Brownian Noise Reduction: Maximizing Privacy Subject to Accuracy Constraints

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Abstract

There is a disconnect between how researchers and practitioners handle privacy-utility tradeoffs. Researchers primarily operate from a *privacy first* perspective, setting strict privacy requirements and minimizing risk subject to these constraints. Practitioners often desire an *accuracy first* perspective, possibly satisfied with the greatest privacy they can get subject to obtaining sufficiently small error. Ligett et al. [2017] have introduced a "noise reduction" algorithm to address the latter perspective. The authors show that by adding correlated Laplace noise and progressively reducing it on demand, it is possible to produce a sequence of increasingly accurate estimates of a private parameter while only paying a privacy cost for the least noisy iterate released. In this work, we generalize noise reduction to the setting of Gaussian noise, introducing the *Brownian mechanism*. The Brownian mechanism works by first adding Gaussian noise of high variance corresponding to the final point of a simulated Brownian motion. Then, at the practitioner's discretion, noise is gradually decreased by tracing back along the Brownian path to an earlier time. Our mechanism is more naturally applicable to the common setting of bounded ℓ_2-sensitivity, empirically outperforms existing work on common statistical tasks, and provides customizable control of privacy loss over the entire interaction with the practitioner. We complement our Brownian mechanism with $\mathrm{ReducedAboveThreshold}$, a generalization of the classical $\mathrm{AboveThreshold}$ algorithm that provides adaptive privacy guarantees. Overall, our results demonstrate that one can meet utility constraints while still maintaining strong levels of privacy.

1 Introduction

Over the past decade, differential privacy has seen industry-wide adoption as a means of protecting sensitive information [Erlingsson et al., 2014, Greenberg, 2016]. By injecting appropriate amounts of noise, differentially private algorithms allow the computation of population-level quantities of interest while guaranteeing individual-level privacy. Of the private mechanisms used in industry, those relating to private empirical risk minimization (ERM) are perhaps the most impactful, in part due to their application in machine learning tasks [Abadi et al., 2016, Song et al., 2013]. Researchers have developed many private ERM mechanisms, ranging from least squares minimzation [Sheffet, 2017, Chaudhuri et al., 2011] to subsampled gradient descent [Abadi et al., 2016, Balle and Wang, 2018, Wang et al., 2019]. Despite this vast literature, most existing results take the same broad approach: they aim to minimize error (statistical risk) subject to strict privacy guarantees. While this strict adherence to privacy constraints may be necessary in some applications, it often provides weak utility guarantees [Fienberg et al., 2010] and can make some learning tasks impossible [Dwork et al., 2009]. Industry applications of differential privacy may desire an *accuracy first* perspective, setting desired risk requirements for models used in production. Privacy may still be a desirable aspect of computation, but it is by no means the only goal; minimizing risk may take center stage.

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The main existing approach to this accuracy-oriented perspective on privacy was given by Ligett et al. [2017]. These authors introduce a *noise reduction mechanism* for gradually releasing a private, high-dimensional parameter. By leveraging a Laplace-based Markov process [Koufogiannis et al., 2017], they construct a mechanism for which the privacy loss of releasing arbitrarily many estimates of a parameter only depends on the privacy loss of the least noisy parameter viewed. This is in contrast to results about the composition of private algorithms, in which privacy degrades according to the total number of parameters witnessed [Dwork et al., 2010, Kairouz et al., 2015, Murtagh and Vadhan, 2016]. The authors also demonstrate how to privately query the utility of observed parameters on private data by coupling their Laplace-based mechanism with $\mathrm{AboveThreshold}$, a classical differentially private algorithm [Dwork and Roth, 2014, Lyu et al., 2017].

While the above mechanism provides significant privacy loss savings over a baseline method that doubles the privacy loss each round, Laplace noise is unfit for many settings in which ℓ_2-sensitivity is used for calibrating noise. Since converting from ℓ_2-sensitivity to ℓ_1-sensitivity[1] incurs a dimension-dependent cost, it is important to develop a noise reduction technique with Gaussian noise.

Contributions and paper outline. We introduce the *Brownian mechanism*, a novel approach for privately releasing a parameter vector subject to accuracy constraints. The Brownian mechanism adds correlated Gaussian noise to a risk-minimizing parameter through a Brownian motion. Noise is then iteratively stripped by moving adaptively backwards along the random walk until a suitable stopping condition is met, such as meeting a target accuracy on a public dataset. In Section 3, we define the Brownian mechanism and characterize its privacy loss. Using machinery from martingale theory, we construct *privacy boundaries* for the Brownian mechanism — upper bounds on privacy loss that hold simultaneously with high probability. In particular, the failure probability of these bounds does not depend on the number of outcomes observed, overcoming a seeming need for a union bound faced by Ligett et al. [2017]. These privacy boundaries yield provable, high-probability bounds on privacy loss under data-dependent stopping conditions.

If private data is used to evaluate risk, then the data-dependent stopping conditions can themselves leak information. To counter this, we introduce $\mathrm{ReducedAboveThreshold}$ in Section 5, a generalization of the classical $\mathrm{AboveThreshold}$ algorithm for privately querying accuracy on sensitive data. We show how to couple $\mathrm{ReducedAboveThreshold}$ and the Brownian mechanism so that a data analyst only ever incurs *twice* the privacy loss they would incur if they had queried accuracy on a public dataset. This is in contrast to the results in Ligett et al. [2017], which note that the privacy loss of $\mathrm{AboveThreshold}$ often dominates the privacy loss incurred from using noise reduction.

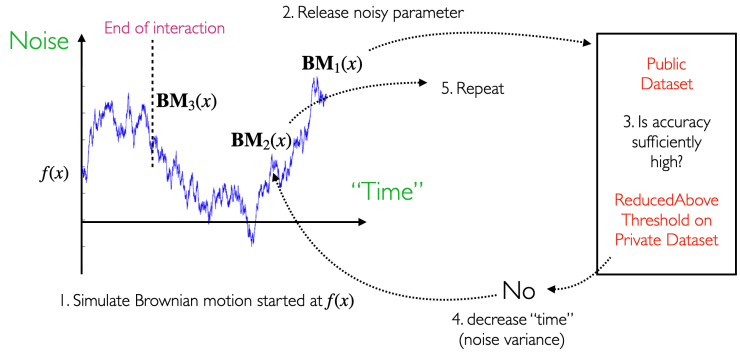


Figure 1: An example of running the Brownian mechanism to gradually release a statistic $f(x)$. First, a very noisy version of the hidden parameter $\mathrm{BM}_1(x)$ is viewed. Then, loss is measured, either on a public dataset, or on a private dataset using a method such as $\mathrm{ReducedAboveThreshold}$. If a target loss is met, the process stops. Otherwise, noise is removed and the process repeats.

We empirically evaluate the Brownian mechanism and $\mathrm{ReducedAboveThreshold}$ in Section 6, finding that the Brownian mechanism can offer privacy loss savings over the Laplace noise reduction method introduced by Ligett et al. [2017]. In our view, these results demonstrate that the Brownian mechanism is a practical, intuitive mechanism for meeting accuracy requirements in private ERM.

[1] The ℓ_p sensitivity of f is defined as $\sup_{x \sim x'} ||f(x) - f(x')||_p$ for $p \geq 1$.

Lastly, we derive other new mechanisms for noise reduction, of independent interest. We generalize the Laplace process of Koufogiannis et al. [2017] to continuous time in Section 4, thus making the Laplace noise reduction mechanism of Ligett et al. [2017] more flexible and adaptive to data-dependent privacy levels. We also briefly mention a noise reduction mechanism for Skellam noise in Appendix E, a discrete distribution used in count queries [Agarwal et al., 2021].

2 Preliminaries

Differential privacy, privacy loss, and ex-post privacy. An algorithm $A : \mathcal{X} \to \mathcal{Y}$ is (ϵ, δ)-differentially private if, for any measurable set $E \subset \mathcal{Y}$ and any neighboring inputs $x \sim x'$,

$$\mathbb{P}(A(x) \in E) \leq e^{\epsilon}\mathbb{P}(A(x') \in E) + \delta. \tag{1}$$

In the above [Dwork et al., 2006], $\sim$ denotes some arbitrary neighboring relation. Typically $x \sim x'$ indicates x and x' differ in one entry, but any other relation suffices. While differential privacy has proven itself a mainstay of private computation, condition (1) is too rigid to allow data analysts to achieve a minimum desired accuracy. In other words, it embraces a *privacy first* perspective, fixing a strict condition in terms of parameters ϵ and δ that must be met. We are interested in the *accuracy first* perspective, setting a target accuracy and correspondingly optimizing privacy parameters.

The above definition of differential privacy is qualitatively focused on bounding the information-theoretic quantity of *privacy loss* [Dwork et al., 2006, 2010, Dwork and Roth, 2014].

Definition 2.1 (**Privacy Loss**). *Let $A : \mathcal{X} \to \mathcal{Y}$ be an algorithm, and fix neighbors $x \sim x'$ in $\mathcal{X}$. Let p^x and $p^{x'}$ be the respective densities of $A(x)$ and $A(x')$ on the space $\mathcal{Y}$. Then, the privacy loss between $A(x)$ and $A(x')$ is the random variable*

$$\mathcal{L}(x, x') := \log\left(\frac{p^x(A(x))}{p^{x'}(A(x))}\right).$$

We think of $A(x)$ as the true outcome, and $\mathcal{L}(x, x')$ measures how much more likely this outcome is under the true input x versus an alternative x'. Privacy loss provides a *probabilistic* definition of privacy. Namely, A is (ϵ, δ)*-probabilistically differentially private* if, for all neighbors $x \sim x'$,

$$\mathbb{P}\left(\mathcal{L}(x, x') > \epsilon\right) \leq \delta. \tag{2}$$

While probabilistic differential privacy is not equivalent to differential privacy [Kasiviswanathan and Smith, 2014], (ϵ, δ)-probabilistically differential privacy implies (ϵ, δ)-differential privacy. Probabilistic differential privacy emerged as a means for studying privacy composition, and has been leveraged in proving many results [Kairouz et al., 2015, Murtagh and Vadhan, 2016, Rogers et al., 2016, Whitehouse et al., 2022]. A natural extension of privacy to the accuracy-oriented regime is *ex-post* privacy, which allows the bound in condition (2) to depend the observed algorithm output.

