A Propagation Property of the Pareto Order with Applications to Multiobjective Optimization

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Abstract

We establish a propagation property of the strict Pareto dominance order $\prec$ on $\mathbb{R}^k$ and present its application for efficient nondominated sorting and Pareto archive maintenance in multiobjective optimization. Precisely, if $u, v \in \mathbb{R}^k$ are mutually nondominated ($(u \not\prec v) \wedge (v \not\prec u)$), then no $q \in \mathbb{R}^k$ can satisfy $u \prec q \prec v$ or $v \prec q \prec u$. We give algebraic and geometric proofs, the latter via containment of strict down-sets (lower orthants). As a corollary, we state and prove a *post-witness* $u \prec q-$ *elimination rule* for Pareto archive S insertion: once a witness $w \in S$ with $q \prec w$ is found, no remaining $u \in S \setminus \{w\}$ can dominate q. We therefore can skip all subsequent "$u \prec q$?" checks and only test "$q \prec u$?" to remove dominated vectors. We provide pseudocode for the resulting archive-insertion routine, and outline extensions to weak, ε-, and noisy dominance. Finally, under a standard random-input model in which points are drawn independently from a continuous distribution on $\mathbb{R}^k$ (general position almost surely), we derive a closed-form expression for the expected post-witness fraction of the remaining "$u \prec q$?" comparisons (over $u \in S \setminus \{w\}$) that become unnecessary once a first witness w with $q \prec w$ is identified. The formula reveals how savings in comparison-checks scale with dimension k (and archive size), justifies witness-first heuristic scanning orders, and provides a reproducible baseline for empirical evaluation of dominance comparison-count reductions in archive insertion and nondominated sorting implementations. This probabilistic baseline complements the deterministic post-witness exclusion guaranteed by the propagation property in mutually nondominated curated archives.

1 Introduction

This work frames optimization through the lens of order theory. A minimization problem[1] is thus defined by a feasible set $\mathcal{F} \subseteq \mathcal{X}$, an objective space Y equipped with a (weak) preference order $\preceq$, and an objective map $F : \mathcal{X} \to Y$ that evaluates each decision.

A feasible decision $x^\star \in \mathcal{F}$ is a *minimizer* precisely when its outcome $F(x^\star)$ is a *minimal element* of the attainable outcome set $F(\mathcal{F}) \subseteq Y$ with respect to the preference order $\preceq$. This order on outcomes can be pulled back to the decision space, inducing a corresponding order $\preceq$ on $\mathcal{F}$:

$$x \preceq y \iff F(x) \preceq F(y) \qquad (x, y \in \mathcal{F}).$$

Equivalently, then, a decision $x^\star$ is a minimizer if and only if it is a minimal element of $\mathcal{F}$ under this induced order. Thus, optimality is fundamentally *minimality on outcomes*, transferred to the decision space by the objective map F.

This perspective immediately clarifies why the solution set of minimizers to F may be a *singleton* or a larger set in terms of cardinality. The distinction hinges on whether the preference order $\preceq$ on Y is total or partial.

When $\preceq$ is a total order on Y, as in standard single-objective optimization where the objective space $Y = \mathbb{R}$ is equipped with($\preceq = \leq$), all outcomes are comparable. This property ensures that if the infimum

[1] We restrict attention to minimization problems without loss of generality, since maximization is equivalent under the reversed (dual) order.

of the attainable set $F(\mathcal{F})$ is reached, the minimal value is unique. While the minimal value is unique, the corresponding *argmin set* of decisions may still contain multiple points:

$$\operatorname{argmin}_{x\in\mathcal{F}} F(x) \;:=\; \{\, x \in \mathcal{F} : F(x) = \min F(\mathcal{F}) \,\}$$

may still contain multiple points: these are *ties*, distinct decisions that achieve the same optimal scalar value.

In contrast, when $\preceq$ is only a partial order on Y, the solution concept becomes inherently set-valued. This is characteristic of multi-objective optimization, where the goal is to simultaneously minimize a set of k conflicting scalar-valued objective functions $(f_1, f_2, \ldots, f_k)$. The objective map is therefore a vector of these functions $F(x) = (f_1(x), f_2(x), \ldots, f_k(x))$, and the objective space $Y = \mathbb{R}^k$ is equipped with the *product (coordinatewise) order*—also called the Pareto order. This order permits *incomparable* outcomes: vectors $u, v \in Y$ that neither $u \preceq v$ nor $v \preceq u$ holds. Such incomparability signifies a fundamental trade-off, where one outcome is superior in some objectives but inferior in others. Consequently, the set $F(\mathcal{F})$ typically possesses not a single least element, but a family of minimal elements. Such family is called the nondominated Pareto front, and the corresponding set of decisions—their preimages in F—is known as the Pareto set, whose elements are termed efficient decisions.[2]

