ON THE DISCRIMINATION-GENERALIZATION TRADEOFF IN GANS

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ABSTRACT

Generative adversarial training can be generally understood as minimizing certain moment matching loss defined by a set of discriminator functions, typically neural networks. The discriminator set should be large enough to be able to uniquely identify the true distribution (*discriminative*), and also be small enough to go beyond memorizing samples (*generalizable*). In this paper, we show that a discriminator set is guaranteed to be discriminative whenever its linear span is dense in the set of bounded continuous functions. This is a very mild condition satisfied even by neural networks with a *single neuron*. Further, we develop generalization bounds between the learned distribution and true distribution under different evaluation metrics. When evaluated with neural or Wasserstein distances, our bounds show that generalization is guaranteed as long as the discriminator set is small enough, regardless of the size of the generator or hypothesis set. When evaluated with KL divergence, our bound provides an explanation on the counter-intuitive behaviors of testing likelihood in GAN training. Our analysis sheds lights on understanding the practical performance of GANs.

1 INTRODUCTION

Generative adversarial networks (GANs) (Goodfellow et al., 2014) and its variants can be generally understood as minimizing certain moment matching loss defined by a set of discriminator functions. Mathematically, GANs minimize the integral probability metric (IPM) (Müller, 1997), that is,

$$\min_{\nu\in\mathcal{G}}\left\{d_{\mathcal{F}}(\hat{\mu}_m,\nu):=\sup_{f\in\mathcal{F}}\left\{\mathbb{E}_{x\sim\hat{\mu}_m}[f(x)]-\mathbb{E}_{x\sim\nu}[f(x)]\right\}\right\}, \tag{1}$$

where $\hat{\mu}_m$ is the empirical measure of the observed data, and $\mathcal{F}$ and $\mathcal{G}$ are the sets of discriminators and generators, respectively.

1. **Wasserstain GAN (W-GAN)** (Arjovsky et al., 2017). $\mathcal{F}=\mathrm{Lip}_1(X):=\{f\colon ||f||_{\mathrm{Lip}}\leq 1\}$. Corresponding to the Wasserstain-1 distance.
2. **MMD-GAN** (Li et al., 2015; Dziugaite et al., 2015; Li et al., 2017a). $\mathcal{F}$ is taken as the unit ball in some Reproducing Kernel Hilbert Spaces (RKHS). Corresponding to the Maximum Mean Discrepency (MMD).
3. **Energy-based GANs** (Zhao et al., 2016). $\mathcal{F}$ is taken as the set bounded continuous functions (Arjovsky et al., 2017). Corresponding to the total variation distance.
4. **f-GAN** (Nowozin et al., 2016). Minimizing the moment matching loss defined over all possible functions plus a regularization term (Liu et al., 2017). See also Section 4.

Due to computational tractability, however, the practical GANs take $\mathcal{F}$ as a parametric function class, typically, $\mathcal{F}_{\text{nn}}=\{f_\theta(x)\colon\theta\in\Theta\}$ where $f_\theta(x)$ is a neural network indexed by parameter θ that take values in $\Theta\subset\mathbb{R}^p$. Consequently, the related $d_{\mathcal{F}_{\text{nn}}}(\mu,\nu)$ is called neural network distance, or neural distance (Arora et al., 2017). Although $d_{\mathcal{F}_{\text{nn}}}(\mu,\nu)$ is meant to be a surrogate, its properties can be fundamentally different or even irrelevant with the original objective functions. For example, in W-GAN, because $\mathcal{F}_{\text{nn}}$ is a much smaller discriminator set than $\mathrm{Lip}_1(X)$, it is unclear from the current GAN literature that if $d_{\mathcal{F}_{\text{nn}}}(\mu,\nu)$ is a discriminative metric in that $d_{\mathcal{F}_{\text{nn}}}(\mu,\nu)=0$ implies $\mu=\nu$, which is critical to ensure the consistency of the learning result. This motivated us to study

the properties of $d_{\mathcal{F}_{\rm nn}}(\mu,\nu)$ with parametric function sets $\mathcal{F}_{\rm nn}$, instead of the original Wasserstein distance or f-divergence.

A more broad question is in developing learning bounds and studying how it depends on the discriminator set $\mathcal{F}$ and the generator set $\mathcal{G}$, under different evaluation metrics of interest. Specifically, assume ν_m is an (approximate) solution of (1), we are interested in obtaining bounds between ν_m and the underlying true distribution μ, under a given evaluation metric $d_{eval}(\mu,\nu_m)$. Existing analysis have been mostly focusing on the case when the evaluation metric coincides with the optimization metric, that is, $d_{eval}(\mu,\nu) = d_{\mathcal{F}}(\mu,\nu)$, which, however, favors smaller discriminator sets that defines *"easier"* evaluation metrics. It is of interest to develop bounds evaluation metrics independent of $\mathcal{F}$ that have clear statistical meaning, such as bounded Lipschitz distance that metrizes weak convergence, and KL divergence which connects to testing likelihood.

Contribution. We show that the role of discriminators $\mathcal{F}$ is best illustrated by the conditions under which $d_{\mathcal{F}}(\mu,\nu)$ metrizes weak convergence (or convergence in distribution), that is,

$$d_{\mathcal{F}}(\mu,\nu_m) \to 0 \quad \text{if and only if} \quad \nu_m \rightharpoonup \mu, \tag{2}$$

for any probability measures μ and ν_m. The choice of $\mathcal{F}$ should strike a balance to achieve (2):

i) $\mathcal{F}$ should be large enough to make $d_{\mathcal{F}}(\mu,\nu)$ *discrimiantive* in that $d_{\mathcal{F}}(\mu,\nu_m) \to 0$ can imply that ν_m converges to μ. Further, with a given metric $d_{eval}(\mu,\nu)$, the discriminator set $\mathcal{F}$ should be large enough so that a small $d_{\mathcal{F}}(\mu,\nu)$ implies a small $d_{eval}(\mu,\nu)$ in certain sense. These are basic requirements in justifying $d_{\mathcal{F}}(\mu,\nu)$ as a valid learning objective function.

ii) $\mathcal{F}$ should also be relatively small so that $\nu_m \rightharpoonup \mu$ implies that $d_{\mathcal{F}}(\mu,\nu_m)$ approaches to zero. This is essential to guarantee that the training and testing loss are similar to each other and hence the algorithm is *generalizable*. Further, in order to obtain a low sample complexity, $\mathcal{F}$ should be sufficiently small so that $d_{\mathcal{F}}(\mu,\nu_m)$ decays with a sufficiently fast rate, preferably $O(1/\sqrt{m})$.

The theme of this work is to characterize the conditions under which i) and ii) hold and develop bounds of $d_{eval}(\mu,\nu_m)$ that characterize the role of discriminators $\mathcal{F}$ and generators $\mathcal{G}$. Our contribution are summarized as follows.

1. We show that a discriminator set $\mathcal{F}$ is discriminative once the linear span of $\mathcal{F}$ is dense in the set of bounded continuous (or Lipschitz) functions. This is a surprisingly mild condition that can satisfied, for example, even for neural networks consists of a *single neuron*. See Section 2.
2. We develop techniques using neural distances $d_{\mathcal{F}}(\mu,\nu)$ to provide upper bounds of different evaluation metrics $d_{eval}(\mu,\nu)$ of interest, including bounded Lipschitz (BL) distance and KL divergence, which provides a key step for developing learning bounds of GANs under these metrics. See Section 2.1.
3. We characterize the generalizability of a discriminator set $\mathcal{F}$ using its Rademacher complexity and put together bounds of between the true distributions μ and GAN estimators ν_m under differen evaluation metrics $d_{eval}(\mu,\nu)$ in Section 3. Under the neural and BL distances, our bounds (Corollary 3.2-3.3) show that a discriminator with vanishing Rademacher complexity effectively acts as a type "universal" regularization term which helps prevent overfitting *regardless of the size of the generator or hypothesis set* $\mathcal{G}$. This a surprising proper of GANs that is in sharp contrast with the typical statistical learning framework in which large hypothesis sets necessarily increase the risk of overfitting.
4. On the other hand, when the KL divergence is used as an evaluation metric, our bound (Corollary 3.5) suggests that the generator and discriminator sets have to be compatible in that the log density ratios of the generators and the true distributions should exist and be included inside the linear span of the discriminator set. The strong condition that log-density ratio should exist partially explains the counter-intuitive behavior of testing likelihood in flow GANs (e.g., Danihelka et al., 2017; Grover et al., 2017).
5. We extend our analysis to study a class of neural f-divergence which is the learning objective of f-GAN. We establish the connection between neural f-divergence and neural distance, and establish similar results on the discriminative and generalization properties.

1.1 NOTATIONS

We use X to denote a subset of $\mathbb{R}^d$. For each function $f\colon X \to \mathbb{R}$, we define the maximum norm as $\|f\|_\infty = \sup_{x\in X} |f(x)|$, and the Lipschitz norm $\|f\|_{\mathrm{Lip}} = \sup\{|f(x) - f(y)|/\|x - y\|\colon x, y \in X,\ \ x \neq y\}$, and the bounded Lipschitz (BL) norm $\|f\|_{\mathrm{BL}} = \max\{\|f\|_{\mathrm{Lip}},\ \|f\|_\infty\}$. The Banach space of continuous functions on X is denoted by $C(X)$, and the set of bounded continuous function is $C_b(X) = \{f \in C(X)\colon \|f\|_\infty < \infty\}$.

The set of Borel probability measures on X is denoted by $\mathcal{P}_{\mathcal{B}}(X)$. In this paper, we assume that all measures involved belongs to $\mathcal{P}_{\mathcal{B}}(X)$, which is sufficient in all practical applications. We denote by $\mathbb{E}_\mu[f]$ the integral of f with respect to probability measure μ. The weak convergence, or convergence in distribution is denoted by $\nu_n \rightharpoonup \nu$. Given a base measures τ (e.g., Lebegure measure), the density of $\mu \in \mathcal{P}_{\mathcal{B}}(X)$, if it exists, is denoted by $\rho_\mu = \frac{\mathrm{d}\mu}{\mathrm{d}\tau}$. We do *not* assume density exists in our main theoretical results.

2 DISCRIMINATIVE PROPERTIES OF NEURAL DISTANCES FOR GAN

As listed in the introduction, many variants of GAN can be viewed as minimizing the integral probability metric (1). Without loss of generality, we assume that the discriminator set $\mathcal{F}$ is even, i.e., $f \in \mathcal{F}$ implies $-f \in \mathcal{F}$. Intuitively speaking, minimizing (1) towards zero corresponds to matching the moments $\mathbb{E}_\mu[f] = \mathbb{E}_\nu[f]$ for all discriminators $f \in \mathcal{F}$. In their original formulation, all these discriminator sets are non-parametric, infinite dimensional, and large enough to guarantee that $d_{\mathcal{F}}(\mu, \nu) = 0$ implies $\mu = \nu$.

In practice, however, the discriminator set is typically restricted to parametric function classes of form $\mathcal{F}_{\mathrm{nn}} = \{f_\theta\colon \theta \in \Theta\}$. When f_θ is a neural network, we call $d_{\mathcal{F}_{\mathrm{nn}}}(\mu, \nu)$ a neural distance following Arora et al. (2017). Neural distances are the actual object function that W-GAN optimizes in practice because they can be practically optimized and can leverage the representation power of neural networks. Therefore, it is of great importance to directly study neural distances, instead of Wasserstein metric, in order to understand practical performance of GANs.

Because the parameter function set $\mathcal{F}_{\mathrm{nn}}$ is much smaller than the non-parametric sets like $\mathrm{Lip}_1(X)$, a key question is whether $\mathcal{F}_{\mathrm{nn}}$ is large enough so that the moment matching on $\mathcal{F}_{\mathrm{nn}}$ (i.e., $d_{\mathcal{F}_{\mathrm{nn}}}(\mu, \nu) = 0$) implies $\mu = \nu$. As we show in the following, it turns out the answer is affirmative once $\mathcal{F}$ is large enough so that its *linear span* (instead of $\mathcal{F}_{\mathrm{nn}}$ itself) forms an universal approximator. This is a rather weak condition, which is satisfied even by very small sets $\mathcal{F}_{\mathrm{nn}}$ such as neural networks with a *single neuron*. We make this concrete in the following.

Definition 2.1. Let $(X, \|\cdot\|)$ be a metric space and $\mathcal{F}$ is a set of functions on X. We say that $d_{\mathcal{F}}(\mu, \nu)$ (and $\mathcal{F}$) is discriminative if

$$d_{\mathcal{F}}(\mu, \nu) = 0 \qquad \Longleftrightarrow \qquad \mu = \nu$$

for any two Borel probability measures $\mu, \nu \in \mathcal{P}_{\mathcal{B}}(X)$. In other words, $\mathcal{F}$ is discriminative if the moment matching on $\mathcal{F}$, that is, $\mathbb{E}_\mu[f] = \mathbb{E}_\nu[f], \forall f \in \mathcal{F}$, implies $\mu = \nu$.

Our key observation is that $\mathbb{E}_\mu[f] = \mathbb{E}_\nu[f], \forall f \in \mathcal{F}$ implies the same holds true for all f in the linear span of $\mathcal{F}$, and hence it is sufficient to require the linear span of $\mathcal{F}$, instead of $\mathcal{F}$ itself, to be large enough to well approximate all the indicator test functions.

