# Refined Dimension-Dependent Analysis for Private Convex Learning and Implications for Fine-Tuning

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## Abstract

Recent works have shown that large pretrained models can be privately fine-tuned to achieve performance approaching non-private models. A common theme in these results is the surprising observation that very large models can achieve favorable privacy-utility trade-offs. This seemingly contradicts known results on the model-size dependence of differentially private convex learning and raises the following research question: When does the performance of differentially private learning not degrade with increasing model size? We identify that the magnitudes of gradients projected onto subspaces is a key factor that determines performance. To precisely characterize this for private convex learning, we introduce a condition on the objective that we term *restricted Lipschitz continuity* and derive refined bounds for the excess empirical and population risks that are dimension-independent under additional conditions. We validate this condition in practice by privately fine-tuning language models and showing that gradients evaluated near a local optimum are mostly controlled by a few principal components. This behavior is similar to conditions under which we obtain dimension-independent bounds in convex settings and provides a possible explanation for recent successes in large-scale private fine-tuning.

## 1 Introduction

Recent works have shown that large publicly pretrained models can be differentially privately fine-tuned on small downstream datasets with performance approaching those attained by non-private models. In particular, past works showed that pretrained BERT [DCLT18] and GPT-2 [RNSS18, RWC+19] models can be fine-tuned to perform well for text classification and generation under a privacy budget of $\varepsilon \in [2, 6]$ [LTLH21, YNB+21]. More recently, it was shown that pretrained ResNets [HZRS16] and vision-Transformers [DBK+20] can be fine-tuned to perform well for ImageNet classification under single digit privacy budgets [DBH+22, MTKC22].

One key ingredient in these successes has been the use of large pretrained models with millions to billions of parameters, and these works generally highlight the importance of two phenomena: (i) large (pretrained) models tend to experience good privacy-utility trade-offs when fine-tuned, and (ii) the trade-off improves with the use of larger models. While the power of scale and pre-training has been demonstrated numerous times in non-private deep learning, one common wisdom in private learning had been that large models tend not to perform well. This intuition was rooted in results in differentially private convex optimization, which predicted that errors would scale proportionally with the dimension of the learning problem.

Consider the problem of differentially private convex *empirical risk minimization* (ERM) as an example. In this problem, we are given a dataset of $n$ examples $\mathcal{D} = \{s_j\}_{j=1}^n \in \mathcal{S}^n$, a convex body

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$\mathcal{K} \subset \mathbb{R}^d$, and the goal is to perform the optimization

$$\text{minimize}_{x \in \mathcal{K}} F(x; \mathcal{D}) = \frac{1}{n} \sum_{j=1}^{n} f(x; s_j)$$

subject to differential privacy, where $f : \mathcal{K} \times \mathcal{S} \to \mathbb{R}$ is some convex loss function. Past works have studied this problem and presented matching upper and lower bounds that are dimension-dependent under the usual Lipschitz (and smoothness) assumptions on the objective [BST14a, CMS11]. These dimension-dependent results suggest that the performance of differentially private convex ERM inevitably degrades with increasing problem dimension — for the worst case functions within the prescribed class of convex losses.

To reconcile the conflict between the dimension-dependent lower bounds for private convex learning and the empirical success of large-scale private fine-tuning, we study the following question both in theory and in practice:

*When does the performance of differentially private learning (via DP-SGD) not degrade with increasing problem size and dimension?*

On the theory front, we identify that the standard dependence on the dimensionality of the ambient space can be replaced by the magnitudes of gradients projected onto subspaces of varying dimensions. We formalize this in a condition that we call *restricted Lipschitz continuity* and derive refined bounds for the excess empirical and population risks when loss functions obey this condition. We show that when the restricted Lipschitz coefficients decay rapidly, both the excess empirical and population risks become dimension-independent.

These theoretical results shed light on the recent success of large-scale private fine-tuning of language models. We empirically show that the gradients of language models during fine-tuning (evaluated near a local optimum) are mostly controlled by a few principle components — a behavior that is similar to conditions under which we obtain dimension-independent bounds for private convex ERM. This provides a possible explanation for the observation that densely fine-tuning need not necessarily have much worse performance than sparsely fine-tuning [LTLH21] and moreover suggests that DP-SGD can be adaptive to problems that are effectively low-dimensional under restricted Lipschitz continuity. We summarize our contributions below.

(1) We introduce a condition on the objective function that we term restricted Lipschitz continuity. This condition generalizes the usual Lipschitz continuity notion and gives rise to refined analyses when magnitudes of gradients projected onto diminishing subspaces decay rapidly.

(2) Under restricted Lipschitz continuity, we present refined bounds on the excess empirical and population risks for DP-SGD when optimizing convex objectives. These bounds generalize previous dimension-independent results [SSTT21] and are broadly applicable to cases where gradients are full rank but most coordinates only marginally influence the objective.

