Cooperative Online Learning with Feedback Graphs

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Abstract

We study the interplay between communication and feedback in a cooperative online learning setting, where a network of communicating agents learn a common sequential decision-making task through a feedback graph. We bound the network regret in terms of the independence number of the strong product between the communication network and the feedback graph. Our analysis recovers as special cases many previously known bounds for cooperative online learning with expert or bandit feedback. We also prove an instance-based lower bound, demonstrating that our positive results are not improvable except in pathological cases. Experiments on synthetic data confirm our theoretical findings.

1 Introduction

Nonstochastic online learning with feedback graphs (Mannor and Shamir, 2011) is a sequential decision-making setting in which, at each decision round, an oblivious adversary assigns losses to all actions in a finite set. What the learner observes after choosing an action is determined by a feedback graph defined on the action set. Unlike bandit feedback, where a learner choosing an action pays and observes the corresponding loss, in the feedback graph setting the learner also observes (without paying) the loss of all neighboring actions in the graph. Special cases of this setting are prediction with expert advice (where the graph is a clique) and multiarmed bandits (where the graph has no edges). The Exp3-SET algorithm (Alon et al., 2017) is known to achieve a regret scaling with the square root of the graph independence number, and this is optimal up to a logarithmic factor in the number of actions. In recommendation systems, feedback graphs capture situations in which a user's reaction to a recommended product allows the system to infer what reaction similar recommendations would have elicited in the same user, see Alon et al. (2017) for more examples.

Online learning has been also investigated in distributed settings, in which a network of cooperating agents solves a common task. At each time step, some agents become active, implying that they are requested to make predictions and pay the corresponding loss. Agents cooperate through a communication network by sharing the feedback obtained by the active agents. The time this information takes to travel the network is taken into account: a message broadcast by an agent is received by another agent after a delay equal to the shortest path between them. Regret in cooperative online learning has been previously investigated only in the full-information setting (Cesa-Bianchi et al., 2020; Hsieh et al., 2022) and in the bandit setting (Cesa-Bianchi et al., 2019; Bar-On and Mansour, 2019a).

In this work we provide a general solution to the problem of cooperative online learning with feedback graphs. In doing so, we generalize previous approaches and also clarify the impact on the regret of the mechanism governing the activation of agents. Under the assumption that agents are stochastically activated, our analysis captures the interplay between the communication graph (over the agents) and the feedback graph (over the actions), showing that the network regret scales with the independence number of the strong product between the communication network and the feedback graph.

More precisely, we design a distributed algorithm, Exp3-α^2, whose average regret R_T/Q (where Q is the expected number of simultaneously active agents) on any communication network N and any feedback graph F is (up to log factors)

$$\frac{R_T}{Q} \overset{\mathcal{O}}{=} \sqrt{\left(\frac{\alpha\big(N^n \boxtimes F\big)}{Q} + 1 + n\right) T} \tag{1}$$

where T is the horizon, n is the diffusion radius (the maximum delay after which feedback is ignored), and $\alpha\big(N^n \boxtimes F\big)$ is the independence number of the strong product between the n-th power N^n of the communication network N and the feedback graph F. We also prove a near-matching instance-dependent lower bound showing that, with the exception of pathological cases, for any pair of graphs (N, F), no algorithm can have regret smaller than $\sqrt{\big(\alpha(N^n \boxtimes F)/Q\big)T}$.

Our results hold for any diffusion radius n, which serves as a parameter to control the message complexity of the protocol. When n is equal to the diameter of the communication network, then every agent can communicate with every other agent. Our protocol is reminiscent of the LOCAL communication model in distributed computing (Linial, 1992; Suomela, 2013), where the output of a node depends only on the inputs of other nodes in a constant-size neighborhood of it, and the goal is to derive algorithms whose running time is independent of the network size. Although our tasks have no completion time, in our model each node is directly influenced only by a constant-size neighborhood around it.

Let $|A|$ and $|K|$ be, respectively, the number of agents and actions. When $Q = |A|$ (all agents are always active) and F is the bandit graph (no edges), then $\alpha(N^n \boxtimes F) = |K|\,\alpha(N^n)$ and we recover the bound $\sqrt{\big(\alpha(N^n)\,|K|\,/\,|A| + 1 + n\big)T}$ of Cesa-Bianchi et al. (2019). When $n = 1$ and F is the expert graph (clique), then $\alpha(N^n \boxtimes F) = \alpha(N)$ and we recover the bound $\sqrt{\big(\alpha(N)/Q + 1\big)T}$ of Cesa-Bianchi et al. (2020)[1]. Interestingly, if all agents were always active in Cesa-Bianchi et al. (2020), the graph topology would become irrelevant in the expert setting, resulting in a simplified regret bound of $\mathcal{O}(\sqrt{T})$, analogous to the case of a clique graph. This starkly contrasts with the bandit case of Cesa-Bianchi et al. (2019), where even when all agents are active simultaneously, the graph topology appears anyway explicitly into the regret bound. This already hints at the fundamental importance of stochastic activations in a partial feedback setting. Finally, in the non-cooperative case (N is the bandit graph), we obtain $\sqrt{|A|\alpha(F)T/Q}$ which, for $|A| = 1$ and $Q = 1$, recovers the bound of Alon et al. (2017). The table below summarizes all known bounds (omitting log factors and setting, for simplicity, $Q = 1$ and $n = 0$).

	N = single node		Any N	
F = clique (experts)	$\sqrt{T}$	(Freund and Schapire, 1997)	$\sqrt{\alpha(N)T}$	(Cesa-Bianchi et al., 2020)
F = no edges (bandits)	$\sqrt{\lvert K\rvert T}$	(Auer et al., 2002)	$\sqrt{\alpha(N)\lvert K\rvert T}$	(Cesa-Bianchi et al., 2019)
Any F	$\sqrt{\alpha(F)T}$	(Alon et al., 2017)	$\sqrt{\alpha(N \boxtimes F)T}$	(this work)

Our lower bound holds irrespectively of the cooperative strategy of the agents, who may have full knowledge of the network topology. On the other hand, our upper bound holds even with the so-called oblivious network interface; i.e., when agents are oblivious to the global network topology and run an instance of the same algorithm using a common initialization and a common learning rate for their updates. In this case, the stochastic activation assumption is also necessary to not incur in a linear regret $R_T = \Omega(T)$ Cesa-Bianchi et al. (2020).