Theorem 2.2. *For a given function set $\mathcal{F}$, define*

$$span\mathcal{F} := \{\alpha_0 + \sum_{i=1}^{n} \alpha_i f_i : \alpha_i \in \mathbb{R}, f_i \in \mathcal{F}, n \in \mathbb{N}\}. \tag{3}$$

Then $d_{\mathcal{F}}(\mu, \nu)$ is discriminative if $span\mathcal{F}$ is dense in the space of bounded continuous functions $C_b(X)$ under the uniform norm $||\cdot||_\infty$, that is, for any $f \in C_b(X)$ and $\epsilon > 0$, there exists an $f_\epsilon \in span\mathcal{F}$ such that $\|f - f_\epsilon\|_\infty \leq \epsilon$. An equivalent way to put is that $C_b(X)$ is included in the closure of $span\mathcal{F}$, that is,

$$cl(span\mathcal{F}) \supseteq C_b(X). \tag{4}$$

Further, (4) *is a sufficient condition for $d_{\mathcal{F}}(\mu, \nu)$ to be discriminative if X is a compact space.*

Remark 2.1. *The basic idea of characterizing probability measures using functions in $C_b(X)$ is closely related to the concept of weak convergence. Recall that a sequence μ_n weakly converges μ, i.e., $\mu_n \rightharpoonup \mu$, iff $\mathbb{E}_{\mu_n}[f] \to \mathbb{E}_\nu[f]$ for all $f \in C_b(X)$. Theorem 2.2 can be viewed as the result of the uniqueness of weakly convergence, that is, if $\mu_n \rightharpoonup \mu$ and $\mu_n \rightharpoonup \nu$, then $\mu = \nu$.*

Remark 2.2. *Because the set of bounded Lipschitz functions $\mathrm{BL}(X) = \{f \in C_b(X) \colon ||f||_{\mathrm{Lip}} < \infty\}$ is dense in $C_b(X)$, the condition in* (4) *can be replaced by a weaker condition*

$$cl(span\mathcal{F}) \supseteq \mathrm{BL}(X).$$

One can define a norm $||\cdot||_{\mathrm{BL}}$ for functions in $\mathrm{BL}(X)$ by $||f||_{\mathrm{BL}} = \max\{||f||_{\mathrm{Lip}},\ ||f||_\infty\}$. This defines the bounded Lipschitz (BL) distance,

$$d_{\mathrm{BL}}(\mu, \nu) = \max_{f \in \mathrm{BL}(X)} \{\mathbb{E}_\mu f - \mathbb{E}_\nu f \colon\ ||f||_{\mathrm{BL}} \leq 1\}.$$

The BL distance is known to metrize weak convergence in sense that $d_{\mathrm{BL}}(\mu, \nu_n) \to 0$ is equivalent to $\nu_n \rightharpoonup \mu$ for all Borel probability measures on $\mathbb{R}^d$; see section 8.3 in Bogachev (2007).

Neural Distances are Discriminative. The key message of Theorem 2.2 is that it is sufficient to require $cl(span\mathcal{F}) \supseteq C_b(X)$ (Condition (4)), which is a much weaker condition than the perhaps more straightforward condition that $cl(\mathcal{F}) \supseteq C_b(X)$. In fact, (4) is met by function sets that are much smaller than what we actually use in practice. For example, it is satisfied by the neural networks with only *a single* neuron, i.e.,

$$\mathcal{F}_{\mathrm{nn}} = \{\sigma(w^T x + b) \colon w \in \mathbb{R}^d, b \in \mathbb{R}\}. \tag{5}$$

This is because its span $span\mathcal{F}_{\mathrm{nn}}$ includes neural networks with infinite numbers of neurons, which are well known to be universal approximators in $C_b(X)$ according to classical theories (e.g., Cybenko, 1989; Hornik et al., 1989; Hornik, 1991; Leshno et al., 1993; Barron, 1993). We recall the following classical result.

Theorem 2.3 (Theorem 1 in Leshno et al. (1993))**.** *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous activation function and $X \subset \mathbb{R}^d$ be any compact set. Let $\mathcal{F}_{\mathrm{nn}}$ be the set of neural networks with a single neuron as defined in* (5)*, then $span\mathcal{F}_{\mathrm{nn}}$ is dense in $C(X)$ if and only if σ is not a polynomial.*

The above result that requires that the parameters $[w, b]$ to take values in $\mathbb{R}^{d+1}$. In practice, however, we can only efficiently search in bounded parameter sets of $[w, b]$ using local search methods like gradient descent. We observe that it is sufficient to replace $\mathbb{R}^{d+1}$ with a bounded parameter set Θ for non-decreasing homogeneous activation function such as $\sigma(u) = \max\{u, 0\}^\alpha$ with $\alpha \in \mathbb{N}$; note that $\alpha = 1$ is the widely used rectified linear unit (ReLU).

Corollary 2.4. *Let $X \subset \mathbb{R}^d$ be any compact set, and $\sigma(u) = \max\{u, 0\}^\alpha$ ($\alpha \in \mathbb{N}$), and $\mathcal{F}_{\mathrm{nn}} = \{\sigma(w^\top x + b) \colon [w, b] \in \Theta\}$. Then $span\mathcal{F}_{\mathrm{nn}}$ is dense in $C_b(X)$ if*

$$\{\lambda\theta : \lambda \geq 0, \theta \in \Theta\} = \mathbb{R}^{d+1}.$$

For the case when $\Theta = \{v \in \mathbb{R}^{d+1} : \|v\|_2 \leq 1\}$,Bach (2017) not only proves that $span\mathcal{F}_{\mathrm{nn}}$ is dense in $BL(X)$ (and thus dense in $C(X)$), but also gives the convergence rate.

Therefore, for ReLU activation functions, $\mathcal{F}_{\mathrm{nn}}$ with bounded parameter sets, like $\{\theta : \|\theta\| \leq 1\}$ or $\{\theta : \|\theta\| = 1\}$ for any norm on $\mathbb{R}^{d+1}$, is sufficient to discriminate any two Borel probability measures. Note that this is not true for some other activation functions such as $\tanh$ or sigmoid, because there is an approximation gap between span$\{\sigma(w^\top x + b) : [w, b] \in \Theta \subset \mathbb{R}^{d+1}\}$ and $C_b(X)$ when $\Theta \subset \mathbb{R}^{d+1}$ is bounded; see e.g., Barron (1993) (Theorem 3). From this perspective, homogeneous activation functions such as ReLU are preferred for as discriminators.

One advantage of using bounded parameter sets Θ is that it makes $\mathcal{F}_{\mathrm{nn}}$ have a bounded Lipschitz norm, that is, $\mathcal{F}_{\mathrm{nn}} \subset \mathrm{Lip}_K(X)$ for some K, and hence upper bounded by Wasserstein distance. In fact, W-GAN uses weight clipping to explicitly enforce boundness $||\theta||_\infty \leq \delta$. But we should point out that the Lipschitz constraint does not help in making $\mathcal{F}$ discriminative since the constraint decreases, instead of enlarge the function set $\mathcal{F}$. Instead, the role of the Lipschitz constraint should be mostly in assuring a generalization bound as we discuss in Section 3. Another related way to justify the Lipschitz constraint is its relation to metrizing weak convergence, as we discuss in the sequel.

Neural Distance and Weak Convergence. If $\mathcal{F}$ is discriminative, then $d_{\mathcal{F}}(\mu,\nu) = 0$ implies $\mu = \nu$. In practice, however, we often can not achieve $d_{\mathcal{F}}(\mu,\nu) = 0$ strictly. Instead, we often have $d_{\mathcal{F}}(\mu,\nu_n) \to 0$ for a sequence of ν_n and want to establish the weak convergence $\nu_n \rightharpoonup \mu$.

Theorem 2.5. *(X, d_X) be any metric space. If $span\mathcal{F}$ is dense in $C_b(X)$, we have $\lim d_{\mathcal{F}}(\mu,\nu_n) = 0$ implies ν_n weakly converges to μ.*

Additionally, if $\mathcal{F}$ is contained in a bounded Lipchitz function space, i.e., there exists $0 < C < \infty$ such that $||f||_{\mathrm{BL}} \leq C$ for all $f \in \mathcal{F}$, then ν_n weakly converges to μ also implies $\lim d_{\mathcal{F}}(\mu,\nu_n) = 0$.

Theorem 10 of Liu et al. (2017) states a similar result for generic adversarial divergences, but does not obtain the specific weak convergence result for neural distances due to lacking of Theorem 2.2. Another difference is that Theorem 10 of Liu et al. (2017) heavily relies on the compactness assumption of X, while our result does not need this assumption. We provide the proof for Theorem 2.5 in the appendix.

When X is compact, Wasserstein distance and the BL distance are equivalent and both metrize weak convergence. As we discussed earlier, the condition $cl(span\mathcal{F}) = C_b(X)$ and $\mathcal{F} \subseteq \mathrm{Lip}_L(X)$ are satisfied by neural networks $\mathcal{F}_{\text{nn}}$ with a single ReLU activations and bounded parameter set Θ. Therefore, the related neural distance $d_{\mathcal{F}_{\text{nn}}}$ is topologically equivalent to the Wasserstein and BL distance, because all of them metrize the weak convergence. This *does not imply*, however, that they are equivalent in the metric sense (or strongly equivalent) since the ratio $d_{\mathrm{BL}}(\mu,\nu)/d_{\mathcal{F}_{\text{nn}}}(\mu,\nu)$ can be unbounded. In general, the neural distance are weaker than the BL distance because of the smaller $\mathcal{F}$. In Section 2.1 (and particularly Corollary 2.8), we draw more discussion on the bounds between BL distance and neural distances.

2.1 Discriminative power of neural distances

Theorem 2.2 characterizes the condition under which a neural distance is discriminative, and shows that even neural networks with a single neuron are sufficient to be discriminative. This does not explain, however, why it is beneficial to use larger and deeper networks as we do in practice. What is missing here is to frame and understand *how discriminative or strong* a neural distance is. This is because even if $d_{\mathcal{F}}(\mu,\nu)$ is discriminative,it can be *relatively weak* in that $d_{\mathcal{F}}(\mu,\nu)$ may be small when μ and ν are very different under standard metrics (e.g., BL distance). Obviously, a larger $\mathcal{F}$ yields a stronger neural distance, that is, if $\mathcal{F} \subset \mathcal{F}'$, then $d_{\mathcal{F}}(\mu,\nu) \leq d_{\mathcal{F}'}(\mu,\nu)$. For example, because it is reasonable to assume that neural networks are bounded Lipschitz when X and Θ are bounded, we can control a neural distance with the BL distance:

$$d_{\mathcal{F}}(\mu,\nu) \leq C d_{\mathrm{BL}}(\mu,\nu),$$

$C := \sup_{f\in\mathcal{F}}\{||f||_{\mathrm{BL}}\} < \infty$. A more difficult question is if we can establish inequalities from the other direction, that is, controlling $d_{\mathrm{BL}}(\mu,\nu)$, or in general a stronger $d_{\mathcal{F}'}(\mu,\nu)$, with a weaker $d_{\mathcal{F}}(\mu,\nu)$ in some way. In this section, we characterizes conditions under which this is possible and develop bounds that allows us to use neural distances to control stronger distances such as BL distance, and even KL divergence. These bounds are used in Section 3 to translate generalization bounds in $d_{\mathcal{F}}(\mu,\nu)$ to that in BL distance and KL divergence.

The core of the discussion involves understanding how $d_{\mathcal{F}}(\mu,\nu)$ can be used to control the difference of the moment $|\,\mathbb{E}_\mu\, g - \mathbb{E}_\nu\, g|$ for g *outside of* $\mathcal{F}$. We address this problem by two steps, first controlling functions in $span\mathcal{F}$, and then functions in $cl(span\mathcal{F})$ that is large enough to include $C_b(X)$ for neural networks.

Controlling Functions in $span\mathcal{F}$. We start with understanding how $d_{\mathcal{F}}(\mu,\nu)$ can bound $|\,\mathbb{E}_\mu\, g - \mathbb{E}_\nu\, g|$ for $g \in span\mathcal{F}$. This can be characterized by introducing a notion of norm on $span\mathcal{F}$.

Proposition 2.6. *For each $g \in span\mathcal{F}$ which can be decomposed into $g = \sum_{i=1}^n w_i f_i + w_0$ as we define in* (3)*, the $\mathcal{F}$-variation norm $||g||_{\mathcal{F},1}$ of g is the infimum of $\sum_{i=1}^n |w_i|$ among all possible decompositions of g, that is,*

$$||g||_{\mathcal{F},1} = \inf\left\{\sum_{i=1}^n |w_i| \colon g = \sum_{i=1}^n w_i f_i + w_0,\ \forall n \in \mathbb{N},\ w_0, w_i \in \mathbb{R},\ f_i \in \mathcal{F}\right\}.$$

Then we have

$$|\,\mathbb{E}_\mu\, g - \mathbb{E}_\nu\, g| \leq ||g||_{\mathcal{F},1}\, d_{\mathcal{F}}(\mu,\nu), \quad \forall g \in span\mathcal{F}.$$

Intuitively speaking, $||g||_{\mathcal{F},1}$ denotes the "minimum number" of functions in $\mathcal{F}$ needed in order to represent g. As $\mathcal{F}$ becomes larger, $||g||_{\mathcal{F},1}$ decreases and $d_{\mathcal{F}}(\mu,\nu)$ can better control $|\,\mathbb{E}_\mu\, g - \mathbb{E}_\nu\, g|$. Precisely, if $\mathcal{F} \subseteq \mathcal{F}'$ then $||g||_{\mathcal{F}',1} \leq ||g||_{\mathcal{F},1}$. Therefore, even adding more neurons in $\mathcal{F}$ may not necessarily increase $span\mathcal{F}$, it decreases $||g||_{\mathcal{F},1}$ and yields a stronger neural distance.

Controlling Functions in $cl(span\mathcal{F})$. A more critical question is how the neural distance $d_{\mathcal{F}}(\mu,\nu)$ can also control the discrepancy $\mathbb{E}_\mu\, g - \mathbb{E}_\nu\, g$ for functions outside of $span\mathcal{F}$ but inside $cl(span\mathcal{F})$. The bound in this case is characterized by a notion of error decay function defined as follows.