(3) Our theory shed light on recent successes of private fine-tuning of language models. We show that gradients near a local optimum obtained through fine-tuning mostly lie in the space spanned by a few principle components — a behavior similar to restricted Lipschitz continuity with decaying coefficients. This provides a possible explanation for recent successes in large-scale private fine-tuning.

## 2 Preliminaries

In this section, we first define the notation used throughout this work. We then state the problems of differentially private empirical risk minimization and differentially private stochastic convex optimization. Finally, we give a brief recap of differentially private stochastic gradient descent, existing dimension-dependent results in the literature, and the seemingly inconsistent results between theory for convex losses and recent empirical findings in language model fine-tuning.

**Notation.** For a non-negative integer $n \in \mathbb{N}_+$, define the shorthand $[n] = \{0, 1, \ldots, n\}$. For a vector $x \in \mathbb{R}^d$, denote its $\ell_2$-norm by $\|x\|_2$ (or $\|x\|$ when it's clear from context). Let $\mathbf{S}^d$ denote the set of $d \times d$ symmetric matrices, and $\mathbf{S}^d_+ \subset \mathbf{S}^d$ the positive semidefinite subset. Given $M \in \mathbf{S}^d$, let

$\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_d(M)$ denote its eigenvalues. Given $A \in \mathbf{S}_+^d$, let $\|x\|_A = (x^\top A x)^{1/2}$ denote the induced Mahalanobis norm. For scalar functions $f, g$, we say $f \lesssim g$ if $f \leq Cg$ for some numerical constant $C$.

### 2.1 Differentially Private Empirical Risk Minimization and Stochastic Convex Optimization

Before stating the theoretical problem of interest, we define the basic concepts of Lipschitz continuity, convexity, and approximate differential privacy.

**Definition 2.1** (Lipschitz Continuity)**.** We say a loss function $h : \mathcal{K} \to \mathbb{R}$ is $G$-Lipschitz with respect to the $\ell_2$ norm if for all $x, x' \in \mathcal{K}$, $|f(x) - f(x')| \leq G\|x - x'\|_2$.

**Definition 2.2** (Convexity)**.** We say a loss function $h : \mathcal{K} \to \mathbb{R}$ is convex if $h(\alpha x + (1-\alpha)y) \leq \alpha h(x) + (1-\alpha)h(y)$, for all $\alpha \in [0,1]$, and $x, y \in \mathcal{K}$.

**Definition 2.3** (Approximate Differential Privacy [DR+14])**.** We say a randomized algorithm $\mathcal{M}$ is $(\varepsilon, \delta)$-differentially private if for all neighboring datasets $\mathcal{D}$ and $\mathcal{D}'$ that differ by a single record and all events $\mathcal{O} \subset \mathrm{range}(\mathcal{M})$, the following expression holds

$$\Pr[\mathcal{M}(\mathcal{D}) \in \mathcal{O}] \leq \exp(\varepsilon) \Pr[\mathcal{M}(\mathcal{D}') \in \mathcal{O}] + \delta.$$

In this work, we study both *differentially private empirical risk minimization* (DP-ERM) for convex objectives and *differentially private stochastic convex optimization* (DP-SCO).

In DP-ERM for convex objectives, we are given a dataset $\mathcal{D} = \{s_j\}_{j \in [n]} \in \mathcal{S}^n$ of $n$ examples. Each per-example loss $f(\cdot; s)$ is convex over a convex body $\mathcal{K} \subseteq \mathbb{R}^d$ and $G$-Lipschitz. We aim to design an $(\varepsilon, \delta)$-DP algorithm that returns a solution $x^{\text{priv}}$ which approximately minimizes the empirical risk

$$F(x; \mathcal{D}) := \frac{1}{n} \sum_{s_j \in \mathcal{D}} f(x; s_j).$$

The term $F(x^{\text{priv}}; \mathcal{D}) - \min_{x \in \mathcal{K}} F(x; \mathcal{D})$ is referred to as the *excess empirical risk* associated with the solution $x^{\text{priv}}$.

In DP-SCO, we assume the per-example loss $f(\cdot; s)$ is convex and $G$-Lipschitz for all $s \in \mathcal{S}$, and we are given $n$ examples drawn i.i.d. from some underlying (unknown) distribution $\mathcal{P}$. The goal is to design an $(\varepsilon, \delta)$-DP algorithm which approximately minimizes the population risk

$$F(x; \mathcal{P}) := \mathop{\mathbb{E}}_{s \sim \mathcal{P}}[f(x; s)].$$

The term $F(x^{\text{priv}}; \mathcal{P}) - \min_{x \in \mathcal{K}} F(x; \mathcal{P})$ is referred to as the *excess population loss* of the solution $x^{\text{priv}}$. Slightly abusing terminology, we refer to the following quantities as the excess empirical and population risks for a randomized solution $x^{\text{priv}}$,

$$\mathbb{E}_{x^{\text{priv}}}\left[F(x^{\text{priv}}; \mathcal{D}) - \min_{x \in \mathcal{K}} F(x; \mathcal{D})\right], \quad \mathbb{E}_{x^{\text{priv}}}\left[F(x^{\text{priv}}; \mathcal{P}) - \min_{x \in \mathcal{K}} F(x; \mathcal{P})\right].$$

