EXPLAINING POINT PROCESSES BY LEARNING INTERPRETABLE TEMPORAL LOGIC RULES

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ABSTRACT

We propose a principled method to learn a set of human-readable logic rules to explain temporal point processes. We assume that the generative mechanisms underlying the temporal point processes are governed by a set of first-order temporal logic rules, as a compact representation of domain knowledge. Our method formulates the rule discovery process from noisy event data as a maximum likelihood problem, and designs an efficient and tractable branch-and-price algorithm to progressively search for new rules and expand existing rules. The proposed algorithm alternates between the rule generation stage and the rule evaluation stage, and uncovers the most important collection of logic rules within a fixed time limit for both synthetic and real event data. In a real healthcare application, we also had human experts (i.e., doctors) verify the learned temporal logic rules and provide further improvements. These expert-revised interpretable rules lead to a point process model which outperforms previous state-of-the-arts for symptom prediction, both in their occurrence times and types.

1 INTRODUCTION

Event sequences with irregular time intervals are ubiquitous in real-world applications. For instance, in healthcare, patient measurements or symptoms are irregularly observed events due to scheduling issues, missed appointments, and changes in disease stages. The inter-event times usually convey rich information regarding the underlying dynamics such as disease progression (Liu et al., 2015). It is of scientific and practical importance to understand the generating mechanisms of these events, as well as the reason and time where such events happen.

In systems where domain knowledge is rich, the generating mechanisms of events can usually be governed by a set of *first-order temporal logic rules*, as a compact representation of domain knowledge. For example, in healthcare, the knowledge "if a sudden fall in blood pressure is observed, vasopressors are required to be applied to patients immediately; and then the blood pressure may return to normal afterwards" may explain why the event "blood pressure returns to normal from the abnormally low level " is observed after the event "vasopressors are used". These domain knowledge can be summarized as a collection of logic rules with *temporal relation constraints*, such as "A happens *before* B", "If A happens, and *after* 5 mins, B can happen", and "If A and B happen *simultaneously*, then *at the same time* C can happen".

A large amount of literature has been devoted to modeling event data, among which *temporal point process* models provide an elegant framework without the need to discretize the time horizon into bins. Temporal point process models treat the inter-event time as random variables and directly model the intensity function (i.e., occurrence rate) of the events. Recently, Li et al. (2020) proposed a unified framework to marry *point process intensity functions* with *temporal logic rules*. This method employs a set of pre-specified temporal logic rules to design the intensity functions to incorporate domain knowledge. The resulting so-called *temporal logic point process* models are inherently interpretable while at the same time being expressive. Using such logically constrained intensity functions, the model can capture nonlinear dependencies, interactions, and various temporal relations between events. It was also shown that, by specifying one or two simple temporal logic rules, many existing parametric models, e.g. the Hawkes process (i.e. self-exciting processes) (Hawkes, 1971a) and the self-correcting process (Isham & Westcott, 1979) are special cases of the proposed temporal logic point process.

However, in (Li et al., 2020) the temporal logic rules are required to be specified by the model builder, but they may not be known beforehand. Models constructed in such a way may suffer from model-misspecification when the pre-specified logic rules are incorrect or missing.

> *Can we automatically discover the temporal logic rules governing the event dynamics based on historical event data alone?*

This is a challenging problem since the space of possible temporal logic formulas is huge, comprising of a combination of a large number of discrete logic variables and all kinds of temporal relations.

Previous inductive logic programming approaches focused mostly on deriving the relations between discrete logic variables, but have largely ignored the crucial temporal information (Dash et al., 2018; Wei et al., 2019). A generalization of these methods by further considering temporal relations in logic variables is largely missing and is in pressing needs due to the recent increasing availability of event sequence data.

To address the aforementioned challenges, we propose TEmporal Logic rule LearnER (TELLER) algorithm for learning temporal logic rules from event sequences. TELLER is inspired by the branch-and-price algorithm (Barnhart et al., 1998), a column generation (CG) algorithm for linear programming (LP) problems where the number of variables are too large to be considered explicitly. In the CG algorithm framework, the original LP problem is solved via two alternating procedures: the master problem and the subproblem, where the master problem is the original problem with only a subset of variables being considered and the subproblem is a new problem created to identify a new variable to be added. Similarly, the TELLER algorithm also alternates between solving a master problem and a subproblem, where the master problem aims to evaluate the current rules by maximizing the likelihood and reweighting these rules as in (Li et al., 2020), and the subproblem is to search and construct a new temporal logic rule (by extending a current rule or adding a new short rule). This procedure repeats until the likelihood can no longer be improved by adding new candidate rules. In this way, TELLER systematically searches through the vast space of potential temporal logic rules and learns the importance weights of the discovered rules to hedge against noise in the data. The overall flow of the TELLER algorithm is illustrated in Fig. 1.

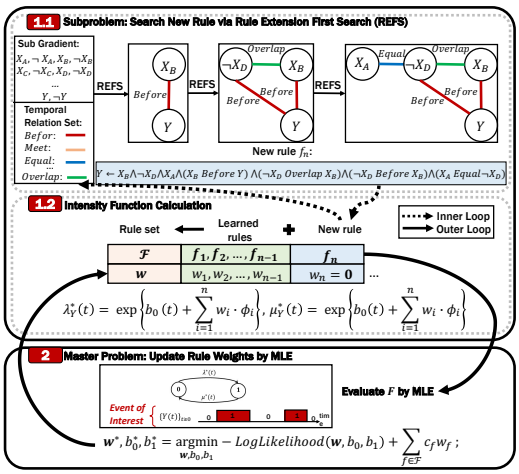

Figure 1: The overall flow of TELLER. It alternates between a rule evaluation stage (outer loop in figure) and a rule proposal stage (inner loop). The set of logic rules and their importance weights are jointly learned.