Definition 2.2 (Ligett et al. [2017]). *Let $A : \mathcal{X} \to \mathcal{Y}$ be an algorithm and $\mathcal{E} : \mathcal{Y} \to \mathbb{R}_{\geq 0}$ a function. We say A is $(\mathcal{E}, \delta)$-ex-post private if, for any neighboring inputs $x \sim x'$, we have*

$$\mathbb{P}\left(\mathcal{L}(x, x') > \mathcal{E}(A(x))\right) \leq \delta.$$

While any algorithm is trivially ex-post private with $\mathcal{E}(A(x)) := \infty$, the goal is to make $\mathcal{E}(A(x))$ as small as possible. We describe theoretical tools for obtaining ex-post privacy guarantees in Section 3, and empirically compute the ex-post privacy distributions of various mechanisms in Section 6.

Background on Noise Reduction. Heuristically, a noise reduction mechanism allows a data analyst to view multiple, increasingly accurate estimates of a risk minimizing parameter while only paying an ex-post privacy cost for the *least* noisy iterate observed. More formally, let $M : \mathcal{X} \to \mathcal{Y}^{\infty}$ be any algorithm mapping databases to sequences of outputs. The indices of the sequence denote the rounds of interaction, with smaller indices indicating greater noise. Let $M_n : \mathcal{X} \to \mathcal{Y}$ give the nth element of the sequence and $M_{1:n} : \mathcal{X} \to \mathcal{Y}^n$ the first n elements.

Definition 2.3 (**Noise Reduction Mechanism**). *We say $M : \mathcal{X} \to \mathcal{Y}^{\infty}$ is a noise reduction mechanism if, for any $n \geq 1$ and any neighboring datasets $x \sim x'$, we have*

$$\mathcal{L}_{1:n}(x, x') = \mathcal{L}_n(x, x'),$$

where $\mathcal{L}_{1:n}(x, x')$ denotes the privacy loss between $M_{1:n}(x)$ and $M_{1:n}(x')$ and $\mathcal{L}_n(x, x')$ is the privacy loss between $M_n(x)$ and $M_n(x')$.

The only noise reduction mechanism in the literature uses a Markov process with Laplace marginals [Koufogiannis et al., 2017] to gradually release a sensitive parameter [Ligett et al., 2017]. As originally presented, this *Laplace Noise Reduction* mechanism is nonadaptive, requiring a data analyst to fix a finite sequence of privacy parameters $(\epsilon_n)_{n\in[K]}$ in advance. Instead of presenting this method as background, we describe it in Section 4, in which we construct an adaptive generalization of this mechanism. We then leverage this generalization as a subroutine in $\mathrm{ReducedAboveThreshold}$, a generalization of $\mathrm{AboveThreshold}$ with adaptive privacy guarantees.

Background on Brownian Motion. We now provide a brief background on Brownian motion, perhaps the best-known example of a continuous time stochastic process [Le Gall, 2016].

Definition 2.4. *A continuous time real-valued process $(B_t)_{t\geq 0}$ is called a standard Brownian motion if (1) $B_0 = 0$, (2) $(B_t)_{t\geq 0}$ has continuous sample paths, (3) (B_t) has independent increments, i.e. $B_{t+s} - B_s$ is independent of B_s for all $s, t \geq 0$, and (4) $B_t \sim \mathcal{N}(0, t)$ for all $t \geq 0$.*

We say a process $(B_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion if each coordinate process is an independent standard Brownian motion.

We use many properties of Brownian motion to construct the Brownian mechanism and analyze its privacy loss in Section 3. One important property of Brownian motion is that it is a continuous time martingale. This property allow us to use time-uniform supermartingale concentration to characterize and bound the privacy loss of the Brownian mechanism at data-dependent stopping times [Howard et al., 2020, 2021]. We do not go into detail about martingale concentration in this background section, but rather defer it to Appendix A. Additionally, $(B_t)_{t\geq 0}$ is a Markov process. This tells us that if we inspect the Brownian motion at times $0 \leq t_1 < t_2 < \cdots < t_n$, then $B_{t_2}, \ldots, B_{t_n}$ can be viewed as a randomized post-processing of B_{t_1} that *does not* depend on B_s for any $s < t_1$. This property allows us to show that the privacy loss of the Brownian mechanism — which adds noise to a parameter via a Brownian motion — only depends on the least noisy parameter observed.

3 The Brownian Mechanism: a Gaussian Noise Reduction Mechanism

The Brownian mechanism works by simulating a Brownian motion starting at some multivariate parameter; this parameter should be thought of as the risk-minimizing output if there were no privacy constraints. The data analyst first observes the random walk at some large time, corresponding to a large amount of noise added to the initial parameter. Then, if so desired, the analyst "rewinds" time to an earlier point on the Brownian path, effectively reducing noise to obtain a more accurate estimate. Due to the Markovian nature of Brownian motion, the analyst will only pay a privacy cost proportional to variance of the random walk at the earliest inspected time. In what follows, when we refer to a sequence $(T_n)_{n\geq 1}$ of *time functions*, we mean a sequence of functions $T_n : (\mathbb{R}^d)^{n-1} \to \mathbb{R}_{\geq 0}$ such that, for all n and $\beta_{1:n} \in (\mathbb{R}^d)^n$, $T_{n+1}(\beta_{1:n}) \leq T_n(\beta_{1:n-1})$.

Definition 3.1. *Let $f : \mathcal{X} \to \mathbb{R}^d$ be a function and $(T_n)_{n\geq 1}$ a sequence of time functions. Let $(B_t)_{t\geq 0}$ be a standard d-dimensional Brownian motion. The Brownian mechanism associated with f and $(T_n)_{n\geq 1}$ is the algorithm $\mathrm{BM} : \mathcal{X} \to (\mathbb{R}^d)^\infty$ given by*

$$\mathrm{BM}(x) := \left(f(x) + B_{T_n(x)}\right)_{n\geq 1},$$

where we set $T_n(x) := T_n\left(f(x) + B_{T_1(x)}, \ldots, f(x) + B_{T_{n-1}(x)}\right)$.

In the context of ERM, one can think of f as computing a risk minimizing parameter associated with a private dataset $x \in \mathcal{X}$. The data analyst uses T_n along with the first $n-1$ iterates to determine how far to rewind time to obtain the nth iterate. Due to the Markovian nature of Brownian motion, we get the following lemma. We include a proof in Appendix B for completeness.

Lemma 3.2. *Let $x \sim x'$ be neighbors. For any $n \geq 1$, let $\mathcal{L}^{\mathrm{BM}}_{1:n}(x, x')$ denote the privacy loss between $\mathrm{BM}_{1:n}(x)$ and $\mathrm{BM}_{1:n}(x')$ and $\mathcal{L}^{\mathrm{BM}}_{n}(x, x')$ the privacy loss between $\mathrm{BM}_n(x)$ and $\mathrm{BM}_n(x')$. Then,*

$$\mathcal{L}^{\mathrm{BM}}_{1:n}(x, x') = \mathcal{L}^{\mathrm{BM}}_{n}(x, x').$$

Lemma 3.2 just tells us that the Brownian mechanism is a noise reduction mechanism, i.e. that the privacy lost by viewing the first n iterates is exactly the privacy lost by viewing the nth iterate in isolation. Thus, we can identify $\mathcal{L}^{\mathrm{BM}}_{1:n}(x, x')$ with $\mathcal{L}^{\mathrm{BM}}_{n}(x, x')$ going forward.

The Brownian mechanism, as defined above, produces an infinite sequence of parameters. In practice, a data analyst will only view finitely many iterates, stopping when some utility condition has been met or a minimum privacy level is reached. We introduce *stopping functions* to model how a data analyst adaptively interacts with noise reduction mechanisms.

Definition 3.3 (Stopping Function). *Let $M : \mathcal{X} \to \mathcal{Y}^\infty$ be a noise reduction mechanism. For $x \in \mathcal{X}$, let $(\mathcal{F}_n(x))_{n\in\mathbb{N}}$ be the filtration given by $\mathcal{F}_n(x) := \sigma(M_i(x) : i \leq n)$.[2] A function $N : \mathcal{Y}^\infty \to \mathbb{N}$ is called a stopping function if for any $x \in \mathcal{X}$, $N(x) := N(M(x))$ is a stopping time with respect to $(\mathcal{F}_n(x))_{n\geq 1}$.*

A stopping function N is just a rule that an analyst uses to decide when to stop viewing parameters. In particular, this rule depends *only* on the observed iterates of the noise reduction mechanism. For instance, N could heuristically be "stop at the first time a parameter achieves an accuracy of 95% on a held-out dataset." If a data analyst uses a stopping function alongside BM, per Definition 2.3, the privacy loss accrued upon stopping is $\mathcal{L}^{\mathrm{BM}}_{N(x)}(x, x')$. In some cases, an analyst may want the stopping function to depend on the underlying private dataset through more than just the released parameters, e.g. they may want their rule to be "stop at the first time a parameter achieves an accuracy of 95% on the private dataset." In this case, additional privacy may be lost due to observing $N(x)$. We detail how to handle this more subtle case in Section 5.