### 2.2 Differentially Private Stochastic Gradient Descent

*Differentially Private Stochastic Gradient Descent* (DP-SGD) [ACG+16, SCS13] is a popular algorithm for DP convex optimization. For the theoretical analysis, we study DP-SGD for *unconstrained* optimization. To facilitate analysis, we work with the $\ell_2$ regularized objective expressed as

$$F_\alpha(x; \mathcal{D}) = \frac{1}{n} \sum_{j=1}^{n} f(x; s_j) + \frac{\alpha}{2} \left\|x - x^{(0)}\right\|_2^2.$$

To optimize this objective, DP-SGD independently samples an example in each iteration and updates the solution by combining the gradient with an isotropic Gaussian whose scale is proportional to $G$, the Lipschitz constant of $f$. Algorithm 1 presents the pseudocode.

It is straightforward to show that Algorithm 1 satisfies differential privacy. The following lemma quantifies the overall privacy spending and builds on a long line of work on accounting the privacy loss of compositions [ACG+16, BBG18].

**Lemma 2.4** ([KLL21, Theorem 3.1])**.** *For $n \geq 10$, $\varepsilon < c_1 T/n^2$ and $1/2 \geq \delta > 0$. Let $\sigma \geq \frac{c_2 G \sqrt{T \log(1/\delta)}}{\varepsilon n}$ for some numerical constants $c_1$ and $c_2$. Then, DP-SGD (Algorithm 1) is $(\varepsilon, \delta)$-differential private.*

**Algorithm 1:** DP-SGD for optimizing regularized finite-sum objectives

**Input:** Initial iterate $x^{(0)}$, dataset $\mathcal{D} = \{s_j\}_{j\in[n]}$, per-step noise magnitude $\sigma$, total number of updates $T$.
**for** $t = 1, \cdots, T$ **do**
  $i_t \sim \text{Uniform}([n])$
  $x^{(t)} = x^{(t-1)} - \eta\Big(\nabla f(x^{(t-1)}; s_{i_t}) + \alpha(x^{(t-1)} - x^{(0)}) + G \cdot \zeta_t\Big), \quad \zeta_t \sim \mathcal{N}(0, \sigma^2 I_d)$
**end**
**Return:** $\overline{x} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T} x^{(t)}$.

### 2.3 Conflicting Results Between Convex Learning and Large-Scale Fine-Tuning

Some of the early works on bounding the excess empirical and population risks for privately optimizing convex objectives focused on a *constrained* optimization setup where iterates are explicitly projected onto a fixed bounded domain after each noisy gradient update. Results in this setting suggested that risks are inevitably dimension-dependent. For instance, it was shown that the excess empirical risk bound $\Theta(GD\sqrt{d\log(1/\delta)}n^{-1}\varepsilon^{-1})$ and excess population risk bound $\Theta(GD(n^{-1/2} + \sqrt{d\log(1/\delta)}n^{-1}\varepsilon^{-1}))$ are tight when privately optimizing convex $G$-Lipschitz objectives in a convex domain of diameter $D$ [BST14a].

However, recent empirical findings in large-scale language model fine-tuning demonstrated that very large models can be optimized with DP-SGD to achieve performance approaching non-private models [LTLH21, YNB+21]. Moreover, fine-tuning more parameters (and hence injecting more noise during training) does not necessarily hurt private fine-tuning performance [LTLH21]. These empirical observations suggest that the error of learning does not degrade substantially (or sometimes negligibly) with the increase in problem dimension, pointing towards a seeming conflict between theory and practice.

In the following, we set out to reconcile this discrepancy. In particular, Section 3 shows that better error dimension-dependence can be achieved for private convex optimization if the objective satisfies additional regularity conditions in addition to Lipschitz continuity. Section 4 verifies our theory and presents initial evidence that private language model fine-tuning exhibits the type of regularity that leads to dimension-independent bounds in our theoretical results.

## 3 Refined Dimension-Dependence Through Restricted Lipschitz Continuity

In this section, we formally introduce the restricted Lipschitz continuity condition and derive improved bounds for the excess empirical and population risks for DP convex optimization.

**Definition 3.1** (Restricted Lipschitz Continuity)**.** We say that the loss function $h : \mathcal{K} \to \mathbb{R}$ is restricted Lipschitz continuous with coefficients $G_0, \ldots, G_d$, if for all $k \in [d]$, there exists an orthogonal projection matrix $P_k$ with rank $k$ such that

$$\|(I - P_k)\nabla h(x)\|_2 \le G_k,$$

for all $x \in \mathcal{K}$ and all subgradients $\nabla h(x) \in \partial h(x)$.

Note that any $G$-Lipschitz function is also trivially restricted Lipschitz continuous with coefficients $G = G_0 = G_1 = \cdots = G_d$, since orthogonal projections never increase the $\ell_2$-norm of a vector (generally, it is easy to see that $G = G_0 \ge G_1 \ge G_2 \ge \cdots \ge G_d = 0$). On the other hand, we expect that a function which exhibits little growth in subspaces of dimension $k$ to have a restricted Lipschitz coefficient $G_{d-k}$ of almost 0.

