Smoothed Online Convex Optimization Based on Discounted-Normal-Predictor

Lijun Zhang[1,2], **Wei Jiang**[1], **Jinfeng Yi**[3], **Tianbao Yang**[4]
[1]National Key Laboratory for Novel Software Technology, Nanjing University, Nanjing, China
[2]Peng Cheng Laboratory, Shenzhen 518055, China
[3]JD AI Research, Beijing, China
[4]Department of Computer Science and Engineering, Texas A&M University, College Station, USA
{zhanglj, jiangw}@lamda.nju.edu.cn, yijinfeng@jd.com, tianbao-yang@tamu.edu

Abstract

In this paper, we investigate an online prediction strategy named as Discounted-Normal-Predictor [Kapralov and Panigrahy, 2010] for smoothed online convex optimization (SOCO), in which the learner needs to minimize not only the hitting cost but also the switching cost. In the setting of learning with expert advice, Daniely and Mansour [2019] demonstrate that Discounted-Normal-Predictor can be utilized to yield nearly optimal regret bounds over any interval, even in the presence of switching costs. Inspired by their results, we develop a simple algorithm for SOCO: *Combining online gradient descent (OGD) with different step sizes sequentially by Discounted-Normal-Predictor*. Despite its simplicity, we prove that it is able to minimize the adaptive regret with switching cost, i.e., attaining nearly optimal regret with switching cost on every interval. By exploiting the theoretical guarantee of OGD for dynamic regret, we further show that the proposed algorithm can minimize the dynamic regret with switching cost in every interval.

1 Introduction

Recently, a variant of online convex optimization (OCO), named as smoothed OCO (SOCO) has received lots of attention in the machine learning community [Goel et al., 2019, Li and Li, 2020]. In each round t, the learner chooses an action $\mathbf{w}_t$ from a convex domain $\mathcal{W}$, and a convex loss function $f_t(\cdot) : \mathcal{W} \mapsto \mathbb{R}$ is selected by an adversary. While in the traditional OCO, the learner only suffers a *hitting* cost $f_t(\mathbf{w}_t)$, in SOCO, it further pays a *switching* cost $m(\mathbf{w}_t, \mathbf{w}_{t-1})$, which penalizes the learner for changing its actions between rounds. SOCO has found wide applications in real-world problems where the change of states brings additional costs [Lin et al., 2012, Kim and Giannakis, 2014, Goel et al., 2017], such as the wear-and-tear cost of switching servers [Lin et al., 2011].

For general convex functions, a natural choice of the switching cost is the distance between successive actions, i.e., $m(\mathbf{w}_t, \mathbf{w}_{t-1}) = \|\mathbf{w}_t - \mathbf{w}_{t-1}\|$. Then, the total loss in the t-th round becomes

$$f_t(\mathbf{w}_t) + \lambda G \|\mathbf{w}_t - \mathbf{w}_{t-1}\| \tag{1}$$

where $\lambda \geq 0$ is the trade-off parameter, and G is the upper bound of the norm of gradients. Here, G is introduced to ensure that the two terms weighted by λ are on the same order. Following the convention of online learning [Cesa-Bianchi and Lugosi, 2006], we choose regret to measure the performance, and meanwhile take into account the switching cost. Let T be the total number of iterations. The standard metric would be *regret with switching cost*:

$$\sum_{t=1}^{T} \big(f_t(\mathbf{w}_t) + \lambda G \|\mathbf{w}_t - \mathbf{w}_{t-1}\| \big) - \min_{\mathbf{w} \in \mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w}) \tag{2}$$

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defined as the difference between the cumulative loss of the learner and that of the best fixed decision in hindsight. It is easy to verify that online gradient descent (OGD) [Zinkevich, 2003] attains an $O(\sqrt{(1+\lambda)T})$ bound for regret with switching cost (c.f. Theorem 6), which can be proved to be optimal [Gradu et al., 2020, Theorem 4].