Our core and main technical contributions are presented in Lemma 1, Theorem 1 for the upper bound, and in Lemma 2 and Theorem 2 for the lower bound. Lemma 1, implies that the second moment of the loss estimates is dominated by the independence number of the strong product between the two graphs. The proof of this result generalizes the analysis of (Cesa-Bianchi et al., 2019, Lemma 3), identifying the strong product as the appropriate notion for capturing the combined effects of the communication and feedback graphs. In Theorem 1, we present a new analysis for the distributed learning setting of the "drift" term arising from the decomposition of network regret. This is obtained by combining Lemma 1 with a regret analysis technique developed in Gyorgy and Joulani (2021) for a single agent. The proof of the lower bound in Theorem 2 builds upon a new reduction to the setting of Lemma 2 that we prove in Appendix A. Lemma 2 contains a lower bound for a single-agent setting with a feedback graph and oblivious adversary where every time step is independently skipped with a known and constant probability q. This reduction is new and necessary, since it is not enough to claim that the average number of rounds played is qT and plug this in the

[1]This is a reformulation of the bound originally proven by Cesa-Bianchi et al. (2020), see Section C of the Supplementary Material for a proof.

lower bound for bandit with feedback graphs. In fact, one needs to build an explicit assignment of $\ell_1, \ldots, \ell_T$ such that by averaging over the random subset of active time steps it is possible to prove the lower bound under the conditions detailed in Lemma 2.

In Section 6, we corroborate our theoretical results with experiments on synthetic data.

2 Further related work

The topic of cooperation in online learning gathered a vast amount of attention in recent years, and many variants of the problem have attracted the interest of the community.

Adversarial losses. A setting closely related to ours is investigated by Herbster et al. (2021). However, they assume that the learner has full knowledge of the communication network—a weighted undirected graph—and provide bounds for a harder notion of regret defined with respect to an unknown smooth function mapping users to actions. Bar-On and Mansour (2019b) bound the individual regret (as opposed to our network regret) in the adversarial bandit setting of Cesa-Bianchi et al. (2019), in which all agents are active at all time steps. Their results, as well as the results of Cesa-Bianchi et al. (2019), have been extended to cooperative linear bandits by Ito et al. (2020). Della Vecchia and Cesari (2021) study cooperative linear semibandits and focus on computational efficiency. Dubey et al. (2020a) show regret bounds for cooperative contextual bandits, where the reward obtained by an agent is a linear function of the contexts. Nakamura et al. (2023) consider cooperative bandits in which agents dynamically join and leave the system.

Stochastic losses. Cooperative stochastic bandits are also an important topic in the online learning community. Kolla et al. (2018) study a setting in which all agents are active at all time steps. In our model, this corresponds to the special case where the feedback graph is a bandit graph (no edges) and the activation probabilities $q(v)$ are equal to 1 for all agents v. More importantly, however, they focus on a stochastic multi-armed bandit problem. Hence, even restricting to the special cases of bandits with simultaneous activation, their algorithmic ideas cannot be directly applied to our adversarial setting. Other recently studied variants of cooperative stochastic bandits consider agent-specific restrictions on feedback (Chen et al., 2021) or on access to arms (Yang et al., 2022), bounded communication (Martínez-Rubio et al., 2019), corrupted communication (Madhushani et al., 2021), heavy-tailed reward distributions (Dubey et al., 2020b), stochastic cooperation models (Chawla et al., 2020), strategic agents (Dubey and Pentland, 2020), and Bayesian agents (Lalitha and Goldsmith, 2021). Finally, Liu et al. (2021) investigate a decentralized stochastic bandit network for matching markets.

3 Notation and setting

Our graphs are undirected and contain all self-loops. For any undirected graph $G = (V, E)$ and all $m \geq 0$, we let $\delta_G(u, v)$ be the *shortest-path distance* (in G) between two vertices $u, v \in V$, G^m the m-th power of G (i.e., the graph with the same set of vertices V of G but in which two vertices $u, v \in V$ are adjacent if and only if $\delta_G(u, v) \leq m$), $\alpha(G)$ the *independence number* of G (i.e., the largest cardinality of a subset I of V such that $\delta_G(u, v) > 1$ for all distinct $u, v \in I$), and $\mathcal{N}^G(v)$ the *neighborhood* $\{u \in V : \delta_G(u, v) \leq 1\}$ of a vertex $v \in V$. To improve readability, we sometimes use the alternative notations $\alpha_m(G)$ for $\alpha\big(G^m\big)$ and $\mathcal{N}_m^G(v)$ for $\mathcal{N}^{G^m}(v)$. Finally, for any two undirected graphs $G = (V, E)$ and $G' = (V', E')$, we denote by $G \boxtimes G'$ their *strong product*, defined as the graph with set of vertices $V \times V'$ in which (v, v') is adjacent to (u, u') if and only if $(v, v') \in \mathcal{N}^G(u) \times \mathcal{N}^{G'}(u')$.

An instance of our problem is parameterized by:

1. A **communication network** $N = (A, E_N)$ over a set A of agents, and a maximum communication delay $n \geq 0$, limiting the communication among agents.
2. A **feedback graph** $F = (K, E_F)$ over a set K of actions.

3. An **activation probability** $q(v) > 0$ for each agent $v \in A$,[2] determining the subset of agents incurring losses on that round. Let $Q = \sum_{v \in A} q(v)$ be the expected cardinality of this subset.
4. A sequence $\ell_1, \ell_2, \ldots : K \to [0,1]$ of **losses**, chosen by an oblivious adversary.

We assume the agents do not know N (see the oblivious network interface assumption introduced later). The only assumption we make is that each agent knows the activation probability $q(v)$ of all agents v located at distance n or less.[3]

The distributed learning protocol works as follows. At each round t, each agent v is activated with a probability $q(v)$, independently of the past and of the other agents. Agents that are not activated at time t remain inactive for that round. Let $\mathcal{A}_t$ be the subset of agents that are activated at time t. Each $v \in \mathcal{A}_t$ plays an action $I_t(v)$ drawn according to its current probability distribution $p_t(\cdot, v)$, is charged the corresponding loss $\ell_t\big(I_t(v)\big)$, and then observes the losses $\ell_t(i)$, for any action $i \in \mathcal{N}_1^F\big(I_t(v)\big)$. Afterwards, each agent $v \in A$ broadcasts to all agents $u \in \mathcal{N}_n^N(v)$ in its n-neighborhood a feedback message containing all the losses observed by v at time t together with its current distribution $p_t(\cdot, v)$; any agent $u \in \mathcal{N}_n^N(v)$ receives this message at the end of round $t + \delta_N(v, u)$.