Our bounds on DP convex optimization will characterize errors through the use of restricted Lipschitz coefficients. We now summarize the main assumptions before presenting the core theoretical results.

**Assumption 3.2.** *The per-example loss function $f(\cdot; s)$ is convex and $G$-Lipschitz continuous for all $s \in \mathcal{S}$. The average loss $F(\cdot; \mathcal{D})$ is restricted Lipschitz continuous with coefficients $\{G_k\}_{k=1}^{d}$.*

### 3.1 Bounds for Excess Empirical Loss

We present the main theoretical result on DP-ERM for convex objectives. The result consists of two pieces. (1) is a general bound that is applicable to any sequence of restricted Lipschitz coefficients. (2) specializes the previous bound to when the sequence of coefficients decays rapidly and demonstrates dimension-independent error scaling. To simplify notation in the following theorems, we let $G_m := 0$ for all integer $m > d$.

**Theorem 3.3** (Excess Empirical Loss)**.** *Let $\delta \in (0, \frac{1}{2}]$ and $\varepsilon \in (0, 10]$. Under Assumption 3.2, for any non-negative integer $k$, setting $T = \Theta(n^2 + d\log^2 d)$, $\sigma = \Theta\left(\frac{\sqrt{T\log(1/\delta)}}{n\varepsilon}\right)$, $\eta = \sqrt{\frac{D^2}{T \cdot G_0^2 \cdot k\sigma^2}}$ and $\alpha = \frac{1}{D}\sqrt{\sum_s s^2 2^s G_{2^{s-1}k}^2}$, DP-SGD (Algorithm 1) is $(\varepsilon, \delta)$-DP, and*

$$\mathbb{E}\left[F(\overline{x}; \mathcal{D}) - \min_x F(x; \mathcal{D})\right] \lesssim \frac{G_0 D\sqrt{k\log(1/\delta)}}{\varepsilon n} + D\sqrt{\sum_{s=1}^{\infty} s^2 2^s G_{2^{s-1}k}^2}, \tag{1}$$

*where $\|x^{(0)} - x^*\|_2 \le D$, $\overline{x}$ is the (random) output of DP-SGD (Algorithm 1), and the expectation is over the randomness of $\overline{x}$.*

*Moreover, if for some $c > 1/2$, we have $G_k \le \frac{G_0}{k^c}$ for all $k \in [d]$, minimizing the right hand side of (1) with respect to $k$ yields*

$$\mathbb{E}\left[F(\overline{x}; \mathcal{D}) - \min_x F(x; \mathcal{D})\right] \lesssim G_0 D \cdot \left(\frac{\sqrt{\log(1/\delta)}}{\varepsilon n}\right)^{2c/(1+2c)}. \tag{2}$$

We include a sketch of the main proof techniques in Section 3.3 and defer the full proof to Appendix A.

*Remark* 3.4. Consider DP-ERM for learning generalized linear models with convex losses. When the (empirical) data covariance is of rank $r < d$, the span of gradients $\text{span}(\{\nabla_x F(x)\})$ is also of rank $r$. Thus, the average loss is restricted Lipschitz continuous with coefficients where $G_{r'} = 0$ for all $r' > r$. Setting $k = r$ in (1) yields a excess empirical risk bound of order $O\left(G_0 D\sqrt{r \cdot \log(1/\delta)}\varepsilon^{-1}n^{-1}\right)$. This recovers a previous dimension-independent result up to constants [SSTT21].

The restricted Lipschitz continuity condition can be viewed as a generalized notion of rank. The result captured in (2) suggests that the empirical loss achieved by DP-SGD does not depend on the problem dimension if the sequence of restricted Lipschitz coefficients decays rapidly. We leverage these insights to build intuition for understanding private fine-tuning of language model in Section 4.

### 3.2 Bounds for Excess Population Loss

For DP-SCO, we make use of the *stability* of DP-SGD to bound its generalization error [BE02], following previous works [BFTT19, BFGT20, SSTT21]. The bound on the excess population loss follows from combining the bounds on the excess empirical risk and the generalization error.