Proposition 2.7. *Given a function g, we say that g is approximated by $\mathcal{F}$ with error decay function $\epsilon(r)$ if for any $r \geq 0$, there exists an $f_r \in span\mathcal{F}$ with $||f_r||_{\mathcal{F},1} \leq r$ such that $||f - f_r||_\infty \leq \epsilon(r)$. Therefore, $g \in cl(span\mathcal{F})$ if and only if $\inf_{r\geq 0} \epsilon(r) = 0$. We have*

$$|\,\mathbb{E}_\mu\, g - \mathbb{E}_\nu\, g| \leq \inf_{r\geq 0}\{2\epsilon(r) + r\; d_{\mathcal{F}}(\mu,\nu)\}.$$

In particular, if $\epsilon(r) = O(r^{-\kappa})$ for some $\kappa > 0$, then $|\,\mathbb{E}_\mu\, g - \mathbb{E}_\nu\, g| = O(d_{\mathcal{F}}(\mu,\nu)^{\frac{\kappa}{\kappa+1}})$.

It requires further efforts to derive the error decay function for specific $\mathcal{F}$ and g. In particular, Proposition 5 of Bach (2017) allows us to derive the decay rate of approximating bounded Lipschitz functions with rectified neurons, yielding a bound between BL distance and neural distance.

Corollary 2.8. *Let X be the unit ball of $\mathbb{R}^d$ under norm $||\cdot||_q$ for some $q \in [2,\infty)$, that is, $X = \{x \in \mathbb{R}^d \colon ||x||_q \leq 1\}$. Consider $\mathcal{F}$ consisting of a single rectified neuron $\mathcal{F} = \{\max(v^\top[x;1],\ 0)^\alpha \colon v \in \mathbb{R}^{d+1},\ ||v||_p = 1\}$ where $\alpha \in \mathbb{N}$, $\frac{1}{p} + \frac{1}{q} = 1$. Then we have*

$$d_{\mathrm{BL}}(\mu,\nu) = \tilde{O}(d_{\mathcal{F}}(\mu,\nu)^{\frac{1}{\alpha+(d+1)/2}}), \tag{6}$$

where $\tilde{O}$ denotes the big-O notation ignoring the logarithm factor.

The result in (6) shows that $d_{\mathcal{F}}(\mu,\nu)$ gives a increasingly weaker bound when the dimension d increases. This is expected because we approximate an non-parametric set with a parametric one.

Likelihood and KL divergence. Maximum likelihood has been the predominant approach in statistical learning, and testing likelihood forms a standard criterion for testing unsupervised models. The recent advances in deep unsupervised learning, however, makes it questionable whether likelihood is the right objective for training and evaluation (e.g., Theis et al., 2015). For example, some recent empirical studies (e.g., Danihelka et al., 2017; Grover et al., 2017) showed a counter-intuitive phenomenon that both the testing and training likelihood (assuming generators with valid densities are used) tend to decrease, instead of increase, as the GAN loss is minimized. A hypothesis for explaining this is that the neural distances used GANs are too weak to be able to control the KL divergence properly. Therefore, from the theoretical perspective, it is desirable to understand under what conditions (even if it is a very strong one), the neural distance can be strong enough to control KL divergence. This can be done by the following simple result.

Proposition 2.9. *Assume μ and ν have positive density functions $\rho_\mu(x)$ and $\rho_\nu(x)$, respectively. Then*

$$\mathrm{KL}(\mu||\nu) + \mathrm{KL}(\nu||\mu) = \mathbb{E}_\mu \log(\rho_\mu/\rho_\nu) - \mathbb{E}_\nu \log(\rho_\mu/\rho_\nu).$$

If $\log(\rho_\mu/\rho_\nu) \in span\mathcal{F}$, then

$$\mathrm{KL}(\mu||\nu) + \mathrm{KL}(\nu||\mu) \leq ||\log(\rho_\mu/\rho_\nu)||_{\mathcal{F},1}\; d_{\mathcal{F}}(\mu,\nu). \tag{7}$$

If $\log(\rho_\mu/\rho_\nu) \in cl(span\mathcal{F})$ with an error decay function $\epsilon(r) = O(r^{-\kappa})$, then

$$\mathrm{KL}(\mu||\nu) + \mathrm{KL}(\nu||\mu) = O(d_{\mathcal{F}}(\mu,\nu)^{\frac{\kappa}{\kappa+1}}). \tag{8}$$

This result shows that we require that the density ratio $\log(\rho_\mu/\rho_\nu)$ exists and behaves nicely in $span\mathcal{F}$ or $cl(span\mathcal{F})$ in order to bound KL divergence with $d_{\mathcal{F}}(\mu,\nu)$. If either μ or ν are empirical measures, the bound is vacuum since $\mathrm{KL}(\mu,\nu) + \mathrm{KL}(\mu,\nu)$ equals infinite, while $d_{\mathcal{F}}(\mu,\nu)$ remains finite once $\mathcal{F}$ is bounded, i.e., $||f||_\infty \leq \Delta < \infty,\ \ \forall f \in \mathcal{F}$. Obviously, this is rather strong condition that is hard to satisfy in practice, because practical data distributions and generators in GANs often have no densities or at least highly peaky densities. We draw more discussion in Corollary 3.5.

3 GENERALIZATION PROPERTY OF GANS

Section 2 suggests that it is better to use larger discriminator sets $\mathcal{F}$ in order to obtain stronger neural distances. However, why do regularization techniques, such as the weight clipping in W-GAN, which effectively shrink the discriminator sets, help GAN training in practice?

The answer to this question has to do with the fact that we observe the true model μ only through an i.i.d. sample of size m (whose empirical measure is denoted by $\hat{\mu}_m$), and hence can only optimize the empirical loss $d_{\mathcal{F}}(\hat{\mu}_m, \nu)$, instead of the exact loss $d_{\mathcal{F}}(\mu, \nu)$. Therefore, generalization bounds are required to control the exact loss $d_{\mathcal{F}}(\mu, \nu)$ when we can only minimize its empirical version $d_{\mathcal{F}}(\hat{\mu}_m, \nu)$. Specifically, let $\mathcal{G}$ be a class of generators that may or may not include the unknown true distribution μ. Assume ν_m minimizes the GAN loss $d_{\mathcal{F}}(\hat{\mu}_m, \nu)$ upto an ϵ ($\epsilon \geq 0$) accuracy, that is,

$$d_{\mathcal{F}}(\hat{\mu}_m, \nu_m) \leq \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\hat{\mu}_m, \nu) \ + \ \epsilon. \tag{9}$$

We are interested in bounding the difference between ν_m and the unknown μ under a proper metric of interest. Depending on what we care about, we may interested in the generalization error in terms of the neural distance $d_{\mathcal{F}}(\mu, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu)$, or other more typical quantities of interest such as BL distance $d_{\mathrm{BL}}(\mu, \nu)$ and KL divergence $\mathrm{KL}(\mu, \nu)$ or the testing likelihood.

In this section, we develop a framework for understanding the generalization proprieties of GANs, showing that the discriminator set $\mathcal{F}$ should be chosen to be small enough to be generalizable, striking a tradeoff with the other requirement that it should be large enough to be discriminative. A surprising result is that the generalization error can be bounded purely by the Rademacher complexity of the discriminator set $\mathcal{F}$ and is independent of the generator set $\mathcal{G}$. Therefore, $\mathcal{G}$ can be chosen to be as large as possible to reduce the model error. Using the bound in (2.1), we also discuss the conditions under which we can bound the negative testing likelihood of GANs.

3.1 GENERALIZATION IN NEURAL DISTANCE

Using the standard derivation and the optimality condition (9), we have (see Appendix E)

$$\begin{aligned} d_{\mathcal{F}}(\mu, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) &\leq 2 \sup_{f \in \mathcal{F}} |\mathbb{E}_{\mu}[f] - \mathbb{E}_{\hat{\mu}_m}[f]| + \epsilon \\ &= 2 d_{\mathcal{F}}(\mu, \hat{\mu}_m) + \epsilon. \end{aligned} \tag{10}$$

This reduces the problem to bounding the discrepancy $d_{\mathcal{F}}(\mu, \hat{\mu}_m) := \sup_{f \in \mathcal{F}} |\mathbb{E}_{\mu}[f] - \mathbb{E}_{\hat{\mu}_m}[f]|$ between the true model μ and its empirical version $\hat{\mu}_m$. This can be achieved by the uniform concentration bounds developed in statistical learning theory (e.g., Vapnik & Vapnik, 1998) and empirical process (e.g., Van de Geer, 2000). In particular, the concentration property related to $\sup_{f \in \mathcal{F}} |\mathbb{E}_{\mu}[f] - \mathbb{E}_{\hat{\mu}_m}[f]|$ can be characterized by the Rademacher complexity of $\mathcal{F}$ (w.r.t. measure μ), defined as

$$R_m^{(\mu)}(\mathcal{F}) := \mathbb{E}\left[\sup_{f \in \mathcal{F}} \left| \frac{2}{m} \sum_i \tau_i f(X_i) \right| \right], \tag{11}$$

where the expectation is taken w.r.t. $X_i \sim \mu$, and Rademacher random variable τ_i: $\mathrm{prob}(\tau_i = 1) = \mathrm{prob}(\tau_i = -1) = 1/2$. Intuitively, $R_m^{(\mu)}(\mathcal{F})$ characterizes the ability of overfitting with pure random labels using functions in $\mathcal{F}$ and hence relates to the generalization bounds. Standard results in learning theory shows that

$$\sup_{f \in \mathcal{F}} |\mathbb{E}_{\mu}[f] - \mathbb{E}_{\hat{\mu}_m}[f]| \leq \Delta R_m^{(\mu)}(\mathcal{F}) + O(\Delta \sqrt{\frac{\log(1/\delta)}{m}}),$$

where $\Delta = \sup_{f \in \mathcal{F}} ||f||_{\infty}$. Combining this with (10), we obtain the following result.

Theorem 3.1. *Assume that all discriminators are bounded by Δ, i.e., $\|f\|_{\infty} \leq \Delta$ for any $f \in \mathcal{F}$. Let $\hat{\mu}_m$ be an empirical measure of an i.i.d. sample of size m drawn from μ. Assume $\nu_m \in \mathcal{G}$ satisfies $d_{\mathcal{F}}(\hat{\mu}_m, \nu_m) \leq \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\hat{\mu}_m, \nu) + \epsilon$. Then with probability at least $1 - \delta$, we have*

$$d_{\mathcal{F}}(\mu, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \leq R_m^{(\mu)}(\mathcal{F}) + 2\Delta \sqrt{\frac{2\log(1/\delta)}{m}} + \epsilon, \tag{12}$$

where $R_m^{(\mu)}(\mathcal{F})$ is the Rademacher complexity of $\mathcal{F}$ defined in (11).

Theorem 3.1 relates the generalization error of GANs to the Rademacher complexity of the discriminator set $\mathcal{F}$. The smaller the discriminator set $\mathcal{F}$ is, the more generalizable the result is. Therefore, the choice of $\mathcal{F}$ should strike a subtle balance between the generalizability and the discriminative power: $\mathcal{F}$ should be large enough to make $d_{\mathcal{F}}(\mu, \nu)$ discriminative as we discuss in Section 2.1, and simultaneously should be small enough to have a small generalization error in (12). It turns out parametric neural discriminators strike a good balance for this purpose, given that it is both discriminative as we show in Section 2.1, and give small generalization bound as we show in the following.

Corollary 3.2. *Let X be the unit ball of $\mathbb{R}^d$ under norm $||\cdot||_2$, that is, $X = \{x \in \mathbb{R}^d \colon ||x||_2 \leq 1\}$. Assume that $\mathcal{F}$ is neural networks with a single rectified linear unit (ReLU) $\mathcal{F} = \{\max(v^\top [x; 1],\ 0) \colon v \in \mathbb{R}^{d+1},\ ||v||_2 = 1\}$. Then with probability at least $1 - \delta$,*

$$d_{\mathcal{F}}(\mu, \nu_m) \leq \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \ + \ \frac{C}{\sqrt{m}} \ + \ \epsilon \tag{13}$$

and

$$d_{\mathrm{BL}}(\mu, \nu_m) = \tilde{O}\left(\left[\inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \ + \ \frac{C}{\sqrt{m}} \ + \ \epsilon \right]^{\frac{1}{(d+3)/2}} \right), \tag{14}$$

where $C = 2\sqrt{2} + 4\sqrt{\log(1/\delta)}$ and $\tilde{O}$ denotes the big-O notation ignoring the logarithm factor.

Note that the three terms in Eqn. (13) takes into account the model error ($\inf_{v \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu)$), sample complexity and generalization error ($C/\sqrt{m}$), and optimization error (ϵ), respectively. Assume zero model error and optimization error, then (1) $d_{\mathcal{F}}(\mu, \nu_m) = O(m^{-1/2})$, which achieves the typical parametric convergence rate; (2) we have $d_{\mathrm{BL}}(\mu, \nu_m) = \tilde{O}(m^{-\frac{1}{d+3}})$, which becomes slower as the dimension d increases. This decrease is because of the non-parametric nature of BL distance, instead of learning algorithm. As we show in Appendix B, we obtain a similar rate of $d_{\mathrm{BL}}(\mu, \nu_m) = O(m^{-\frac{1}{d}})$, even if we directly use BL distance as the learning objective.

Similar results can be obtained for general parametrized discriminator class as follows.

Corollary 3.3. *Under the condition of Theorem 3.1, we further assume that (1) $\mathcal{F} = \{f_\theta : \theta \in \Theta \subset [-1, 1]^p\}$ is a parametric function class with p parameters in a bounded set Θ and that (2) every f_θ is L-Lipschitz continuous with respect to the parameters θ. Then with probability at least $1 - \delta$, we have*

$$d_{\mathcal{F}}(\mu, \nu_m) \leq \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \ + \ \frac{C}{\sqrt{m}} \ + \ 2\epsilon, \tag{15}$$

where $C = 2\Delta(\sqrt{2p \log(Lp/\epsilon)} + \sqrt{2\log(1/\delta)})$.

This result can be easily applied to neural discriminators, since neural networks $f_\theta(x)$ are generally Lipschitz w.r.t. the parameter θ, once the input domain X is bounded.