However, regret is not suitable for changing environments in which the best decision may vary over time. To address this limitation, new performance measures, including adaptive regret and dynamic regret have been proposed [Zhang, 2020, Cesa-Bianchi and Orabona, 2021]. Adaptive regret enforces the algorithm to have a small regret over every interval [Hazan and Seshadhri, 2007, Daniely et al., 2015], which essentially measures the performance w.r.t. a changing comparator. By extending adaptive regret to SOCO, we obtain *adaptive regret with switching cost*:

$$\text{A-R-S}(T,\tau) = \max_{r \leq T+1-\tau} \text{R-S}(r, r+\tau-1) \tag{3}$$

where τ is the length of the interval, and

$$\text{R-S}(r,s) = \sum_{t=r}^{s} \big(f_t(\mathbf{w}_t) + \lambda G \|\mathbf{w}_t - \mathbf{w}_{t-1}\| \big) - \min_{\mathbf{w} \in \mathcal{W}} \sum_{t=r}^{s} f_t(\mathbf{w})$$

is the regret with switching cost over interval $[r,s]$. To cope with changing environments, dynamic regret directly compares the learner against a sequence of comparators $\mathbf{u}_1, \ldots, \mathbf{u}_T \in \mathcal{W}$ [Zinkevich, 2003]. Similarly, we can incorporate the switching cost into dynamic regret, and obtain *dynamic regret with switching cost*:

$$\sum_{t=1}^{T} \big(f_t(\mathbf{w}_t) + \lambda G \|\mathbf{w}_t - \mathbf{w}_{t-1}\| \big) - \sum_{t=1}^{T} f_t(\mathbf{u}_t). \tag{4}$$

In the literature, there are only two works that investigate adaptive regret with switching cost. However, one of them relies on strong convexity [Gradu et al., 2020], and the other one makes use of sophisticated techniques [Zhang et al., 2022]. There also exist preliminary studies on dynamic regret with switching cost, but are limited to the special case that $\lambda = 1/G$ [Zhang et al., 2021a] or follow different settings [Li et al., 2018, Goel et al., 2019]. Besides, all of previous works only target one performance measure. Notice that for OCO, we do have algorithms that are able to minimize adaptive regret and dynamic regret simultaneously [Zhang et al., 2020, Cutkosky, 2020].

In this paper, we develop a simple algorithm for SOCO based on Discounted-Normal-Predictor [Kapralov and Panigrahy, 2010, 2011], which is designed for online bit prediction and can be used to combine two experts. In a recent study, Daniely and Mansour [2019] reveal that Discounted-Normal-Predictor automatically controls the switching cost, and with the help of projection, can be extended to support adaptive regret. Motivated by their observations, we first analyze a variant of Discounted-Normal-Predictor with conservative updating (abbr. DNP-cu), and prove that it suffers a small loss on every interval, even in the presence of switching costs. To ensure adaptivity, we choose conservative updating instead of projection because the former one can be analyzed more easily. Then, we create multiple OGD with different step sizes, and combine them sequentially by DNP-cu. Theoretical analysis shows that the proposed method achieves an $O(\sqrt{(1+\lambda)\tau \log T})$ bound for adaptive regret with switching cost. Furthermore, based on the fact that OGD is also equipped with dynamic regret bounds, we establish nearly optimal guarantees for dynamic regret with switching cost in every interval. Specifically, for any interval $[r,s]$ with length τ, our method attains an $O(\sqrt{(1+\lambda)\tau(1+P_{r,s}) \log T})$ bound for dynamic regret with switching cost, where

$$P_{r,s} = \sum_{t=r}^{s} \|\mathbf{u}_t - \mathbf{u}_{t+1}\| \tag{5}$$

is the path-length of an arbitrary comparator sequence $\mathbf{u}_r, \ldots, \mathbf{u}_s \in \mathcal{W}$.

Compared with state-of-the-art results [Zhang et al., 2021a, 2022], this paper has the following advantages.

1. Our $O(\sqrt{(1+\lambda)\tau \log T})$ adaptive regret bound is on the same order as that of Zhang et al. [2022], but our method is more simple. In contrast, their algorithm pieces together several techniques, including online coin betting, geometric covering intervals, and the reduction from unconstrained online learning to constrained online learning.

2. Although Zhang et al. [2021a] establish an $O(\sqrt{T(1+P_{1,T})})$ dynamic regret bound with switching cost, they only consider the case that $\lambda = 1/G$ and the whole interval $[1, T]$. By comparison, our dynamic regret bound holds for any $\lambda \geq 0$ and any interval.
3. To the best of our knowledge, this is the first effort to minimize both adaptive regret and dynamic regret, under the setting of SOCO.