Each loss observed by an agent v (either directly or in a feedback message) is used to update its local distribution $p_t(\cdot, v)$. To simplify the analysis, updates are postponed, i.e., updates made at time t involve only losses generated at time $t - n - 1$. This means that agents may have to store feedback messages for up to $n + 1$ time steps before using them to perform updates.

The online protocol can be written as follows.

At each round $t = 1, 2, \ldots$

1. Each agent v is independently activated with probability $q(v)$;
2. Each active agent v draws an action $I_t(v)$ from K according to its current distribution $p_t(\cdot, v)$, is charged the corresponding loss $\ell_t\big(I_t(v)\big)$, and observes the set of losses $\mathcal{L}_t(v) = \big\{\big(i, \ell_t(i)\big) : i \in \mathcal{N}_1^F\big(I_t(v)\big)\big\}$
3. Each agent v broadcasts to all agents $u \in \mathcal{N}_n^N(v)$ the feedback message $\big(t,\, v,\, \mathcal{L}_t(v),\, p_t(\cdot, v)\big)$, where $\mathcal{L}_t(v) = \varnothing$ if $v \notin \mathcal{A}_t$
4. Each agent v receives the feedback message $\big(t - s,\, u,\, \mathcal{L}_{t-s}(u),\, p_{t-s}(\cdot, u)\big)$ from each agent u such that $\delta_N(v, u) = s$, for all $s \in [n]$

Similarly to Cesa-Bianchi et al. (2019), we assume the feedback message sent out by an agent v at time t contains the distribution $p_t(\cdot, v)$ used by the agent to draw actions at time t. This is needed to compute the importance-weighted estimates of the losses, $b_t(i, v)$, see (3).

The goal is to minimize the *network regret*

$$R_T = \max_{i \in K} \mathbb{E}\left[\sum_{t=1}^{T} \sum_{v \in \mathcal{A}_t} \ell_t\big(I_t(v)\big) - \sum_{t=1}^{T} |\mathcal{A}_t|\, \ell_t(i)\right] \tag{2}$$

where the expectation is taken with respect to the activations of the agents and the internal randomization of the strategies drawing the actions $I_t(v)$. Since the active agents $\mathcal{A}_t$ are chosen i.i.d. from a fixed distribution, we also consider the average regret

$$\frac{R_T}{Q} = \frac{1}{Q}\mathbb{E}\left[\sum_{t=1}^{T} \sum_{v \in \mathcal{A}_t} \ell_t\big(I_t(v)\big)\right] - \min_{i \in K} \sum_{t=1}^{T} \ell_t(i)$$

where $Q = \mathbb{E}\big[|\mathcal{A}_t|\big] > 0$ for all t.

[2] We assume without loss of generality that $q(v) \neq 0$ for all agents $v \in A$. The definition of regret (2) and all subsequent results could be restated equivalently in terms of the restriction $N' = (A', E'_N)$ of the communication network N, where $A' = \{v \in A : q(v) > 0\}$ and for all $u, v \in A'$, $(u, v) \in E'_N$ if and only if $(u, v) \in E'_N$.

[3] This assumption can be relaxed straightforwardly by assuming that each agent v only knows $q(v)$, which can then be broadcast to the n-neighborhood of v as the process unfolds.

In our setting, each agent locally runs an instance of the same online algorithm. We do not require any ad-hoc interface between each local instance and the rest of the network. In particular, we make the following assumption (Cesa-Bianchi et al., 2020).

Assumption 1 (Oblivious network interface)**.** *An online algorithm* Alg *is run with an* oblivious network interface *if:*

1. *Each agent v locally runs a local instance of* Alg
2. *All local instances use the same initialization and the same strategy for updating the learning rates*
3. *All local instances make updates while being oblivious to whether or not their host node v was active and when*

This assumption implies that each agent's instance is oblivious to both the network topology and the location of the agent in the network. Its purpose is to show that communication improves learning rates even without any network-specific tuning. In concrete applications, one might use ad-hoc variants that rely on the knowledge of the task at hand, and decrease the regret even further. However, the lower bound we prove in Theorem 2 shows that in most instances the regret cannot be greatly decreased even when agents have full knowledge of the graph.

4 Upper bound

In this section, we introduce Exp3-α^2 (Algorithm 1), an extension of the Exp3-Coop algorithm by Cesa-Bianchi et al. (2019), and analyze its network regret when run with an oblivious network interface.

Algorithm 1: Exp3-α^2 (Locally run by each agent $v \in A$)

input: learning rates $\eta_1(v), \eta_2(v) \dots$
for $t = 1, 2, \dots, n+1$ **do**
  if v is active in this round, draw $I_t(v)$ from K uniformly at random
for $t \geq n+2$ **do**
  if v is active in this round, draw $I_t(v)$ from K according to $p_t(\cdot, v)$ in (3)

An instance of Exp3-α^2 is locally run by each agent $v \in A$. The algorithm is parameterized by its (variable) learning rates $\eta_1(v), \eta_2(v), \dots$, which, in principle, can be arbitrary (measurable) functions of the history. In all rounds t in which the agent is active, v draws an action $I_t(v)$ according to a distribution $p_t(\cdot, v)$. For the first $n+1$ rounds t, $p_t(\cdot, v)$ is the uniform distribution over K. During all remaining time steps t, the algorithm computes exponential weights using all the available feedback generated up to (and including) round $t-n-1$. More precisely, for any action $i \in K$,

$$
\begin{aligned}
p_t(i,v) &= w_t(i,v) / \left\| w_t(\cdot,v) \right\|_1 \\
w_t(i,v) &= \exp\big(-\eta_t(v) \textstyle\sum_{s=1}^{t-n-1} \widehat{\ell}_s(i,v)\big) \\
\widehat{\ell}_s(i,v) &= \ell_s(i) B_s(i,v) / b_s(i,v) \\
B_s(i,v) &= \mathbb{I}\big\{\exists u \in \mathcal{N}_n^N(v) : u \in \mathcal{A}_s, I_s(u) \in \mathcal{N}_1^F(i)\big\} \\
b_s(i,v) &= 1 - \textstyle\prod_{u \in \mathcal{N}_n^N(v)} \big(1 - q(u) \sum_{j \in \mathcal{N}_1^F(i)} p_s(j,u)\big)
\end{aligned}
\tag{3}
$$