With the basic result in Theorem 3.1, we can also discuss the learning bounds of GANs with choices of non-parametric discriminators. Making use of Rademacher complexity of bounded sets in a RKHS (e.g., Lemma 22 in Bartlett & Mendelson (2003)), we give the learning bound of MMD-based GANs (Li et al., 2015; Dziugaite et al., 2015) as follows. We present the results for Wasserstain distance and total variance distance in Appendix B, and highlight the advantages of using parametric neural discriminators.

Corollary 3.4. *Under the condition of Theorem 3.1, we further assume that $\mathcal{F} = \{f \in \mathcal{H} \colon \|f\|_{\mathcal{H}} \leq 1\}$ where $\mathcal{H}$ is a RKHS whose positive definite kernel $k(x, x')$ satisfies $k(x, x) \leq C_k < +\infty$ for all $x \in X$. Then with probability at least $1 - \delta$,*

$$d_{\mathcal{F}}(\mu, \nu_m) \leq \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \ + \ \frac{C}{\sqrt{m}} \ + \ \epsilon, \tag{16}$$

where $C = 2\left(1 + \sqrt{2\log(1/\delta)}\right)\sqrt{C_k}$.

Remark 3.1 (Comparisons with results in Arora et al. (2017))**.** *Arora et al. (2017) also discussed the generalization properites of GANs under a similar framework. In particular, they developed bounds of form $|d_{\mathcal{F}}(\mu, \nu) - d_{\mathcal{F}}(\hat{\mu}_m, \hat{\nu}_m)|$ where $\hat{\mu}_m$ and $\hat{\nu}_m$ are empirical versions of the target*

distribution μ and ν with sample size m. Our framework is similar, but considers bounding the quantity $d_{\mathcal{F}}(\mu, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu)$, which is of more direct interest. In fact, our Eqn. (10) *shows that our generalization error can be bounded by the generalization error studied in Arora et al. (2017), and thus our results are similar. Our framework allows us to understand the generalization properties of GANs more clearly, and observes the surprising fact that the generalization of GANs is independent with the hypothesis set $\mathcal{G}$.*

The Role of the Generator Set $\mathcal{G}$. Compared with the discriminator set $\mathcal{F}$ that involves a subtle trade-off, the role of the generator set, or the hypothesis set $\mathcal{G}$ only contributes to the model error $\inf_{v \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu)$, and does not influence the generalization error. This violates the conventional intuition in the traditional statistical learning frameworks, where larger hypothesis set always leads to higher risk of overfitting. For example, the generalizability of the maximum likelihood estimator $\max_{\nu \in \mathcal{G}} \mathbb{E}_{\hat{\mu}_m}[\log \rho_\nu(x)]$ is determined by the size of the hypothesis set $\mathcal{G}$, forming the classical trade-off between the model error and overfitting risk. However, under the GAN framework, one does not have to scarify generalizability for model error as long as the discriminator set is small enough. This seems a critical difference, perhaps advantage of GANs over the traditional learning frameworks.

Bounding the KL Divergence and Testing Likelihood. However, the above result depends on the evaluation metric we use, which are $d_{\mathcal{F}}(\mu, \nu)$ or $d_{\mathrm{BL}}(\mu, \nu)$. If we are interested evaluating the model using even stronger metrics, such as KL divergence or equivalently testing likelihood, then the generator set $\mathcal{G}$ enters the scene in a more subtle way, in that a larger generator set $\mathcal{G}$ should be companioned with a larger discriminator set $\mathcal{F}$ in order to provide meaningful bounds on KL divergence. This is illustrated in the following result obtained by combining Theorem 3.1 and Proposition 2.9.

Corollary 3.5. *Assume both the true μ and all the generators $\nu \in \mathcal{G}$ have positive densities ρ_μ and ρ_ν, respectively. Assume $\mathcal{F}$ consists of bounded functions with $\Delta := \sup_{f \in \mathcal{F}} ||f||_\infty < \infty$.*

Further, assume the discriminator set $\mathcal{F}$ is compatible *with the generator set $\mathcal{G}$ in the sense that $\log(\rho_\nu/\rho_\mu) \in span\mathcal{F}$, $\forall \nu \in \mathcal{G}$, with a compatible coefficient defined as*

$$\Lambda_{\mathcal{F},\mathcal{G}} := \sup_{\nu \in \mathcal{G}} ||\log(\rho_\nu/\rho_\mu)||_{\mathcal{F},1} < \infty.$$

Then

$$\mathrm{KL}(\mu, \nu_m) \leq \Lambda_{\mathcal{F},\mathcal{G}} \left(2R_m^{(\mu)}(\mathcal{F}) + 2\Delta\sqrt{2\log(1/\delta)/m} + \Delta \inf_{\nu \in \mathcal{G}} \sqrt{\mathrm{KL}(\mu, \nu)} + \epsilon\right). \tag{17}$$

Different from the earlier bounds, the bound in (17) depends on the compatibility coefficient $\Lambda_{\mathcal{F},\mathcal{G}}$ that casts a more interesting trade-off on the choice of the generator set $\mathcal{G}$: the generator set $\mathcal{G}$ should be small and have well behaved density functions to ensure a small $\Lambda_{\mathcal{F},\mathcal{G}}$, while should be large enough to have a small model error $\inf_{\nu \in \mathcal{G}} \sqrt{\mathrm{KL}(\mu, \nu)}$. Related, the discriminator set should be large enough to include all density ratios $\log(\rho_\mu/\rho_\nu)$ in a ball of radius $\Lambda_{\mathcal{F},\mathcal{G}}$ of $span\mathcal{F}$, and should also be small to have a low Rademacher complexity $R_m^{(\mu)}(\mathcal{F})$. Obviously, one can also extend Corollary 3.5 using (8) in Proposition 2.7, to allow $\log(\rho_\mu/\rho_\nu) \in cl(span\mathcal{F})$ in which case the compatibility of $\mathcal{G}$ and $\mathcal{F}$ should be mainly characterized by the error decay function $\epsilon(r)$. $\mathrm{KL}(\mu, \nu_m) = \mathbb{E}_\mu[\log p_\mu] - \mathbb{E}_\mu[\log p_{\nu_m}]$ is the difference between the testing likelihood $\mathbb{E}_\mu[\log p_{\nu_m}]$ of estimated model ν_m and the best possible testing likelihood $\mathbb{E}_\mu[\log p_\mu]$. Therefore, Corollary 3.5 also provides a bound for testing likelihood. Unfortunately, the conditions in Corollary 3.5 is rather strong, in that it requires that both the true distribution μ and the generators ν to have positive densities and the log-density ratios $\log(\rho_\mu/\rho_\nu)$ is well-behaved. In practical applications of computer vision, however, both μ and ν tend to concentrate on local regions or sub-manifolds of X, with very peaky densities, or even no valid densities; this causes the compatibility coefficient $\Lambda_{\mathcal{F},\mathcal{G}}$ very large, or infinite, making the bound in (17) loose or vacuum. This provides a potential explain for some of the recent empirical findings (e.g., Danihelka et al., 2017; Grover et al., 2017) that the negative testing likelihood is uncorrelated with the GAN loss functions, or even increases as the GAN training progress. The underlying reason here is that the neural distance is not strong enough to provide meaningful bound for KL divergence. See Appendix G for an illustration using toy examples.

4 NEURAL ϕ-DIVERGENCE

We have been focusing on GANs that are based on integral probability metrics (IPM), which cover W-GANs and MMD-GANs. f-GAN is another broad family of GANs that are based on minimizing f-divergence (also called ϕ-divergence) (Nowozin et al., 2016), which includes the original GAN by Goodfellow et al. (2014). [1] However, ϕ-divergence has substantially different properties from IPM (see e.g., Sriperumbudur et al. (2009)), and is not defined as the intuitive moment matching form as IPM. In this section, we extend our analysis to ϕ-divergence by interpreting it as a form of *penalized moment matching*. Similar to the case of IPM, we analyze the *neural ϕ-divergence* that restricts the discriminators to parametric sets $\mathcal{F}$ for practical computability, and establish its discriminative and generalizable properties under mild conditions that practical f-GANs satisfy.

Assume that μ and ν are two distributions on X. Given a convex, lower-semicontinuous univariate function ϕ that satisfies $\phi(1) = 0$, the related ϕ-divergence is $d_\phi(\mu \,||\, \nu) = \mathbb{E}_\nu \left[\phi\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)\right]$. If ϕ is strictly convex, then a standard derivation based on Jensen's inequality shows that ϕ-divergence is nonnegative and discriminative: $d_\phi(\mu \,||\, \nu) \geq \phi(1) = 0$ and the equality holds iff $\mu = \nu$. Different choices of ϕ recover popular divergences as special cases. For example, $\phi(t) = (t-1)^2$ recovers Pearson χ^2 divergence, and $\phi(t) = (u+1)\log((u+1)/2) + u\log u$ gives the Jensen-Shannon divergence used in the vanilla GAN Goodfellow et al. (2014).

In this work, we find it helps to develop intuition by introducing another convex function $\psi(t) := \phi(t+1)$, defined by shifting the input variable of ϕ by $+1$; the ϕ-divergence becomes

$$d_\phi(\mu \,||\, \nu) = \mathbb{E}_\nu \left[\psi\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu} - 1\right)\right] = \int_X \rho_\nu(x)\psi\left(\frac{\rho_\mu(x)}{\rho_\nu(x)} - 1\right)\tau(\mathrm{d}x), \tag{18}$$

where we should require that $\psi(0) = 0$; in right hand side of (18), we assume ρ_μ and ρ_ν are the density functions of μ and ν, respectively, under a base measure τ. The key advantage of introducing ψ is that it gives a suggestive variational representation that can be viewed as a regularized moment matching. Specially, assume ψ^* is the convex conjugate of ψ, that is, $\psi^*(t) = \sup_y\{yt - \psi(y)\}$. By standard derivation, we can show that

$$d_\phi(\mu \,||\, \nu) \geq \sup_{f\in\mathcal{A}} \left(\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f] - \Psi_{\nu,\psi^*}[f]\right), \quad \text{with} \quad \Psi_{\nu,\psi^*}[f] := \mathbb{E}_{x\sim\nu}[\psi^*(f(x))], \tag{19}$$

where $\mathcal{A}$ is *the class of all functions* $f : X \to \mathrm{dom}(\psi^*)$ where $\mathrm{dom}(\psi^*) = \{t\colon \psi^*(t) \in \mathbb{R}\}$, and the equality holds if $\varphi^*(\frac{\rho_\mu(x)}{\rho_\nu(x)} - 1) \in \mathcal{A}$ where φ is the inverse function of $\psi^{*\prime}$. In (19), the term $\Psi_{\nu,\psi^*}[f]$, as we show in Lemma 4.1 in sequel, can be viewed as a type of complexity penalty on f that ensures the supreme is finite. This is in contrast with the IPM $d_\mathcal{F}(\mu,\nu)$ in which the complexity constraint is directly imposed using the function class $\mathcal{F}$, instead of a regularization term.

Lemma 4.1. *Assume $\psi\colon \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ is a convex, lower-semicontinuous function with conjugate ψ^* and $\psi(0) = 0$. The penalty $\Psi_{\nu,\psi^*}[f]$ in* (19) *has the following properties*

i) $\Psi_{\nu,\psi^*}[f]$ is a convex functional of f, and $\Psi_{\nu,\psi^*}[f] \geq 0$ for any f.

ii) There exists a constant $b_0 \in \mathbb{R}\cup\{\infty\}$ such that $\psi^*(b_0) = 0$. Further, if ψ is strictly convex, then $\Psi_{\nu,\psi^*}[f] = 0$ implies $f(x) = b_0$ all most surely under measure ν.

In practice, it is impossible to numerically optimize over the class of all functions in (19). Instead, practical f-GANs restrict the optimization to a parametric set $\mathcal{F}$ of neural networks, yielding the following *neural ϕ-divergence*:

$$d_{\phi,\mathcal{F}}(\mu \,||\, \nu) = \sup_{f\in\mathcal{F}} \left(\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f] - \Psi_{\nu,\psi^*}[f]\right). \tag{20}$$

Note that this can be viewed as a generalization of the $\mathcal{F}$-related IPM $d_\mathcal{F}(\mu,\nu)$ by considering $\psi^* = 0$. However, the properties of the neural ϕ-divergence can be significantly different from that of $d_\mathcal{F}(\mu,\nu)$. For example, $d_{\phi,\mathcal{F}}(\mu \,||\, \nu)$ is not even guaranteed to be non-negative for arbitrary discriminator sets $\mathcal{F}$ because of the negative regularization term. Fortunately, we can still establish the non-negativity and discriminative property of $d_{\phi,\mathcal{F}}(\mu \,||\, \nu)$ under certain weak conditions on $\mathcal{F}$. Moreover, the property that $d_\mathcal{F}(\mu,\nu) = 0$ implies moment matching on $\mathcal{F}$, which is the key step to

[1] In this section, we call it ϕ-divergence because f has been used for discriminators.

establish the discriminative power, is not necessarily true for neural divergence. Fortunately, it turns out that $d_{\phi,\mathcal{F}}(\mu \,||\, \nu) = 0$ implies moment matching on features defined by the last linear layer of discriminators.