**Theorem 3.5** (Excess Population Loss)**.** *Let $\delta \in (0, \frac{1}{2}]$, $\varepsilon \in (0, 10]$. Under Assumption 3.2, for any non-negative integer $k$, by setting $T = \Theta(n^2 + d\log^2 d)$, $\sigma = \Theta\left(\frac{\sqrt{T\log(1/\delta)}}{n\varepsilon}\right)$, $\eta = \sqrt{\frac{D^2}{T \cdot G_0^2 (T/n + k\sigma^2)}}$ and $\alpha = \frac{\sqrt{\sum_s s^2 2^s G_{2^{s-1}k}^2}}{D}$, DP-SGD (Algorithm 1) is $(\varepsilon, \delta)$-DP, and*

$$\mathbb{E}\left[F(\overline{x}; \mathcal{P}) - \min_x F(x; \mathcal{P})\right] \lesssim \frac{G_0 D}{\sqrt{n}} + \frac{G_0 D\sqrt{k\log(1/\delta)}}{\varepsilon n} + D\sqrt{\sum_{s=1}^{\infty} s^2 2^s G_{2^{s-1}k}^2},$$

*where $\|x^{(0)} - x^*\|_2 \le D$, $\overline{x}$ is the (random) output of DP-SGD (Algorithm 1), and the expectation is over the randomness of $\overline{x}$.*

*Moreover, if for some $c > 1/2$, we have $G_k \le \frac{G_0}{k^c}$ for all $k \in [d]$, minimizing with respect to $k$ yields*

$$\mathbb{E}\left[F(\overline{x}; \mathcal{P}) - \min_x F(x; \mathcal{P})\right] \lesssim \frac{G_0 D}{\sqrt{n}} + G_0 D \cdot \left(\frac{\sqrt{\log(1/\delta)}}{\varepsilon n}\right)^{2c/(1+2c)}.$$

We refer the reader to Appendix B for the full proof.

*Remark* 3.6. Our result on DP-SCO also recovers the DP-SCO rank-dependent result for learning generalized linear models with convex losses [SSTT21].

*Remark* 3.7. When $c > 1/2$, $\varepsilon = \Theta(1)$ and $\delta = 1/\mathrm{poly}(n)$, the population loss matches the (non-private) informational-theoretical lower bound $\Omega(G_0 D/\sqrt{n})$ [AWBR09].

### 3.3 Overview of Proof Techniques

The privacy guarantees in Theorems 3.3 and 3.5 follow directly from Lemma 2.4. It suffices to prove the utility guarantees. We give an outline of the main proof techniques and defer full proofs to the supplementary material. The following is a proof sketch for Theorem 3.3.

By convexity, the error term of SGD is upper bounded as follows

$$f_j(x^{(t)}) - f_j(x^*) \leq \nabla f_j(x^{(t)})^\top (x^{(t)} - x^*), \tag{3}$$

where $j \in [n]$ is the random index sampled at iteration $t$. By definition of $G_k$, we know that there is a $k$ dimensional subspace $U$ such that the gradient is small orthogonal to $U$. Naively, one decomposes the gradient $\nabla f_j(x^{(t)}) = g_1 + g_2$, where $g_1 \in U$ and $g_2 \in U^\perp$. Since $g_1$ lies in a $k$ dimensional subspace, we can follow existing arguments on DP-SGD to bound $g_1^\top (x^{(t)} - x^*)$. Unfortunately, this argument does not give a dimension-independent bound. Although $\|g_2\|_2 \leq G_k$ (which can be small for large $k$), the term $\|x^{(t)} - x^*\|_2$ is as large as $\Omega(\sqrt{d})$ with high probability due to the isotropic Gaussian noise we add in DP-SGD. Therefore, $|g_2^\top (x^{(t)} - x)|$ can be as large $\Omega(\sqrt{d} G_k)$.

One natural way to derive a good bound on $\|x^{(t)} - x^*\|_2$ is through projection. However, previous lower bounds in constrained optimization already suggest that DP projected-SGD cannot evade the curse of dimensionality [BST14b], since projections tend to destroy the low-rank structure.

Our crucial idea is to partition the whole space $\mathbb{R}^d$ into $\lceil \log(d/k) \rceil + 1$ different orthogonal sub-spaces, expressing the error term $\nabla f_j(x^{(t)})^\top (x^{(t)} - x^*)$ as the sum of terms where each term corresponds to a subspace. Fix $k \in [d]$, and consider the following subspaces: Let $U_0 = \mathrm{range}(P_k)$, $U_s = \mathrm{range}(P_{2^s k}) \backslash U_{s-1}$ for $s = 1, 2, \cdots, \lfloor \log(d/k) \rfloor$, and $U_S = \mathbb{R}^d \backslash U_{S-1}$ where $S = \lfloor \log(d/k) \rfloor + 1$. Here, $P_i$ is the orthogonal projection matrix with rank $i$ promised by $G_i$ in Assumption 3.2. Let $Q_s$ be the orthogonal projection to the subspace $U_s$, and observe that $\mathrm{rank}(Q_s) \leq 2^s k$ and $\|Q_s \nabla F(x)\|_2 \leq G_{2^{s-1}k}$ for all $x$ and all $s \geq 1$. Rewriting the right hand side of (3) with this decomposition yields

$$f_j(x^{(t)}) - f_j(x^*) \leq \left( Q_0 \nabla f_j(x^{(t)}) + \sum_s Q_s \nabla f_j(x^{(t)}) \right)^\top (x^{(t)} - x^*).$$