The following theorem characterizes the privacy loss of the Brownian mechanism.

Theorem 3.4. *Let* BM *be the Brownian mechanism associated with $(T_n)_{n\geq 1}$ and a function f. For neighbors $x \sim x'$, the privacy loss between $\mathrm{BM}_{1:n}(x)$ and $\mathrm{BM}_{1:n}(x')$ is given by*

$$\mathcal{L}^{\mathrm{BM}}_{1:n}(x, x') = \mathcal{L}^{\mathrm{BM}}_{n}(x, x') = \frac{||f(x) - f(x')||_2^2}{2T_n(x)} + \frac{||f(x) - f(x')||_2}{T_n(x)} W_{T_n(x)},$$

where $(W_t)_{t\geq 0}$ is a standard, univariate Brownian motion. Suppose f has ℓ_2-sensitivity at most Δ_2. Then, letting $a^+ := \max(0, a)$, we have

$$\mathcal{L}^{\mathrm{BM}}_{n}(x, x') \leq \frac{\Delta_2^2}{2T_n(x)} + \frac{\Delta_2}{T_n(x)} W^+_{T_n(x)}.$$

Theorem 3.4 also holds when a deterministic time n is replaced by $N(x)$, where N is a stopping function. The above theorem can be viewed as a process-level equivalent of the well-known fact that the privacy loss of the Gaussian mechanism has an uncentered Gaussian distribution [Balle and Wang, 2018]. We prove the Theorem 3.4 in Appendix B. Given the clean characterization of privacy loss above, we now show how to construct high-probability, time-uniform privacy loss bounds. We start by defining the notion of *privacy boundaries*.

Definition 3.5. *A function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a δ-privacy boundary for the Brownian mechanism associated with time functions $(T_n)_{n\geq 1}$ if for any neighboring datasets $x \sim x'$, we have*

$$\mathbb{P}\left(\exists n \geq 1 : \mathcal{L}^{\mathrm{BM}}_{n}(x, x') > \psi(T_n(x))\right) \leq \delta.$$

Since the privacy loss of BM can be expressed as a deterministic function of a Brownian motion, we can apply results from martingale theory to construct general families of privacy boundaries.

Theorem 3.6. *Assume the same setup as in Theorem 3.4. Let $\delta > 0$ and f be a function with ℓ_2-sensitivity Δ_2. The following classes of functions form δ-privacy boundaries.*

1. ***(Mixture boundary)*** *For any $\rho > 0$, ψ^M_ρ given by*

$$\psi^M_\rho(t) := \frac{\Delta_2^2}{2t} + \frac{\Delta_2}{t}\sqrt{2(t+\rho)\log\left(\frac{1}{\delta}\sqrt{\frac{t+\rho}{\rho}}\right)}.$$

2. ***(Linear boundary)*** *For any $a, b > 0$ such that $2ab = \log(1/\delta)$, $\psi^L_{a,b}$ given by*

$$\psi^L_{a,b}(t) := \frac{\Delta_2}{t}\left(\frac{\Delta_2}{2} + b\right) + \Delta_2 a.$$

[2] This definition can be extended to allow for N to depend on independent, external randomization, but we omit this for simplicity.

We prove Theorem 3.6 in Appendix B and we plot the boundaries in Figure 4.

Privacy boundaries serve a dual purpose for the Brownian mechanism. First, since time-uniform concentration bounds are also valid at arbitrary stopping times [Howard et al., 2021], privacy boundaries can be leveraged to provide ex-post privacy guarantees. Second, in many settings, it may be more natural for a data analyst to adaptively specify target privacy levels instead of noise levels. This is, for instance, the case in our experiments in Section 6. By inverting privacy boundaries, data analysts can compute the proper amount of noise to remove at each step to meet target privacy levels.

We make the above precise in the following corollary of Theorem 3.6. In what follows, when we refer to a sequence $(\mathcal{E}_n)_{n\geq 1}$ of *privacy functions*, we mean a sequence of functions $\mathcal{E}_n : (\mathbb{R}^d)^{n-1} \to \mathbb{R}_{\geq 0}$ such that, for all n and $\beta_{1:n} \in (\mathbb{R}^d)^n$, $\mathcal{E}_{n+1}(\beta_{1:n}) \geq \mathcal{E}_n(\beta_{1:n-1})$.

Corollary 3.7. *Let N be a stopping function, as in Definition 3.3. If ψ is a δ-privacy boundary for* BM*, we have*

$$\sup_{x\sim x'} \mathbb{P}\left(\mathcal{L}^{\mathrm{BM}}_{N(x)}(x, x') \geq \psi\left(T_{N(x)}(x)\right)\right) \leq \delta,$$

i.e. the algorithm $\mathrm{BM}_{1:N(\cdot)}(\cdot)$ is $\big(\psi(T_{N(\cdot)}(\cdot)), \delta\big)$-ex post private, where $(\cdot)$ denotes a positional argument for an input $x \in \mathcal{X}$. Further, let $(\mathcal{E}_n)_{n\geq 1}$ be a sequence of privacy functions, and define

$$T_n(\beta_{1:n-1}) := \inf\left\{t \geq 0 : \psi(t) \geq \mathcal{E}_n(\beta_{1:n-1})\right\}.$$

Then $\mathrm{BM}_{1:N(\cdot)}(\cdot)$ is $(\mathcal{E}_{N(\cdot)}(\cdot), \delta)$-ex post private, where $\mathcal{E}_n(x)$ is defined analogously to $T_n(x)$.

Again, N should be thought of as a stopping rule based on parameter accuracy. $\mathcal{E}_n$ should be thought of as a rule for choosing the nth privacy parameter given the first $n-1$ iterates of BM.

4 An Adaptive, Continuous-Time Extension of Laplace Noise Reduction

Here, we generalize the original noise reduction mechanism of Ligett et al. [2017]. We first describe the original Laplace-based Markov process of Koufogiannis et al. [2017]. Fix any positive integer K and any finite, increasing sequence of times $(t_n)_{n\in[K]}$. Let $(\zeta_n)_{n=0}^K$ be the d-dimensional process given by $\zeta_0 = 0$ and

$$\zeta_n = \begin{cases} \zeta_{n-1} & \text{with probability } \left(\frac{t_{n-1}}{t_n}\right)^2 \\ \zeta_{n-1} + \mathrm{Lap}(t_n) & \text{otherwise.} \end{cases} \tag{3}$$

Koufogiannis et al. [2017] show that $\zeta_n \sim \mathrm{Lap}(t_n)$ and that $(\zeta_n)_{n=0}^K$ is Markovian. Ligett et al. [2017] use the above process to construct a noise reduction mechanism. Namely, they define the the *Laplace Noise Reduction* mechanism associated with $f : \mathcal{X} \to \mathbb{R}^d$ and $(t_n)_{n\in[K]}$ to be the algorithm $\mathrm{LNR} : \mathcal{X} \to (\mathbb{R}^d)^K$ given by $\mathrm{LNR}(x) := (f(x) + \zeta_K, \ldots, f(x) + \zeta_1)$. If $t_n := \Delta_1/\epsilon_n$, then releasing nth component $\mathrm{LNR}_n(x)$ in isolation is equivalent to running the classical Laplace mechanism with privacy level ϵ_n.

We now extend the process $(\zeta_n)_{n\in[K]}$ to a continuous time process with the same finite-dimensional distributions. Let $\eta > 0$ be arbitrary, and let $(P_t)_{t\geq\eta}$ be an inhomogeneous Poisson process with intensity function $\lambda(t) := \frac{2}{t}$. For $n \geq 1$, let $\mathcal{T}_n := \inf\{t \geq \eta : P_t \geq n\}$ be the nth jump of $(P_t)_{t\geq\eta}$ and set $\mathcal{T}_0 := \eta$. Noting that P_t must be a nonnegative integer, define the process $(Z_t)_{t\geq\eta}$ by

$$Z_t := \sum_{n=0}^{P_t} \mathrm{Lap}(\mathcal{T}_n). \tag{4}$$

It is immediate that $(Z_t)_{t\geq\eta}$ is Markovian. We show in Appendix D that $Z_t \sim \mathrm{Lap}(t)$. With $(Z_t)_{t\geq\eta}$, one can make LNR fully adaptive, meaning that the times $(t_n)_{n\in[K]}$ at which it is invoked need not be prespecified, and can depend on the underlying input database x by using time functions.

Definition 4.1. *Let $f : \mathcal{X} \to \mathbb{R}^d$ be a function and $(T_n)_{n\geq 1}$ a sequence of time functions. Let $(Z_t)_{t\geq\eta}$ be the process defined in Equation* (4)*. The Laplace noise reduction mechanism associated with f and $(T_n)_{n\geq 1}$ is the algorithm $\mathrm{LNR} : \mathcal{X} \to (\mathbb{R}^d)^\infty$ given by*

$$\mathrm{LNR}(x) := \left(f(x) + Z_{T_n(x)}\right)_{n\geq 1},$$

If the analyst would prefer instead to specify privacy functions $(\mathcal{E}_n)_{n\geq 1}$, they can do so by leveraging the corresponding time functions $T_n(x) := \Delta_1/\mathcal{E}_n(x)$, where $\mathcal{E}_n(x)$ is defined analogously to $T_n(x)$. We leverage LNR in our experiments in Section 6 and the process $(Z_t)_{t\geq 0}$ as a subroutine in constructing ReducedAboveThreshold. LNR admits the following trivial ex-post privacy guarantee.