More specifically, the proposed TELLER is computationally efficient and with guarantees. It includes the candidate rules one by one in a tractable and progressive fashion, and is able to find a near-optimal set of rules. The hypothesized logic rules are in disjunctive normal form (DNF, OR-of-ANDs) with temporal relation constraints, and can be of various lengths. Given the temporal logic point process modeling framework, we show that the resulting objective function, i.e. likelihood, is convex, which guarantees an optimal performance for our search algorithm. A set of logic rules and their importance weights are jointly learned by maximizing the likelihood. The most important collection of logic rules are guaranteed to be discovered within a fixed time limit.

Furthermore, the uncovered rules by TELLER will not only shed light on *when* and *what* events would happen, but also *why* some events would happen at specific times. Our method facilitates the human-readable knowledge exchange between experts and point process models. On the one hand, the mined temporal logic rules may supplement or refine the existing domain knowledge; on the other hand, human experts can easily provide feedback to modify the learned models via logic rules, as a way to add the safety of using these models in high-stake tasks like healthcare and autonomous driving. In our real experiment related to medicine, TELLER is used to learn the explanatory temporal logic rules regarding the choice and arrangement of drugs in improving patients' health status. Doctors were invited to justify the rules and they confirmed that these discovered logic

rules are consistent with the pathogenesis and have captured the most important factors in affecting patients' health status.

2 BACKGROUND

This section introduces preliminaries about first-order temporal logic and point processes. Due to space limitation, the related work is summarized in Appendix A.

2.1 FIRST-ORDER TEMPORAL LOGIC RULES

First-order temporal logic is a form of symbolized reasoning in which each statement is a composition of temporal predicates and their relations. More details can be found below.

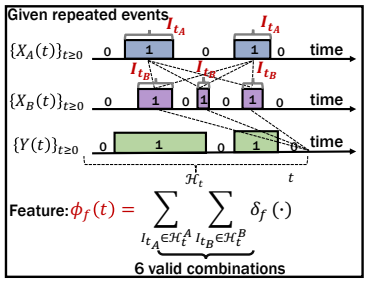

Figure 2: Demonstration of the feature constructions of TLPP using a simple logic formula $f : Y \leftarrow A \wedge B$. The main idea is to use the specified f as a template to gather valid combinations of the body predicate events from history to construct feature.

Temporal predicate. First define a set of *static predicates* $\mathcal{X} = \{X_1, \ldots, X_d\}$, where each static predicate $X_i(c)$ is a logic random variable that defines the *property* or *relation* of entities, such as Smokes(c) or Friend(c, c'), where c and c' are the entities. By adding a temporal dimension to predicate X_i, we obtain a *temporal predicate* $\{X_i(c,t)\}_{t\geq 0}$, which can be viewed as a continuous-time stochastic process. Given observations, each temporal predicate will be grounded as a list of ordered 0-1 events, which take True (1) and False (0) in an alternating way with the state transition times recorded. For example, a temporal predicate $\{\text{NormalBloodPressure}(c,t)\}_{t\geq 0}$, where entity c is referring to the patient, will take value 1 or 0 at any time t to indicate whether the patient's blood pressure is normal or abnormal, respectively, with the state transition time being stochastic (as demonstrated in Fig. 2).

To simplify the notation and better illustrate the main ideas of this paper, we will temporally drop the dependency of predicates on entities. Hence, we will write $X(c,t)$ as $X(t)$ instead and write the grounded temporal predicate as $\{x(t)\}_{t\geq 0}$.

Temporal relation. We will use the *interval-based temporal relations* originally introduced in Allen's seminal paper (Allen, 1990) to describe the temporal relations that can exist in any two temporal predicates. There are 13 types of possible temporal relations, including *Before, Meets, Overlap* and so on (see Appendix B for a comprehensive illustration), and we denote this set by $\mathcal{R}$. To evaluate these temporal relations, we need to define the *time interval* using the state transit in/out times. For example, at time t_A, the temporal predicate $\{X_A(t)\}_{t\geq 0}$ takes state $x_A = 0$ or 1, we define $I_{t_A} = (t_{A_1}, t_{A_2}]$, $t_A \in (t_{A_1}, t_{A_2}]$, as a time interval where t_{A_1} is the transition time that this predicate enters the state x_A and t_{A_2} is the transition time that this predicate leaves the state x_A. For any two temporal predicates, given their time intervals, $I_{t_A} = (t_{A_1}, t_{A_2}]$ and $I_{t_B} = (t_{B_1}, t_{B_2}]$, their temporal relation function, denoted as $R_{A\ type\ B}(I_{t_A}, I_{t_B}) \in \mathcal{R}$, is a logic function defined over the time intervals and can be evaluated by plugging in the specific time intervals. For example, $R_{A\ Before\ B} = \mathbb{1}\{t_{B_1} > t_{A_2}\}$ and $R_{A\ Equals\ B} = \mathbb{1}\{t_{A_1} = t_{B_1}\}\mathbb{1}\{t_{A_2} = t_{B_2}\}$. All these quantitative definitions of temporal relations are also illustrated in Appendix B.

Temporal logic formula. A first-order temporal logic rule f is defined as logical connectives of *temporal predicates* and their *temporal relations*. The generic form is

$$f : (\neg)Y(t_y) \leftarrow \bigwedge\nolimits_{X_u \in \mathcal{X}_f^+} X_u(t_u) \bigwedge\nolimits_{X_v \in \mathcal{X}_f^-} \neg X_v(t_v) \bigwedge\nolimits_{X_u, X_v \in \mathcal{X}_f} R_{u\ type\ v}(I_{t_u}, I_{t_v}) \tag{1}$$

where each temporal predicate and their temporal relations can be evaluated at times t_y, t_u and t_v along with their corresponding time intervals I_{t_u} and I_{t_v}. In the above formula, negation $(\neg)$ means the head predicate Y can take a negative sign; $\mathcal{X}_f = \mathcal{X}_f^+ \cup \mathcal{X}_f^-$ is the set of predicates defined in f, where $\mathcal{X}_f^-$ is the set of predicates as negation in the formula f, and $\mathcal{X}_f^+ = \mathcal{X}_f \setminus \mathcal{X}_f^-$. We assume $\leftarrow$ has a causal direction, and the body part of the formula indicate the evidence to be gathered from history to deduce the state of the head predicate. By this assumption, it is only valid to consider the time points such that $t_y \geq t_u, t_v$.