The event $B_s(i,v)$ indicates whether an agent in the n-neighborhood of v played at time s an action in the 1-neighborhood of i. If $B_s(i,v)$ occurs, then agent v can use $\widehat{\ell}_s(i,v)$ to update the local estimate for the cumulative loss of i. Note that $\widehat{\ell}_s(i,v)$ are proper importance-weighted estimates, as $\mathbb{E}_{s-n}\big[\widehat{\ell}_s(i,v)\big] = \ell_s(i)$ for all $v \in A$, $i \in K$, and $s > n$. The notation $\mathbb{E}_{s-n}$ denotes conditioning with respect to any randomness in rounds $1, \dots, s-n-1$. Note also that when $q(u) = 1$ for all $u \in A$ and F is the edgeless graph, the probabilities $p_t(i,v)$ in (3) correspond to those computed by Exp3-Coop (Cesa-Bianchi et al., 2019).

Before analyzing our cooperative implementation of Exp3-α^2, we present a key graph-theoretic result that helps us characterize the joint impact on the regret of the communication network and the feedback graph.

Our new result relates the variance of the estimates of eq. (3) to the structure of the communication graph given by the strong product of N^n and F.

Lemma 1. *Let $N = (A, E_N)$ and $F = (K, E_F)$ be any two graphs, $n \geq 0$, $\big(q(v)\big)_{v\in A}$ a set of numbers in $(0, 1]$, $Q = \sum_{v\in A} q(v)$, and $\big(p(i, v)\big)_{i\in K, v\in A}$ a set of numbers in $(0, 1]$ such that $\sum_{i\in K} p(i, v) = 1$ for all $v \in A$. Then,*

$$\sum_{v\in A}\sum_{i\in K} \frac{q(v)p(i,v)}{1-\prod_{u\in\mathcal{N}_n^N(v)}\big(1-q(u)\sum_{j\in\mathcal{N}_1^F(i)}p(j,u)\big)} \leq \frac{1}{1-e^{-1}}\left(\alpha\big(N^n \boxtimes F\big)+Q\right)$$

Proof. Let $\boldsymbol{w} = \big(w(i, v)\big)_{(i,v)\in K\times A}$ where $w(i, v) = q(v)p(i, v)$, and for all $(i, v) \in K \times A$ set $W(i, v) = \sum_{(j,u)\in\mathcal{N}_1^F(i)\times\mathcal{N}_n^N(v)} w(j, u)$. Define also, for all $\boldsymbol{c} = \big(c(j, u)\big)_{(j,u)\in K\times A} \in [0, 1]^{|K|\times|A|}$ and $(i, v) \in K \times A$

$$f_{\boldsymbol{c}}(i,v) = 1 - \prod_{u\in\mathcal{N}_n^N(v)}\left(1-\sum_{j\in\mathcal{N}_1^F(i)} c(j,u)\right)$$

Then we can write the left-hand side of the statement of the lemma as

$$\sum_{(i,v)\in K\times A}\frac{w(i,v)}{f_{\boldsymbol{w}}(i,v)} = \underbrace{\sum_{(i,v)\in K\times A\,:\,W(i,v)<1}\frac{w(i,v)}{f_{\boldsymbol{w}}(i,v)}}_{\text{(I)}} + \underbrace{\sum_{(i,v)\in K\times A\,:\,W(i,v)\geq 1}\frac{w(i,v)}{f_{\boldsymbol{w}}(i,v)}}_{\text{(II)}}$$

and proceed by upper bounding the two terms (I) and (II) separately. For the first term (I), using the inequality $1 - x \leq e^{-x}$ (for all $x \in \mathbb{R}$) with $x = w(j, u)$, we can write, for any $(i, v) \in K \times A$,

$$f_{\boldsymbol{w}}(i,v) \geq 1-\exp\left(-\sum_{u\in\mathcal{N}_n^N(v)}\sum_{j\in\mathcal{N}_1^F(i)} w(j,u)\right) = 1-\exp\big(-W(i,v)\big)$$

Now, since in (I) we are only summing over $(i, v) \in K \times A$ such that $W(i, v) < 1$, we can use the inequality $1 - e^{-x} \geq (1 - e^{-1})\, x$ (for all $x \in [0, 1]$) with $x = W(i, v)$, obtaining $f_{\boldsymbol{w}}(i, v) \geq (1 - e^{-1})W(i, v)$, and in turn

$$\text{(I)} \leq \sum_{(i,v)\in K\times A\,:\,W(i,v)<1}\frac{w(i,v)}{(1-e^{-1})W(i,v)} \leq \frac{1}{1-e^{-1}}\sum_{(i,v)\in K\times A}\frac{w(i,v)}{W(i,v)} \leq \frac{\alpha\big(N^n\boxtimes F\big)}{1-e^{-1}}$$

where in the last step we used a known graph-theoretic result—see Lemma 3 in the Supplementary Material. For the second term (II): for all $v \in A$, let $r(v)$ be the cardinality of $\mathcal{N}_n^N(v)$. Then, for any $(i, v) \in K \times A$ such that $W(i, v) \geq 1$,