Proposition 4.2. *Let* LNR *be associated with* $(T_n)_{n\geq 1}$ *and a function* f*, and suppose* f *has* ℓ_1*-sensitivity* Δ_1*. If* N *is stopping function, the algorithm* $\mathrm{LNR}_{1:N(\cdot)}(\cdot)$ *is* $\Delta_1/T_{N(\cdot)}(\cdot)$*-ex post private.*

5 Privately Checking if Accuracy is Above a Threshold

In Section 3 we presented the Brownian mechanism, characterized its privacy loss, and showed how to obtain ex-post privacy guarantees for arbitrary stopping functions. In particular, these stopping functions could be based on the accuracy of the observed iterates on public held-out data.

However, it may be desirable to privately check the accuracy of observed iterates directly on the hidden dataset $x \in \mathcal{X}$. Ligett et al. [2017] were able to accomplish this goal by coupling LNR with AboveThreshold, a classical algorithm for privately answering threshold queries [Dwork and Roth, 2014, Lyu et al., 2017]. In the context of ERM, AboveThreshold works by iteratively checking whether the empirical risk of each parameter is below a target threshold, stopping at the first such occurrence. As Ligett et al. [2017] note, the downside to AboveThreshold is that it requires a prefixed privacy level. In empirical studies, the authors found this pre-selected privacy cost dominated the ex-post privacy guarantees, showing little benefit to using noise reduction.

Below, we construct ReducedAboveThreshold, a generalization of AboveThreshold which provides ex-post privacy guarantees. We show how to couple BM with ReducedAboveThreshold to obtain tighter ex-post privacy guarantees than coupling with AboveThreshold would permit. In particular, if BM is run using parameters $(\epsilon_n)_{n\geq 1}$ and ReducedAboveThreshold indicates the Nth parameter obtains sufficiently high accuracy, the privacy loss of the net procedure will be at most $2\epsilon_N$ — only twice the privacy loss that would be accrued by testing on public data.

Algorithm 1 ReducedAboveThreshold (via Laplace Noise Reduction)

Require: Algorithm $\mathrm{Alg} : \mathcal{X} \to \mathcal{Y}^\infty$, parameter $\epsilon_{\max} > 0$, threshold τ, database $x \in \mathcal{X}$, utility $u : \mathcal{Y} \times \mathcal{X} \to \mathbb{R}$ where $u(\beta, \cdot)$ is Δ-sensitive $\forall \beta$, privacy functions $(\mathcal{E}_n)_{n\geq 1}$ with $\mathcal{E}_n \leq \epsilon_{\max}\ \forall n$.
for $n \geq 1$ **do**
  $\epsilon_n := \mathcal{E}_n(\mathrm{Alg}_{1:n-1}(x)), T_n := 2\Delta/\epsilon_n$
  $\zeta_n := Z_{T_n}$, where $(Z_t)_{t\geq\eta}$ in Eq. (4) defines the LNR mechanism with $\eta := 2\Delta/\epsilon_{\max}$.
  $\xi_n \sim \mathrm{Lap}\left(\frac{4\Delta}{\epsilon_n}\right)$
  if $u(\mathrm{Alg}_n(x), x) + \xi_n \geq \tau + \zeta_n$ **then**
    Print 1 and HALT
  else
    Print 0

τ should be seen as a target accuracy, Alg as a mechanism for releasing a parameter (e.g. BM, LNR), and u as evaluating the accuracy of $\mathrm{Alg}_n(x)$ on x. $\epsilon_{\max}$ is an arbitrarily large maximum privacy level, used to prevent the user from examining (Z_t) at arbitrarily small times. The above generalizes to sequences of thresholds $(\tau_n)_{n\geq 1}$ and sequences $(u_n)_{n\geq 1}$ of functions $u_n : \mathcal{Y}^n \times \mathcal{X} \to \mathbb{R}$ that are Δ-sensitive in their second argument, but the added generality yields only marginal benefits. When $\mathcal{E}_n = \epsilon$ for all n, Algorithm 1 recovers AboveThreshold as a special case. The intuition behind ReducedAboveThreshold is that by gradually removing Laplace noise from the threshold, a data analyst can ensure that privacy of the whole procedure only depends on the magnitude of Laplace noise added when the algorithm halts. The following characterizes the privacy loss of Algorithm 1.

Theorem 5.1. *For any* $n \geq 1$ *and neighboring datasets* $x \sim x'$*, let* $\mathcal{L}^{\mathrm{Alg}}_{1:n}(x, x')$ *denote the privacy between* $\mathrm{Alg}_{1:n}(x)$ *and* $\mathrm{Alg}_{1:n}(x')$*. For any* $x \in \mathcal{X}$*, define* $N(x)$ *to be the first round where* ReducedAboveThreshold *run on input* $x \in \mathcal{X}$ *outputs 1, that is*

$$N(x) := \inf\{n \geq 1 : \mathrm{ReducedAboveThreshold}_n(x) = 1\}.$$

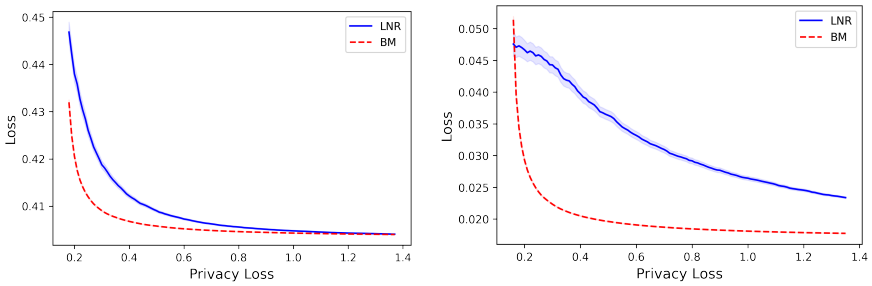
Then, the privacy loss between $\mathrm{ReducedAboveThreshold}(x)$ *and* $\mathrm{ReducedAboveThreshold}(x')$*, denoted* $\mathcal{L}^{\mathrm{RAT}}(x, x')$*, is bounded by*

$$\mathcal{L}^{\mathrm{RAT}}(x, x') \leq \mathcal{L}^{\mathrm{Alg}}_{1:N(x)}(x, x') + \mathcal{E}_{N(x)}(\mathrm{Alg}_{1:N(x)-1}(x)).$$

We prove Theorem 5.1 in Appendix C, where we also provide a utility guarantee for ReducedAboveThreshold. This utility guarantee, much like the utility guarantee for AboveThreshold, is in practice weak as it derives from a union bound. Using Theorem 5.1, we can simply choose $\mathrm{Alg} = \mathrm{BM}$ as a means of adaptively generating parameters. The following corollary, which follows immediately from the above theorem, provides the ex-post privacy guarantees of combining ReducedAboveThreshold and BM.

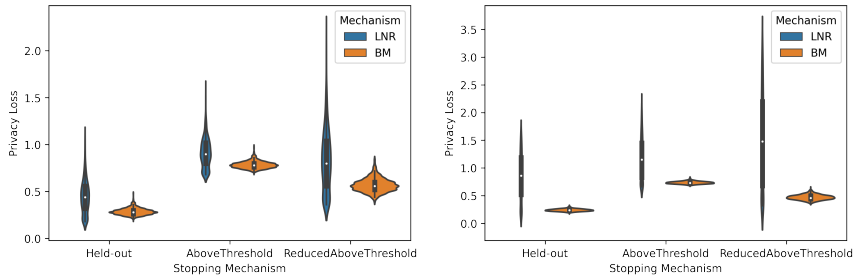
Corollary 5.2. *Let* BM *be the Brownian mechanism associated with a function f, decreasing time functions $(T_n)_{n\geq 1}$, and a a δ-privacy boundary ψ. Let* ReducedAboveThreshold *be run with privacy functions $(\psi(T_n))_{n\geq 1}$, threshold τ, and algorithm* BM*. Then,* ReducedAboveThreshold *is $\big(2\psi(T_{N(\cdot)}(\cdot)), \delta\big)$-ex post private.*

6 Experiments



(a) Regularized Logistic Regression (b) Ridge Regression

Figure 2: Privacy loss plotted against loss (respectively regularized logistic and ridge loss) for the statistical tasks of regularized logistic regression and ridge regression.



(a) Regularized Logistic Regression (b) Ridge Regression

Figure 3: Empirical privacy loss distributions for logistic regression and ridge regression with loss assessed either (left) on the training data treated as a public, held-out dataset, (middle) via AboveThreshold, or (right) via ReducedAboveThreshold.

Choice of tasks: We compare the performance of BM and LNR on the tasks of regularized logistic regression via output perturbation [Chaudhuri et al., 2011] and ridge regression via covariance perturbation [Smith et al., 2017].[3] For logistic regression, we leveraged the KDD-99 dataset [KDD, 1999] with $d = 38$ features, predicting whether network events can be classified as "normal" or "malicious". For ridge regression, we used the Twitter dataset [Kawala et al., 2013] with $d = 77$ features to predict log-popularity of posts. In each case, we ran our experiments on $n = 10,000$ randomly sub-sampled data points. In order to guarantee bounded sensitivity, we normalized each data point to have unit ℓ_2 norm. We note that this aspect differs from the experimentation conducted by Ligett et al. [2017], who normalized by the *maximum* ℓ_2 norm, a non-private operation.

[3]The two tasks use the logistic loss $\ell(y, z) := \log(1+\exp(-yz))$ and the squared loss $\ell(y, z) := \frac{1}{2}(z-y)^2$. The regularized loss on a dataset $\mathcal{D} := \{(x_i, y_i)\}_{i\in[n]}$ is $L(\beta, \mathcal{D}) := \frac{1}{n}\sum_{i=1}^{n} \ell(y_i, \beta^T x_i) + \frac{\lambda||\beta||_2^2}{2}$.