Computationally, the central goal in multiobjective optimization is to construct a faithful finite approximation of the Pareto nondominated front: a *finite, well-distributed subset of mutually minimal elements* (with respect to the coordinatewise/Pareto order) that approximates the typically infinite Pareto nondominated front. Two broad methodological paradigms are common. *Population-based* methods (e.g., evolutionary and swarm procedures) generate batches of candidates and maintain an external *Pareto archive* throughout the run Deb et al. (2002); Knowles & Corne (2000); Fonseca & Fleming (1995). *Deterministic* methods (e.g., scalarization sweeps, decomposition, branch-and-bound) systematically explore the feasible region while maintaining the same archive abstraction Miettinen (1999); Bechikh et al. (2015).

Across both paradigms, three routines dominate computation: (i) *Archive maintenance:* given a finite set $A \subset \mathbb{R}^k$, maintain its *minimal elements* under the product order; when a new point z arrives, update by $A \leftarrow Min(A \cup \{z\})$ (remove elements that cease to be minimal; keep z only if it is minimal in the enlarged set). (ii) *Nondominated sorting:* given a finite set $S \subseteq \mathbb{R}^k$, repeatedly extract the minimal elements, delete them, and iterate on the remainder to obtain successive layers Deb et al. (2002); Jensen (2003); Zhang et al. (2014). (iii) *Preference handling (optional):* when a single design must be chosen, apply a secondary rule (e.g., scalarization or rank) to select from the current minimal set Miettinen (1999); Bechikh et al. (2015).

The principal cost driver in (i)–(ii) is the number of pairwise *order tests* between vectors: with N points and k objectives, naive procedures perform on the order of N^2 vector-to-vector comparisons (each involving up to k coordinate checks), which quickly becomes a bottleneck as N or k grows Deb et al. (2002); Jensen (2003). Empirical studies confirm that comparison counts dominate wall-clock time and compare variants of nondominated sorting on this basis Long et al. (2021); Zhang et al. (2014).

Thus, it remains an open area of research to *identify and spotlight* structural properties of the Pareto order that allow many dominance relations to be *inferred* from a few. The propagation proposition proved in this paper is precisely one such property that realizes this task.

1.1 Contributions

This paper makes three kinds of contributions—structural, algorithmic, and quantitative—each stated and proved formally.

1. **Structural result (propagation).** We prove a propagation property of the strict Pareto order: if $u \parallel v$ then neither $u \prec q \prec v$ nor $v \prec q \prec u$ can occur. This is formalized in Theorem 1 and equivalently in the "no two-step bridge" Corollary 1. We provide three proofs—an algebraic argument, a one-line proof via transitivity, and a geometric proof using strict down-sets—providing intuition to the order-theoretic structure underpinning this property.

[2] See Section 2 for formal definitions and notation.

2. **Algorithmic consequence (post-witness $u \prec q$—elimination rule).** Specializing the structural result to mutually nondominated archives yields a one-sided dominance checks pruning rule: once a witness w with $q \prec w$ is found, no remaining u can satisfy $u \prec q$. This is stated in Corollary 2 and, for fully nondominated archives, in the archive specialization Corollary 3. We give a drop-in insertion routine (Algorithm 1) and prove its correctness (Theorem 2); the rule preserves worst-case asymptotics while strictly reducing dominance checks whenever a witness appears.

3. **Typical-case analysis (random inputs).** Under an independent and identically distributed (i.i.d.) general-position continuous model, we derive a closed-form expression for the expected *post-witness pruning rate*—the expected fraction of remaining "$u \prec q$?" tests eliminated after the first witness. This quantitative estimate is given in Proposition 1; it predicts how savings scale with dimension of the underlying Euclidean space $\mathbb{R}^k$ and archive size $S \subseteq \mathbb{R}^k$, and it justifies witness-first scanning orders as an implementation heuristic.

To the best of our knowledge, all formal statements above—Theorems 1 and 2, Corollaries 1 to 3, and Proposition 1—and their proofs have not been explicitly mentioned in the literature.

1.2 Organization

Section 2 fixes notation and recalls basic notions from order theory and multiobjective optimization.

Section 3 states and proves the structural result *(Propagation under strict Pareto dominance, Theorem 1)*, gives its "no two-step bridge" corollary *(Corollary 1)*, and provides algebraic/geometric proofs and a visualization. This realizes **Contribution 1**.