Finally, we would like to emphasize the strength of Discounted-Normal-Predictor, which, in our opinion, was not getting enough attention. The results of Daniely and Mansour [2019] and this paper demonstrate that Discounted-Normal-Predictor provides an elegant way (and a very different methodology) to minimize both adaptive regret and dynamic regret, with or without switching cost, under the setting of learning with expert advice (LEA) or OCO.

2 Related Work

We briefly review the related work on OCO and SOCO, as well as Discounted-Normal-Predictor.

2.1 Online Convex Optimization (OCO)

During the past decades, OCO has been extensively studied, and various algorithms have been proposed to minimize the regret, i.e., (2) with $\lambda = 0$ [Shalev-Shwartz, 2011, Hazan, 2016]. It is well-known that OGD [Zinkevich, 2003] achieves an $O(\sqrt{T})$ bound which is minimax optimal [Abernethy et al., 2008]. We can obtain tighter regret bounds if the loss function satisfies specifical curvature properties, such as strong convexity [Shalev-Shwartz et al., 2007], exponential concavity [Hazan et al., 2007, Orabona et al., 2012], and smoothness [Srebro et al., 2010, Chiang et al., 2012].

Adaptive regret has been explored under the setting of LEA [Littlestone and Warmuth, 1994, Freund et al., 1997, Adamskiy et al., 2012, György et al., 2012, Luo and Schapire, 2015] and OCO [Hazan and Seshadhri, 2007, 2009, Jun et al., 2017b, Zhang et al., 2019, 2021b, Wan et al., 2022]. We follow the definition of Daniely et al. [2015], which is given by (3) with $\lambda = 0$. For general convex functions, Jun et al. [2017a] establish an $O(\sqrt{\tau \log T})$ bound, which holds for any interval length τ.

Dynamic regret is proposed by Zinkevich [2003], and its definition can be found by setting $\lambda = 0$ in (4). Existing works have analyzed two types of dynamic regret: (i) the general version where the comparator sequence is arbitrary [Hall and Willett, 2013, Zhang et al., 2018b, Zhao et al., 2020a, Baby and Wang, 2021], and (ii) the worst-case version where $\mathbf{u}_t$ is chosen as the minimizer of $f_t(\cdot)$ [Jadbabaie et al., 2015, Besbes et al., 2015, Yang et al., 2016, Mokhtari et al., 2016, Zhang et al., 2017, Zhao and Zhang, 2021]. For general convex functions, Ader attains an optimal $O(\sqrt{T(1+P_{1,T})})$ bound, where $P_{1,T}$ is the path-length of $\mathbf{u}_1, \ldots, \mathbf{u}_T$ [Zhang et al., 2018b].

Both adaptive regret and dynamic regret are designed for changing environments, but our understanding of their relationship is quite limited. Currently, we know that it is possible to bound the worst-case dynamic regret by adaptive regret [Zhang et al., 2018c], and minimize adaptive regret and the general dynamic regret simultaneously [Zhang et al., 2020, Cutkosky, 2020].

2.2 Smoothed Online Convex Optimization (SOCO)

The research of SOCO is motivated by real-world applications where the switching cost plays a crucial role [Lin et al., 2011, 2012]. Besides regret, competitive ratio is another popular metric for SOCO, and a large number of algorithms have been proposed to yield dimension-free competitive ratio. However, all of them are limited to the *lookahead* setting where the learner can observe the hitting cost $f_t(\cdot)$ before deciding its action $\mathbf{w}_t$, and rely on strong conditions [Bansal et al., 2015, Antoniadis and Schewior, 2018, Chen et al., 2018, Goel et al., 2019, Goel and Wierman, 2019, Argue et al., 2020a]. We note that SOCO is closely related to convex body chasing (CBC) [Friedman and Linial, 1993, Antoniadis et al., 2016, Bansal et al., 2018, Argue et al., 2019, Bubeck et al., 2019, 2020, Argue et al., 2020b, Sellke, 2020]. For more details, please refer to Lin et al. [2020].

In the study of online control, Gradu et al. [2020] and Zhang et al. [2022] have developed adaptive algorithms for OCO with memory, which can be applied to bounding adaptive regret with switching cost. However, Gradu et al. [2020] need to assume the hitting cost is strongly convex. Although Zhang et al. [2022] successfully deliver an $O(\sqrt{(1+\lambda)\tau \log T})$ bound, their method is rather complex.