Theorem 4.1. *Assume $\mathcal{F}$ includes the constant function $b_0 \in \mathbb{R}$, which satisfies $\psi^*(b_0) = 0$ as defined in Lemma 4.1. We have*

i) $0 \leq d_{\phi,\mathcal{F}}(\mu \,||\, \nu) \leq d_{\mathcal{F}}(\mu, \nu)$ for $\forall \mu, \nu$. As a result,

$$d_{\mathcal{F}}(\mu, \nu) = 0 \quad \textit{implies} \quad d_{\phi,\mathcal{F}}(\mu \,||\, \nu) = 0.$$

In other words, moment matching on $\mathcal{F}$ is a sufficient condition of zero neural ϕ-divergence.

ii) Further, we assume $\mathcal{F}$ has the following form:

$$\mathcal{F} \supseteq \{\sigma(\alpha f_0 + c_0) \colon \forall |\alpha| \leq \alpha_{f_0}, \ \textit{and} \ f_0 \in \mathcal{F}_0\}, \tag{21}$$

where $\mathcal{F}_0$ is any function set, and $\alpha_{f_0} > 0$ is positive number associated with each $f_0 \in \mathcal{F}_0$, and c_0 is a constant and $\sigma \colon \mathbb{R} \to \mathbb{R}$ is any function that satisfies $\sigma(c_0) = b_0$ and $\sigma'(c_0) > 0$. Here σ can be viewed as the activation of the last layer of a deep neural network whose earlier layers are specified by $\mathcal{F}_0$. Assume $\psi^*(y)$ is differentiable at $y = b_0$. Then

$$d_{\phi,\mathcal{F}}(\mu \,||\, \nu) = 0 \quad \textit{implies} \quad d_{\mathcal{F}_0}(\mu \,,\, \nu) = 0.$$

In other words, moment matching on $\mathcal{F}_0$ is a necessary condition of zero neural ϕ-divergence.

iii) $cl(span\mathcal{F}_0) \supseteq C_b(X)$ is a sufficient condition for $d_{\phi,\mathcal{F}}$ to be discriminative, i.e., $d_{\phi,\mathcal{F}}(\mu \,||\, \nu) = 0$ implies $\mu = \nu$.

Condition (21) defines a commonly used structure of $\mathcal{F}$ that naturally satisfied by the f-GANs used in practice; in particular, the output activation function σ plays the role of ensuring the output of $\mathcal{F}$ respects the input domain of the convex function ψ^*. For example, the vanilla GAN has $\psi^* = -\log(1 - \exp(t)) - t$ with an input domain of $(-\infty, 0)$, and activation function is taken to be $\sigma(t) = -\log(1 + \exp(-t))$. See Table 2 of Nowozin et al. (2016) for the list of output activation functions related to commonly used ψ.

Similar to Theorem 2.5, under the conditions of Theorem 4.1, we have $\lim_n d_{\phi,\mathcal{F}}(\mu \,||\, \nu_n) = 0$ implies $\nu_n \rightharpoonup \mu$. See the exact statement in Theorem C.1 in Appendix C.

Remark 4.1. *Our results on neural ϕ-divergence can in general extended to the more unified framework of Liu et al. (2017) in which divergences of form $\max_f \mathbb{E}_{(x,y)\sim\mu\otimes\nu}[f(x, y)]$ are studied. We choose to focus on ϕ-divergence because of its practical importance. Our Theorem 4.1 i) can be viewed as a special case of Theorem 4 of Liu et al. (2017) and our Theorem 4.1 ii) is related to Theorem 5 of Liu et al. (2017). However, note that Theorem 5 of Liu et al. (2017) requires a rather counter-intuitive condition, while our condition in Theorem 4.1 ii) is clear and satisfied by all ϕ-divergence listed in Nowozin et al. (2016).*

Similar to the case of neural distance, we can establish generalization bounds for neural ϕ-divergence.

Theorem 4.2. *Assume that $\|f\| \leq \Delta$ for any $f \in \mathcal{F}$. $\hat{\mu}_m$ is an empirical distribution with m samples from μ, and $\nu_m \in \mathcal{G}$ satisfies $d_{\phi,\mathcal{F}}(\hat{\mu}_m \,||\, \nu_m) \leq \inf_{\nu\in\mathcal{G}} d_{\phi,\mathcal{F}}(\hat{\mu}_m \,||\, \nu) + \epsilon$. Then with probability at least $1 - \delta$, we have*

$$d_{\phi,\mathcal{F}}(\mu \,||\, \nu_m) \leq \inf_{\nu\in\mathcal{G}} d_{\phi,\mathcal{F}}(\mu \,||\, \nu) + R_m^{(\mu)}(\mathcal{F}) + 2\Delta\sqrt{\frac{2\log(1/\delta)}{m}} + \epsilon, \tag{22}$$

where $R_m^{(\mu)}(\mathcal{F})$ is the Rademacher complexity of $\mathcal{F}$.

With Theorem 4.2, we obtain generalization bounds for difference choices of $\mathcal{F}$, as we had in section 3. We list those results in Appendix C.

5 Related work

There is a surge of research interest in GANs; however, most of the work has been empirical in nature. There has been some theoretical literature on understanding GANs, including the discrimination and generalization properties of GANs.

The discriminative power of GANs is typically justified by assuming that the discriminator set $\mathcal{F}$ has enough capacity. For example, Goodfellow et al. (2014) assumes that the optimal discriminator $\frac{p_{\text{data}(x)}}{p_{\text{data}(x)}+p_{\text{g}(x)}} \in \mathcal{F}$. Similar capacity assumptions have been made in nearly all other GANs to prove its discriminative power; see, e.g., Zhao et al. (2016); Nowozin et al. (2016); Arora et al. (2017). However, discriminators are in practice taken as certain parametric function class, like neural networks, which violates these capacity assumptions. The universal approximation property of neural networks is used to justify the discriminative power empirically. In this work, we show that the GAN loss is discriminative if $span\mathcal{F}$ can approximate any continuous functions. This condition is very weak and can be satisfied even when none of the discriminators is close to the optimal discriminator. The MMD-based GANs (Li et al., 2015; Dziugaite et al., 2015; Li et al., 2017a) avoid the parametrization of discriminators by taking advantage of the close-form solution of the optimal discriminator in the non-parametric RKHS space. Therefore, the capacity assumption is satisfied in MMD-based GANs, and their discriminative power is easily justified.

Liu et al. (2017) defines a notion of adversarial divergences that includes a number of GAN objective functions. They show that if the objective function is an adversarial divergence with some additional conditions, then using a restricted discriminator family has a moment-matching effect. Our treatment of the neural divergence was directly inspired by them. We refer to Remark 4.1 for a detailed comparison. Liu et al. (2017) also show that for objective functions that are strict adversarial divergence, convergence in the objective function implies weak convergence. However, they do not provide a condition under which a adversarial divergence is strict. A major contribution of our work is to fill this gap, and to provide such a condition that is sufficient and necessary.

Dziugaite et al. (2015) studies generalization error, defined as $d_{\mathcal{F}}(\mu, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu)$ in our notation, for MMD-GAN in terms of fat-shattering dimension. Moreover, Dziugaite et al. (2015) obtains a generalization bound that incorporates the complexity of the hypothesis set $\mathcal{G}$. Although their overall error bound is still $O(m^{-1/2})$, their work shows the possibility to sharpen our $\mathcal{G}$-independent bound. Arora et al. (2017) studies the generalization properties of GANs through the quantity $d_{\mathcal{F}}(\mu, \nu) - d_{\mathcal{F}}(\hat{\mu}_m, \hat{\nu}_m)$ (in our notations). The main difference between our work and Arora et al. (2017) is the definition of generalization error; see more discussions in Remark 3.1. Moreover, Arora et al. (2017) allows only polynomial number of samples from the generated distribution because the training algorithm should run in polynomial time. We do not consider this issue because in this work we only study the statistical properties of the objective functions and do not touch the optimization method. Finally, Arora et al. (2017) showed that the GAN loss can approach its optimal value even if the generated distribution has very low support, and (Arora & Zhang, 2017) showed empirical evidence for this problem. Our result is consistent with their results because our generalization error is measured by the neural distance/divergence.

Finally, there has been some other lines of research on understanding GANs. Li et al. (2017b) study the dynamics of GAN's training and find that: a GAN with an optimal discriminator provably converges, while a first order approximation of the discriminator leads to unstable dynamics and mode collapse. Lei et al. (2017) study WGAN and optimal transportation by convex geometry, and provide a close-form formula for the optimal transportation map. Hu et al. (2017) provide a new formulation of GANs and variational autoencoders (VAEs), and thus unify the most two popular methods to train deep generative models. We'd like to mention other recent interesting research on GANs, e.g., (Guo et al., 2017; Sinn & Rawat, 2017; Nock et al., 2017; Mescheder et al., 2017; Tolstikhin et al., 2017).

6 Conclusions

We studied the discrimination and generalization properties of GANs with parameterized discriminator class such as neural networks. A neural distance is guaranteed to be discriminative whenever the linear span of its discriminator set is dense in the set of bounded continuous functions. On the other hand, a neural divergence is discriminative whenever the linear span of features defined by the last linear layer of its discriminators is dense in the set of bounded continuous functions. We also provide a generalization bound for GANs in different evaluation metrics. In terms of neural or Wasserstein distances, our bounds show that generalization is guaranteed as long as the discriminator set is small enough, regardless of the size of the generator or hypothesis set. This raises an interesting discrimination-generalization balance in GANs. Fortunately, several GAN methods in

practice already choose their discriminator set at the sweet point, where both the discrimination and generalization hold. Finally, our generalization bound in KL divergence provides an explanation on the counter-intuitive behaviors of testing likelihood in GAN training.

There are several directions that we would like to explore in the future. First of all, in this paper, we do not talk about methods to compute the neural distance/divergence. This is typically a non-concave maximization problem and is extremely difficult to solve. Many methods have been proposed to solve this kind of minimax problems, but both stable training methods and theoretical analysis of these algorithms are still missing. Secondly, our generalization bound depends purely on the discriminator set. It's possible to obtain sharper bounds by incorporating structural information from the generator set. Finally, we would like to extend our analysis to conditional GANs (see, e.g., Mirza & Osindero (2014); Springenberg (2015); Chen et al. (2016); Odena et al. (2016)), which have demonstrated impressive performance (Reed et al., 2016a;b; Zhang et al., 2017).

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A DISCRIMINATIVE IN $\mathcal{P}_{\tau,p}(X)$

In Theorem 2.2, we show that the integral probability metric (IPM) $d_{\mathcal{F}}$ is discriminative in the Borel measure set $\mathcal{P}_{\mathcal{B}}(X)$ if $span\mathcal{F}$ is dense in the space of bounded continuous functions $C_b(X)$. There is considerable interest in discontinuous function class approximation, such as neural networks with the threshold activation function $\sigma(u) = \mathbf{1}_{u>0}$. Discontinuous functions are not used as often as continuous ones due to the lack of good training algorithms, but they are of theoretical interest because of their close relationship to classical perceptrons Minsky & Papert (1988). In this discontinuous scenario, we have a similar result as below.

Given a Borel probability measure τ on X, we can also define the Banach space $L^p_\tau(X) = \{f : \mathbb{E}_\tau[|f|^p] \leq \infty\}$, the set of probability measures $\mathcal{P}_{\tau,p}(X) = \{\mu \in \mathcal{P}_{\mathcal{B}}(X) : \frac{\mathrm{d}\mu}{\mathrm{d}\tau} \in L^p_\tau(X)\}$ and the subspace of Borel measures $M_{\tau,p}(X) = \{\mu \in M_{\mathcal{B}}(X) : \frac{\mathrm{d}\mu}{\mathrm{d}\tau} \in L^p_\tau(X)\}$. Here, $\frac{\mathrm{d}\mu}{\mathrm{d}\tau}$ is the Radon-Nikodym derivative.

Let $p \in (1, \infty]$ and $q = \frac{p}{p-1}$. Similar to Definition 2.1, we define that $\mathcal{F} \subset L^q_\tau(X)$ is discriminative in $\mathcal{P}_{\tau,p}(X)$ if for any two probability measures $\mu, \nu \in \mathcal{P}_{\tau,p}(X)$,

$$\mathbb{E}_\mu[f] = \mathbb{E}_\nu[f] \quad \forall f \in \mathcal{F}$$

implies $\mu = \nu$. Similar to Theorem 2.2, we have that $\mathcal{F} \subset L^q_\tau(X)$ is discriminative in $\mathcal{P}_{\tau,p}(X)$ if and only if span$(\mathcal{F} \cup \{\mathbf{1}\})$ is dense in $L^q_\tau(X)$.

Different function classes (e.g., $C_b(X)$ or $L^q_\tau(X)$) can discriminate different probability measures (e.g., $\mathcal{P}_{\mathcal{B}}(X)$ or $\mathcal{P}_{\tau,p}(X)$). For GAN's applications where we want to generate high dimensional objects (e.g., images) from low dimensional hidden variables (e.g., a Gaussian random noise), the induced probability measures on the high dimensional space is not resolutely continuous with respect to its Lebesgue measure. Moreover, the empirical distribution of data points is not resolutely continuous with respect to its Lebesgue measure, either. Therefore, a discriminator function class whose span is only dense in $L^q_\tau(X)$ is not enough. However, if the generator function is continuous, the induced probability measures are Borel. The empirical distribution of data points is a Borel measure, too. Therefore, a discriminator function class whose span is dense in $C_b(X)$ is sufficient to discriminate the data distribution and the generated distributions.

B GENERALIZATION ERROR OF OTHER DISCRIMINATOR SETS $\mathcal{F}$

With the basic result in Theorem 3.1, we discuss the learning bounds of GANs with other choices of non-parametric discriminator sets $\mathcal{F}$. This allows us to highlight the advantages of using parametric neural discriminators. For simplicity, we assume zero model error and optimization so that the bound is solely based on the generalization error $d_{\mathcal{F}}(\mu, \hat{\mu}_m)$ between μ and its empirical version $\hat{\mu}_m$.

1. Bounded Lipschitz distance, $\mathcal{F} = \{f \in C(X) : ||f||_{\mathrm{BL}} \leq 1\}$, which is equivalent to Wasserstein distance when X is compact. When X is a convex bounded set in $\mathbb{R}^d$, we have $R^{(\mu)}_m(\mathcal{F}) \leq m^{-1/d}$ for $d > 2$ (see Corollary 12 in Sriperumbudur et al. (2009)), and hence $d_{\mathrm{BL}}(\mu, \nu) = O(m^{-1/d})$. This is comparable with Corollary 3.2.