$$\begin{aligned}
1-f_{\boldsymbol{w}}(i,v) &\leq \max\left\{1-f_{\boldsymbol{c}}(i,v)\;:\;\boldsymbol{c}\in[0,1]^{|K|\times|A|},\;\sum_{(j,u)\in\mathcal{N}_1^F(i)\times\mathcal{N}_n^N(v)} c(j,u)\geq 1\right\}\\
&= \max\left\{\prod_{u\in\mathcal{N}_n^N(v)}\left(1-\sum_{j\in\mathcal{N}_1^F(i)} c(j,u)\right)\;:\;\boldsymbol{c}\in[0,1]^{|K|\times|A|},\;\sum_{u\in\mathcal{N}_n^N(v)}\sum_{j\in\mathcal{N}_1^F(i)} c(j,u)=1\right\}\\
&\leq \max\left\{\prod_{u\in\mathcal{N}_n^N(v)}\big(1-C(u)\big)\;:\;\boldsymbol{C}\in[0,1]^{|A|},\;\sum_{u\in\mathcal{N}_n^N(v)} C(u)=1\right\}\\
&= \max\left\{\prod_{u\in\mathcal{N}_n^N(v)}\big(1-C(u)\big)\;:\;\boldsymbol{C}\in[0,1]^{|A|},\;\sum_{u\in\mathcal{N}_n^N(v)}\big(1-C(u)\big)=r(v)-1\right\}\\
&\leq \left(1-\frac{1}{r(v)}\right)^{r(v)} \leq e^{-1}
\end{aligned}$$

where the first equality follows from the definition of $f_{\boldsymbol{c}}(i,v)$ and the monotonicity of $x \mapsto 1-x$, the second-to-last inequality is implied by the AM-GM inequality (Lemma 4), and the last one comes from $r(v) \geq 1$ (being $v \in \mathcal{N}_n^N(v)$). Hence

$$\begin{aligned}(\mathrm{II}) &= \sum_{(i,v)\in K\times A\,:\,W(i,v)\geq 1} \frac{w(i,v)}{f_{\boldsymbol{w}}(i,v)} \leq \sum_{(i,v)\in K\times A\,:\,W(i,v)\geq 1} \frac{w(i,v)}{1-e^{-1}} \\ &\leq \frac{1}{1-e^{-1}} \sum_{i\in K}\sum_{v\in A} w(i,v) = \frac{1}{1-e^{-1}} \sum_{v\in A} q(v) \sum_{i\in K} p(i,v) = \frac{Q}{1-e^{-1}}\end{aligned}$$

□

For a slightly stronger version of this result, see Lemma 6 in the Supplementary Material. By virtue of Lemma 1, we can now show the main result of this section.

Theorem 1. *If each agent $v \in A$ uses adaptive learning rates equal to $\eta_t(v) = 0$ for $t \leq n+1$, $\eta_t(v) = \sqrt{\log(K)/\sum_{s=1}^t X_s(v)}$ with $X_t(v) = n + \sum_{i\in K} \frac{p_t(i,v)}{b_t(i,v)}$ for $t > n+1$, the average network regret of* Exp3-α^2 *playing with an oblivious network interface can be bounded as*

$$\frac{R_T}{Q} \overset{\widetilde{\mathcal{O}}}{=} \sqrt{\log(K)\left(n+1+\frac{\alpha\left(N^n \boxtimes F\right)}{Q}\right)T}\,. \tag{4}$$

Proof. Let $i^\star \in \operatorname{argmin}_{i\in K} \mathbb{E}\big[\sum_{t=1}^T |\mathcal{A}_t|\,\ell_t(i)\big]$ where the expectation is with respect to the random sequence $\mathcal{A}_t \subseteq A$ of agent activations. We write the network regret R_T as a weighted sum of single agent regrets $R_T(v)$:

$$R_T = \sum_{v\in A} q(v) R_T(v) = \sum_{v\in A} q(v)\mathbb{E}\left[\sum_{t=1}^T \sum_{i\in K} \widehat{\ell}_t(i,v)p_t(i,v) - \widehat{\ell}_t(i^\star,v)\right],$$

where the expectation is now only with respect to the random draw of the agents' actions, and it is separated from the activation probability $q(v)$. Fix any agent $v \in A$. Exp3-α^2 plays uniformly for the first $n+1$ rounds, and each agent, therefore, incurs a linear regret in this phase. For $t > n+1$ we borrow a decomposition technique from (Gyorgy and Joulani, 2021), for any sequence $\big(\widetilde{p}_t(\cdot,v)\big)_{t>n+1}$ of distributions over K, the above expectation can be written as

$$\mathbb{E}\left[\sum_{t=n+2}^T \sum_{i\in K} \widehat{\ell}_t(i,v)\widetilde{p}_{t+1}(i,v) - \widehat{\ell}_t(i^\star,v)\right] + \sum_{t=n+2}^T \mathbb{E}\left[\sum_{i\in K} \widehat{\ell}_t(i,v)p_t(i,v)\left(1-\frac{\widetilde{p}_{t+1}(i,v)}{p_t(i,v)}\right)\right]. \tag{5}$$

Take now $\widetilde{p}_t(\cdot,v)$ as the (full-information) exponential-weights updates with non-increasing step-sizes $\eta_{t-1}(v)$ for the sequence of losses $\widehat{\ell}_t(\cdot,v)$. That is, $\widetilde{p}_1(\cdot,v)$ is the uniform distribution over K, and for any time step t and action $i \in K$, $\widetilde{p}_{t+1}(i,v) = \widetilde{w}_{t+1}(i,v)/\left\|\widetilde{w}_{t+1}(\cdot,v)\right\|_1$, where $\widetilde{w}_{t+1}(i,v) = \exp\big(-\eta_t(v)\sum_{s=1}^t \widehat{\ell}_s(i,v)\big)$. With this choice, the first term in (5) is the "look-ahead" regret for the iterates $\widetilde{p}_{t+1}(\cdot,v)$ (which depend on $\widehat{\ell}_t(\cdot,v)$ at time t), while the second one measures the drift of $p_t(\cdot,v)$ from $\widetilde{p}_{t+1}(\cdot,v)$.