Parameter settings: We set the regularization parameter to be $\lambda = 0.05$ and note that the ℓ_2 and ℓ_1-sensitivity for the output perturbation of logistic regression are respectively $\frac{2}{n\lambda}$ and $\frac{2\sqrt{d}}{n\lambda}$. Likewise, for covariance perturbation in ridge regression, the ℓ_2-sensitivities for privately releasing $X^T X$ and $X^T y$ are both 2.0, and the corresponding ℓ_1-sensitivities for releasing these quantities are $2.0d$ and $2.0\sqrt{d}$ respectively [Ligett et al., 2017, Chaudhuri et al., 2011]. We set the failure probability for BM to be $\delta = 10^{-6}$, and in each task map privacy parameters (ϵ_n) to times (t_n) using the linear privacy boundary $\psi^L_{a,b}$ optimized for tightness at $\epsilon = 0.3$.

Experiments: For each task, we conducted two experiments. In the first experiment, we plotted guaranteed (in the case of LNR) or high-probability (in the case of BM) privacy loss on the x-axis against average loss (either logistic or ridge) on the y-axis. We computed the average loss curve for each mechanism over 1,000 trials, and have included point-wise valid 95% confidence intervals.

In the second experiment, we plotted the empirical privacy loss distributions for BM and LNR under the stopping conditions of loss being at most 0.41 for logistic regression and 0.025 for ridge regression. For each mechanism, we evaluated this empirical distribution using three approaches for testing empirical loss: treating the training data as a held-out dataset, using $\mathrm{AboveThreshold}$, and using our mechanism, $\mathrm{ReducedAboveThreshold}$. In $\mathrm{AboveThreshold}$, we set the privacy parameter to be fixed at $\epsilon = 0.5$. In $\mathrm{ReducedAboveThreshold}$, we took the sequence of privacy parameters to be the same as the sequence of privacy parameters used by BM and LNR. We once again computed these empirical distributions over 1,000 runs of each mechanism.

Findings: The findings of the two experiments are summarized in Figure 2 and Figure 3. For both tasks, BM obtains significant improvements in loss over LNR near the privacy loss level that was optimized for. For both tasks, the privacy loss distribution for BM has lower median privacy loss than that of LNR. In addition, the privacy loss distribution for BM is more tightly concentrated around the median, indicating more consistent performance. The privacy loss distribution for LNR has a heavy tail, demonstrating that many runs do not attain the target loss until high privacy loss costs are incurred. Comparing $\mathrm{ReducedAboveThreshold}$ and $\mathrm{AboveThreshold}$, we see that the privacy loss distribution for $\mathrm{ReducedAboveThreshold}$ has higher variance than that of $\mathrm{AboveThreshold}$. However, $\mathrm{ReducedAboveThreshold}$ attains a significantly lower median level of privacy loss when coupled with BM. This latter point reflects the observations of Ligett et al. [2017], who note that when $\mathrm{AboveThreshold}$ is used to determine stopping conditions on private data, it contributes the bulk of the privacy loss to the empirical distributions. On the other hand, our figures demonstrate that $\mathrm{ReducedAboveThreshold}$ results in a more mild privacy loss at target stopping conditions.

7 Conclusion

In this paper, we constructed the Brownian mechanism (BM), a novel approach to noise reduction that adds noise to a hidden parameter via a Brownian motion. We not only precisely characterized the privacy loss of the Brownian mechanism, but also bounded it through applying machinery from continuous time martingale theory. We then demonstrated how the utility of the iterates produced by BM can be assessed on private data via $\mathrm{ReducedAboveThreshold}$, a generalization of the classical $\mathrm{AboveThreshold}$ algorithm. This was itself accomplished by a continuous-time generalization of the original Laplace noise reduction (LNR) mechanism. Last, we empirically demonstrated that BM outperforms LNR on common statistical tasks, such as regularized logistic and ridge regression.

We comment on several limitations and open problems related to our work. We considered noise reduction mechanisms in the setting of one-shot privacy, in which only a single mechanism is run on private data. Traditional composition results, such as those for fixed privacy parameters [Dwork et al., 2010, Kairouz et al., 2015, Murtagh and Vadhan, 2016] or adaptively selected parameters [Rogers et al., 2016, Feldman and Zrnic, 2021, Whitehouse et al., 2022] are not directly applicable to algorithms satisfying ex-post privacy; additional machinery needs to be developed to handle composition in this case. Finally, noise reduction is currently only applicable to output perturbation methods; it remains open to see how to combine noise reduction with other prominent methods for private computation, such as objective perturbation.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] **We construct the Brownian mechanism is Section 3, show how to query utility on private datasets in a manner that provides adaptive privacy guarantees in Section 5, and conduct experiments showing the Brownian mechanism outperforms Laplace noise reduction in Section 6**.
   (b) Did you describe the limitations of your work? [Yes] **In the Conclusion (Section 7) we discuss how traditional composition results for differential privacy do not directly apply to mechanisms with ex-post privacy guarantees. We leave it as future work to address composition in this setting. Additionally, in the same section, we mention noise reduction is only applicable to output perturbation methods. We leave it as future work to generalize to other modes of private computation, such as objective perturbation.**
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] **There are no negative societal impacts.**
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] **All theoretical results have a full set of assumptions stated.**
   (b) Did you include complete proofs of all theoretical results? [Yes] **The proofs of all non-trivial results are fully contained in the appendices. Appendix B contains the proofs for Section 3, Appendix C contains the proofs for Section 5, and Appendix D contains proofs for Section 4.**
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] **The code is commented and contained in the supplement.**
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] **These details are included in the paper and code.**
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] **We included 95% confidence intervals on the relevant experiments. See the graphs in Section 6**
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A] **Experiments were lightweight and run on a laptop.**
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] **The datasets are properly attributed in the Bibliography.**
   (b) Did you mention the license of the assets? [N/A] **The datasets leveraged in this work are public, and hence do not require licensing.**
   (c) Did you include any new assets either in the supplemental material or as a URL? [No]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [No] **we only use well-known, broadly used, publicly available datasets, and there is no known offensive content or personal information to the best of our knowledge.**
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

A Background on Martingale Concentration

In this section, we provide a background on the basics of martingale concentration needed throughout this paper. Central to all results in this section is Ville's inequality [Ville, 1939], which can be viewed as a time-uniform version of Markov's inequality for martingales.

Lemma A.1 (**Ville's Inequality** [Ville, 1939])**.** *Let $(X_t)_{t\geq 0}$ be a nonnegative supermartingale with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$. Then, for any confidence parameter $\delta \in (0,1)$, we have*

$$\mathbb{P}\left(\exists t \geq 0 : X_t \geq \frac{\mathbb{E}X_0}{\delta}\right) \leq \delta.$$

While standard Brownian motion $(B_t)_{t\geq 0}$ is not a nonnegative supermartingale, geometric Brownian motion given by $Y_t^\lambda := \exp\left(\lambda B_t - \frac{\lambda^2}{2}t\right)$ is a nonnegative martingale for any $\lambda \in \mathbb{R}$, and hence Lemma A.1 can be applied. In fact, the probability in the lemma above becomes exactly δ when it is applied to a nonnegative martingale with continuous paths like Y_t^λ. From Ville's inequality, the following *line-crossing inequality* for Brownian motion can be obtained.

Lemma A.2 (Line-Crossing Inequality)**.** *For $\delta \in (0,1)$ and $a, b > 0$ satisfying $e^{-2ab} = \delta$, we have*

$$\mathbb{P}\left(\exists t \geq 0 : B_t \geq at + b\right) = \delta.$$

A proof of the above fact can be found in any standard book on continuous time martingale theory [Le Gall, 2016, Durrett, 2019]. The above also follows from a special case of the more general time-uniform Chernoff bound presented in Howard et al. [2020].

The above inequality can be seen as optimizing the tightness of the time-uniform boundary at one pre-selected point in time. However, due to the adaptive nature of the Brownian mechanism presented in Section 3, it is sometimes desirable to construct a time-uniform boundary which sacrifices tightness at a fixed point in time to obtain greater tightness over all of time.

The *method of mixtures* provides one such approach for constructing tighter time-uniform boundaries [Kaufmann and Koolen, 2021, Howard et al., 2021]. We discuss this concept briefly in the context of Brownian motion. Observe that, since $(Y_t^\lambda)_{t\geq 0}$ is a nonnegative martingale, for any probability measure π on $\mathbb{R}$, the process $(X_t^\pi)_{t\geq 0}$ given by

$$X_t^\pi := \int_{\mathbb{R}} Y_t^\lambda \pi(d\lambda)$$

is also nonnegative martingale. By appropriately choosing the probability measure π and applying Ville's inequality, one obtains the following concentration inequality [Howard et al., 2021].

Lemma A.3 (Mixture Inequality)**.** *Let $\rho > 0$ and $\delta \in (0,1)$ be arbitrary. Then,*

$$\mathbb{P}\left(\exists t \geq 0 : B_t \geq \sqrt{2(t+\rho)\log\left(\frac{1}{\delta}\sqrt{\frac{t+\rho}{\rho}}\right)}\right) = \delta.$$

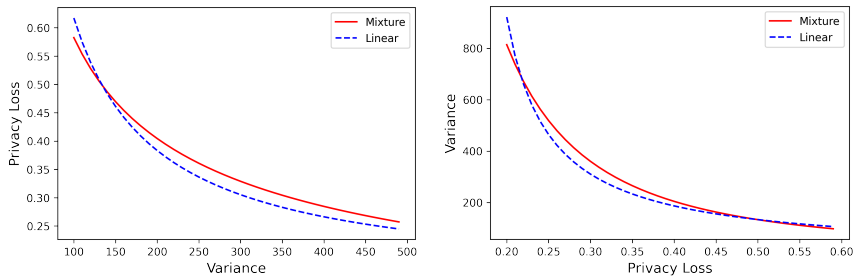
We leverage Lemmas A.2 and A.3 to construct the privacy boundaries in Theorem 3.6 in Appendix B.