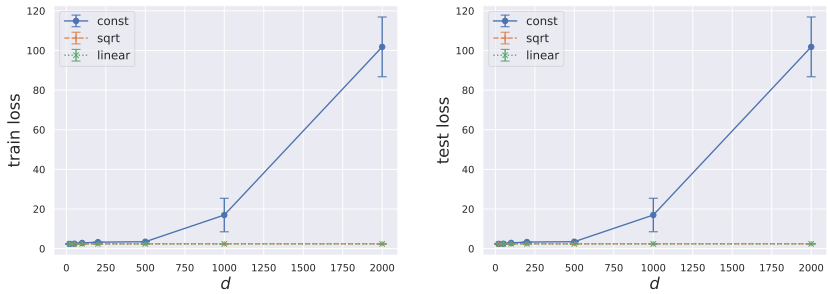
On the one hand, if $G_k$ decays quickly, $\|\mathbb{E}_j[Q_s \nabla f_j]\|_2$ will be small for large $s$. On the other hand, since the noise in DP-SGD is isotropic, we expect $\|Q_s(x^{(t)} - x^*)\|_2$ to be small for small $s$ where $Q_s$ is an orthogonal projection onto a small subspace. Thus, for each $s$, $\nabla f_j(x^{(t)})^\top Q_s (x^{(t)} - x^*)$ is small either due to a small gradient (small $Q_s \nabla f_j$ in expectation over random index) or small noise (small $Q_s(x^{(t)} - x^*)$). More formally, in Lemma A.1, we show that for any projection matrix $Q$ with rank $r$, $\|Q(x^{(t)} - x^{(0)})\|_2$ can be upper bounded by a term that depends only on $r$ (rather than $d$).

## 4 Numerical Experiments

The aim of this section is twofold. In Section 4.1, we study a synthetic example that matches our theoretical assumptions and show that DP-SGD attains dimension-independent empirical and population loss when the sequence of restricted Lipschitz coefficients decays rapidly—even when gradients span the entire ambient space. In Section 4.2, we study a stylized example of privately fine-tuning large language models. Building on the previous theory, we provide insights as to why dense fine-tuning can yield good performance.

### 4.1 Synthetic Example: Estimating the Generalized Geometric Median

We privately estimate the geometric median which minimizes the average Mahalanobis distance. Specifically, let $x_i \in \mathbb{R}^d$ for $i \in [n]$ be feature vectors drawn i.i.d. from some distribution $P_x$, each




(a) empirical loss (b) (estimated) population loss

Figure 1: The empirical and population losses grow with increasing problem dimension when the sequence of restricted Lipschitz coefficients remain constant. On the other hand, these losses remain almost constant when the sequence of restricted Lipschitz coefficients decays rapidly. Error bars represent one standard deviation over five runs of DP-SGD with the same hyperparameters which were tuned on separate validation data. For the same $A$, the optimal training error $\min_{x\in\mathbb{R}^d} F(x)$ is the same for problem instances with different dimensions (thus errors do not scale if learning was non-private). Each training run was performed with $\varepsilon = 2$, $\delta = 10^{-6}$, and $n = 10000$.

of which is treated as an individual record. Denote the entire dataset as $\mathcal{D} = \{x_i\}_{i=1}^n$. Subject to differential privacy, we perform the following optimization

$$\min._{x\in\mathbb{R}^d} F_\alpha(x) = \frac{1}{n}\sum_{i=1}^n f_i(x) + \frac{\alpha}{2}\left\|x - x^{(0)}\right\|_2^2 = \frac{1}{n}\sum_{i=1}^n \|x - x_i\|_A + \frac{\alpha}{2}\left\|x - x^{(0)}\right\|_2^2, \quad (4)$$

where we adopt the shorthand $f_i(x) = f(x; x_i) = \|x - x_i\|_A$. When $A = I_d$, the problem reduces to estimating the usual geometric median (commonly known as center of mass).

For this example, individual gradients are bounded since $\|\nabla f_i(x)\|_2 = \|A(x - x_i)/\|x - x_i\|_A\|_2 \leq \lambda_1(A^{1/2}) = G_0$. More generally, the restricted Lipschitz constants of $F(x)$ are the eigenvalues of $A^{1/2}$, since

$$\|Q_k \nabla F(x)\|_2 = \left\| Q_k A^{1/2} \frac{1}{n}\sum_{i=1}^n \frac{A^{1/2}(x - x_i)}{\|x - x_i\|_A} \right\|_2 \leq \|Q_k A^{1/2}\|_2 = \lambda_{k+1}(A^{1/2}) = G_k,$$

where $Q_k = I - P_k$ is chosen to be the rank $(d - k)$ orthogonal projection matrix that projects onto the subspace spanned by the bottom $(d - k)$ eigenvectors of $A^{1/2}$.