Using an argument from (Joulani et al., 2020, Theorem 3),[4] we deterministically bound the first term in (5):

$$\sum_{t=n+2}^T \sum_{i\in K} \widehat{\ell}_t(i,v)\widetilde{p}_{t+1}(i,v) - \widehat{\ell}_t(i^\star,v) \leq \frac{\ln|K|}{\eta_T(v)}\,. \tag{6}$$

The subtle part is now to control the second term in (5). To do so, fix any $t > n+1$. Note that for all $i \in K$,

$$w_t(i,v) = \exp\left(-\eta_t(v)\sum_{s=1}^{t-n-1} \widehat{\ell}_s(i,v)\right) \geq \exp\left(-\eta_t(v)\sum_{s=1}^{t} \widehat{\ell}_s(i,v)\right) = \widetilde{w}_{t+1}(i,v)$$

[4]We use (Joulani et al., 2020, Theorem 3) with $p_t = 0$ for all $t \in [T]$, $r_0 = (1/\eta_0(v))\sum_i p_i \ln(p_i)$, $r_t(p) = (1/\eta_t(v) - 1/\eta_{t-1}(v))\sum_i p_i \ln(p_i)$ for all $t \in [T]$, and dropping the Bregman-divergence terms due to the convexity of r_t.

(using $\ell_s(i,v) \geq 0$ for all s,i,v), which in turn, using the inequality $e^x \geq 1+x$ (for all $x \in \mathbb{R}$), yields

$$\frac{\widetilde{p}_{t+1}(i,v)}{p_t(i,v)} \geq \frac{\widetilde{w}_{t+1}(i,v)}{w_t(i,v)} = \exp\left(-\eta_t(v)\sum_{s=t-n}^{t}\widehat{\ell}_s(i,v)\right) \geq 1-\eta_t(v)\sum_{s=t-n}^{t}\widehat{\ell}_s(i,v)$$

Thus, we upper bound the second expectation in (5) by

$$\sum_{i\in K}\mathbb{E}\left[\eta_t(v)\,\widehat{\ell}_t(i,v)p_t(i,v)\sum_{s=t-n}^{t-1}\widehat{\ell}_s(i,v)\right] + \sum_{i\in K}\mathbb{E}\left[\eta_t(v)\,\widehat{\ell}_t(i,v)^2 p_t(i,v)\right] =: g_t^{(1)}(v) + g_t^{(2)}(v) \tag{7}$$

We study the two terms $g_t^{(1)}(v)$ and $g_t^{(2)}(v)$ separately. Let then $\mathcal{H}_t = \mathcal{H}_t(v)$ be the σ-algebra generated by the activations of agents and the actions drawn by them at times $1,\dots,t-1$, and let also indicate $\mathbb{E}_t = \mathbb{E}[\cdot \mid \mathcal{H}_t]$.

First, we bound $g_t^{(1)}(v)$ in (7) using the fact that $p_t(\cdot,v)$, $\eta_t(v)$ and $b_s(\cdot,v)$ (for all $s \in \{t-n,\dots,t\}$) are determined by the randomness in steps $1,\dots,t-n-1$. We use the tower rule and take the conditional expectation inside since all quantities apart from $B_{t-n}(i,v),\dots,B_t(i,v)$ are determined given $\mathcal{H}_{t-n}$, we rewrite the expression as

$$g_t^{(1)}(v) = \mathbb{E}\left[\sum_{i\in K}\eta_t(v)p_t(i,v)\sum_{s=t-n}^{t-1}\frac{\ell_t(i)}{b_t(i,v)}\frac{\ell_s(i)}{b_s(i,v)}\mathbb{E}_{t-n}\big[B_t(i,v)B_s(i,v)\big]\right].$$

Conditional on $\mathcal{H}_{t-n}$ the Bernoulli random variables $B_s(i,v)$, and $B_t(i,v)$ for $s=t-n,\dots,t$ are independent. This follows because the feedbacks at time s are missing at time t for $s=t-n,\dots,t-1$, and therefore, from the independent activation of agents and the fact that the only other source of randomness is the independent internal randomization of the algorithm, they are independent random variables, implying

$$\mathbb{E}_{t-n}\big[B_t(i,v)B_s(i,v)\big] = b_t(i,v)b_s(i,v)$$

for $s=t-n,\dots,t-1$. Using $\ell_t(i),\ell_s(i) \leq 1$, we then get

$$g_t^{(1)}(v) \leq \mathbb{E}\left[\sum_{i\in K}\eta_t(v)p_t(i,v)n\right] = \mathbb{E}[\eta_t(v)n].$$

With a similar argument, we also get

$$g_t^{(2)}(v) = \mathbb{E}\left[\sum_{i\in K}\frac{\eta_t(v)\ell_t(i)^2 p_t(i,v)}{b_t(i,v)^2}\mathbb{E}_{t-n}\Big[B_t(i,v)\Big]\right] \leq \mathbb{E}\left[\sum_{i\in K}\eta_t(v)\frac{p_t(i,v)}{b_t(i,v)}\right]$$

Finally, the single agent regret for each agent $v \in A$ is bounded by

$$\begin{aligned} R_T(v) &\leq \mathbb{E}\left[\frac{\ln|K|}{\eta_T(v)}\right] + (n+1) + \sum_{t=n+2}^{T}\mathbb{E}\left[\eta_t(v)\left(n+\sum_{i\in K}\frac{p_t(i,v)}{b_t(i,v)}\right)\right] \\ &= \mathbb{E}\left[\frac{\ln|K|}{\eta_T(v)}\right] + (n+1) + \sum_{t=n+2}^{T}\mathbb{E}\left[\eta_t(v)X_t(v)\right] \end{aligned}$$

where, in the second line, we defined $X_t(v) = \mathbb{I}\{t > n+1\}\left(n + \sum_{i\in K}\frac{p_t(i,v)}{b_t(i,v)}\right)$.

We now take $\eta_t(v) = \sqrt{\log(K)\big/\sum_{s=1}^{t}X_s(v)}$, and we use a standard inequality stating that for any $a_t > 0$, $\sum_{t=1}^{T}a_t\Big/\sqrt{\sum_{s=1}^{t}a_s} \leq 2\sqrt{\sum_{t=1}^{T}a_t}$. Applying this inequality for a_t equal to $X_t(v)$ we have

$$R_T(v) \leq (n+1) + \mathbb{E}\left[\sqrt{\log(K)\sum_{s=1}^{T}X_s(v)}\right] + \mathbb{E}\left[\sum_{t=1}^{T}\frac{\sqrt{\log(K)}X_t(v)}{\sqrt{\sum_{s=1}^{t}X_s(v)}}\right]$$

$$\leq (n+1) + 3\mathbb{E}\left[\sqrt{\log(K)\sum_{t=1}^{T} X_t(v)}\right]$$

Multiplying by $q(v)$, summing over agents $v \in A$ we obtain

$$\begin{aligned}
R_T = \sum_v q(v) R_T(v) &= Q(n+1) + 3Q\frac{\sum q(v)}{Q}\sqrt{\sum_{t=1}^{T}\log(K)\left(n + \sum_{i\in K}\frac{p_t(i,v)}{b_t(i,v)}\right)} \\
&\leq Q(n+1) + 3Q\sqrt{\frac{\log(K)}{Q}\sum_{t=1}^{T}\left(nQ + \sum_v\sum_{i\in K}\frac{p_t(i,v)q(v)}{b_t(i,v)}\right)} \\
&\leq Q(n+1) + 3Q\sqrt{\log(K)\left(n+1+\frac{\alpha\left(N^n \boxtimes F\right)}{Q(1-e^{-1})}\right)T}
\end{aligned}$$

where the first inequality follows from Jensen's inequality and the second from Lemma 1. □