Section 4 develops the algorithmic consequences: the post-witness $u \prec q$ elimination rule for archive insertion *(Corollary 2)* and its archive specialization *(Corollary 3)*; a drop-in insertion algorithm *(Algorithm 1)* with a correctness proof *(Theorem 2)*; and a quantitative analysis under i.i.d. continuous sampling, yielding the expected post-witness pruning rate *(Proposition 1)*. These address **Contributions 2–3**.

Section 5 discusses the paper, and Section 6 concludes.

2 Preliminaries: orders, dominance, and down-sets

This section fixes notation and recalls the order-theoretic notions used throughout: (pre)orders, dominance relations on outcomes and decisions, and down-sets (order ideals). We keep the presentation self-contained and aligned with the optimization viewpoint.

For basic order theory we follow Davey & Priestley (2002); Grätzer (2011); for multiobjective optimization and Pareto efficiency, Miettinen (1999); Ehrgott (2005); algorithmic context draws on Deb (2001); Zhang et al. (2014).

2.1 Orders and preorders

A *binary relation* $\preceq$ on a set Y is a subset of $Y \times Y$. We write $u \preceq v$ for $(u, v) \in \preceq$ and define its *strict part* by

$$u \prec v \iff \big(u \preceq v \ \wedge\ \neg(v \preceq u)\big).$$

A relation $\preceq$ is a

- *preorder* if it is reflexive and transitive;
- *partial order* (poset) if it is a preorder and antisymmetric ($u \preceq v$ and $v \preceq u \Rightarrow u = v$);
- *total (linear) order* if it is a partial order and *complete* ($u \preceq v$ or $v \preceq u$ for all $u, v \in Y$).

Given a preorder $\preceq$, the *indifference* relation $u \sim v \iff (u \preceq v \ \wedge \ v \preceq u)$ is an equivalence relation; the quotient $Y/\sim$ with the induced relation is a poset.

For $S \subseteq Y$:

- $m \in S$ is *minimal* if there is no $s \in S$ with $s \prec m$;
- $\ell \in S$ is *least* if $\ell \preceq s$ for all $s \in S$.

Every least element is minimal; the converse need not hold in partial orders.[3] We write $Min(S)$ for the set of all minimal elements of S.

2.2 Product orders on $\mathbb{R}^k$

The canonical order on $\mathbb{R}^k$ used in multiobjective optimization is the *product (coordinatewise) order*. This is a partial order $\preceq$ defined by componentwise inequality:

$$\forall\, u, v \in \mathbb{R}^k : \quad u \preceq v \iff \forall\, i \in \{1, \dots, k\},\ u_i \leq v_i, \tag{1a}$$

$$u \prec v \iff \big(u \preceq v \ \wedge \ u \neq v\big)$$

$$\iff \Big(\forall\, i,\ u_i \leq v_i\Big) \ \wedge \ \Big(\exists\, j,\ u_j < v_j\Big). \tag{1b}$$

Here equation 1b is the strict counterpart of the weak-order equation 1a.

2.3 Dominance on outcomes and on decisions

Let $(Y, \preceq)$ be the objective space with $Y = \mathbb{R}^k$ endowed with the canonical coordinatewise *partial* order $\preceq$ in equation 1, and let $F : \mathcal{X} \to Y$ be the objective map. This order encodes the outcome–level notion of "no worse than" (weak dominance). Pulling it back to the decision space defines a *preorder* on decisions:

$$x \preceq y \iff F(x) \preceq F(y), \qquad x \prec y \iff \big(x \preceq y \ \wedge \ \neg(y \preceq x)\big). \tag{2}$$

Commonly, this coordinatewise product order is called Pareto order. Under this order, for any two vectors $u, v \in Y$ exactly one of the following four mutually exclusive relations holds:

- **Strict dominance**: $u \prec v$ (read as "u dominates v"). Formally, $u \prec v \iff \big(u \preceq v \ \wedge \ u \neq v\big)$. This means u is better than or equal to $v$in all objectives $\{f_i\}_{i=1}^k$ and strictly better in at least one.
- **Inverse strict dominance:** $v \prec u$ (read as "u is dominated by v").
- **Equality:** $u = v$. Note that for this partial order, indifference $(u \preceq v \wedge v \preceq u)$ is equivalent to equality.
- **Strict incomparability:** $u \parallel v$. This holds if neither $u \prec v$ nor $v \prec u$ is true. Incomparability represents a trade-off: one vector is strictly better in at least one component, while the other is strictly better in at least one different component.