To verify our theory, we study the optimization and generalization performance of DP-SGD for minimizing (4) under Mahalanobis distances induced by different $A$ as the problem dimension grows. The optimization performance is measured by the final training error, and the generalization performance is measured by the population quantity $\mathbb{E}_{x\sim P_x, \overline{x}}[\|\overline{x} - x\|_A]$, where $\overline{x}$ denotes the random output of DP-SGD. We study the dimension scaling behavior for $A$ being one of

$$A_{\text{const}} = \text{diag}(1, \ldots, 1), \quad A_{\text{sqrt}} = \text{diag}(1, 1/\sqrt{2}, \ldots, 1/\sqrt{d}), \quad A_{\text{linear}} = \text{diag}(1, 1/2, \ldots, 1/d),$$

where $\text{diag} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ maps vectors onto square matrices with inputs on the diagonal. In all cases, the span of gradients $\text{span}(\{\nabla F(x)\})$ is the ambient space $\mathbb{R}^d$, since $A$ is of full rank. To ensure the distance from the initial iterate $\beta^{(0)} = 0$ to the optimum is the same for problem instances of different dimensions, we let feature vectors $\{x_i\}_{i=1}^n$ take zero values in any dimension $k > d_{\text{min}}$, where $d_{\text{min}}$ is the dimension of the smallest problem in our experiments. Our theoretical bounds suggest that when the sequence of restricted Lipschitz coefficients is constant (when $A = A_{\text{const}}$), the excess empirical loss grows with the problem dimension, whereas when the sequence of $k$th-Lipschitz constants rapidly decays with $k$ (when $A = A_{\text{sqrt}}$ or $A = A_{\text{linear}}$), the excess empirical loss does not grow beyond a certain problem dimension. Figure 1 empirically captures this phenomenon. We include additional experimental setup details in Appendix C.

### 4.2 Why Does Dense Fine-Tuning Work Well for Pretrained Language Models?

Stated informally, our bounds in Theorem 3.5 imply that DP-SGD obtains dimension-independent errors if gradients (mostly) reside in a subspace much smaller than the ambient space. Inspired by these results for the convex case, we now turn to study dense language model fine-tuning [LTLH21] and provide a possible explanation for their recent intriguing success — fine-tuning gigantic parameter vectors frequently results in moderate performance drops compared to non-private learning.

In the following, we present initial evidence that gradients of the fine-tuning objective evaluated near an (approximate) local optimum mostly lie in a small subspace. We design subsequent experiments to work under a simplified setup. Specifically, we fine-tune DistilRoBERTa [SDCW19, LOG+19] under $\varepsilon = 8$ and $\delta = 1/n^{1.1}$ for sentiment classification on the SST-2 dataset [SPW+13]. We reformulate the label prediction problem as templated text prediction [LTLH21], and fine-tune only the query and value matrices in attention layers.

We focus on fine-tuning these specific parameter matrices due to the success of LoRA for non-private learning [HSW+21] which focuses on adapting the attention layers. Unlike LoRA, we fine-tune all parameters in these matrices rather than focusing on low-rank updates. This gives a setup that is lightweight enough to run spectral analyses computationally tractably but retains enough parameters ($\approx 7$ million) such that a problem of similar scale outside of fine-tuning results in substantial losses in utility.[1] For our setup, DP-SGD obtains a dev set accuracy approximately of $90\%$ and $92\%$, privately and non-privately, respectively. These numbers are similar to previous results obtained with the same pretrained model [YNB+21, LTLH21]. We include the full experimental protocol and additional results in Appendix D.

To provide evidence for the small subspace hypothesis, we sample gradients near a local optimum and study their principal components. Specifically, we "over-train" by privately fine-tuning for $r = 2 \times 10^3$ updates and collect all the non-privatized average clipped gradients along the optimization trajectory. While fine-tuning for 200 and 2k updates have similar final dev set performance under our hyperparameters, the increased number of steps allows us to collect more gradients around the converged solution. This yields a gradient matrix $H \in \mathbb{R}^{r \times p}$, where $p \approx 7 \times 10^6$ is the size of the parameter vector. We perform PCA for $H$ with the orthogonal iteration algorithm [Dem97] and visualize the set of estimated singular values $\sigma_i(H) = \lambda_i(H^\top H)^{1/2}$ in terms of both (i) the density estimate, and (ii) their relation with the rank. Figure 2 (a) shows the top 1000 singular values sorted and plotted against their rank and the least squares fit on log-transformed inputs and outputs. The plot displays few large singular values which suggests that gradients are controlled through only few principal directions. The linear fit suggests that singular values decay rapidly (at a rate of $k^{-0.6}$).

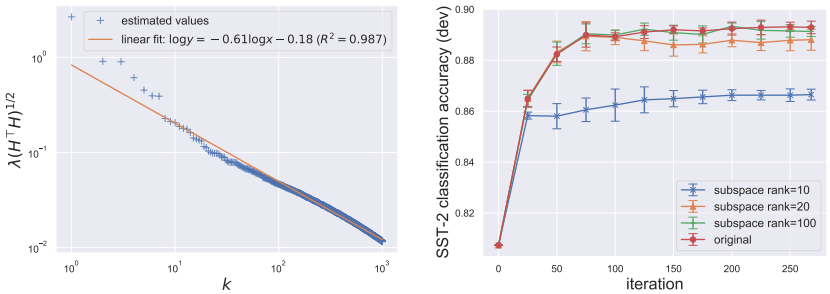
To study the effects that different principal components have on fine-tuning performance, we further perform the following re-training experiment. Given the principal components, we privately re-fine-tune with gradients projected onto the top $k \in \{10, 20, 100\}$ components. Note that this projection applies only to the (non-privatized) average clipped gradients and the isotropic DP noise is still applied to all dimensions. Figure 2 (b) shows that the original performance can be attained by optimizing within a subspace of only dimension $k = 100$, suggesting that most of the dimensions of the 7 million parameter vector encode a limited learning signal.