   This bound is tight. Assume that μ is the uniform distribution on X. A simple derivation (similar to Lemma 1 in Arora et al. (2017)) shows that $d_{\mathcal{F}}(\mu, \hat{\mu}_m) \geq c(1 - m\exp(-\Omega(d)))$ for some constant only depending on X. Therefore, one must need at least $m = \exp(\Omega(d))$ samples to reduce $d_{\mathcal{F}}(\mu, \hat{\mu}_m)$, and hence the generalization bound, to $O(\epsilon)$.
2. Total variation (TV) distance, $\mathcal{F} = \{f \in C(X) : \|f\| \leq \min\{1, \Delta\}\}$. It is easy to verify that $R^{(\mu)}_m(\mathcal{F}) = 2$. Therefore, Eqn. (12) cannot guarantee generalization even when we have infinite number of samples, i.e., $m \to \infty$.

   The estimate given in Eqn. (12) is tight. Assume that μ is the uniform distribution on X. It is easy to see that $d_{\mathrm{TV}}(\mu, \hat{\mu}_m) = 2$ almost surely. Therefore, ν_m is close to $\hat{\mu}_m$ implies that it is order 1 away from μ, which means that generalization does not hold in this case.

With the statement that training with the TV distance does not generalize, we mean that training with TV distance does not generalize in TV distance. More precisely, even if the training loss on

empirical samples is very small, i.e., $\mathrm{TV}(\hat{\mu}_m, \nu_m) = O(\epsilon)$, the TV distance to the unknown target distribution can be large, i.e., $d_{TV}(\mu, \nu_m) = O(1)$. However, this does not imply that training with TV distance is useless, because it is possible that training with a stronger metric leads to asymptotic vanishing in a weaker metric. For example, $d_{TV}(\hat{\mu}_m, \nu_m) = O(\epsilon)$ implies $d_{\mathcal{F}_{\mathrm{nn}}}(\hat{\mu}_m, \nu_m) = O(\epsilon)$, and thus a small $d_{\mathcal{F}_{\mathrm{nn}}}(\mu, \nu_m)$.

Take the Wasserstein metric as another example, even though we only establish $d_W(\mu, \nu_m) = O(m^{-1/d})$ (assuming zero model error ($\mu \in \mathcal{G}$) and optimization $\epsilon = 0$), it does not eliminate the possibility that the weaker neural distance has a faster convergence rate $d_{\mathcal{F}_{\mathrm{nn}}}(\mu, \nu_m) = O(m^{-1/2})$. From the practical perspective, however, TV and Wasserstein distances are less clearly favorable than neural distance because the difficulty of calculating and optimizing them.

C More results on neural divergence

Theorem C.1. *Let (X, d_X) be a compact metric space and $\mathcal{F}_0 \subset C(X)$ satisfy $cl(span\mathcal{F}_0) \supseteq C(X)$. Further, we assume $\mathcal{F}$ has the following form:*

$$\mathcal{F} \supseteq \{\sigma(\alpha f_0 + c_0) \colon \forall |\alpha| \leq \alpha_{f_0}, \ \text{ and } \ f_0 \in \mathcal{F}_0\},$$

where $\mathcal{F}_0$ is any function set, and $\alpha_{f_0} > 0$ is positive number associated with each $f_0 \in \mathcal{F}_0$, and c_0 is a constant and $\sigma \colon \mathbb{R} \to \mathbb{R}$ is any function that satisfies $\sigma(c_0) = b_0$ and $\sigma'(c_0) > 0$. Then if $\lim_{n\to\infty} d_{\phi,\mathcal{F}}(\mu \,||\, \nu_n) = 0$, ν_n converges to μ in the distribution sense.

Further, if there exists $C > 0$ such that $\mathcal{F} \subset \{f \in C(X) : \|f\|_{Lip} \leq C\}$, we have

$$\lim_{n\to\infty} d_{\phi,\mathcal{F}}(\mu \,||\, \nu_n) = 0 \Longleftrightarrow \nu_n \rightharpoonup \mu.$$

Notice that we assume that (X, d_X) be a compact metric space here for simplicity. A non-compact result is available but its proof is messy and non-intuitive.

Proof. The first half is a direct application of Theorem 4.1 and Theorem 10 in Liu et al. (2017).

For the second half, we have

$$d_{\phi,\mathcal{F}}(\mu \,||\, \nu_n) \leq d_{\mathcal{F}}(\mu, \nu_n) \leq C d_W(\mu, \nu_n),$$

where we use Theorem 4.1 i) in the first inequality and the Lipschitz condition of $\mathcal{F}$ in the second ineqaulity. Since d_W metrizes the weak convergence for compact X, we obtain $d_W(\mu, \nu_n) \to 0$ and thus $d_{\phi,\mathcal{F}}(\mu \,||\, \nu_n) \to 0$. □

For neural divergence with bounded parameter space and Lipschitz continuous discriminators, we have the following result.

Corollary C.2. *Under the condition of Theorem 4.2, we further assume that (1) $\mathcal{F} = \mathcal{F}_{nn} = \{f_\theta : \theta \in \Theta \subset [-1, 1]^p\}$ is a parametric function class with p parameters in a bounded set Θ and that (2) every f_θ is L-Lipschitz continuous with respect to the parameters θ. Then with probability at least $1 - \delta$, we have*

$$d_{\phi,\mathcal{F}_{nn}}(\mu \,||\, \nu_m) \leq \inf_{\nu \in \mathcal{G}} d_{\phi,\mathcal{F}_{nn}}(\mu \,||\, \nu) + \sqrt{C}m + 2\epsilon, \tag{23}$$

where $C = 2\Delta(\sqrt{2p\log(Lp/\epsilon)} + \sqrt{2\log(1/\delta)})$.

Proof. The proof is the same with that of Corollary 3.3. □

D Proof of results in section 2

Proof of Theorem 2.2. In one direction, suppose that $\mathbb{E}_\mu[f] = \mathbb{E}_\nu[f]$ for all $f \in \mathcal{F}$ ($\mathbb{E}_\mu[\mathbf{1}] = \mathbb{E}_\nu[\mathbf{1}] = 1$ holds by definition) and that $cl(\mathrm{span}(\mathcal{F} \cup \{\mathbf{1}\})) = C_b(X)$. Since $\mathbb{E}_\mu[f]$ is continuous in $C_b(X)$ for any probability measure μ, we have $\mathbb{E}_\mu[f] = \mathbb{E}_\nu[f]$ for all $f \in C_b(X)$. Then we have $\mu = \nu$. One has several methods to prove the last step. One is to take $f(x) = e^{iw^T x}$ for any

$w \in \mathbb{R}^d$ and to use the fact that the characteristic function uniquely determines a Borel probability measure. Here, we provide a longer but more elementary proof.

Let $h(t) = \max\{1-t, 0\}$ for $t \geq 0$ and $F \subset X$ be a closed set. For any $k \in \mathbb{N}$, define $f_k(x) = h(kd(x, F))$ and $F_k = \{x \in X : d(x, F) \leq 1/k\}$. It's easy to verify that (1) f_k is continuous, (2) F_k is a closed set and that (3) $\mathbf{1}_F \leq f_k \leq \mathbf{1}_{F_k}$. Then we have

$$\mu(F) \leq \mathbb{E}_\mu[f_k] = \mathbb{E}_\nu[f_k] \leq \nu(F_k),$$

and similarly, $\nu(F) \leq \mu(F_k)$ for all $k \in \mathbb{N}$. Since $F_1 \supset \cdots \supset F_k \supset \cdots \supset F$ and $\cap_{k=1}^\infty F_k = F$, we have $\lim_{k\to\infty} \mu(F_k) = \mu(F)$ and $\lim_{k\to\infty} \nu(F_k) = \nu(F)$. Therefore, we have $\mu(F) = \nu(F)$. Remind that any Borel probability measure μ is regular, i.e., $\mu(A) = \sup\{\mu(F) : F \subset A \text{ is closed}\} = \inf\{\mu(G) : G \supset A \text{ is open}\}$ for any Borel set A [2]. Therefore, we have $\mu = \nu$.

In the other direction, suppose that $\mathcal{F} \subset C_b(X)$ is discriminative in $\mathcal{P}_\mathcal{B}(X)$. Assume that $cl(\text{span}(\mathcal{F} \cup \{\mathbf{1}\}))$ is a strictly closed subspace of $C_b(X)$. Take $g \in C_b(X) \backslash cl(\text{span}(\mathcal{F}))$ and $\|g\|_\infty = 1$. By the Hahn-Banach theorem, there exists a bounded linear functional $L : C(X) \to \mathbb{R}$ such that $L(f) = 0$ for any $f \in cl(\text{span}(\mathcal{F} \cup \{\mathbf{1}\}))$ and $L \neq \mathbf{0}$. Thanks to the Riesz representation theorem for compact metric spaces, there exists a signed, regular Borel measure $m \in M_\mathcal{B}(X)$ such that

$$L(f) = \int_m f \quad \forall f \in C_b(X).$$

Suppose $m = \mu - \nu$ are the Hahn decomposition of m, where μ and ν are two nonnegative Borel measures. Then we have $L(f) = \int_\mu f - \int_\nu f$ for any $f \in C_b(X)$. Thanks to $L(\mathbf{1}) = 0$, we have $0 < \mu(X) = \nu(X) < \infty$. We can assume that μ and ν are Borel probability measures. (Otherwise, we can use the normalized nonzero linear functional $L/\mu(X)$ whose Hahn decomposition consists of two Borel probability measures.) Since $L(f) = 0$ for any $f \in cl(\text{span}(\mathcal{F}))$, we have $\int_\mu f = \int_\nu f$ for any $f \in \mathcal{F}$. Since $\mathcal{F} \subset C_b(X)$ is discriminative, we have $\mu = \nu$ and thus $L = \mathbf{0}$, which leads to a contradiction. □

Proof of Corollary 2.4. Thanks to $\{\lambda\theta : \lambda \geq 0, \theta \in \Theta\} = \mathbb{R}^{d+1}$, for any $[w, b] \in \mathbb{R}^{n+1}$, there exists $[w_0, b_0] \in \Theta$ and $\lambda > 0$ such that

$$\sigma(w^\top x + b) = \sigma(\lambda(w_0^\top x + b_0)) = \lambda^\alpha \sigma(w_0^\top x + b_0),$$

where we used $\sigma(u) = \max\{u, 0\}^\alpha$ in the last step. Therefore, we have

$$span\mathcal{F}_{\text{nn}} \subset span\{\sigma(w^\top x + b) \colon [w, b] \in \mathbb{R}^{d+1}\}.$$

Thanks to Theorem 2.3, we know that $span\mathcal{F}_{\text{nn}}$ is dense in $C_b(X)$. □

Proof of Theorem 2.5. Given a function $g \in C_b(X)$, we say that g is approximated by $\mathcal{F}$ with error decay function $\epsilon(r)$ if for any $r \geq 0$, there exists $f_r \in span\mathcal{F}$ with $||f_r||_{\mathcal{F},1} \leq r$ such that $||f - f_r||_\infty \leq \epsilon(r)$. Obviously, $\epsilon(r)$ is an non-increasing function w.r.t. r. Thanks to $cl(span\mathcal{F}) = C_b(X)$, we have $\lim_{r\to\infty} \epsilon(r) = 0$. Now denote $r_n := d_\mathcal{F}(\mu, \nu_n)^{-1/2}$ and correspondingly $f_n := f_{r_n}$. We have

$$\begin{aligned} |\mathbb{E}_\mu g - \mathbb{E}_{\nu_n} g| &\leq |\mathbb{E}_\mu g - \mathbb{E}_\mu f_n| + |\mathbb{E}_\nu g - \mathbb{E}_\nu f_n| + |\mathbb{E}_\mu f_n - \mathbb{E}_{\nu_n} f_n| \\ &\leq 2\epsilon(r_n) + r_n\, d_\mathcal{F}(\mu, \nu_n). = 2\epsilon(r_n) + 1/r_n. \end{aligned}$$

If $\lim_{n\to\infty} d_\mathcal{F}(\mu, \nu_n) = 0$, we have $\lim_{n\to\infty} r_n = \infty$. Thanks to $\lim_{r\to\infty} \epsilon(r) = 0$, we prove that $\lim_{n\to\infty} |\mathbb{E}_\mu g - \mathbb{E}_{\nu_n} g| = 0$. Since this holds true for any $g \in C_b(X)$, we conclude that ν_n weakly converges to μ.

If $\mathcal{F} \subseteq \text{BL}_C(X)$ for some $C > 0$, we have $d_\mathcal{F}(\mu, \nu) \leq C d_{BL}(\mu, \nu)$ for any μ, ν. Because the bounded Lipschitz distance (also called FortetMourier distance) metrizes the weak convergence, we obtain that $\nu_n \rightharpoonup \mu$ implies $d_{BL}(\mu, \nu_n) \to 0$, and thus $d_\mathcal{F}(\mu, \nu_n) \to 0$. □

[2] This can be proved by the fact that these sets form a σ-algebra.