B Proofs From Section 3

Here, we prove the results from Section 3. We start by showing that BM is in fact a noise-reduction mechanism, per the condition in Definition 2.3.

Proof of Lemma 3.2. Let $x \in \mathcal{X}$ be a fixed input dataset and x' a neighboring dataset. For any increasing sequence of times $t_1 < t_2 < \cdots < t_n$ and point $y \in \mathbb{R}^d$, let $p^y_{t_1,\ldots,t_n}$ denote the joint density of $(y + B_{t_1}, \ldots, y + B_{t_n})$. Since p^y is just the joint density for a Brownian motion started at y, applying Bayes rule and the Markov property, it follows that, for any $\beta_1, \ldots, \beta_n \in \mathbb{R}^d$,

$$p^y_{t_1,\ldots,t_n}(\beta_1, \ldots, \beta_n) = p^y_{t_1}(\beta_1) p^{\beta_1}_{t_2-t_1,\ldots,t_n-t_1}(\beta_2, \ldots, \beta_n). \tag{5}$$



(a) Variance of Noise vs. Privacy Loss (b) Privacy Loss vs. Variance of Noise

Figure 4: A comparison of the linear and mixture boundaries, both optimized for tightness at $\epsilon = 0.5$. The first plot directly plots the corresponding bounds as in Theorem 3.6. The second plot inverts the boundaries, showing the variance necessary to meet a target privacy level.

The strong Markov property Durrett [2019] will yield that the above is true even when the selected times are random. Note that $\mathcal{L}^{\mathrm{BM}}_{1:n}(x, x')$, the privacy loss between $\mathrm{BM}_{1:n}(x)$ and $\mathrm{BM}_{1:n}(x')$, is given by

$$\mathcal{L}^{\mathrm{BM}}_{1:n}(x, x') = \log\left(\frac{p^{f(x)}_{T_n(x),\ldots,T_1(x)}(\mathrm{BM}_n(x), \ldots, \mathrm{BM}_1(x))}{p^{f(x')}_{T_n(x),\ldots,T_1(x)}(\mathrm{BM}_n(x), \ldots, \mathrm{BM}_1(x))}\right).$$

Likewise, $\mathcal{L}^{\mathrm{BM}}_{n}(x, x')$, the privacy loss between $\mathrm{BM}_n(x)$ and $\mathrm{BM}_n(x')$, is given by

$$\mathcal{L}^{\mathrm{BM}}_{n}(x, x') = \log\left(\frac{p^{f(x)}_{T_n(x)}(\mathrm{BM}_n(x))}{p^{f(x')}_{T_n(x)}(\mathrm{BM}_n(x))}\right).$$

Applying Equation (5), we have

$$\begin{aligned}
\mathcal{L}^{\mathrm{BM}}_{1:n}(x, x') &= \log\left(\frac{p^{f(x)}_{T_n(x),\ldots,T_1(x)}(\mathrm{BM}_n(x), \ldots, \mathrm{BM}_1(x))}{p^{f(x')}_{T_n(x),\ldots,T_1(x)}(\mathrm{BM}_n(x), \ldots, \mathrm{BM}_1(x))}\right) \\
&= \log\left(\frac{p^{f(x)}_{T_n(x)}(\mathrm{BM}_n(x))}{p^{f(x')}_{T_n(x)}(\mathrm{BM}_n(x))}\right) + \log\left(\frac{p^{\mathrm{BM}_n(x)}_{T_{n-1}(x)-T_n(x),\ldots,T_1(x)-T_n(x)}(\mathrm{BM}_{n-1}(x), \ldots, \mathrm{BM}_1(x))}{p^{\mathrm{BM}_n(x)}_{T_{n-1}(x)-T_n(x),\ldots,T_1(x)-T_n(x)}(\mathrm{BM}_{n-1}(x), \ldots, \mathrm{BM}_1(x))}\right) \\
&= \log\left(\frac{p^{f(x)}_{T_n(x)}(\mathrm{BM}_n(x))}{p^{f(x')}_{T_n(x)}(\mathrm{BM}_n(x))}\right) = \mathcal{L}^{\mathrm{BM}}_{n}(x, x'),
\end{aligned}$$

proving the desired result.

□

We now prove Theorem 3.4, which gives a closed form characterization of the Brownian mechanism. In what follows, we leverage the same notation for the density of Brownian motion as in the above proof.

Proof of Theorem 3.4. The second statement of the theorem is trivial and follows from our assumption of bounded ℓ_2 sensitivity. Hence, we only prove the first statement below.

Without loss of generality, and for the sake of simplicity, we can consider the function $g(y) := f(y) - f(x)$, as then $g(x) = 0$. Observe that, for $y \in \mathcal{X}$, the random vector $g(y) + B_t$ has Lebesgue density

$$p^{g(y)}_t(\beta) \propto \exp\left(-\frac{1}{2t}||\beta - g(y)||_2^2\right).$$

Consequently, the privacy loss can be written as

$$\begin{aligned}
\mathcal{L}_n^{\mathrm{BM}}(x, x') &= \log\left(\frac{p^0_{T_n(x)}(B_{T_n(x)})}{p^{g(x')}_{T_n(x)}(B_{T_n(x)})}\right) \\
&= \frac{1}{2T_n(x)}\left(-||B_{T_n(x)}||_2^2 + ||B_{T_n(x)} - g(x')||_2^2\right) \\
&= -\frac{1}{T_n(x)}\langle B_{T_n(x)}, g(x')\rangle + \frac{1}{2T_n(x)}||g(x')||_2^2 \\
&= -\frac{||g(x')||_2}{T_n(x)}\left\langle B_{T_n(x)}, \frac{g(x')}{||g(x')||_2}\right\rangle + \frac{1}{2T_n(x)}||g(x')||_2^2 \\
&= -\frac{||g(x')||_2}{T_n(x)}\left\langle B_{T_n(x)}, \frac{g(x')}{||g(x')||_2}\right\rangle + \frac{1}{2T_n(x)}||g(x')||_2^2 \\
&= -\frac{||g(x')||_2}{T_n(x)}W_{T_n(x)} + \frac{1}{2T_n(x)}||g(x')||_2^2.
\end{aligned}$$

Note that the last inequality follows from the fact that if $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion and $z \in \mathbb{R}^d$ is a unit vector under the ℓ_2 norm, then the process $W_t := \langle z, B_t\rangle$ is a standard Brownian motion. Noting that $g(x') = f(x') - f(x)$ and that $(-W_t)_{t\geq 0}$ is also a Brownian motion furnishes the result.

□

We now use the characterization of privacy loss in Theorem 3.4 alongside the time-uniform concentration results for continuous time martingales found in Appendix A to construct two general families of privacy boundaries. We now prove Theorem 3.6.

Proof of Theorem 3.6. Recall from Theorem 3.4 that we have the following bound

$$\mathcal{L}_n^{\mathrm{BM}}(x, x') \leq \frac{\Delta^2}{2T_n(x)} + \frac{\Delta}{T_n(x)}W^+_{T_n(x)},$$

where, for a random variable X, the random variable X^+ is given by $\max(X, 0)$. First, by leveraging Lemma A.3, we see that, with probability at least $1-\delta$, for all $n \in \mathbb{N}$, we have

$$\mathcal{L}_n^{\mathrm{BM}}(x, x') \leq \frac{\Delta^2}{2T_n(x)} + \frac{\Delta}{T_n(x)}\sqrt{2(T_n(x)+\rho)\log\left(\frac{1}{\delta}\sqrt{\frac{T_n(x)+\rho}{\rho}}\right)} = \psi^M_\rho(T_n(x)),$$

proving that ψ^M_ρ is a valid δ-privacy boundary. Likewise, by Lemma A.2, we have that

$$\mathcal{L}_n^{\mathrm{BM}}(x, x') \leq \frac{\Delta^2}{2T_n(x)} + \frac{\Delta}{T_n(x)}(aT_n(x)+b) = \frac{\Delta}{T_n(x)}\left(\frac{\Delta}{2}+b\right) + \Delta a = \psi^L_{a,b}(T_n(x)),$$

showing $\psi^L_{a,b}$ is a valid δ-privacy boundary.

□

C Proofs From Section 5

In this appendix, we provide proofs of the results in Section 5. We start by proving the privacy guarantees for $\mathrm{ReducedAboveThreshold}$.

Proof of Theorem 5.1. For $\mathrm{ReducedAboveThreshold}$ as described in Algorithm 1, on the event $\{N(x) = n\}$, all information leaked about the underlying private dataset is contained in $\mathrm{Alg}_{1:n}(x)$ and $\alpha_{1:n}(x)$, where $\alpha_n(x)$ is defined to be the nth bit output by $\mathrm{ReducedAboveThreshold}$. For any $y \in \mathcal{X}$, let $q^y_{1:n}$ denote the joint density of $(\mathrm{Alg}_{1:n}(y), \alpha_{1:n}(y))$, $p^y_{1:n}$ the marginal density of $\mathrm{Alg}_{1:n}(y)$, and $p^y_{1:n}(\cdot \mid \cdot)$ the conditional pmf of $\alpha_{1:n}(y)$ given the observed values of $\mathrm{Alg}_{1:n}(y)$.