While these empirical results present encouraging insights for the dimension-independent performance of fine-tuning, we acknowledge that this is not a complete validation of the restricted Lipschitz continuity condition and fast decay of coefficients (even locally near the optimum). We leave a more thorough analysis with additional model classes and fine-tuning tasks to future work.

## 5 Related Work

Differentially private empirical risk minimization (DP-ERM) and its close cousin differentially is private stochastic convex optimization (DP-SCO) are arguably the most well-studied areas of DP [CMS11, KST12, BST14a, SCS13, WYX17, FTS17, BFTT19, MRTZ17, ZZMW17, WLK+17, FKT20, INS+19, BFGT20, STT20, LL21, AFKT21, BGN21, GTU22, GLL22]. Tight dependence on the number of model parameters and the number of data samples are known for both DP-ERM [BST14a] and DP-SCO errors [BFTT19]. In particular, it is known that for the error on general

[1] For instance, an off-the-shelf ResNet image classifier has 10 to 20+ million parameters. A plethora of works report large performance drops when training these models from scratch [YZCL21, LWAFF21, DBH+22].




(a) singular values decay with rank

(b) retrain in fixed subspace

Figure 2: Gradients near a local optimum are controlled by a few principle components. *Left:* Singular values decay rapidly with their rank. *Right:* Retraining with gradients projected onto a subspace is sufficient to recover original performance.

convex losses, an explicit polynomial dependence on the number of parameters in the model (i.e. $d$) is necessary. However, [JT14, STT20] showed that if all the gradients of the underlying loss function lie in a fixed low-rank subspace $M$, the dependence on $d$ can be replaced by $\mathsf{rank}(M)$ (which can be significantly smaller). We extend this line of work to show that under a weaker assumption called restricted Lipschitz continuity (mentioned earlier) one can obtain analogous error guarantees that are independent of any explicit dependence on $d$, but do not require the gradients of the loss to strictly lie in any fixed low-rank subspace $M$. As a consequence, we use our results to provide a plausible explanation why in recent empirical works it has been observed that training a larger model under DP is at times more advantageous in terms of utility than training a smaller model [LTLH21, YNB+21].

In a complementary line of work, [AGM+21, LVS+21, ADF+21, KDRT21, YZCL21] designed variants of DP-SGD that either explicitly or implicitly control the subspace in which the gradients are allowed to reside. They demonstrated improved dependence of the error on dimensions/model parameters if the true gradients lie in a "near" low-rank subspace. Our results are incomparable to this line of work, because of two reasons: i) Our algorithm is vanilla DP-SGD, and does not track the gradient subspace either explicitly or implicitly, and hence does not change the optimization landscape. The improved dependence on dimensions is just an artifact of the analysis. ii) Our analytical results do not need the existence of any public data to obtain tighter dependence on dimensions. All prior works mentioned above need the existence of public data to demonstrate any improvement.

## 6 Conclusion & Limitation

We made an attempt to reconcile two seemingly conflicting results: (1) in private convex optimization, errors are predicted to scale proportionally with the dimension of the learning problem; while (2) in empirical works on large-scale private fine-tuning through DP-SGD, privacy-utility trade-offs become better with increasing model size. We introduced the notion of restricted Lipschitz continuity, with which we gave refined analyses of DP-SGD for DP-ERM and DP-SCO. When the magnitudes of gradients projected onto diminishing subspaces decay rapidly, our analysis showed that excess empirical and population losses of DP-SGD are independent of the model dimension. Through preliminary experiments, we gave empirical evidence that gradients of large pretrained language models near a local optimum (obtained through fine-tuning) mostly lie in the subspace spanned by a few principal components. Our theoretical and empirical results together give a possible explanation for recent successes in large-scale differentially private fine-tuning.

Given our improved upper bounds on the excess empirical and population risks for differentially private convex learning, it is instructive to ask if such bounds are tight in the mini-max sense. We leave answering this inquiry to future work. In addition, while we have presented encouraging empirical evidence that fine-tuning gradients mostly lie in a small subspace, more work is required study the robustness of this phenomenon with respect to the model class and fine-tuning problem. Overall, we hope that our work leads to more research on understanding conditions under which DP learning does not degrade with increasing problem size, and more generally, how theory can inform and explain the practical successes of differentially private deep learning.

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## Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] This work presents theory which characterizes the performance of existing algorithms. This work would not lead to any direct societal impact.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
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   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]
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   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]
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