Pareto efficiency and fronts. A feasible point $x^\star$ is (Pareto) *efficient* or Pareto optimal/minimal if there is no $x \in \mathcal{F}$ with $x \prec x^\star$, equivalently $F(x^\star) \in Min\big(F(\mathcal{F})\big)$ under $\prec$. The *Pareto set* is

$$\mathcal{P} \ = \ \Big\{x \in \mathcal{F} : \ \nexists\, y \in \mathcal{F} \text{ with } y \prec x\Big\},$$

and its image $F(\mathcal{P})$ is the *nondominated Pareto front*.

[3] In a total order, any nonempty S has *at most* one minimal element; if it exists, it is automatically the least element. In partial orders, S can have many minimal elements.

Standing convention. From now on, unless stated otherwise, "Pareto dominance" means the *strict* relation $u \prec v$. "Incomparable" means *strict incomparability*, denoted $u \parallel v$, i.e., $u \not\prec v$ and $v \not\prec u$. When the non-strict order is intended, we will write $u \preceq v$, and $u \parallel_{\preceq} v$ explicitly.

2.4 Down-sets (order ideals) and strict down-sets

Let $(Y, \preceq)$ be a poset and write its strict part by $x \prec y \iff (x \preceq y \text{ and } x \neq y)$. For $u \in Y$:

$$\downarrow u \;:=\; \{\, v \in Y : v \preceq u \,\} \qquad \text{(weak/principal down-set)},$$

$$\downarrow_{\prec} u \;:=\; \{\, v \in Y : v \prec u \,\} \qquad \text{(strict down-set)}.$$

Clearly,

$$\downarrow_{\prec} u \;= \downarrow u \setminus \{u\}.$$

Closure under moving downward. If $q \in \downarrow u$ and $w \preceq q$, then $w \in \downarrow u$. Likewise, if $q \in \downarrow_{\prec} u$ and $w \prec q$, then $w \in \downarrow_{\prec} u$. (Both follow from transitivity.)

Explicit forms in $(Y, \preceq) = (\mathbb{R}^k, \leq)$. With the product order, for $u \in \mathbb{R}^k$,

$$\downarrow u \;=\; \{\, z \in \mathbb{R}^k : z_i \leq u_i \;\forall i \,\} \;=\; u - \mathbb{R}^k_+,$$

$$\downarrow_{\prec} u \;=\; \{\, z \in \mathbb{R}^k : z_i \leq u_i \;\forall i,\; z \neq u \,\} \;=\; \big(u - \mathbb{R}^k_+\big) \setminus \{u\},$$

i.e., the lower $\leq$-orthant under u, with the corner point u removed in the strict case.

Intersection and coordinatewise meet. For $u, v \in \mathbb{R}^k$, let $(u \wedge v)_i := \min\{u_i, v_i\}$. Then

$$\downarrow u \cap \downarrow v \;= \downarrow (u \wedge v).$$

If $u \parallel v$ (strict incomparability), then $u \wedge v \neq u$ and $u \wedge v \neq v$, hence

$$\downarrow_{\prec} u \cap \downarrow_{\prec} v \;=\; \{\, z : z \leq u \wedge v \,\},$$

so the strict down-sets still overlap (they coincide on the orthant under $u \wedge v$).

Connection to nondominated sorting. For a finite $S \subset \mathbb{R}^k$, an element $x \in S$ is *minimal* (Pareto-nondominated) iff

$$\neg\exists\, y \in S \text{ with } y \prec x \quad\iff\quad \big(S \setminus \{x\}\big) \cap \downarrow_{\prec} x \;=\; \varnothing.$$

Thus the first nondominated layer is $L_1 = Min(S) := \{x \in S : \; (S \setminus \{x\}) \cap \downarrow_{\prec} x = \emptyset\}$. Removing it and iterating constructs subsequent layers:

$$L_{t+1} \;=\; Min\Big(S \setminus \bigcup_{i=1}^{t} L_i\Big), \qquad t \geq 1.$$

(Using $\prec$ or $\preceq$ here is equivalent on finite S: "no strict dominator" $\Leftrightarrow$ "minimal in $\preceq$."). This perspective provides algebraic shortcuts for nondominated sorting, which we exploit later.

3 Main result: propagation under the strict Pareto order

In this section, we prove a propagation property of the strict product (Pareto) order on $\mathbb{R}^k$ and its symmetric counterpart. The result is established via concise proofs—algebraic and geometric (via strict down-sets)—and is visualized in the $k = 2$ case. We also derive an immediate corollary that lays the foundation for the algorithmic consequences developed later.