2.2 TEMPORAL LOGIC POINT PROCESS

Temporal Point Process (TPP) provides an elegant tool to capture the dynamics of the event sequences. It is characterized by conditional intensity function, denoted by $\lambda(t|\mathcal{H}_t)$, where history $\mathcal{H}_t$ is the

knowledge of times of all events. By definition, we have $\lambda(t|\mathcal{H}_t)dt = \mathbb{E}[N([t, t+dt])|\mathcal{H}_t]$, where $N([t, t+dt])$ denotes the number of points falling in an interval $[t, t+dt]$. Here we aim to use TPP to model these 0-1 events and use the intensity function to capture the 0-1 transition rate.

Given observed sequences of data, each predicate in $\mathcal{X}$ will be grounded as a list of ordered 0-1 events. The main idea of *temporal logic point process* (TLPP) (Li et al., 2020) is to use the temporal logic rules (1) as templates to selectively choose *combinations of events* from history as evidence to infer the transition rate of these 0-1 events. In this way, the structure of the intensity function will be informed by the pre-specified temporal logic rules. Below we provide more details.

Logic-informed intensity function. Let $\mathcal{H}_t$ include all the historical trajectories of the grounded predicates in $\mathcal{X}$ up to t. TLPP leverages the rule set to model the state transition dynamics of the head predicates. Define $\lambda^*(t) := \lambda(t|\mathcal{H}_t)$ as the transition intensity for $\{Y(t)\}_{t\geq 0}$ to transit from 0 to 1, given history up to t; and $\mu^*(t) := \mu(t|\mathcal{H}_t)$ vice versa. Assume the generating process of the 0-1 events are governed by the set of temporal logic rules $\mathcal{F} = \{f_1, f_2, \dots\}$.

First consider one formula f, to incorporate its knowledge in intensity construction, we define a *formula effect* (FE), which aims to gather only the effective combinations of the historical predicate events defined by f as evidence to reason about the transition rate of the head predicate. To ease the notation, define the predicate index set of $\mathcal{X}_f$ as U, then the *formula effect* of f is computed as

$$\delta_f\left(t \mid y(t), \{x_u(t_u), I_{t_u}\}_{u\in U} \in \mathcal{H}_t\right) := f\left(1-y(t), \{x_u(t_u), I_{t_u}\}_{u\in U}\right) - f\left(y(t), \{x_u(t_u), I_{t_u}\}_{u\in U}\right) \tag{2}$$

where $y(t)$ is the observed head predicate state at t, $1-y(t)$ is its counterfactual state, $\{x_u(t_u), I_{t_u}\}_{u\in U} \in \mathcal{H}_t$ is one historical combination of the body predicates (including their states and time intervals) in f, and $f(\cdot)$ is the clausal form of (1), which is an alternative expression to ease the evaluation of the rule. For example, $Y \leftarrow X$ is logically equivalent to its clausal form $\neg Y \vee X$. FE, which is essentially a difference between the what-if scenario and the true scenario, answers the question such as "should $Y(t)$ transit its state given logic formula f is true". One can check that the sign of FE can only be 1, -1 or 0, which can be interpreted as: $\text{sign}(\text{FE}) = 1$ indicates a positive effect for the head to transit, -1 indicates negative effect, and 0 means no effect. Only the non-zero FE will refer to an effective combination. Let $\mathcal{H}_t^u$ be the historical trajectory specific to predicate u up to t, and one can aggregate all the valid (i.e., non-zero) FEs from history by a summation over all combinations of the temporal predicate states and their intervals, i.e.,

$$\phi_f(t) = \sum\nolimits_{\{(x_u(t_u), I_{t_u})\in\mathcal{H}_t^u\}_{u\in U}} \delta_f(t \mid y(t), \{x_u(t_u), I_{t_u}\}_{u\in U}), \tag{3}$$

where $\phi_f(t)$ is the *feature* informed by logic formula f. One can refer to Fig. 2 for an illustration.

For each $f \in \mathcal{F}$, one can build the rule-informed and history-dependent features $\phi_f(t)$ as above, and assume these rules are connected in disjunctive normal form (i.e., OR-of-ANDs) to deduce Y, then the transition functions of $\{Y(t)\}_{t\geq 0}$ are modeled as monotonically increasing and non-negative functions of the weighted sum of these features, i.e.,

$$\lambda^*(t) = \exp\left(b_0 + \sum\nolimits_{f\in\mathcal{F}} w_f \cdot \phi_f(t)\right), \qquad \mu^*(t) = \exp\left(b_1 + \sum\nolimits_{f\in\mathcal{F}} w_f \cdot \phi_f(t)\right) \tag{4}$$

where $\boldsymbol{w} = [w_f]_{f\in\mathcal{F}} \geq 0$ are the weight parameters associated with each rule $f \in \mathcal{F}$, and b_0, b_1 are the spontaneous intensity, which are distinct for the dual intensities. Weight parameters will be shared by the dual intensities, but the calculated features ϕ_f through (2) and (3) will always have opposite signs. One can think of the formula weight w_f as the confidence level put on f. The higher the weight, the more influence that the formula will have on the intensity. The rationale of TLPP is that the intensity functions are constructed in a way that the yielding 0-1 event sequences will enable the set of logic rules to be more satisfied.

Likelihood function and its convexity. Given $\{\mathcal{H}_t\}_{0<t<T}$, one can write out the likelihood of the data in terms of the intensity function $\lambda^*(t)$ and $\mu^*(t)$. Suppose a realization of $\{Y(t)\}_{0<t<T} = \{Y(0) = 0, Y(t_1) = 1, \dots, Y(t_n) = 1, t_n < T < t_{n+1}\}$, the likelihood and the log likelihood are (see proof in Appendix C),

$$\begin{aligned}\mathcal{L}(\boldsymbol{w}, b_0, b_1) =& \lambda^*(t_1)\exp\left(-\int_0^{t_1}\lambda^*(s)ds\right)\cdot\mu^*(t_2)\exp\left(-\int_{t_1}^{t_2}\mu^*(s)ds\right)\cdots\exp\left(-\int_{t_n}^{T}\mu^*(s)ds\right),\\ \ell(\boldsymbol{w}, b_0, b_1) =& \log\mathcal{L}(\boldsymbol{w}, b_0, b_1)\end{aligned} \tag{5}$$

Note that $-\ell$ is convex w.r.t $\boldsymbol{w}$, b_0 and b_1. This is true because the intensity function has a functional form of Eq. (4), and this turns the log likelihood function into a generalized linear model (GLM) with Poisson observations and log link (Fahrmeir & Tutz, 2013). It is well-known that negative log likelihood loss of a GLM is a convex function w.r.t the model parameters. This convexity property allows us to obtain guarantees for the performance of the TELLER algorithm.