Note that in Theorem 1, every agent tunes the learning rate $\eta_t(v)$ using available information at time t. This allows the network regret to adapt to the unknown parameters of the problems such as the time horizon T, the independence number $\alpha\big(N^n \boxtimes F\big)$, and the total activation mass Q on A. This approach improves over the doubling trick approach introduced in Cesa-Bianchi et al. (2019) since we do not need to restart the algorithm.

5 Lower bound

In this section, we prove that not only the upper bound in Theorem 1 is optimal in a *minimax* sense (i.e., that it is attained for some pairs of graphs (N, F)), but it is also tight in an *instance-dependent* sense, for *all* pairs of graphs belonging to a large class.

Definition 1. *Let $\mathscr{G}$ be the class of all pairs of graphs (N, F) such that $\alpha(N \boxtimes F) = \alpha(N)\alpha(F)$.*

Many sufficient conditions guaranteeing that $(N, F) \in \mathscr{G}$ are known in the graph theory literature: see, e.g., Hales (1973, Section 3), Acín et al. (2017, Theorem 6), and Rosenfeld (1967, Theorem 2). To the best of our knowledge, a full characterization of $\mathscr{G}$ is still a challenging open problem in graph theory that goes beyond the scope of this paper. It is easy to verify that if (either N or) F is a clique or an edgeless graph, then $(N, F) \in \mathscr{G}$. We remark that these instances cover in particular both the bandit and the full-info case that were previously only studied individually, and analyzed with *ad hoc* techniques. For some further discussion on $\mathscr{G}$, we refer the interest reader to Appendix B.1.

The proof of the lower bound in Theorem 2 exploits a reduction to a setting we introduce in Lemma 2. In this lemma, we state that in a single-agent setting with a feedback graph, if each one of T time step is independently skipped with a known and constant probability q, the learner's regret is $\Omega\big(\sqrt{\alpha(F)qT}\big)$. In particular, skipped rounds do to count towards regret.

Lemma 2. *For any feedback graph F, for any online learning algorithm, for any $q > 0$, and for any $T \geq \max\big\{0.0064 \cdot \alpha(F)^3, \frac{1}{q^3}\big\}$, there exists an assignment of T losses $\ell_1, \dots, \ell_T$ such that, if each round is independently skipped with probability q, then the regret of the algorithm satisfies*

$$R_T(q) \overset{\Omega}{=} \sqrt{\alpha(F)qT}\ .$$

The proof of this lemma can be found in Appendix A. We can now prove our lower bound.

Theorem 2. *For any choice of n, any pair of graphs $(N^n, F) \in \mathscr{G}$, and all $Q \in (0, \alpha_n(N)]$, we have that for $T \geq \max\left\{0.0064 \cdot \alpha(F)^3, \alpha(N)^3/Q^3\right\}$ the following lower bound holds true*

$$\inf\sup R_T \overset{\Omega}{=} \sqrt{Q\alpha\big(N^n \boxtimes F\big)T}$$

where the inf *is over all player's strategies, the* sup *is over all assignments of losses $\ell_1, \dots, \ell_T$ and activation probabilities $\big(q(v)\big)_{v\in A}$ such that $Q = \sum_{v\in A} q(v)$.*

Proof. Fix any (N, n, F) and $Q \in (0, \alpha_n(N)]$ as in the statement of the theorem. Then $\alpha\big(N^n \boxtimes F\big) = \alpha_n(N)\alpha(F)$. Let $\mathcal{I} \subset A$ be a set of $\alpha_n(N)$ agents such that $\delta_N(u, v) > n$ for all $u, v \in \mathcal{I}$. Define the activation probabilities $q(v) = Q/\alpha_n(N) \le 1$ for all $v \in \mathcal{I}$ and $q(v) = 0$ for all $v \in A \setminus \mathcal{I}$. Let also $T_v = \big\{t \in [T] : v \in \mathcal{A}_t\big\}$ for all $v \in A$, and note that the expected number of rounds $\mathbb{E}\big[|T_v|\big]$ that each agent v is activated is $(Q/\alpha_n(N))T$ if $v \in \mathcal{I}$, zero otherwise. By construction, no communication occurs among agents in $\mathcal{I}$. Furthermore, since each agent in $\mathcal{I}$ is activated independently with probability $q(v) = Q/\alpha_n(N)$, we can use the result of Lemma 2, which states that for $T \ge \max\big\{0.0064 \cdot \alpha(F)^3, \alpha_n(N)^3/Q^3\big\}$ we have

$$\begin{aligned}\inf \sup R_T &\ge \inf \sup_{\boldsymbol{\ell}} \max_{i \in K} \mathbb{E}\left[\sum_{v \in \mathcal{I}} \sum_{t \in T_v} \Big(\ell_t\big(I_t(v)\big) - \ell_t(i)\Big)\right] \\ &\overset{\Omega}{=} \alpha_n(N)\sqrt{\alpha(F)(Q/\alpha_n(N))T} = \sqrt{Q\big(\alpha(F)\alpha_n(N)\big)T} = \sqrt{Q\alpha\big(N^n \boxtimes F\big)T}\end{aligned}$$