Theorem 1 (Propagation under strict Pareto dominance)**.** *Let $u, v, q \in \mathbb{R}^k$ and assume $u \parallel v$.*

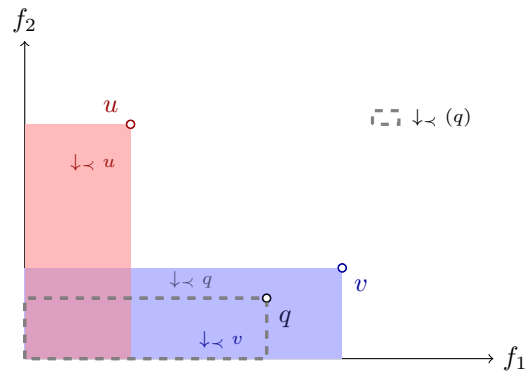

Figure 1: Strict down-sets (punctured $\leq$-orthants) for incomparable u (red) and v (blue). Given $q \prec v$, we have $\downarrow_{\prec} q \subset \downarrow_{\prec} v$. If $u \prec q$ also held, then $\downarrow_{\prec} u \subset \downarrow_{\prec} q \subset \downarrow_{\prec} v$ and $u \in \downarrow_{\prec} q \subset \downarrow_{\prec} v$, forcing $u \prec v$—contradiction.

(a) If $q \prec v$, then $u \not\prec q$. Consequently (since $q \neq u$ would imply $u \prec v$), either $q \prec u$ or $q \parallel u$.

(b) Symmetrically, if $q \prec u$, then $v \not\prec q$. Consequently, either $q \prec v$ or $q \parallel v$.

One-line proof via transitivity. Pareto dominance is transitive: if $x \prec y$ and $y \prec z$, then $x_i \leq y_i \leq z_i$ for all i and $x \neq z$, hence $x \prec z$. Thus, if $q \prec v$ and $u \prec q$ both held, we would get $u \prec v$, contradicting $u \parallel v$. □

Algebraic proof. Since $u \parallel v$, there exist indices n, m with $v_n < u_n$ and $u_m < v_m$. Suppose $q \prec v$. Then $q_i \leq v_i$ for all i, so in particular $q_n \leq v_n < u_n$. If $u \prec q$ held, we would need $u_i \leq q_i$ for all i, which is impossible at $i = n$. Hence $u \not\prec q$. Equality $q = u$ would force $u \prec v$, contradicting $u \parallel v$ thus, the remaining possibilities are $q \prec u$ (which can happen if, e.g., $q_m \leq u_m$) or strict-incomparability $q \parallel u$ (if, e.g., $q_m > u_m$). The symmetric statement follows by exchanging u and v. □

Remark 1. *(i) The assumption $u \parallel v$ rules out equality and guarantees "crossing coordinates" (some coordinate where u is smaller and another where v is smaller).*

(ii) The conclusion uses only strict relations: we prove $u \not\prec q$ (not the stronger $u \not\preceq q$), which is the sharp statement when one insists on strict dominance throughout.

Corollary 1 (No two-step bridge between incomparable anchors u, v)**.** *If $v \parallel u$, the chains $u \prec q \prec v$ and $v \prec q \prec u$ are impossible.*

Proof. Immediate from transitivity of $\prec$: $u \prec q \prec v$ would imply $u \prec v$, contradicting $v \parallel u$. □

Geometric proof via strict down-sets. Let $\downarrow_{\prec} x := \{z \in \mathbb{R}^k : z \prec x\}$ denote the strict down-set of x. We use two elementary facts (both immediate from the definition and transitivity of $\prec$):

(i) If $x \prec y$, then $\downarrow_{\prec} x \subset \downarrow_{\prec} y$ and $x \in \downarrow_{\prec} y$.

(ii) If $x \parallel y$, then neither $\downarrow_{\prec} x \subset \downarrow_{\prec} y$ nor $\downarrow_{\prec} y \subset \downarrow_{\prec} x$.

Assume $u \parallel v$ and $q \prec v$. Then by (i), $\downarrow_{\prec} q \subset \downarrow_{\prec} v$. If, for contradiction, $u \prec q$, then again by (i),

$$\downarrow_{\prec} u \subset \downarrow_{\prec} q \subset \downarrow_{\prec} v \quad \text{and} \quad u \in \downarrow_{\prec} q \subset \downarrow_{\prec} v.$$

The latter inclusion is exactly $u \prec v$, contradicting $u \parallel v$. Hence $u \not\prec q$. This proves (a). Part (b) is identical with u and v interchanged. □

Remark 2 (Contrapositive and closure view)**.** *The contrapositive of the corollary is: if $u \prec q \prec v$, then $v \nparallel u$ (indeed $u \prec v$).*

In "closure" terms with strict *down-sets $\downarrow_{\prec} x := \{z : z \prec x\}$:*

$$q \prec v \;\Rightarrow \downarrow_{\prec} q \subset \downarrow_{\prec} v, \qquad u \prec q \;\Rightarrow \downarrow_{\prec} u \subset \downarrow_{\prec} q.$$

Hence $u \prec q \prec v \Rightarrow \downarrow_{\prec} u \subset \downarrow_{\prec} v$. But $u \parallel v$ forbids either strict containment $\downarrow_{\prec} u \subset \downarrow_{\prec} v$ or $\downarrow_{\prec} v \subset \downarrow_{\prec} u$ (there are "crossing coordinates"), so $u \prec q$ cannot occur when $q \prec v$.