3 TELLER

We will design our TELLER algorithm with the backgrounds detailed in the previous section. Specifically, TELLER uses TLPP as the backbone to evaluate the likelihood of the grounded 0-1 temporal predicate sequences. The likelihood will be maximized so that the set of temporal logic formulas and their weight parameters are jointly learned. Typically, this learning problem is combinatorial in nature and requires enumerating exponentially

large set of combinations of the predicates and their signs. Our problem is even more challenging compared to traditional inductive logic programming cases due to the sequential properties of the data and the presence of temporal information. To address this challenge, TELLER solves the problem progressively by alternating between a *temporal rule proposal stage* and a *rule evaluation stage*, as illustrated in Fig. 1. We will start our exposition with the formulation of the original, or the mater rule learning problem.

3.1 Temporal Logic Rule Learning Problem (Master Problem)

We aim to uncover the set of temporal logic rules $\mathcal{F}$ based on the following optimization

$$\text{P}_{\text{Master}}: \quad \boldsymbol{w}^*, b_0^*, b_1^* = \underset{\boldsymbol{w}, b_0, b_1}{\operatorname{argmin}} -\ell(\boldsymbol{w}, b_0, b_1) + \sum\nolimits_{f \in \bar{\mathcal{F}}} c_f w_f; \quad (6)$$
$$s.t. \quad w_f \geq 0, \quad f \in \bar{\mathcal{F}}$$

where $\bar{\mathcal{F}}$ is the complete and exponentially large collection of *all possible* temporal logic formulas that can be created by the set of the pre-specified temporal predicates and their temporal relations. Specifically, the algorithm needs to determine 1) the assignment $\mathcal{X}_f$, $\mathcal{X}_f^+$, and $\mathcal{X}_f^-$; 2) the important temporal relations defined over the involved predicates in f; and 3) the weight parameter w_f as in (4). The regularization penalties related to the rule complexity are incorporated into the objective function to force sparsity and to trade off the model accuracy against rule simplicity. Rule complexity coefficient c_f can be the rule length. The sparsity of $\boldsymbol{w}^*$ will indicate the collection of rules learned and these non-zero weight rules will compose $\mathcal{F}$.

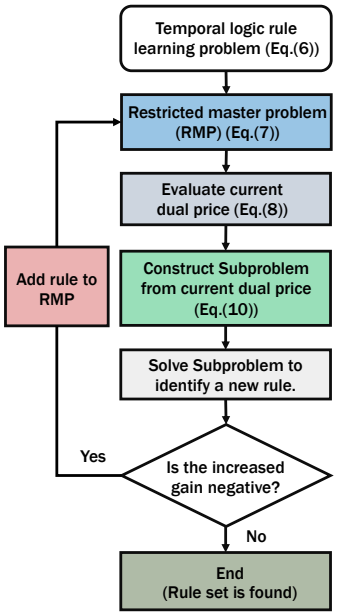

Figure 3: TELLER Algorithm

Key Idea: Although we can write the rule learning problem as a regularized convex optimization above, the set of variables (the w_fs) we are optimizing on is exponentially large and can not be optimized simultaneously in a tractable way. The idea of TELLER is to start from a restricted master problem (RMP), where the search space is smaller and the solution is tractable but suboptimal. TELLER will gradually improve this solution by iteratively expanding the search space until the solution approximates the optimum. The idea is inspired by the fact that most of the candidate rules in $\bar{\mathcal{F}}$ will not be in $\mathcal{F}$ and will lead to a zero value weight in the optimal solution. We do not want to waste resources by including these redundant rules when we expand the search space to optimize (6). TELLER will always generate the rule that has the potential to improve the objective function. Moreover, suppose the runtime of the algorithm has exceeded the preset time limit, one can safely stop the algorithm with the guarantee that the most important rules have been considered.

Such a progressive algorithm then gives rise to two technical questions that need to be addressed: **1)** What should be the criterion used to determine the new rule to add? **2)** How to construct new candidate temporal logic rules? We will tackle these questions one by one in the following two sections.

3.2 Criterion to Add Rules

Restricted master problem (RMP). TELLER first replaces the original rule set $\bar{\mathcal{F}}$ by a small subset $\mathcal{F}_0 \subset \bar{\mathcal{F}}$ and gradually expands this subset to produce a nested sequence of subsets $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \cdots$. It does so by adding candidate rules identified by subproblems (we will elaborate on this later). Note that the initial rule set $\mathcal{F}_0$ can be an empty set or any pre-defined small set.