Proof of Proposition 2.6. Let $g = \sum_{i=1}^n w_i f_i + w_0$. Then we have

$$|\mathbb{E}_\mu g - \mathbb{E}_\nu g| = \sum_{i=1}^n |w_i||\mathbb{E}_\mu f_i - \mathbb{E}_\nu f_i| \leq \left(\sum_{i=1}^n |w_i|\right) d_{\mathcal{F}}(\mu, \nu).$$

The result is obtain by taking infimum over all possible w_i. □

Proof of Proposition 2.7. For any $r \geq 0$, we have

$$|\mathbb{E}_\mu g - \mathbb{E}_\nu g| \leq |\mathbb{E}_\mu g - \mathbb{E}_\mu f_r| + |\mathbb{E}_\nu g - \mathbb{E}_\nu f_r| + |\mathbb{E}_\mu f_r - \mathbb{E}_\nu f_r| \leq 2\epsilon(r) + r\, d_{\mathcal{F}}(\mu, \nu).$$

Taking the infimum on $r > 0$ on the right side gives the result. □

Proof of Corollary 2.8. Proposition 5 of Bach (2017) shows that for any bounded Lipschitz function g that satisfies $||g||_{\mathrm{BL}} := \max\{||g||_\infty, ||g||_{\mathrm{Lip}}\} \leq \eta$, we have $\epsilon(r) = O(\eta(r/\eta)^{-1/(\alpha+(d-1)/2)} \log(r/\eta))$. Using Proposition 2.7, we get

$$|\mathbb{E}_\mu g - \mathbb{E}_\nu g| \leq \tilde{O}(||g||_{\mathrm{BL}}\, d_{\mathcal{F}}(\mu, \nu)^{\frac{1}{\alpha+(d+1)/2}}),$$

The result follows $\mathrm{BL}(\mu, \nu) = \sup_g \{|\mathbb{E}_\mu g - \mathbb{E}_\nu g| \colon\ ||g||_{\mathrm{BL}} \leq 1\}$. □

E PROOF OF RESULTS IN SECTION 3

Proof of Equation 10 Using the standard derivation and the optimality condition (9), we have

$$\begin{aligned}
&d_{\mathcal{F}}(\mu, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \\
&= d_{\mathcal{F}}(\mu, \nu_m) - d_{\mathcal{F}}(\hat{\mu}_m, \nu_m) + d_{\mathcal{F}}(\hat{\mu}_m, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \\
&\leq d_{\mathcal{F}}(\mu, \nu_m) - d_{\mathcal{F}}(\hat{\mu}_m, \nu_m) + \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\hat{\mu}_m, \nu) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) + \epsilon.
\end{aligned}$$

Therefore, we obtain

$$d_{\mathcal{F}}(\mu, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \leq 2 \sup_{\nu \in \mathcal{G}} |d_{\mathcal{F}}(\mu, \nu) - d_{\mathcal{F}}(\hat{\mu}_m, \nu)| + \epsilon.$$

Combining with the definition (1), we obtain

$$d_{\mathcal{F}}(\mu, \nu_m) - \inf_{\nu \in \mathcal{G}} d_{\mathcal{F}}(\mu, \nu) \leq 2 \sup_{f \in \mathcal{F}} |\mathbb{E}_\mu[f] - \mathbb{E}_{\hat{\mu}_m}[f]| + \epsilon.$$

Proof of Theorem 3.1. Consider the function

$$h(X_1, X_2, \ldots, X_m) = \sup_{f \in \mathcal{F}} |\mathbb{E}_\mu[f] - \mathbb{E}_{\hat{\mu}_m}[f]|.$$

Since f takes values in $[-\Delta, \Delta]$, changing X_i to another independent copy X_i' can change h by no more than $4\Delta/m$. McDiarmid's inequality implies that with probability at least $1 - \delta$,

$$\sup_{f \in \mathcal{F}} |\mathbb{E}_\mu[f] - \mathbb{E}_{\hat{\mu}_m}[f]| \leq \mathbb{E}\left[\sup_{f \in \mathcal{F}} |\mathbb{E}_\mu[f] - \mathbb{E}_{\hat{\mu}_m}[f]|\right] + 2\Delta\sqrt{\frac{2\log(1/\delta)}{m}}.$$

Standard argument on Rademacher complexity gives

$$\mathbb{E}\left[\sup_{f \in \mathcal{F}} |\mathbb{E}_\mu[f] - \mathbb{E}_{\hat{\mu}_m}[f]|\right] \leq 2\,\mathbb{E}_{\boldsymbol{\tau}, \boldsymbol{X}}\left[\sup_{f \in \mathcal{F}} \left|\frac{1}{m}\sum_i \tau_i f(X_i)\right|\right] := R_m(\mathcal{F}).$$

Combining the two estimates above and Eqn. (10), we conclude the proof. □

Proof of Corollary 3.2. Part of the proof is from Proposition 7 in Bach (2017). More accurately, the discriminator set we use here is

$$\mathcal{F} = \left\{ \sum_{i=1}^{n} w_i \max(v_i^\top [x;1],\ 0) \colon \sum_{i=1}^{n} |w_i| \le 1, \quad \|v_i\|_2 = 1 \forall 1 \le i \le n \right\}$$

for a fix $n \in \mathbb{N}$. Since $\|x\|_2 \le 1$ and $\|v\|_2 \le 1$, it is easy to see that $\|f\|_\infty \le \sqrt{2}$ for all $f \in \mathcal{F}$.

We want to estimate $R_m^{(\mu)}(\mathcal{F})$ and then use Theorem 3.1 to prove the result. First, it's easy to verify that

$$\sup_{f \in \mathcal{F}} \left| \frac{2}{m} \sum_i \tau_i f(X_i) \right| = \sup_{\|v\|_2 = 1} \left| \frac{2}{m} \sum_i \tau_i \max(v^\top [X_i;1], 0) \right|.$$

Then we have

$$\begin{aligned} R_m^{(\mu)}(\mathcal{F}) &= \mathbb{E} \left[\sup_{\|v\|_2=1} \left| \frac{2}{m} \sum_i \tau_i \max(v^\top [X_i;1], 0) \right| \right] \\ &\le \mathbb{E} \left[\sup_{\|v\|_2=1} \left| \frac{2}{m} \sum_i \tau_i v^\top [X_i;1] \right| \right] = \frac{2}{m} \mathbb{E} \left[\left\| \sum_i \tau_i [X_i;1] \right\|_2 \right], \end{aligned}$$

where we use the 1-Lipschitz property of $\max(x, 0)$ and Talagrand's contraction lemma (Ledoux & Talagrand, 2013) in the inequality step. From Kakade et al. (2009), we get the Rademacher complexity of linear functions

$$\mathbb{E} \left[\left\| \sum_i \tau_i [X_i;1] \right\|_2 \right] \le \sqrt{2m}.$$

Therefore, we obtain

$$R_m^{(\mu)}(\mathcal{F}) \le \frac{2\sqrt{2}}{\sqrt{m}}.$$

Combined with $\|f\|_\infty \le \sqrt{2}$ and Theorem 3.1, we finish the proof. □

Proof of Corollary 3.3. Given a sample $(X_1, \ldots, X_m)$, we first examine the sample Rademacher complexity $\hat{R}_m(\mathcal{F}) := \mathbb{E}_{\boldsymbol{\tau}} \left[\sup_{f \in \mathcal{F}} \left| \frac{2}{m} \sum_i \tau_i f(X_i) \right| \right]$. Since $\mathcal{F}$ is even (i.e., $f \in \mathcal{F}$ implies $-f \in \mathcal{F}$), we have $\hat{R}_m(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\tau}} \left[\sup_{f \in \mathcal{F}} \frac{2}{m} \sum_i \tau_i f(X_i) \right]$.

Let $\hat{\Theta}$ be a finite set such that every point in Θ is within distance $\epsilon/2L$ of a point in $\hat{\Theta}$. Since $\Theta \subset [-1, 1]^p$, standard constructions give an $\hat{\Theta}$ satisfying $\log |\hat{\Theta}| \le p \log(Lp/\epsilon)$. Now for the finite set $\{(f_\theta(X_1), \ldots, f_\theta(X_m)) : \theta \in \hat{\Theta}\}$, we can bound its Rademacher complexity by the Massart Lemma:

$$\mathbb{E}_{\boldsymbol{\tau}} \left[\sup_{f_\theta : \theta \in \hat{\Theta}} \frac{2}{m} \sum_i \tau_i f_\theta(X_i) \right] \le 2\Delta \sqrt{\frac{2 \log |\hat{\Theta}|}{m}} \le 2\Delta \sqrt{\frac{2p \log(Lp/\epsilon)}{m}}. \tag{24}$$

Now for any $\theta \in \Theta$, we can find $\hat{\theta} \in \hat{\Theta}$ such that $\|\theta - \hat{\theta}\| \le \epsilon/2L$. Therefore,

$$\begin{aligned} \frac{2}{m} \sum_i \tau_i f_\theta(X_i) &\le \frac{2}{m} \sum_i \tau_i f_{\hat{\theta}}(X_i) + \left| \frac{2}{m} \sum_i \tau_i f_\theta(X_i) - \frac{2}{m} \sum_i \tau_i f_{\hat{\theta}}(X_i) \right| \\ &\le \frac{2}{m} \sum_i \tau_i f_{\hat{\theta}}(X_i) + \frac{2}{m} \sum_i L * \frac{\epsilon}{2L} \\ &= \frac{2}{m} \sum_i \tau_i f_{\hat{\theta}}(X_i) + \epsilon. \end{aligned}$$

Therefore, we have

$$\mathbb{E}_{\boldsymbol{\tau}} \left[\sup_{f_\theta : \theta \in \Theta} \frac{2}{m} \sum_i \tau_i f_\theta(X_i) \right] \le \mathbb{E}_{\boldsymbol{\tau}} \left[\sup_{f_\theta : \theta \in \hat{\Theta}} \frac{2}{m} \sum_i \tau_i f_\theta(X_i) \right] + \epsilon. \tag{25}$$

Combining Eqn. (24) and (25), we obtain

$$\hat{R}_m(\mathcal{F}) \equiv \mathbb{E}_{\boldsymbol{\tau}}\left[\sup_{f_\theta:\theta\in\Theta}\frac{2}{m}\sum_i \tau_i f_\theta(X_i)\right] \leq 2\Delta\sqrt{\frac{2p\log(Lp/\epsilon)}{m}} + \epsilon.$$

Since the estimate above holds for an arbitrary sample, we conclude that $R_m(\mathcal{F}) \leq 2\Delta\sqrt{\frac{2p\log(Lp/\epsilon)}{m}} + \epsilon$. Combining this result with Theorem 3.1, we finish the proof. □

Proof of Corollary 3.4. Lemma 22 in Bartlett & Mendelson (2003) shows that if $\sup_{x\in X} k(x,x) \leq C_k \leq +\infty$, we have $R_m^{(\mu)}(\mathcal{F}) \leq 2\sqrt{C_k/m}$ for any $\mu \in \mathcal{P}_{\mathcal{B}}(X)$. Also, note that $f(x) \leq ||f||_H\sqrt{k(x,x)} \leq ||f||_H\sqrt{C_k}$. Combined with Theorem 3.1, we conclude the proof. □

Proof of Corollary 3.5. Use Proposition 3.1 and note that $\mathrm{KL}(\mu,\nu_m) \leq \Lambda_{\mathcal{F},\mathcal{G}}\ d_{\mathcal{F}}(\mu,\nu)$ and $d_{\mathcal{F}}(\mu,\nu) \leq \Delta\mathrm{TV}(\mu,\nu) \leq \Delta\sqrt{2\mathrm{KL}(\mu,\nu)}$ by Pinsker's inequality. □

F Proof of results in section 4

Proof of Lemma 4.1. i) It is obvious that $\Psi_{\nu,\psi^*}[f]$ is convex given that f^* is convex. By the convex conjugate, we have $\psi(t) = \sup_y \{ty - \psi^*(y)\}$. Take $t = 0$ and note that $\psi(0) = 0$, then we have $\psi^*(y) \geq 0,\ \forall y$. This proves $\Psi_{\nu,\psi^*}[f] \geq 0$.

ii) If ψ is strictly convex, then ψ^* is also strictly convex. This implies there exists at most a single value b_0 such that $\psi^*(c) = 0$. Given that $\psi^*(y) \geq 0$ for $\forall y$, we arrive that $\mathbb{E}_{x\sim\nu}[\psi^*(f(x))] = 0$ implies $\psi^*(f(x)) = 0$ almost surely under $x \sim \nu$, which then implies $f(x) = b_0$ almost surely. □

Proof of Theorem 4.1. i) because $b_0 \in \mathcal{F}$ and $\psi^*(b_0) = 0$, we have $d_{\phi,\mathcal{F}}(\mu\,||\,\nu) \geq \mathbb{E}_\mu[b_0] - \mathbb{E}_\nu[b_0] - \Psi_{\nu,\psi^*}[b_0] = 0$. Because $\Psi_{\nu,\psi^*}[f] \geq 0$, we obtain $d_{\phi,\mathcal{F}}(\mu\,||\,\nu) \leq d_{\mathcal{F}}(\mu,\nu)$ by comparing (20) with $d_{\mathcal{F}}(\mu,\nu) = \sup_{f\in\mathcal{F}}\{\mathbb{E}_\mu f - \mathbb{E}_\nu f\}$.

ii), note that $d_{\psi,\mathcal{F}}(\mu\,||\,\nu) = 0$ implies $\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f] \leq \Psi_{\nu,\psi^*}[f],\quad \forall f \in \mathcal{F}$. Therefore,

$$\mathbb{E}_\mu[\sigma(\alpha f_0 + c_0)] - \mathbb{E}_\nu[\sigma(\alpha f_0 + c_0)] \leq \Psi_{\nu,\psi^*}[\sigma(\alpha f_0 + c_0)],\quad \forall f_0 \in \mathcal{F}_0,\quad |\alpha| \leq \alpha_{f_0}.$$

This implies that

$$\frac{1}{\alpha}(\mathbb{E}_{x\sim\mu}[\sigma(\alpha f_0(x) + c_0)] - \mathbb{E}_{x\sim\nu}[\sigma(\alpha f_0(x) + c_0)]) \leq \frac{1}{\alpha}\,\mathbb{E}_{x\sim\nu}[\psi^*(\sigma(\alpha f_0(x) + c_0))]. \tag{26}$$

By the differentiability assumptions,

$$\lim_{\alpha\to 0}\frac{\sigma(\alpha f_0(x) + c_0) - \sigma(c_0)}{\alpha} = \sigma'(c_0)f(x),$$
$$\lim_{\alpha\to 0}\frac{\psi^*(\sigma(\alpha f_0(x) + c_0)) - \psi^*(\sigma(c_0))}{\alpha} = \psi^{*\prime}(b_0)\sigma'(c_0)f_0(x) = 0,$$

where we used the fact that $\psi^*(\sigma(c_0)) = \psi^*(b_0) = 0$ and $\psi^{*\prime}(b_0) = 0$ because b_0 is a differentiable minimum point of ψ^*. Taking the limit of $\alpha \to 0$ on both sides of (26), we get

$$\sigma'(c_0)[\mathbb{E}_{x\sim\mu}[f_0(x)] - \mathbb{E}_{x\sim\nu}[f_0(x)]] \leq 0,\quad \forall f_0 \in \mathcal{F}_0.$$

Because $\sigma'(c_0) > 0$ by assumption, this implies $\mathbb{E}_{x\sim\mu}[f_0(x)] - \mathbb{E}_{x\sim\nu}[f_0(x)]$. The same argument applies to $-f_0$, and we thus we finally obtain $\mathbb{E}_{x\sim\mu}[f_0(x)] = \mathbb{E}_{x\sim\nu}[f_0(x)]$.

iii) Combining Theorem 2.2 and the last point, we directly get the result. □

G Inconsistency between GAN's loss and testing likelihood

In this section, we will test our analysis of the consistency of GAN objective and likelihood objective on two toy datasets, e.g., a 2D Gaussian dataset and a 2D 8-Gaussian mixture dataset.