4 Algorithmic consequence for nondominated sorting: a post-witness $u \prec q$–elimination rule for Pareto archive insertion

In this section, we formalize the dominance-comparison elimination rule suggested by Theorem 1, then give an insertion algorithm that implements this rule and prove its correctness under the strict Pareto dominance, and discuss practical variants (weak or ε-dominance; noisy comparisons); first we fix terminology.

In this section, we first define the notion of a *witness*, then formalize the post-witness $u \prec q$ elimination implied by Theorem 1. We give an insertion algorithm for a mutually nondominated archive that implements this rule and prove its correctness under the strict Pareto order, noting practical variants (weak or ε-dominance; noisy comparisons).

Terminology (witness). Let $S \subseteq \mathbb{R}^k$ be mutually nondominated and let $q \in \mathbb{R}^k$. Any $w \in S$ with $q \prec w$ is called a *witness* for q in S; the set of all witnesses is

$$W(q; S) := \{\, w \in S : \; q \prec w \,\}.$$

When scanning S in some order, the *first witness* is the first element of $W(q; S)$ encountered by the scan (if any). Once a witness is found, the post-witness $u \prec q$–elimination rule applies to all remaining u that are incomparable with that witness (see Corollary 2).

Corollary 2 (post-witness $u \prec q$– elimination rule for archive maintenance)**.** *Let $S \subseteq \mathbb{R}^k$ be a set of mutually nondominated points (no two related by $\prec$). Given a candidate $q \in \mathbb{R}^k$, suppose there exists $v \in S$ with $q \prec v$. Then for every $u \in S$ incomparable with v (i.e., $v \parallel u$) we have $u \not\prec q$. Consequently, all dominance checks of the form "$u \prec q$?" for such u can be safely skipped.*

Proof. Immediate from Theorem 1(a) with the triple (v, u, q). □

Corollary 3 (Archive case: global pruning after first witness)**.** *If S is mutually nondominated and the scan discovers a* witness *$w \in S$ with $q \prec w$, then $w \parallel u$ for every $u \in S \setminus \{w\}$ (by mutual nondominance). Hence, by Corollary 2, no remaining u can dominate q, and* all subsequent *tests of the form "$u \prec q$?" may be omitted.*

Application to the standard incremental Pareto archive insertion routine. In incremental Pareto archive maintenance, a typical insertion of q proceeds by (i) rejecting q if it is dominated, and otherwise (ii) removing any $a \in S$ dominated by q. The post-witness $u \prec q$- elimination rule prunes tests in step (i) *after* the first witness w with $q \prec w$ is found: once such a w exists, no remaining u can dominate q (Corollary 3), so the scan thereafter only needs to test whether q dominates u in order to remove u.

Theorem 2 (Correctness of Algorithm 1)**.** *Let $S \subseteq \mathbb{R}^k$ be mutually nondominated and let $q \in \mathbb{R}^k$. Algorithm 1 returns S* iff *there exists $a \in S$ with $a \prec q$; otherwise it returns*

$$S' \;=\; (S \setminus R) \cup \{q\}, \qquad R \;=\; \{\, a \in S : q \prec a \,\},$$

and S' is mutually nondominated.

Proof. Rejection occurs only when the scan encounters some $a \in S$ with $a \prec q$, so the "if and only if" is immediate. If accepted, then any a seen *before* the first witness w with $q \prec w$ failed $a \prec q$, and any u incomparable with w *after* that point cannot satisfy $u \prec q$ by Corollary 2; comparable u are explicitly tested, hence no survivor dominates q, and we remove exactly $R = \{a : q \prec a\}$, so S' is mutually nondominated. □

Algorithm 1 Insert q into a nondominated archive S using the post-witness $u \prec q$- elimination rule