Algorithm 1 Discounted-Normal-Predictor

Require: Two parameters: n and Z

Set $x_1 = 0$, and $\rho = 1 - 1/n$
for $t = 1, \ldots, T$ **do**
  Predict $g(x_t)$
  Receive b_t
  Set $x_{t+1} = \rho x_t + b_t$
end for

Specifically, they make use of online coin betting [Orabona and Pál, 2016], geometric covering intervals [Daniely et al., 2015], and the reduction from unconstrained online linear optimization to constrained online linear optimization [Cutkosky and Orabona, 2018].

For dynamic regret with switching cost, Zhang et al. [2021a] extend Ader to support switching costs, and prove an optimal $O(\sqrt{T(1+P_{1,T})})$ bound, but they only consider the case that $\lambda = 1/G$. Other works on dynamic regret with switching cost are incomparable to our paper, because they either rely on strong convexity [Li et al., 2018, Li and Li, 2020], set the switching cost as the squared distance [Goel et al., 2019], or assume an upper bound of the total switching cost is given in advance [Chen et al., 2018, Zhao et al., 2020b].

2.3 Discounted-Normal-Predictor

Following the terminology of Kapralov and Panigrahy [2010], we introduce Discounted-Normal-Predictor in the context of the bit prediction problem. Let $b_1, \ldots, b_T$ be an adversarial sequence of bits, where $b_t \in [-1, 1]$ can take real values. In each round t, the algorithm is required to output a confidence level $c_t \in [-1, 1]$, then observes the value of b_t, and finally gets a *payoff* $c_t b_t$. The goal is to maximize the cumulative payoff of the algorithm $\sum_{t=1}^{T} c_t b_t$.

Let $n > 0$ be a parameter for the interval length, and define the discount factor $\rho = 1 - 1/n$. As the name suggests, Discounted-Normal-Predictor maintains a discounted deviation $x_t = \sum_{j=1}^{t-1} \rho^{t-1-j} b_j$ at each round t, and the prediction is determined by $g(x_t)$ for a confidence function $g(\cdot)$ defined as

$$g(x) = \operatorname{sign}(x) \cdot \min\left(Z \cdot \operatorname{erf}\left(\frac{|x|}{4\sqrt{n}}\right) e^{\frac{x^2}{16n}}, 1\right) \tag{6}$$

where $Z > 0$ is a parameter, and $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds$ is the error function. The complete procedure is summarized in Algorithm 1. For any $Z \leq 1/e$, Kapralov and Panigrahy [2010, Theorem 14] have proved that Discounted-Normal-Predictor satisfies

$$\sum_{t=1}^{T} g(x_t) b_t \geq \max\left(\left|\sum_{j=1}^{T} b_j\right| - O\big(\sqrt{T \log(1/Z)}\big), -O\big(Z\sqrt{T}\big)\right)$$

where we set $n = T$ in Algorithm 1. By choosing $Z = o(1/T)$, we observe that it has $O(\sqrt{T \log T})$ regret against the strategy that predicts the majority bit (whose payoff is $|\sum_j b_j|$), as well as a subconstant $o(1)$ loss. To address the problem of learning with two experts, we can define b_t as the difference between the losses of experts and restrict $c_t \in [0, 1]$. It follows that the algorithm suffers $O(\sqrt{T \log T})$ and $o(1)$ regret w.r.t. the two experts, respectively [Kapralov and Panigrahy, 2010, Lemma 15]. Discounted-Normal-Predictor can be applied to the general setting of N experts, by aggregating experts one by one. We note that the problem of trading off regret to the best expert for regret to the "special" expert stems from the study of Even-Dar et al. [2007], and is later investigated by Sani et al. [2014].

Kapralov and Panigrahy [2010, Theorem 5] also investigate a variant of the adaptive regret, which uses an infinite window with geometrically decreasing weighting. To this end, they propose a conservative updating rule to control the value of the deviation x_t. Let $U(n) = O(\sqrt{n \log(1/Z)})$ be a constant such that $|g(x)| = 1$ for $|x| \geq U(n)$. The current bit b_t is utilized to update x_t only when the confidence of the algorithm is low i.e., $|x_t| < U(n)$ or when the algorithm predicts incorrectly i.e., $g(x_t) b_t < 0$. Otherwise, they will ignore b_t and set $x_{t+1} = \rho x_t$. However, it remains open

whether their conservative updating works with the standard adaptive regret [Daniely et al., 2015], which is answered affirmatively by our Theorem 1.