For each $\mathcal{F}_k$, $k = 0, 1, \ldots$, TELLER solves the following restricted master problem by replacing $\bar{\mathcal{F}}$ with $\mathcal{F}_k$

$$\text{P}_{\text{Restricted}}: \ \boldsymbol{w}_{(k)}^*, b_{0,(k)}^*, b_{1,(k)}^* = \underset{\boldsymbol{w}, b_0, b_1}{\operatorname{argmin}} \ -\ell(\boldsymbol{w}, b_0, b_1) + \sum_{f \in \mathcal{F}_k} c_f w_f; \ s.t. \ w_f \geq 0, \ f \in \mathcal{F}_k. \quad (7)$$

This can be regarded as the *rule evaluation stage*, where all rules in the current set will be reweighed. Note that an *optimal* solution to the *restricted* master problem above can be extended to a *feasible* solution to the original problem by setting the weights of all missing $f \in \bar{\mathcal{F}} \setminus \mathcal{F}_k$ to zero. The *optimality* of this extended solution can be verified by leveraging the convexity property of the original problem (6). More specifically, Let the Lagrangian of the original problem be $L(\boldsymbol{w}, b_0, b_1, \boldsymbol{\nu}) = -\ell(\boldsymbol{w}, b_0, b_1) + \sum_{f \in \bar{\mathcal{F}}} c_f w_f - \sum_{f \in \bar{\mathcal{F}}} \nu_f w_f$, where $\nu_f \geq 0$ is the Lagrange multiplier associated with the non-negativity constraints of w_f. Since it is a convex problem, strong duality holds under mild conditions. Suppose $\boldsymbol{w}^*, b_0^*, b_1^*$ is primal optimal, and $\boldsymbol{\nu}^*$ is dual optimal, then $-\ell(\boldsymbol{w}^*, b_0^*, b_1^*) = \inf_{\boldsymbol{w}, b_0, b_1} L(\boldsymbol{w}, b_0, b_1, \boldsymbol{\nu}^*)$. This implies the *complementary slackness*, which will lead to the following condition.

Evaluate current dual price. Compute the dual price for each constraint in the *original problem (6)* as:

$$\nu_{f,(k)} = -\left.\frac{\partial \ell(\boldsymbol{w}, b_0, b_1)}{\partial w_f}\right|_{\boldsymbol{w}_{(k)}^*, b_{0,(k)}^*, b_{1,(k)}^*} + c_f, \quad (8)$$

where subscript (k) means the price depends on the current optimal solution of the RMP, and subscript f refers to the constraint $w_f \geq 0$. The optimality condition says: if the extended solution of $\boldsymbol{w}^*_{(k)}$ is optimal to the original problem, it must satisfy the following condition:

$$w^*_{f,(k)} > 0 \;\Rightarrow\; \nu_{f,(k)} = 0; \qquad w^*_{f,(k)} = 0 \;\Rightarrow\; \nu_{f,(k)} \geq 0 \qquad \forall f \in \bar{\mathcal{F}}, \tag{9}$$

where $\nu_{f,(k)}$ is computed via (8). For the missing rules, we have $w_{f,(k)} = 0$ automatically, and if they are the optimal solution to the original problem, the computed $\nu_{f,(k)} \geq 0$ must hold. A more detailed description of complementary slackness and optimality condition is given in Appendix D.

So, the criterion to add rules is to search over the missing rules and find the rule that most *violates* the condition, i.e., leads to the most negative ν_f as computed by (8). Adding it to $\mathcal{F}_{k+1}$ will most reduce the objective function. Assuming the computed minimal $\nu_f \geq 0$, it is guaranteed that the current solution to the restricted problem is also optimal to the original problem. A subproblem will be constructed and optimized to identify a new rule, a process which will be discussed next.

3.3 Propose a New Temporal Logic Rule

Construct a subproblem from current dual price. A subproblem is formulated to propose a new (or missing) temporal logic rule which can potentially improve the optimal value of the RMP, i.e. having a negative increased gain. The increased gain of a missing rule is defined as the possible change in the objective function per unit when it is included in the rule set, and it is computed by taking the partial derivative of the objective function w.r.t the rule weight. Note the subproblem itself is a minimization, which aims to find the most negative increased gain.

Given the optimal solution $\boldsymbol{w}^*_{(k)}, b^*_{0,(k)}, b^*_{1,(k)}$ for the restricted master problem (7), a subproblem is constructed by taking the partial derivative w.r.t the weights. Note the log likelihood can be regarded as a function of the dual conditional intensities, and we can write it as $\ell(\lambda^*, \mu^*)$. Using the chain rule for gradient, the subproblem is to optimize over the features,

$$\min_{\phi_f \in \Phi} \nu(\phi_f | \boldsymbol{w}^*_{(k)}, b^*_{0,(k)}, b^*_{1,(k)}) = -\left(\frac{\partial \ell(\lambda^*, \mu^*)}{\partial \lambda^*} \frac{\partial \lambda^*}{\partial w_f} + \frac{\partial \ell(\lambda^*, \mu^*)}{\partial \mu^*} \frac{\partial \mu^*}{\partial w_f} \right)\Bigg|_{\boldsymbol{w}^*_{(k)}, b^*_{0,(k)}, b^*_{1,(k)}} + c_f, \tag{10}$$

where $\phi_f \in \Phi$ is the rule-based feature, λ^* and μ^* are functions of ϕ_f, and the conditional intensities are time and history dependent (see Eq. (4)). In our formulation, this subproblem objective happens to be the current dual price. The objective can be hard to express in a closed-form but will be easily evaluated numerically. Minimizing over the feature space Φ is essentially searching through a combinatorial space of the temporal predicates and their temporal relations to determine a rule f, so that its yielding feature will minimize the objective function.

Solve subproblem to identify a new rule. As discussed above, if the optimal value of the subproblem is negative, we have identified a feature (i.e., rule) to enter the rule set to construct $\mathcal{F}_{k+1}$; if the optimal value of the above problem is non-negative, we have proven that the current solution to the RMP is also optimal to the original problem. Note that the subproblem would not generate the same feature or rule more than once, since the optimal Lagrangian multipliers in the RMP are non-negative. However, explicitly solving subproblem (10) requires enumerating all possible conjunctions of the input predicates, their signs, and all possible pairwise temporal relations. Even worse, the construction of feature ϕ_f requires evaluating all valid combinations of predicate events from history, which will add an extra burden to the computation.

Speedup subproblem. To make the subproblem tractable, we adopt a couple of principles. 1) Theoretically, the *optimality of the subproblem can be sacrificed*. In fact, any solution to (10) that has *a negative objective* will generate a rule with negative increased gain and can be added to the rule set. This provides performance guarantee for speedup algorithms. 2) Practically, *prior knowledge in rule structures can be leveraged*. Our prior knowledge consists of 3 parts: *sparsity*, *heredity*, and *expert's preference* in rule templates. By *sparsity*, we mean the uncovered rule set is small and the rule lengths are generally short. In our algorithm, the maximal rule length can be pre-specified as H, which means we do not need to consider rules with length exceeding H. By *heredity*, we mean if the conjunctions of the input predicates is important, at least one of the involved input predicates will show some significance. Heredity justifies that we can gradually grow rules from the existing short segments in the current rule set.