Require: S is mutually nondominated. DOMINATES is strict Pareto dominance on vectors.

```
function INCOMPARABLE(u, v)
    return ¬DOMINATES(u, v)  and  ¬DOMINATES(v, u)
end function
w ← None                                  ▷ first archive point w with q ≺ w (a witness), if any
R ← ∅                                     ▷ points to remove (dominated by q)
for each a ∈ S do
    if w ≠ None  and  INCOMPARABLE(a, w) then
        skip testing DOMINATES(a, q)      ▷ by Corollary. 2: a ⊀ q
    else
        if DOMINATES(a, q) then
            return S                      ▷ q is rejected (no witness was found earlier)
        end if
    end if
    if DOMINATES(q, a) then
        R ← R ∪ {a}
        if w = None then
            w ← a                         ▷ first witness found: indeed q ≺ w
        end if
    end if
end for
return (S \ R) ∪ {q}
```

Back-of-the-envelope post-witness $u \prec q$–elimination rate (random inputs). The following estimate is useful for gauging behavior when scanning a *randomly generated* set (e.g., during front-by-front sorting of a random cloud), where pairwise comparabilities can still occur.

Proposition 1 (Heuristic post-witness pruning rate under i.i.d. continuous sampling)**.** *Assume the following sampling model, used solely for estimation: points in $\mathbb{R}^k$ are sampled independently; for each coordinate $i \in \{1, \dots, k\}$, the random variables are independent across i, identically distributed across points, and have continuous marginals. In particular, for independent scalar copies U_i, V_i we have $\Pr(U_i = V_i) = 0$ and $\Pr(U_i < V_i) = \Pr(V_i < U_i) = \frac{1}{2}$.*

Let $U, V \in \mathbb{R}^k$ be independent points drawn from this model. Then

$$\Pr\big(U \text{ and } V \text{ are comparable under } \prec\big) = 2^{\,1-k}, \qquad \Pr(U \parallel V) = 1 - 2^{\,1-k}.$$

Interpreting the post-witness scenario heuristically by treating the first witness w and each remaining u as independent draws from the same model (i.e., ignoring the bias from conditioning on $q \prec w$), the expected fraction of the remaining "$\cdot \prec q$?" checks that can be omitted equals $1 - 2^{\,1-k}$.

Proof. Let $U = (U_1, \dots, U_k)$ and $V = (V_1, \dots, V_k)$ be independent $\mathbb{R}^k$–valued random vectors such that, for each i, U_i and V_i are i.i.d. with a continuous distribution and the coordinates are independent across i. Then $\Pr(U_i = V_i) = 0$ and, by exchangeability, $\Pr(U_i < V_i) = \Pr(V_i < U_i) = \frac{1}{2}$. Hence

$$\Pr(U \le V) = \Pr(U_1 \le V_1, \dots, U_k \le V_k) = \prod_{i=1}^{k} \Pr(U_i \le V_i) = 2^{-k}.$$

Because $\Pr(U = V) = 0$, we have $\Pr(U \prec V) = \Pr(U \le V,\, U \ne V) = 2^{-k}$. By symmetry, $\Pr(V \prec U) = 2^{-k}$. The events $\{U \prec V\}$ and $\{V \prec U\}$ are disjoint, so

$$\Pr(U \text{ and } V \text{ comparable under } \prec) = \Pr(U \prec V) + \Pr(V \prec U) = 2^{1-k},$$

and therefore $\Pr(U \parallel V) = 1 - 2^{1-k}$.

For the post-witness pruning fraction, if we approximate the discovered witness w as an independent generic draw (ignoring the conditioning on $q \prec w$) and model each remaining u as an i.i.d. draw independent of w, then $\Pr(u \parallel w) = 1 - 2^{1-k}$ by the calculation above. Consequently, the expected fraction of omitted "$\cdot \prec q$?" checks is $1 - 2^{1-k}$. □

Remark 3 (Archives vs. i.i.d. samples; scope of the proxy)**.** *The i.i.d. estimate in Proposition 1 is intended only as a baseline for* uncurated *point sets sampled independently from a continuous distribution on $\mathbb{R}^k$. In a* mutually nondominated archive *S, once a witness w with $q \prec w$ is found, Corollary 3 implies that every remaining $u \in S \setminus \{w\}$ is incomparable with w; hence* all *subsequent tests of the form "$u \prec q$?" may be omitted deterministically. If exact duplicates are permitted, then under* strict *dominance duplicates are still incomparable with w and cannot dominate q; under* weak *dominance they may be removed by equality tests. In either case, the elimination rule remains sound. Real archives typically violate the i.i.d. assumptions: nondominance filtering induces dependencies across points, and optimization dynamics introduce correlations across both points and coordinates. Consequently, the i.i.d. comparability probability 2^{1-k} should* not *be used to predict pruning rates in curated archives; it serves only as a typical-case heuristic for genuinely i.i.d. samples.*