Daniely and Mansour [2019] extend Discounted-Normal-Predictor to support the switching cost and the adaptive regret. The confidence function is modified slightly as[1]

$$g(x) = \Pi_{[0,1]}\left[\tilde{g}(x)\right] \tag{7}$$

where

$$\tilde{g}(x) = \sqrt{\frac{n}{8}} Z \cdot \operatorname{erf}\left(\frac{x}{\sqrt{8n}}\right) e^{\frac{x^2}{16n}}, \tag{8}$$

and $\Pi_{[0,1]}[\cdot]$ denotes the projection operation onto the set $[0,1]$. First, by a more careful analysis, they demonstrate that Discounted-Normal-Predictor has similar regret bounds even in the presence of switching costs. Second, they introduce a projection operation to prevent x_t from being too large, and then derive tight bounds for the standard adaptive regret. Specifically, the updating rule for x_t becomes

$$x_{t+1} = \Pi_{[-2,U(n)+2]}\left[\rho x_t + b_t\right] \tag{9}$$

where

$$U(n) = \tilde{g}^{-1}(1) \leq \sqrt{16n \log \frac{1}{Z}}. \tag{10}$$

Our work is inspired by Daniely and Mansour [2019], but with the following differences.

- While Daniely and Mansour [2019] consider the setting of LEA, we investigate OCO.
- To bound the adaptive regret, Daniely and Mansour [2019] introduce the projection operation, as shown in (9). In contrast, we make use of the conservative updating [Kapralov and Panigrahy, 2010], and provide a more simple analysis.
- We study not only the adaptive regret, but also the dynamic regret. Our algorithm is equipped with nearly optimal bounds for both metrics, in the presence of switching costs.

3 Main Results

We take Discounted-Normal-Predictor with conservative updating (DNP-cu) as our meta-algorithm, and present its performance by taking the switching cost into consideration. Then, we use DNP-cu to combine multiple OGD sequentially, and show its adaptive regret and dynamic regret with switching cost. Due to space limitations, all the proofs are deferred to Appendix A.

3.1 The Meta-algorithm

To consist with previous studies, we describe DNP-cu from the perspective of bit prediction, and require the prediction to lie in $[0,1]$ so that it can be used later as a meta-algorithm to combine experts. The detailed procedure is summarized in Algorithm 2. Compared with the original algorithm [Kapralov and Panigrahy, 2010], we make two modifications.

1. We choose the confidence function $g(\cdot)$ in (7), whose property has been revealed by Daniely and Mansour [2019, Lemmas 18 and 19] more formally.
2. We adapt the updating rule to the fact that $g(\cdot)$ belongs to $[0,1]$ instead of $[-1,1]$. Specifically, we perform the standard updating in Line 6, when the confidence is low i.e., $x_t \in [0, U(n)]$, or when the algorithm predicts incorrectly i.e.,

$$x_t < 0 \& b_t > 0 \text{ or } x_t > U(n) \& b_t < 0.$$

   Otherwise, we follow Line 8, which only shrinks $\mathbf{x}_t$ and ignores b_t.

To analyze the performance of DNP-cu, we consider both the payoff and the switching cost, referred to as *reward* below. We have the following theorem which exploits the possibility that the magnitude of the bit sequence may be smaller than 1.

[1] Their definition of the error function drops the constant $\frac{2}{\sqrt{\pi}}$.

Algorithm 2 Discounted-Normal-Predictor with conservative updating (DNP-cu)

Require: Two parameters: n and Z

Set $x_1 = 0$, $\rho = 1 - 1/n$, and $U(n) = \tilde{g}^{-1}(1)$
for $t = 1, \ldots, T$ **do**
  Predict $g(x_t)$ where $g(\cdot)$ is defined in (7)
  Receive b_t
  if $x_t \in [0, U(n)]$ or $x_t < 0 \& b_t > 0$ or $x_t > U(n) \& b_t < 0$ **then**
    Set $x_{t+1} = \rho x_t + b_t$
  else
    Set $x_{t+1} = \rho x_t$
  end if
end for