Search algorithms. With the prior knowledge above, we can design search algorithms in a way like depth-first search (DFS) or breadth-first search (BFS). In our setting, they are named as rule-extension-first-search (REFS) and rule-addition-first-search (RAFS), respectively. REFS will always extend the existing rules until the length reaches H, while RAFS will first add all possible rules with length 1, and then consider the rules with length 2 and so on. Here we describe details of REFS, and leave RAFS for Appendix F. As illustrated in Fig. 1, a REFS is used to construct a new temporal logic rule. Specifically, it starts by constructing a rule with length one (i.e., one body predicate). This is achieved by scanning the predicate set $\mathcal{X}$ and temporal relation set $\mathcal{R}$, then enumerating the signs and possible temporal relations with the head predicate. The subproblem objective is used to score these segments and select the most negative one. The process continues to extend current segments in the same way, until the rule length reaches H, or the minimal subproblem objective becomes positive. In this way, the computational complexity of our subproblem is $\mathcal{O}(d)$, where d is the number of predicates. For TELLER, once new rules are added to the rule set, the model will be re-fitted and the model parameters will be re-weighted according to Eq. (7). Longer rules might outweigh their short segments after the rule evaluation stage and thus the length of an important rule can be automatically determined.

In addition, we can use *expert's preference* in rule templates to solve the subproblems. Human experts can provide preferred rule templates (rather than a rule set) without too much effort. Such rule templates will effectively reduce the rule searching space, since it masks out many irrelevant predicates and their combinations. In medicine dataset, for example, doctors are interested in how a treatment affects the evolution of patient health status. To reflect the doctor's conjectures, rule template can be defined as "symptoms $\leftarrow$ drugs $\wedge$ symptoms".

3.4 Discussion on Scalability of the Algorithm

As a recap, our TELLER works as follows. The RMP is solved in (7) to update optimal rule weights, from which dual prices are evaluated for each constraint in the original problem. This information is then utilized in constructing the objective function of the subproblem as in (8). The subproblem can then be solved using the proposed efficient search schemes. The overall algorithm can be regarded as an *optimization-based heuristic* approach. TELLER alternates between solving the restricted master problem and the subproblem until no negative increased gain rules are identified. An algorithm flowchart in displayed in Fig. 3 and detailed algorithm boxes are in Appendix E.

The previous discussion provides a theoretical framework to find an (near-) optimal temporal logic rule set based on the point process model. For large problems with thousands of event sequences and hundreds of predicates, this type of progressive algorithm tends to be hard to terminate, and obtaining the optimality condition is almost impossible; this is the so-called tailing-off effect (Lübbecke & Desrosiers, 2005; Savelsbergh, 2002). Therefore, we need other techniques to make TELLER scale with the data and the predicates. For example, we may condition an early termination to the algorithm if a fixed time limit is exceeded. In practice, setting the time limit is proven to work well and the most important rules are likely to be discovered. For large problems, we may also choose to take random subset of sequences and random subset of predicates for evaluation. When solving for the RMP, we can use a stochastic gradient descent-type of optimization to approximate the gradient using a batch of sequences. By doing so, we enable the RMP to scale with the data (sequences). Furthermore, we can warm-start the model parameters for consecutive RMPs, i.e. we initialize $\boldsymbol{w}_{(k+1)}, b_{0,(k+1)}, b_{1,(k+1)}$ using the optimal solution $\boldsymbol{w}^*_{(k)}, b^*_{0,(k)}, b^*_{1,(k)}$. When solving the subproblems, besides the efficient search scheme, we may also choose to take a subset of the predicates in searching to further accelerate the computation.

4 Experiments

We evaluate the rule discovery ability and model prediction accuracy of TELLER on synthetic and real event data. For synthetic data, where the ground-truth temporal logic rules are known, the accuracy can be evaluated by comparing the uncovered results to the ground truth. For real data, we focus on high-stakes applications where interpretable models are merited. We consider a healthcare treatment understanding problem and a crime pattern learning problem (all the crime results are in Appendix H and I). We will show the transparency, prediction accuracy, and flexibility of our learned rule-informed temporal point process models in these applications.

4.1 Baselines

We considered the following SOTA baselines: 1) Recurrent Marked Temporal Point Processes (RMTPP) (Du et al., 2016b), the first neural point process (NPP) model, where the intensity function is modeled by a Recurrent Neural Network (RNN); 2) Neural Hawkes Process (NHP) (Mei & Eisner, 2016), an improved variant of RMTPP by constructing a continuous-time LSTM; 3) Transformer Hawkes Process (THP) (Zuo et al., 2020), a NPP model with a self-attention mechanism; 4) Tree-Regularized GRU (TR-GRU) (Wu et al., 2018a), a deep time-series model with a designed tree regularizer to add model interpretability; 5) Recurrent Point Process Network (RPPN) (Xiao et al., 2019), where the intensity is modeled by two interleaved RNNs with attention mechanism; 6) CAUSE (Zhang et al., 2020b), a NPP model with Granger causality statistic learned. We also considered other widely used parametric/nonparametric TPPs: 1) Hawkes Process with an exponential kernel (HExp) (Lewis & Mohler, 2011); 8) Inhomogeneous Poisson Process (IPP), where the intensity is a sum of k weighted Gaussian kernels: $\hat{\lambda}(t) = \sum_{i=1}^{k} \alpha_i \left(2\pi\sigma_i^2\right)^{-1/2} \exp\left(-\left(t-c_i\right)^2/\sigma_i^2\right)$, where c_i and σ_i are fixed center and standard deviation, and α_i is the weight for kernel i.