Remark 4 (Extensions: weak, ε-, and noisy dominance)**.** *The propagation/elimination argument relies only on down-set inclusion for the same relation used in both the witness and the tests; i.e., incomparability and strictness must be taken w.r.t. the same base relation. It therefore extends to any dominance relation on $\mathbb{R}^k$ whose (i) strict part induces proper down-set inclusion and (ii) incomparability is defined with respect to the same relation used in the tests. Three common cases:*

(a) Weak dominance. *If the algorithm uses weak dominance $\preceq$ (and its strict part $\prec$), the post-witness $u \prec q$- elimination rule remains valid with a* weak *witness: if $q \preceq v$ and $v \parallel u$ (both w.r.t. $\preceq$), then $u \not\prec q$. Indeed, $u \prec q \preceq v$ would imply $u \preceq v$, contradicting $v \parallel u$.*

(b) ε-dominance (additive or multiplicative). *Let $\prec_\varepsilon$ denote an ε-dominance relation (e.g., additive: $u \prec_\varepsilon v \iff u_i \leq v_i - \varepsilon_i$ for all i, with strict part $\prec_\varepsilon$). Then down-sets are shifted orthants and satisfy $q \prec_\varepsilon v \Rightarrow \downarrow_\varepsilon q \subset \downarrow_\varepsilon v$. Consequently, if $q \prec_\varepsilon v$ and $v \parallel_\varepsilon u$, one cannot have $u \prec_\varepsilon q$.*

(c) Noisy/stochastic comparisons.

Suppose each pairwise dominance test returns a (possibly noisy) label with error probability at most δ, and repeated tests are independent. To bound the chance of an unsafe *comparison-test skips, require r independent confirmations of the witness relation (e.g., $q \prec v$) before pruning incomparable points. Then the probability of making a wrong skip decision is at most δ^r. Choosing $r \geq \lceil \log(\alpha)/\log(\delta) \rceil$ ensures a per-skip error probability $\leq \alpha$.*

5 Discussion

The propagation property formalizes an order-theoretic exclusion principle: once two anchors v and u are mutually nondominated, moving strictly "downward" from one anchor cannot land in the other's strict down-set. This down-set inclusion view uses only the product order; it is independent of any metric or inner-product structure.

Two operational regimes should be distinguished:

(i) Archive insertion (curated sets): Since a Pareto archive is mutually nondominated by construction (duplicates aside), the discovery of a witness $w \in S$ with $q \prec w$ guarantees that every remaining $u \in S \setminus \{w\}$ is incomparable with w; hence, by the propagation property, no such u can satisfy $u \prec q$. Therefore, all further tests of the form "$u \prec q$?" may be deterministically omitted. Only tests of the form "$q \prec u$?" remain necessary to remove dominated points.

(ii) randomly generated sets (e.g., random point clouds or early passes in front-by-front sorting): In these settings, pairwise comparabilities are still possible. The post-witness $u \prec q$— elimination rule remains sound,

but its quantitative effect is probabilistic and well captured by the i.i.d. heuristic derived earlier, which serves as a guide to the expected rate of prunable comparisons.

These observations motivate certain design choices. For instance, data structures that spatially cluster points likely to be incomparable with a given witness (such as orthant-based trees or coarse grid decompositions) may help surface large prunable neighborhoods once the first witness is found, reducing the number of checks of the form "$u \prec q$?".

In performance evaluation, metrics such as the number of pairwise comparisons and wall-clock time should be prioritized, as they typically dominate the computational cost in both sorting and archiving routines.

Finally, because the propagation argument relies solely on down-set inclusion, it extends naturally—provided the same base relation is used for both witnesses and tests—to weak dominance, ε-dominance (additive or multiplicative), and approximate or noisy comparisons with appropriate safeguards.

6 Conclusion

We have formalized and proved a propagation property of the Pareto order: when two points are mutually nondominated, a third point that dominates one cannot be dominated by the other. Instantiating this property yields a simple post-witness $u \prec q$— elimination rule with a strong guarantee for archive insertion: after the first witness is found, all remaining checks of the form "$u \prec q$?" (duplicates aside) can be eliminated while preserving correctness. We provided a correctness proof of the post-witness $u \prec q$- elimination rule and pseudocode of its application to the Pareto archive insertion routine, and we discussed extensions to weak, ε-, and noisy dominance.

For intuition in settings where sets are not yet curated, we gave an i.i.d. heuristic indicating that expected pruning grows with dimension; we underscored that this back-of-the-envelope calculation is not a model of archives, where the post-witness pruning is deterministic. Future work includes systematic empirical studies quantifying comparison savings and runtime in both archive updates and front-by-front sorting across objective counts, correlations, and generator regimes, and integrating the rule into modern sorters and archives.

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