G.1 A 2D GAUSSIAN EXAMPLE

The underlying ground-truth distribution is a 2D Gaussian with mean $(0.5, -0.5)$ and covariance matrix $\frac{1}{128}\begin{bmatrix}17 & 15\\ 15 & 17\end{bmatrix}$. We take 10^5 samples as training dataset, and 1000 samples as testing dataset.

For a 2D Gaussian distribution, we use the following generator

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1 & \\ l & 1\end{bmatrix}\begin{bmatrix}e^{s_1} & \\ & e^{s_2}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}, \tag{27}$$

where $\boldsymbol{z} = \begin{bmatrix}z_1\\ z_2\end{bmatrix}$ is a standard 2D normal random vector, $l \in \mathbb{R}$, $\boldsymbol{s} = \begin{bmatrix}s_1\\ s_2\end{bmatrix} \in \mathbb{R}^2$ and $\boldsymbol{b} = \begin{bmatrix}b_1\\ b_2\end{bmatrix} \in \mathbb{R}^2$ are trainable parameters.

We train the generative model by WGAN with weight clipping. In the first experiment, the discriminator set is neural network with one hidden layer and 500 hidden neurons, i.e.,

$$\mathcal{F}_{\text{nn}} = \{\sum_{i=1}^{500} \alpha_i \max(\boldsymbol{w}_i^\top [x; 1],\ 0) : -0.05 \le \boldsymbol{\alpha} \le 0.05, -0.05 \le \boldsymbol{w}_i \le 0.05 \quad \forall i\}.$$

Motivated by Corollary 3.5, in the second experiment, we take the quadratic polynomials as discriminator set, i.e.,

$$\mathcal{F}_{\text{quad}} = \{x^\top A x + \boldsymbol{b}^\top x : -0.05 \le A \le 0.05, -0.05 \le \boldsymbol{b} \le 0.05\}.$$

We plot their results in Figure 1. We can see that both training loss (neural distance) converge to 0 and that the testing log likelihood is increasing during the training, which is consistent with the increase of the **negative** training loss. However, we can see that the quadratic polynomial discriminators $\mathcal{F}_{\text{quad}}$ leads to higher log likelihood and better generative model after finishing the training. This is expected because Corollary 3.5 guarantees that the log likelihood is bounded by the GAN loss (up to a constant), while it is not true for $\mathcal{F}_{\text{nn}}$.

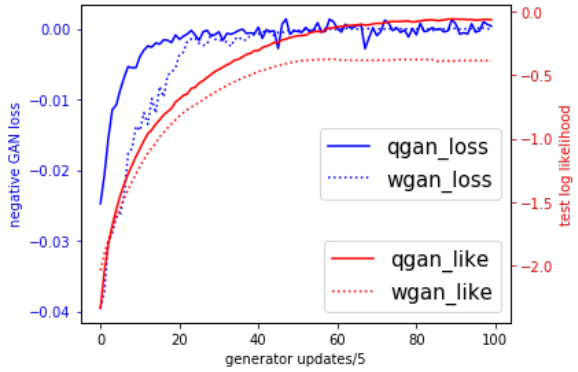

Figure 1: Negative GAN losses and testing likelihood. qgan: WGAN with quadratic polynomials as discriminator. wgan: WGAN with neural discriminator.

We can also maximize the likelihood on the training dataset to train the model, and we show its result in Figure 3. We can see that WGAN with quadratic polynomials as discriminators and maximal likelihood training leads to similar results. However, directly maximizing the likelihood converges much faster than the WGAN in this example.

In this simple Gaussian examples, the WGAN loss and the testing log likelihood is consistent. We indeed observe that by carefully choosing the discriminator set (as suggested in Corollary 3.5), the testing likelihood can be simultaneously optimized as we optimize the GAN objective.

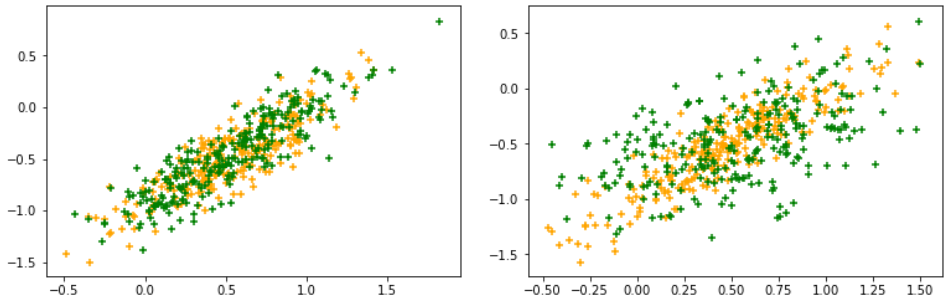

Figure 2: Samples from trained generators. Left: WGAN with quadratic polynomials as discriminator. Right: WGAN with neural discriminator.

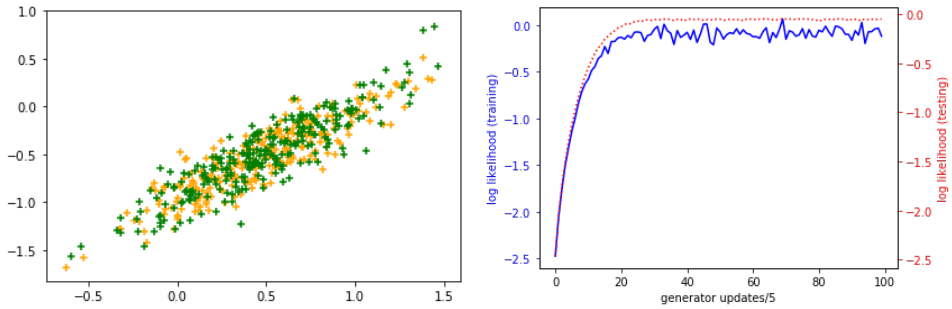

Figure 3: Upper: maximal likelihood estimates, trained with SGD. Middle: negative training loss. Right: log likelihood on testing dataset.

G.2 An example of 2D 8-Gaussian mixture

The underlying ground truth distribution is a 2D Gaussian mixture with 8 Gaussians and with equal weights. Their centers distributed equally on the circle centered at the origin and with radius $\sqrt{2}$, and their standard deviations are all 0.01414. We take 10^5 samples as training dataset, and 1000 samples as testing dataset. We show one batch (256) of training dataset and the testing dataset in Figure 4. We remind that the density of the ground-truth distribution is very singular.

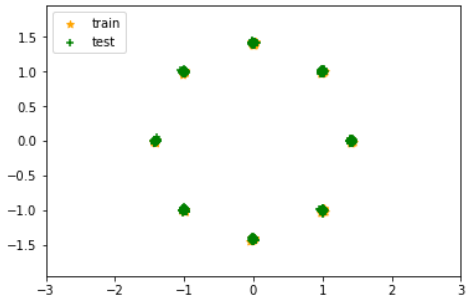

Figure 4: Samples from training and testing datasets.

We still use Eqn. (27) as the generator for a single Gaussian component. Our generator assume that there are 8 Gaussian components and they have equal weights, and thus our generator does not have any modeling error. The training parameters are eight sets of scaling and biasing parameters in Eqn. (27), each for one Gaussian component.

We first training the model by WGAN with clipping. We use an MLP with 4 hidden layers and relu activations as the discriminator set. We show the result in Figure 5. We can see that the generator's samples are nearly indistinguishable from the real samples. However, the GAN loss and the log likelihood are not consistent. In the initial stage of training, both the **negative** GAN loss and likelihood are increasing. As the training goes on, the generator's density gets more and more singular, the likelihood behaves erratically in the latter stage of training. Although the **negative** GAN loss is still increasing, the log likelihood oscillates a lot, and in fact over half of time the log likelihood is $-\infty$. We show the generated samples at intermediate steps in Figure 6, and we indeed see that the likelihood start to oscillate inviolately when the generator's distribution gets singular.

This inconsistency between GAN loss and likelihood is observed by other works. The reason for this consistency is that the neural discriminators are not a good approximation of the singular density ratios.

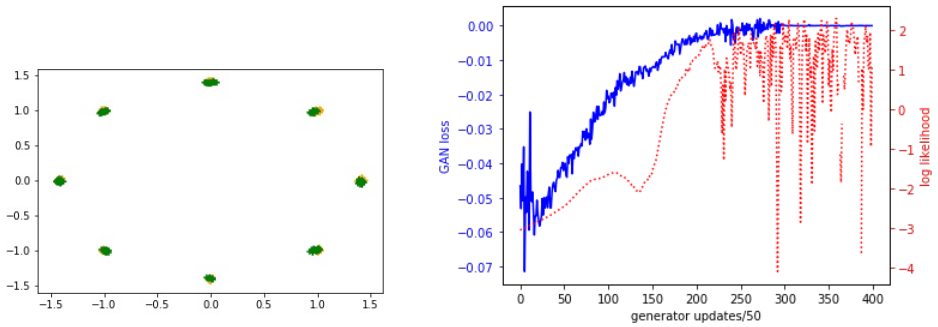

Figure 5: Left: samples from training dataset and samples from generator. Right: negative training loss and log likelihood (on testing dataset).

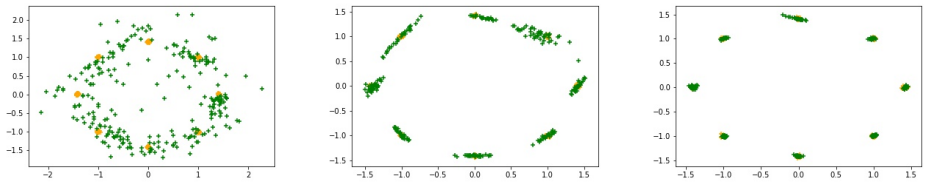

Figure 6: Left to right: generated samples at step 100, 200 and 300, respectively.

We also trained the model by maximizing likelihood on the training dataset. We show the result in Figure 7. We can see that the maximal likelihood training got stuck in a local minimal, and fail to exactly recover all 8 components. The log likelihood on training and testing dataset are consistent as expected. Although the log likelihood (≈ 2.7) obtained by maximizing likelihood is higher than that (≈ 2.0) obtained by WGAN training, its generator is obviously worse than what we obtained in WGAN training. The reason for this is that the negative log-likelihood loss has many local minima and each of them are deeper than those in WGAN. Maximizing likelihood is much easier to get trapper in a local minimum.

The FlowGAN proposed to combine the WGAN loss and the log likelihood to solve the inconsistency problem. We showed the FlowGAN result on this dataset in Figure 8. We can see that training by FlowGAN indeed makes the training loss and likelihood consistent. However, FlowGAN got stuck in a local minimum as maximizing likelihood did, which is not desirable.

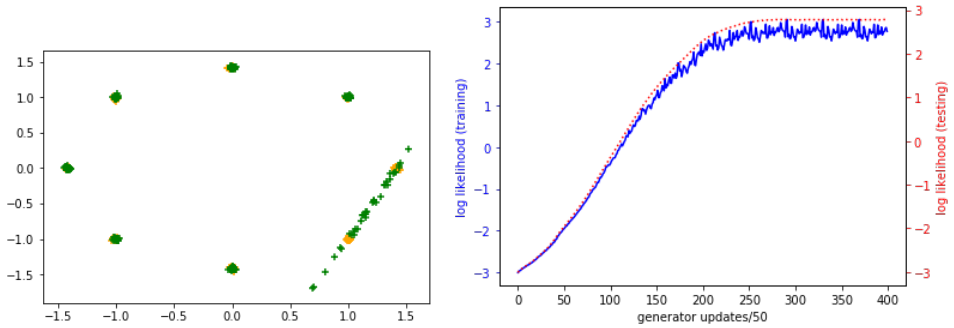

Figure 7: Left: samples from training dataset and samples from generator. Right: log likelihood on training and testing dataset.

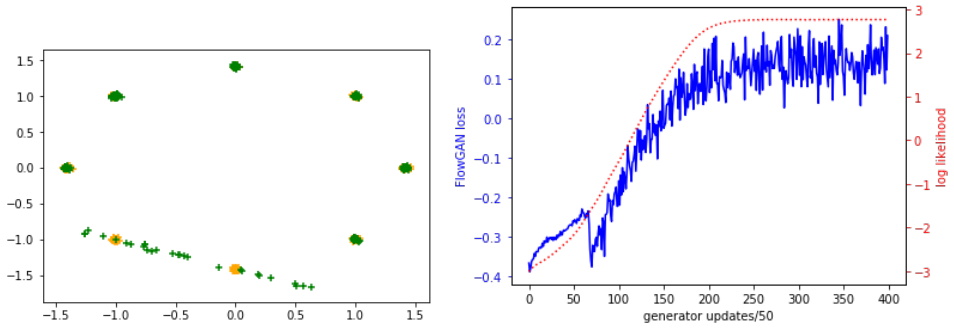

Figure 8: Left: samples from training dataset and samples from generator. Right: negative FlowGAN loss and log likelihood on testing dataset.