Theorem 1 *Suppose $Z \leq \frac{1}{e}$ and $n \geq \max\{8e, 16 \log \frac{1}{Z}\}$. For any bit sequence $b_1, \ldots, b_T$ such that $|b_t| \leq \mu \leq 1$, the cumulative reward of Algorithm 2 over any interval $[r, s]$ with length τ satisfies*

$$\begin{aligned} &\sum_{t=r}^{s} \left(g(x_t) b_t - \frac{1}{\mu} |g(x_t) - g(x_{t+1})| \right) \\ \geq &\max \left(0, \sum_{t=r}^{s} b_t - \frac{\tau}{n} \left(U(n) + 2\mu \right) - U(n) - \mu \right) - U(n) - \mu - Z\tau \end{aligned} \tag{11}$$

where $U(n)$ is defined in (10). Furthermore, for intervals starting from 1, we have

$$\sum_{t=1}^{s} \left(g(x_t) b_t - \frac{1}{\mu} |g(x_t) - g(x_{t+1})| \right) \geq \max \left(0, \sum_{t=1}^{s} b_t - \frac{\tau(U(n) + 2\mu)}{n} - U(n) \right) - Z\tau. \tag{12}$$

And the change of successive predictions satisfies

$$|g(x_t) - g(x_{t+1})| \leq \mu \sqrt{\frac{1}{n} \log \frac{1}{Z}} + \frac{Z\mu}{4}. \tag{13}$$

Remark: First, the above theorem reveals that DNP-cu controls the switching cost automatically, and the trade-off between the payoff and the switching cost is determined by the magnitude of the bit sequence. Second, thanks to the conservative updating, we are able to bound the cumulative reward over any interval, which can be exploited to support adaptive regret.

Recall that in the total loss (1), the trade-off parameter λ could be arbitrary. As a result, to utilize DNP-cu for SOCO, we also need a flexible way to balance the payoff and the switching cost. To this end, we introduce λ and define the reward in the t-round as $g(x_t)b_t - \lambda|g(x_t) - g(x_{t-1})|$. Then, our goal is to maximize

$$\sum_{t=r}^{s} \left(g(x_t) b_t - \lambda |g(x_t) - g(x_{t-1})| \right) \tag{14}$$

for every interval $[r, s]$.[2] Based on Theorem 1, a straightforward way is to multiply the bit sequence by $1/\lambda$, and then pass it to Algorithm 2. But in this way, the lower bound will scale linearly with λ. To improve the dependence on λ, we will multiply the bit sequence by $1/\sqrt{\lambda}$ [Daniely and Mansour, 2019], and have the following corollary based on (11) and (13) of Theorem 1.[3]

Corollary 2 *Suppose $Z \leq \frac{1}{e}$, $n \geq \max\{8e, 16 \log \frac{1}{Z}\}$, and $b_1, \ldots, b_T$ is a bit sequence such that $|b_t| \leq 1$. Running Algorithm 2 over the* scaled *bit sequence*

$$\frac{b_1}{\max(\sqrt{\lambda}, 1)}, \ldots, \frac{b_T}{\max(\sqrt{\lambda}, 1)},$$

[2] We actually focus on $\sum_{t=r}^{s} \left(g(x_t) b_t - \lambda |g(x_t) - g(x_{t+1})| \right)$, which differs from (14) by a constant factor.

[3] We omit (12), because it is not used in the subsequent analysis.

for any interval $[r, s]$ with length τ, we have

$$\begin{aligned}&\sum_{t=r}^{s}\left(g(x_t)b_t - \lambda|g(x_t) - g(x_{t+1})|\right)\\ \geq &\max\left(0, \sum_{t=r}^{s} b_t - \frac{\max(\sqrt{\lambda}, 1)U(n)(\tau + n)}{n} - \frac{2\tau + n}{n}\right)\\ &- \max(\sqrt{\lambda}, 1)U(n) - 1 - \max(\sqrt{\lambda}, 1)Z\tau.\end{aligned} \tag{15}$$

And the change of successive predictions satisfies

$$|g(x_t) - g(x_{t+1})| \leq \frac{1}{\max(\sqrt{\lambda}, 1)}\left(\sqrt{\frac{1}{n}\log\frac{1}{Z}} + \frac{Z}{4}\right). \tag{16}$$