Table 1: Model properties and interpretability comparison of all baselines and TELLER

Model	RMTPP	NHP	THP	TR-GRU	RPPN	CAUSE	HExp	IPP	**TELLER**
Interpretable				✓		✓	✓	✓	✓
Flexible	✓	✓	✓	✓	✓	✓			✓
Parsimonious							✓	✓	✓
Pairwise interaction						✓	✓		✓
Higher-order interaction									✓

In Tab. 1, we compared TELLER with these baselines in terms of the following model properties: 1) Interpretable: model parameters and prediction results are understandable to humans; 2) Flexible: model is expressive to capture sophisticated non-linear dependencies; 3) Parsimonious: model parameters are of small or medium size; 4) Pairwise interaction: model can identify the mutual influence of the events; 5) Higher-order interaction: model can identify the composite effect of a subset of events to another event.

4.2 SYNTHETIC DATA

We first verify the rule discovery ability of TELLER on synthetic datasets where the ground truth is known. The synthetic events are generated from TLPPs as follows. We specify a set of ground truth temporal logic rules as well as their rule weights. The event (transition) intensity functions are built according to Eq. (4). Roll-out predicate event samples are generated via thinning algorithm (Lewis & Shedler, 1979) and are subject to the pre-defined temporal logic rules. We prepared 12 synthetic datasets by specifying various rule lengths, weights, and temporal relations. See Appendix G for detailed information.

Table 2: Learned Rules on Synthetic Dataset-1.

True Rules (# 3) v.s. Learned Rules (REFS)
2400 Sequences:
E ← A ∧ (A Before E)
E ← B ∧ C ∧ (B Before E) ∧ (C Before E)
E ← C ∧ D ∧ (C Before D) ∧ (D Equal E)
600 Sequences:
E ← A ∧ (A Before E)
E ← A ∧ D ∧ (A Equal D) ∧ (A Before E)
E ← A ∧ ¬B ∧ (A Before E) ∧ (¬B Before A)
E ← B ∧ C ∧ (B Before E) ∧ (C Before E)
E ← B ∧ C ∧ (B Before E) ∧ (C Before B)
E ← A ∧ C ∧ (A Before E) ∧ (C Before E)
E ← C ∧ A ∧ (A Equal C) ∧ (A Before E)
E ← A ∧ ¬C ∧ (A Before ¬C) ∧ (¬C Before E)

TELLER with both the REFS and RAFS subproblem search scheme is tested on all 12 datasets. Consistent results show an accurate performance of our algorithm in terms of both rule discovery and parameter learning. We list all the discovered rules based on dataset-1 using 2400 and 600 sequences respectively in Tab. 2, where the correct rules are marked in blue. For dataset-1, the body predicate set is {A, B, C, D}, and the head predicate is E. Given 600 sequences, TELLER successfully discovered 2 out of 3 truth rules along with several noise rules; whereas, given 2400 sequences, TELLER accurately uncovered all 3 ground truth rules without including any noise rules. The results indicate that TELLER is capable of mining hidden rules from noisy event data. Given more samples, the learning performance will improve as a result. More results are reported in Fig. 4, where 6 synthetic datasets with 2400 sequences are used

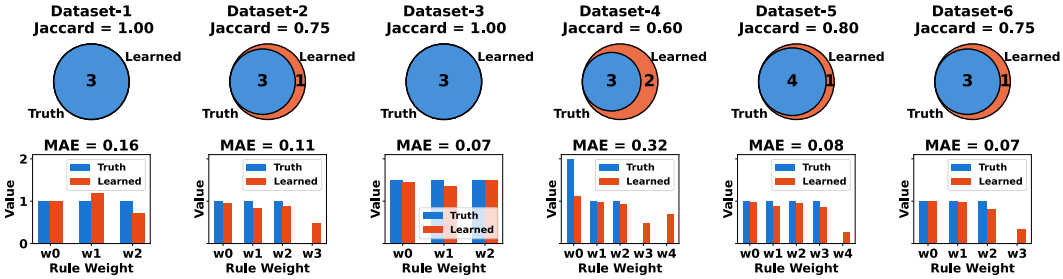

Figure 4: Rule discovery and weight learning results of TELLER-REFS on 6 synthetic datasets (2.4K seqs).

for evaluation. Each figure in the top row uses a Venn diagram to demonstrate the true rule set and the learned rule set, from which the Jaccard similarity score (i.e., the area of the intersection divided by the area of their union) is calculated. From the results, we see that TELLER successfully discovered almost all the true rules, only including one or two noise rules in some challenging tasks. Each figure in the bottom row compares the true rule weights with the learned rule weights, with the Mean Absolute Error (MAE) reported. One can observe that almost all truth rule weights are accurately learned and the included noise rule weights are relatively small. The results demonstrate that TELLER is stable and accurate under various settings. In dataset-4, we crafted a long and complex rule with 3 body predicates to challenge TELLER, and it is still able to correctly discover this rule, while the weight is slightly underestimated. More information about the experiment settings can be found in Appendix G.

4.3 REAL DATA: UNDERSTAND TREATMENT STRATEGIES FROM MIMIC-III

MIMIC-III is an electronic health record dataset of patients admitted to the intensive care unit (ICU) (Johnson et al., 2016). We focused on patients diagnosed with sepsis (Saria, 2018; Raghu et al., 2017; Peng et al., 2018), which is one of the major causes of mortality in ICU. Evidence suggests that the treatment strategy remains uncertain – it is unclear how to use intravenous fluids and vasopressors to support the circulatory system of patients. And there exists clinical controversy about when and how to use these two groups of drugs to reduce the side effect. The goal of our experiment is to implement TELLER to automatically learn the explanatory temporal logic rules regarding these two groups of drugs from patient data in ICU as well as their importance weights. Our results will shed light on the strategy of choice and arrangement of these drugs and indicate which rules play a more critical role in improving patients' health status.

Table 3: MIMIC-III: Event prediction results.

