0$ imposed for $t \in [0, 1]$. The Burgers' equation is given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ is the velocity field, and ν is the viscosity coefficient.

Diffusion Equation The diffusion equation includes a time-dependent source term and is defined over the spatial domain $x \in [-1, 1]$ and temporal interval $t \in [0, 1]$. The initial condition is specified as $u(x, 0) = \sin(\pi x)$, with Dirichlet boundary conditions $u(-1, t) = 0$ and $u(1, t) = 0$. The diffusion equation is expressed as:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = e^{-t}\left(\sin(\pi x) - \pi^2 \sin(\pi x)\right),$$

where $u(x, t)$ is the solution.

Poisson Equation The Poisson equation is defined over the spatial domain $x \in [0, 1]$ and $y \in [0, 1]$. The Poisson equation is expressed as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y),$$

where $u(x, y)$ is the solution, and $f(x, y)$ is the source term given by:

$$f(x, y) = 2\pi^2 \sin(\pi x)\sin(\pi y).$$

C.2 IMPACT OF ACTIVATION FUNCTION ON PINN

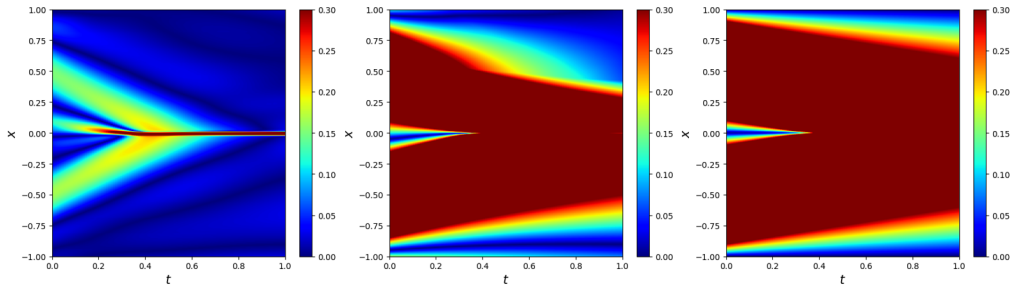
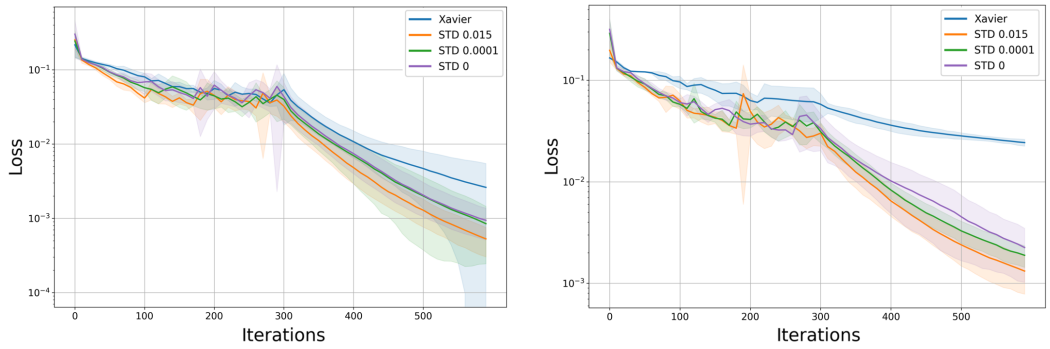


Figure 14: Absolute error for the Burgers' equation with varying activation functions (from left: tanh, ReLU, sigmoid). The FFNN has 30 hidden layers (32 nodes each) and is trained for 300 iterations using Adam followed by 300 iterations using L-BFGS.

C.3 σ_z FOR BURGERS' EQUATION



(a) Same dimension (b) Varying dimensions

Figure 15: Here, STD refers to σ_z. (a) shows the PINN loss for the Burgers' equation, using an FFNN with 30 layers and 32 nodes in each hidden layer. (b) shows the PINN loss for an FFNN with 30 layers, where the hidden layers alternate between 64 and 32 nodes, repeated 15 times. Each experiment was repeated 10 times with different random seeds.

C.4 ABSOLUTE ERROR FOR BURGERS' EQUATION

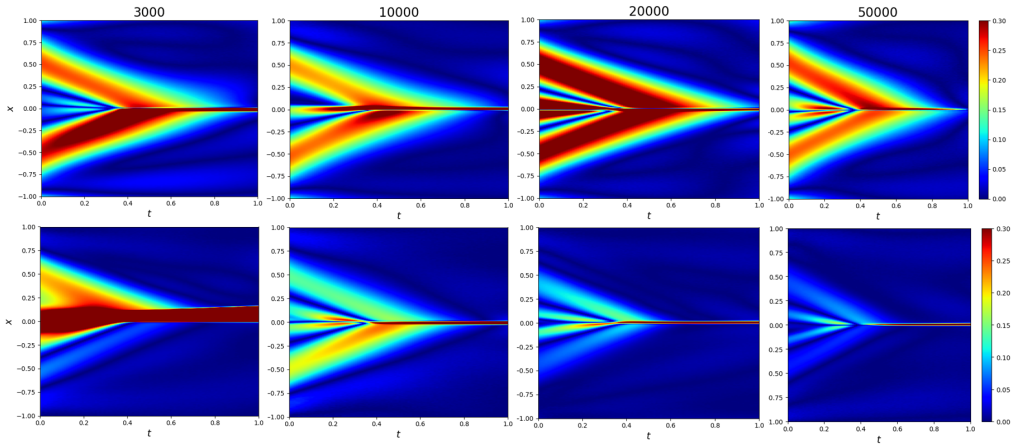


Figure 16: Absolute error between the exact solution and the PINN-predicted solution for the Burgers' equation with varying numbers of collocation points (3000, 10000, 20000, 50000) using **(upper row)** Xavier and **(lower row)** the proposed initialization. The FFNN has 30 hidden layers (32 nodes each) and is trained for 300 iterations using Adam followed by 300 iterations using L-BFGS. The color bar ranges from 0 to 0.3.