As such, for any neighboring datasets $x \sim x'$, on the event $\{N(x) = n\}$, the privacy loss of $\mathrm{ReducedAboveThreshold}$, denoted by $\mathcal{L}^{\mathrm{RAT}}(x, x')$, is given by

$$\begin{aligned}
\mathcal{L}^{\mathrm{RAT}}_{1:n}(x, x') &= \log\left(\frac{q^x_{1:n}(\mathrm{Alg}_{1:n}(x), \alpha_{1:n}(x))}{q^{x'}_{1:n}(\mathrm{Alg}_{1:n}(x), \alpha_{1:n}(x))}\right) \\
&= \log\left(\frac{p^x_{1:n}(\mathrm{Alg}_{1:n}(x))}{p^{x'}_{1:n}(\mathrm{Alg}_{1:n}(x))}\right) + \log\left(\frac{p^x_{1:n}(\alpha_{1:n}(x) \mid \mathrm{Alg}_{1:n}(x))}{p^{x'}_{1:n}(\alpha_{1:n}(x) \mid \mathrm{Alg}_{1:n}(x))}\right) \\
&= \log\left(\frac{p^x_{1:n}(\mathrm{Alg}_{1:n}(x))}{p^{x'}_{1:n}(\mathrm{Alg}_{1:n}(x))}\right) + \log\left(\frac{p^x_{1:n}(0^{n-1}1 \mid \mathrm{Alg}_{1:n}(x))}{p^{x'}_{1:n}(0^{n-1}1 \mid \mathrm{Alg}_{1:n}(x))}\right) \\
&= \mathcal{L}^{\mathrm{Alg}}_n(x, x') + L_n(x, x'),
\end{aligned}$$

where $0^{n-1}1$ denotes the string of $n-1$ 0's followed by a single 1. In the last line we leverage the definition of the privacy loss between $\mathrm{Alg}_{1:n}(x)$ and $\mathrm{Alg}_{1:n}(x')$ and define

$$L_n(x, x') := \log\left(\frac{p^x_{1:n}(0^{n-1}1 \mid \mathrm{Alg}_{1:n}(x))}{p^{x'}_{1:n}(0^{n-1}1 \mid \mathrm{Alg}_{1:n}(x))}\right).$$

Now, to finish the result, it suffices to prove that, for any n, $L_n(x, x') \leq \mathcal{E}_n(\mathrm{Alg}_{1:n-1}(x))$. Without loss of generality, we can assume all thresholds take the same value τ across rounds, as we can always define the shifted function $u'_n(\mathrm{Alg}_{1:n}(x), x) := u_n(\mathrm{Alg}_{1:n}(x), x) - \tau_n + \tau$. To prove our desired inequality, we proceed largely in the same way as the proof of $\mathrm{AboveThreshold}$ found in Lyu et al. [2017], noting that conditioning on $\mathrm{Alg}_{1:n}(x)$ serves to fix the utility functions $u_1(\mathrm{Alg}_1(x), \cdot), \ldots, u_n(\mathrm{Alg}_{1:n}(x), \cdot)$ and the privacy levels $\mathcal{E}_1, \mathcal{E}_2(\mathrm{Alg}_1(x)), \ldots, \mathcal{E}_n(\mathrm{Alg}_{1:n-1}(x))$. For simplicity, going forward, we refer to the former quantities as $u_1(\cdot), \ldots, u_n(\cdot)$ and the latter quantities just as $\epsilon_1, \ldots, \epsilon_n$. The only remaining caveat that we must take care in handling variable amount of noise on the threshold introduced by LNR. Going forward, let $\mathbb{P}_{1:n}$ denote the conditional probability $\mathbb{P}(\cdot \mid \mathrm{Alg}_{1:n}(x))$. First, observe that we can write the numerator of $L_n(x, x')$ as

$$p^x\left(0^{n-1}1 \mid \mathrm{Alg}_{1:n}(x)\right) = \int_{\mathbb{R}^n} g^\tau_{1:n}(s_1, \ldots, s_n)\left(\prod_{i=1}^{n-1} \mathbb{P}_{1:n}\left(u_i(x) + \xi_i < s_i\right)\right)\mathbb{P}_{1:n}\left(u_n(x) + \xi_n \geq s_n\right) d\vec{s},$$

where $g^\tau_{1:n}$ represents the density for the joint distribution of $(\tau + Z(2\Delta/\epsilon_m))_{m=1}^n$, where $(Z(t))$ is as in Equation (4). We now need three inequalities. The first two are standard from the analysis of Lyu et al. [2017], so we do not provide a proof. The third inequality is a product of our novel $\mathrm{ReducedAboveThreshold}$ mechanism, and hence we provide a proof. The inequalities are as follows:

1. For $i < n$ and fixed s_i, $\mathbb{P}_{1:n}(u_i(x) + \xi_i < s_i) \leq \mathbb{P}_{1:n}(u_i(x') + \xi_i < s_i + \Delta)$,
2. for $i = n$ and any s_n, $\mathbb{P}_{1:n}(u_n(x) + \xi_n \geq s_n) \leq e^{\epsilon_n/2}\mathbb{P}_{1:n}(u_n(x') + \xi_n \geq s_n + \Delta)$, and
3. for any $s_{1:n} \in \mathbb{R}^n$, $g^\tau_{1:n}(s_1, \ldots, s_n) \leq e^{\epsilon_n/2} g^\tau_{1:n}(s_1 + \Delta, \ldots, s_n + \Delta)$.

We now prove the third inequality. We have that

$$\begin{aligned}
\frac{g^\tau_{1:n}(s_1, \ldots, s_n)}{g^{\tau-\Delta}_{1:n}(s_1, \ldots, s_n)} &= \frac{g^\tau_n(s_n) g^\tau_{1:n-1}(s_1, \ldots, s_{n-1} \mid s_n)}{g^{\tau-\Delta}_n(s_n) g^{\tau-\Delta}_{1:n-1}(s_1, \ldots, s_{n-1} \mid s_n)} \\
&= \frac{g^\tau_n(s_n)}{g^{\tau-\Delta}_n(s_n)} \leq e^{\epsilon_n/2},
\end{aligned}$$

where the first equality follows from applying Bayes rule to the joint densities of the noisy thresholds, and the second equality follows from the fact that $(Z(t))$ forms a Markov process. This in particular implies that the density conditional density given the nth threshold satisfies $g^a_{1:n-1}(s_1, \ldots, s_{n-1} \mid s_n) = g^b_{1:n-1}(s_1, \ldots, s_{n-1} \mid s_n)$ for all $a, b \in \mathbb{R}$. The last inequality follows from examining the ratio of densities of $\mathrm{Lap}(\tau, \epsilon_n/2)$ and $\mathrm{Lap}(\tau - \Delta, \epsilon_n/2)$ random variables. Now, observe that by a simple shift of parameters we have

$$g^{\tau-\Delta}_{1:n}(s_1, \ldots, s_n) = g^\tau_{1:n}(s_1 + \Delta, \ldots, s_n + \Delta).$$

Plugging this in, we have

$$
\begin{aligned}
&p^x\left(0^{n-1}1 \mid \mathrm{Alg}_{1:n}(x)\right) \\
&= \int_{\mathbb{R}^n} g^{\tau}_{1:n}(s_1,\ldots,s_n)\left(\prod_{i=1}^{n-1}\mathbb{P}_{1:n}(u_i(x)+\xi_i<s_i)\right)\mathbb{P}_{1:n}(u_n(x)+\xi_n\geq s_n)d\vec{s} \\
&\leq e^{\epsilon_n/2}\int_{\mathbb{R}^n} g^{T-\Delta}_{1:n}(s_1,\ldots,s_n)\left(\prod_{i=1}^{n-1}\mathbb{P}_{1:n}(u_i(x)+\xi_i<s_i)\right)\mathbb{P}_{1:n}(u_n(x)+\xi_n\geq s_n)d\vec{s} \\
&\leq e^{\epsilon_n}\int_{\mathbb{R}^n} g^{\tau-\Delta}_{1:n}(s_1,\ldots,s_n)\left(\prod_{i=1}^{n-1}\mathbb{P}_{1:n}(u_i(x')+\xi_i<s_i+\Delta)\right)\mathbb{P}(u_n(x')+\xi_n\geq s_n+\Delta)d\vec{s} \\
&= e^{\epsilon_n}\int_{\mathbb{R}^n} g^{\tau}_{1:n}(s_1,\ldots,s_n)\left(\prod_{i=1}^{n-1}\mathbb{P}_{1:n}(u_i(x')+\xi_i<s_i)\right)\mathbb{P}_{1:n}(u_n(x')+\xi_n\geq s_n)d\vec{s} \\
&= e^{\epsilon_n}p^{x'}\left(0^{n-1}1 \mid \mathrm{Alg}_{1:n}(x)\right).
\end{aligned}
$$

Rearranging furnishes the desired result.

□

We can also prove a corresponding utility guarantee for $\mathrm{ReducedAboveThreshold}$. As mentioned earlier, this utility guarantee is naive in the sense that it is derived from a union bound. Thus, instead of plotting the utility guarantee in our experiments in Section 6, we instead plot empirically observed loss/accuracy. Additionally, for the utility guarantee to hold, the sequence of privacy functions $(\mathcal{E}_n)_{n\geq 1}$ must be constant functions, i.e. $\mathcal{E}_n = \epsilon_n$ for each n. We now state the formal, high-probability utility guarantee in the following proposition.