Method	LowUrine (MAE)	Survival (AUC)
RMTPP	1.983	0.796
NHP	1.684	0.802
THP	1.545	0.824
TR-GRU	2.666	0.811
RPPN	1.640	0.767
CAUSE	1.712	0.740
HExp	2.578	0.882
IPP	2.472	0.784
TELLER	**1.266**	**0.930**

Predicates and dataset statistics. We define 62 predicates, including two groups of drugs (i.e., intravenous fluids and vasopressors) and lab measurements. The variables involved are similar to (Saria, 2018). Among all these predicates we are interested in reasoning about two predicates. So we define them as head predicates: 1) LowUrine and 2) Survival. We treat real-time urine as a head predicate since low urine is the direct

indicator of bad circulatory systems and is a important signal for septic shock. A complete table of predicates can be found in Appendix H. In our experiment, lab measurement variables are converted to binary values (according to the normal range used in medicine) with the transition times recorded. For drug predicates, they are recorded as 1 when they were applied to the patient. We extracted 4298 patient sequences, and randomly chose 80% of them for training and the remaining for testing. The average time horizon is 392.69 hours and the average #events per sequence is 79.03. Detailed experiment setup and hyperparameters can be found in Appendix H.2.

Evaluation metrics and results. We use different evaluation metrics for the two head predicates: 1) `LowUrine` is evaluated by predicting the time of transitions from state 0 to state 1 (this is related to the appearance of septic shock in medicine), for which the performance is measured by Mean Absolute Error (MAE) – the lower the better. 2) `Survival` is evaluated by predicting whether the patient is survived at the end of treatment, for which the performance is measured by AUC – the bigger the better. The performances of our model and all baselines are reported in Tab. 3, from which one can observe that our model outperforms all baselines in this experiment.

Discovered temporal logic rules. Tab. 4 shows the uncovered explanatory temporal logic rules and their weights learned by TELLER. Here, we use `LowUrine` as the head predicate and `NormalUrine` as its negation. For each discovered rule, the sign of the head predicate will be automatically determined by the algorithm. We invited human experts (i.e., doctors in ICU) to justify the correctness of the result. Experts confirmed that these discovered logic rules are consistent with the pathogenesis of sepsis, and have captured the most important factors in affecting real-time urine output and survival. Experts are also interested in interpreting the clinical meaning of the discovered rules. They found that Rule 1 to Rule 4 capture the major lab measurements that usually emerge together with extremely low urine. Rule 5 seemed hard to interpret in the beginning by experts, since they thought it was slightly counter-intuitive. They later concluded that, in some extreme cases, even after using Crystalloid (one type of intravenous fluids) as the treatment, the urine can remain low, and thus Rule 5 is still likely to be observed. Rules 6-9 shed light on drug selection. For example, reflected in Rule 7, using Crystalloid and Phenylephrine (a type of vasopressors) together will play an evident role in triggering the urine to a normal level. In Rules 6, 8, and 9, using Crystalloid yields a weight of 4.14, using Colloid (a type of intravenous fluids) yields a weight of 1.97, and using the combinations of the two yields a weight of 1.83, which can be regarded as an incremental contribution in addition to using them separately.

Expert-Model interaction. Our TELLER can naturally involve human experts in the learning loop. The interaction with experts can be summarized as follows: 1) Experts can provide a predicate set, their interested head predicates, an initial rule set, and desired rule templates to the model; 2) One can run TELLER to obtain preliminary results for experts to check. Experts can delete unreasonable or noise rules and give the refined rule set to the model before continuing the learning process. In the MIMIC experiment, experts provided advice for predicates definition and the desired rule templates. We ran TELLER from an empty initial rule set and generated 12 rules. Experts verified the correctness of all the rules and deleted the following ones: `LowUrine` $\leftarrow$ `HighSpO2`; `LowUrine` $\leftarrow$ `NormalSodium`; `LowUrine` $\leftarrow$ `NormalSodium` $\wedge$ `Water`. We were informed that the first rule is contradictory with Pathophysiology common sense, while the second and third rules are correct but irrelevant with the treatment strategy. We deleted these 3 rules and re-implemented TELLER to refine the rule set and rule weights. The final output is reported in Tab. 4.

Table 4: Learned Rules with `LowUrine` as the head predicate.

Weight	Rule
1.07	**Rule 1:** LowUrine $\leftarrow$ LowSysBP $\wedge$ (LowSysBP Before LowUrine)
0.89	**Rule 2:** LowUrine $\leftarrow$ LowSodim $\wedge$ (LowSodim Before LowUrine)
1.16	**Rule 3:** LowUrine $\leftarrow$ HighCreatinine $\wedge$ Equal(HighCreatinine, LowUrine)
1.98	**Rule 4:** LowUrine $\leftarrow$ HighBUN $\wedge$ (HighBUN Before LowUrine)
1.10	**Rule 5:** LowUrine$\leftarrow$ HighBUN $\wedge$ Crystalloid $\wedge$ (HighBUN Before LowUrine) $\wedge$ (Crystalloid Before LowUrine)
4.14	**Rule 6:** NormalUrine $\leftarrow$ LowUrine $\wedge$ Crystalloid $\wedge$ (LowUrine Equal NormalUrine) $\wedge$ (Crystalloid Equal NormalUrine)
3.13	**Rule 7:** NormalUrine $\leftarrow$ LowUrine $\wedge$ Phenylephrine $\wedge$ (LowUrine Equal NormalUrine) $\wedge$ (Phenylephrine Equal NormalUrine)
1.83	**Rule 8:** NormalUrine $\leftarrow$ LowUrine $\wedge$ Crystalloid $\wedge$ Colloid $\wedge$ (Colloid Equal NormalUrine) $\wedge$ (Crystalloid Equal NormalUrine) $\wedge$ (LowUrine Equal NormalUrine)
1.97	**Rule 9:** NormalUrine $\leftarrow$ LowUrine $\wedge$ Colloid $\wedge$ (LowUrine Equal NormalUrine) $\wedge$ (Colloid Equal NormalUrine)

5 Conclusion

We propose a promising TELLER algorithm to learn interpretale temporal logic rules to explain point processes. TELLER alternates between a rule generation stage and a rule evaluation stage, and can uncover the most important collection of logic rules with guarantees. To our best knowledge, this is the first work that automates the discovery of the temporal logic rules based on point process models. Our method adds transparency and interpretability to event models. We also demonstrate the accurate rule discovery and event prediction results of TELLER using synthetic and real experiments.

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