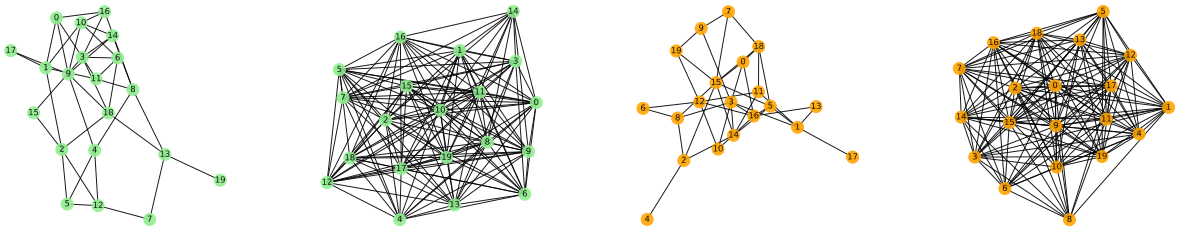
where the second supremum is only over losses $\ell_1, \dots, \ell_T$ (for our choice of activation probabilities) and in the next step we invoked the lower bound of Lemma 2 for $|\mathcal{I}| = \alpha_n(N)$ independent instances v of single-agent learning with a feedback graph $G = F$. □

6 Experiments

To empirically appreciate the impact of cooperation, we run a number of experiments on synthetic data.

For each choice of N and F, we compare Exp3-α^2run on N and F against a baseline which runs Exp3-α^2 on N' and F, where N' is an edgeless communication graph. Hence, the baseline runs A independent instances on the same feedback graph.

In our experiments, we fix the time horizon ($T = 10{,}000$), the number of arms ($K = 20$), and the number of agents ($A = 20$). We also set the delay δ_N to 1. The loss of each action is a Bernoulli random variable of parameter $1/2$, except for the optimal action which has parameter $1/2 - \sqrt{K/T}$. The activation probabilities $q(v)$ are the same for all agents $v \in A$, and range in the set $\{0.05, 0.5, 1\}$. This implies that $Q \in \{1, 10, 20\}$. The feedback graph F and the communication graph N are Erdős–Rényi random graphs of parameters p_N, $p_F \in \{0.2, 0.8\}$. For each choice of the parameters, the same realization of N and F was kept fixed in all the experiments, see Figure 1.



(a) (b) (c) (d)

Figure 1: The random instances of N (leftmost graphs) and F (rightmost graphs) used in our experiments. The sparse graphs are Erdős–Rényi of parameter 0.2, the dense graphs are Erdős–Rényi of parameter 0.8.

In each experiment, Exp3-α^2 and our baseline are run on the same realization of losses and agent activations. Hence, the only stochasticity left is the internal randomization of the algorithms. Our results are averages of 20 repetitions of each experiment with respect to this randomization.

Figure 2 summarizes the results of our experiments in terms of the average regret R_T/Q. See Appendix D for the actual learning curves. Recall that our upper bound (1) scales with the quantity $\sqrt{\alpha(N \boxtimes F)/Q}$.

- Note that our algorithm (blue dots) is never worse than the baseline (red dots). This is consistent with the fact that N for the baseline is the edgeless graph, implying that $\alpha(N \boxtimes F) = A\,\alpha(F)$.

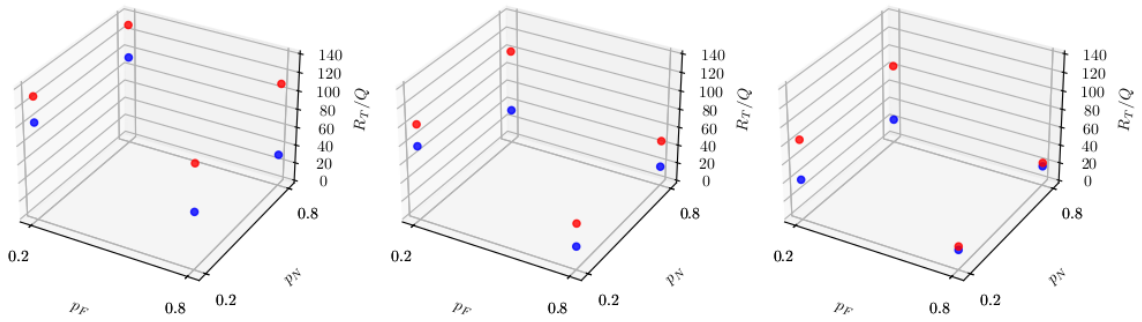


Figure 2: Average regret of Exp3-α^2 (blue dots) against the baseline (red dots). The X-axis and the Y-axis correspond to the parameters p_F and p_N of the Erdős–Rényi graph, the Z-axis is the average regret R_T/Q. The three plots correspond to increasing values (from left to right) of activation probability: $q = 0.05$ (leftmost plot), $q = 0.5$ (central plot), $q = 1$ (rightmost plot).

- Consistently with (1), the average performance gets worse when $Q \to 1$.[5]
- By construction, the performance of the baseline in each plot remains constant when p_N varies in $\{0.2, 0.8\}$. On the other hand, our algorithm is worse when N is sparse because $\alpha(N \boxtimes F)$ increases.
- The performance of both algorithms is worse when F is sparse because, once more, $\alpha(N \boxtimes F)$ increases.

7 Conclusions

In this work, we nearly-characterize the minimax regret in cooperative online learning with feedback graphs, showing that the dependence on $\alpha\big(N^n \boxtimes F\big)$ in our bounds is tight in all but a few, pathological instances. In a bandit setting, when all agents are active at all time steps, previous works showed that communication speeds up learning by reducing the variance of loss estimates. On the opposite end of the spectrum, in full-information settings, updating non-active agents was shown to improve regret. These results left open the question of which updates would help in intermediate settings and why. In this paper, we prove that both types of updates help local learners across the entire experts-bandits spectrum (Theorem 1). We stress that this strategy crucially depends on the stochasticity of the activations. Indeed, Cesa-Bianchi et al. (2020) disproved the naive intuition that more information automatically translates into better bounds, showing how using all the available data can lead to linear regret in the case of adversarial activations with oblivious network interface.

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[5]Also the baseline, whose agents learn in isolation, gets worse when Q decreases. Indeed, when $Q = 1$ agents get only to play for $T/|A|$ time steps each, and together achieve a network regret R_T that scales with $\sqrt{|A|\alpha(F)}$, as predicted by our analysis.

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