Proposition C.1. *Let $(p_n)_{n\geq 1}$ be a sequence of non-negative numbers such that $\sum_{i=1}^{\infty} p_i = 1$, and let $\gamma \in (0,1)$ be a confidence parameter. Define the sequence of parameters $(\eta_n)_{n\geq 1}$ by*

$$\eta_n := \frac{4\Delta}{\epsilon_n}\left(\log\left(\frac{2}{\gamma}\right) - \log(p_n)\right).$$

Then, if $N(x)$ is the time define in Theorem 5.1, with probability at least $1-\gamma$, we have

$$u_{N(x)}(x) \geq \tau_{N(x)} - \eta_{N(x)}.$$

Proof. The above utility guarantee follows from applying two simple union bounds. First, we have

$$\mathbb{P}\left(\bigcup_{n\geq 1}\{|\xi_n| \geq \eta_n/2\}\right) \leq \sum_{n\geq 1}\mathbb{P}(|\xi_n| \geq \eta_n/2) = \sum_{n\geq 1}\exp\left(\frac{-\epsilon_n\eta_n}{4\Delta}\right) = \frac{\gamma}{2}\sum_{n\geq 1} p_n = 1.$$

Second, we have that

$$\mathbb{P}\left(\bigcup_{n\geq 1}\{|\zeta_n| \geq \eta_n/2\}\right) \leq \sum_{n\geq 1}\mathbb{P}(|\zeta_n| \geq \eta_n/2) = \sum_{n\geq 1}\exp\left(\frac{-\epsilon_n\eta_n}{2\Delta}\right) \leq \frac{\gamma}{2}\sum_{n\geq 1} p_n = 1.$$

Thus, with probability at least $1-\gamma$, we have simultaneously for all $n \geq 1$ that $|\xi_n| \leq \eta_n/2$ and $|\zeta_n| \leq \eta_n/2$. Thus, with the same probability, on round $N(x)$, we have

$$u_{N(x)}(x) \geq \tau_{N(x)} - \eta_{N(x)}.$$

□

D Proofs From Section 4

We first prove that the process defined in Equation (4) has Laplace marginal distributions. In what follows, we sometimes use the notation $Z(t)$ interchangeably with Z_t for convenience.

Theorem D.1. *Let $(Z_t)_{t\geq\eta}$ be the process defiend in Equation* (4)*. Then, for any $t \geq \eta$, we have*

$$Z_t \sim \mathrm{Lap}(t).$$

Proof. Recall that if $X \sim \mathrm{Lap}(s)$, then X has characteristic function φ_s given by

$$\varphi_s(\lambda) = \frac{1}{1+\lambda^2 s^2}.$$

Let ϕ denote the characteristic function of $Z_t - Z_\eta$. Since Z_η and $Z_t - Z_\eta$ are independent, to show $Z_t \sim \mathrm{Lap}(t)$, it suffices to show that

$$\phi(\lambda) = \frac{\varphi_t(\lambda)}{\varphi_\eta(\lambda)} = \frac{1+\lambda^2\eta^2}{1+\lambda^2 t^2}.$$

Now, observe that the inhomogenous Poisson process $(P_t)_{t\geq\eta}$ can be written as $(\widetilde{P}(e^{t/2}))_{t\geq\log(\eta^2)}$ where $\widetilde{P}$ is a homogeneous Poisson process with rate $\lambda = 1$ on $[\log(\eta^2), \infty)$. In terms of the process $\widetilde{P}$, we can consider the process $(\widetilde{Z}_t)_{t\geq\log(\eta^2)}$ given by

$$\widetilde{Z}_t = \sum_{n\leq\widetilde{P}_t} \mathrm{Lap}\left(e^{\widetilde{\mathcal{T}}_n/2}\right),$$

where $\widetilde{\mathcal{T}}_n := \inf\{t \geq \log(\eta^2) : \widetilde{P}_t \geq n\}$ and $\widetilde{\mathcal{T}}_0 = \log(\eta^2)$. It is easy to see that $\widetilde{Z}(\log(t^2)) - \widetilde{Z}(\log(\eta^2)) =_d Z_t - Z_\eta$.

Leveraging this identity, it follows that we have

$$\begin{aligned}
\phi(\lambda) &= \mathbb{E}\left[e^{i\lambda(Z_t - Z_\eta)}\right] = \mathbb{E}\left[e^{i\lambda(\widetilde{Z}(\log(t^2)) - \widetilde{Z}(\log(\eta^2)))}\right] \\
&= \sum_{n=0}^{\infty} \frac{\eta^2}{t^2} \frac{\left[\log(t^2/\eta^2)\right]^n}{n!} \int_{\log(\eta^2)\leq u_1<u_2<\cdots<u_n\leq\log(t^2)} f^{(n)}(u_1,\ldots,u_n) \prod_{i=1}^{n} \mathbb{E}\left[e^{i\lambda \mathrm{Lap}(e^{u_i/2})}\right] d\mathbf{u} \\
&= \frac{\eta^2}{t^2} \sum_{n=0}^{\infty} \int_{\log(\eta^2)\leq u_1<u_2<\cdots<u_n\leq\log(t^2)} \prod_{i=1}^{n} \frac{1}{1+\lambda^2 e^{u_i}} d\mathbf{u}. \qquad (6)
\end{aligned}$$

In the above, $f^{(n)}(u_1,\ldots,u_n) := \frac{n!}{[\log(t^2/\eta^2)]^n}$ is the distribution of the order statistics $(U_{(1)},\ldots,U_{(n)})$ of n i.i.d. random variables that are uniform on $[\log(\eta^2), \log(t^2)]$. Essentially, what we have done is *first* conditioned of the number of Poisson arrivals that occur in the interval $[\log(\eta^2), \log(t^2)]$. Then, on the event $\{N(t) = n\}$, we condition again on the location of the n arrivals, which we know to be uniformly distributed across the time interval. Once the arrival locations are known, we can compute the conditional characteristic function, which is the the product of characteristic functions as illustrated in the integral above.

Now, we show inductively that

$$\int_{\log(\eta^2)\leq u_1<u_2<\cdots<u_n\leq\log(t^2)} \prod_{i=1}^{n} \frac{1}{1+\lambda^2 e^{u_i}} d\mathbf{u} = \frac{1}{n!}\left[\log\left(\frac{t^2}{\eta^2}\frac{1+\lambda^2\eta^2}{1+\lambda^2 t^2}\right)\right]^n.$$

The base case of $n = 1$ is trivially true. Now, we have that

$$\begin{aligned}
&\int_{\log(\eta^2)\leq u_1<u_2<\cdots<u_n\leq\log(t^2)} \prod_{i=1}^{n} \frac{1}{1+\lambda^2 e^{u_i}} d\mathbf{u} \\
&= \int_{u_1=\log(\eta^2)}^{\log(t^2)} \frac{1}{1+\lambda^2 e^{u_1}} \int_{u_1<u_2<\cdots<u_n} \prod_{i=2}^{n} \frac{1}{1+\lambda^2 e^{u_i}} d\mathbf{u}_{-1} du_1 \\
&= \frac{1}{(n-1)!} \int_{u=\log(\eta^2)}^{\log(t^2)} \frac{1}{1+\lambda^2 e^{u}} \left[\log\left(\frac{t^2}{e^u}\frac{1+\lambda^2 e^u}{1+\lambda^2 t^2}\right)\right]^{n-1} du \\
&= \frac{1}{n!} \int_{\log(\eta^2)}^{\log(t^2)} \frac{d}{du} \left[\log\left(\frac{t^2}{e^u}\frac{1+\lambda^2 e^u}{1+\lambda^2 t^2}\right)\right]^{n} du \\
&= \frac{1}{n!} \left[\log\left(\frac{t^2}{\eta^2}\frac{1+\lambda^2 \eta^2}{1+\lambda^2 t^2}\right)\right]^{n}.
\end{aligned}$$

Leveraging this identity and picking up from the expression for $\phi(\lambda)$ in Equation (6), we have that

$$\begin{aligned}
\phi(\lambda) &= \frac{\eta^2}{t^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\log\left(\frac{t^2}{\eta^2}\frac{1+\lambda^2 \eta^2}{1+\lambda^2 t^2}\right)\right]^{n} \\
&= \frac{\eta^2}{t^2} \exp\left(\log\left(\frac{t^2}{\eta^2}\frac{1+\lambda^2 \eta^2}{1+\lambda^2 t^2}\right)\right) \;=\; \frac{1+\lambda^2\eta^2}{1+\lambda^2 t^2}.
\end{aligned}$$

This proves the desired result. □

The above proof can also be leveraged to show that, for any finite fixed sequence of times $(t_n)_{n\in[K]}$, $(Z(t_1), \ldots, Z(t_K))$ has the same distribution as $(\zeta_1, \ldots, \zeta_K)$, where $(\zeta_n)_{n\in[K]}$ is the Laplace process associated with times $(t_n)_{n\in[K]}$ as outlined in Equation (3). This justifies that the process $(Z(t))_{t\geq\eta}$ is in fact a continuous time generalization of the aforementioned discrete time process.

E Skellam Noise Reduction

Lastly, we briefly discuss how to generate a noise reduction mechanism for Skellam noise [Agarwal et al., 2021]. Recall that a random variable X has a Skellam distribution with parameters λ_1 and λ_2 if $X =_d Y_1 - Y_2$, where $Y_1 \sim \text{Poisson}(\lambda_1)$ and $Y_2 \sim \text{Poisson}(\lambda_2)$ are independent Laplace random variables. For succinctness, we write $X \sim \text{Skell}(\lambda_1, \lambda_2)$.

Let $(P_1(t))_{t\geq 0}$ and $(P_2(t))_{t\geq 0}$ be two independent, homogeneous Poisson process with rates λ_1 and λ_2 respectively. Observe that the continuous time process $(X_t)_{t\geq 0}$ given by $X_t := P_1(t) - P_2(t)$ is clearly Markovian, has independent increments, and has $X_t \sim \text{Skell}(t\lambda_1, t\lambda_2)$. Thus, $(X_t)_{t\geq 0}$ can be used to define a Skellam noise reduction mechanism by releasing $(f(x) + X_{T_n(x)})_{n\geq 1}$ for some sequence of time functions $(T_n)_{n\geq 1}$.