Remark: From the definition of $U(n)$ in (10), we have

$$\begin{aligned}\sum_{t=r}^{s}\left(g(x_t)b_t - \lambda|g(x_t) - g(x_{t+1})|\right) &= -O\left(\sqrt{(1+\lambda)n\log\frac{1}{Z}} + \sqrt{(1+\lambda)}Z\tau\right)\\ &\overset{Z=O(1/T)}{=} -O\left(\sqrt{(1+\lambda)n\log T}\right).\end{aligned} \tag{17}$$

for any interval $[r, s]$. Notice that the bound in (17) is independent of the interval length τ, so it holds even when τ is *larger* than n. On the other hand, for any interval $[r, s]$ whose length is no larger than n, i.e., $\tau/n \leq 1$, we also have the following regret bound of Algorithm 2 w.r.t. the baseline strategy which always outputs 1:

$$\sum_{t=r}^{s}\left(g(x_t)b_t - \lambda|g(x_t) - g(x_{t+1})|\right) \overset{Z=O(1/T)}{=} \sum_{t=r}^{s} b_t - O\left(\sqrt{(1+\lambda)n\log T}\right). \tag{18}$$

As elaborated in the analysis (c.f. Lemma 3), when using Algorithm 2 to combine multiple algorithms, there are two issues that need to be addressed.

1. We do not destroy the theoretical guarantee of early algorithms, which is ensured by (17).
2. We can inherit the theoretical guarantee of the current algorithm, which is archived by (18).

3.2 Smoothed OGD

We introduce the following common assumptions for OCO [Shalev-Shwartz, 2011].

Assumption 1 *All the functions f_t's are convex over the domain $\mathcal{W}$.*

Assumption 2 *The gradients of all functions are bounded by G, i.e.,*

$$\max_{\mathbf{w}\in\mathcal{W}} \|\nabla f_t(\mathbf{w})\| \leq G, \ \forall t \in [T]. \tag{19}$$

Assumption 3 *The diameter of the domain $\mathcal{W}$ is bounded by D, i.e.,*

$$\max_{\mathbf{w},\mathbf{w}'\in\mathcal{W}} \|\mathbf{w} - \mathbf{w}'\| \leq D. \tag{20}$$

Without loss of generality, we assume

$$f_t(\mathbf{w}) \in [0, GD], \ \forall \mathbf{w} \in \mathcal{W}, \ t \in [T], \tag{21}$$

since we can always redefine $f_t(\mathbf{w})$ as $f_t(\mathbf{w}) - \min_{\mathbf{w}\in\mathcal{W}} f_t(\mathbf{w})$ which belongs to $[0, GD]$ according to (19) and (20).

First, we explain how to use DNP-cu to combine predictions of two algorithms designed for OCO. Let $\mathcal{A}^1$ and $\mathcal{A}^2$ be two online learners, and denote their predictions in the t-th round by $\mathbf{w}_t^1$ and $\mathbf{w}_t^2$, respectively. Let $\mathcal{A}$ be a meta-algorithm which outputs a convex combination of $\mathbf{w}_t^1$ and $\mathbf{w}_t^2$, i.e.,

$$\mathbf{w}_t = (1 - w_t)\mathbf{w}_t^1 + w_t\mathbf{w}_t^2 \tag{22}$$

where the weight $w_t \in [0, 1]$. We have the following lemma regarding the meta-regret of $\mathcal{A}$.

Algorithm 3 Combiner

Require: Three parameters: M, n and Z
Require: Two algorithms: $\mathcal{A}^1$ and $\mathcal{A}^2$
Let $\mathcal{D}$ be an instance of DNP-cu, i.e., Algorithm 2, with parameter n and Z
Receive $\mathbf{w}_1^1$ and $\mathbf{w}_1^2$ from $\mathcal{A}^1$ and $\mathcal{A}^2$ respectively
Receive the prediction w_1 from $\mathcal{D}$
for $t = 1, \ldots, T$ **do**
    Predict $\mathbf{w}_t$ according to (22)
    Send the loss function $f_t(\cdot)$ to $\mathcal{A}^1$ and $\mathcal{A}^2$
    Receive $\mathbf{w}_{t+1}^1$ and $\mathbf{w}_{t+1}^2$ from $\mathcal{A}^1$ and $\mathcal{A}^2$ respectively
    Send the bit $\frac{\ell_t}{\max(\sqrt{\lambda},1)}$ to $\mathcal{D}$, where ℓ_t is defined in (28)
    Receive the prediction w_{t+1} from $\mathcal{D}$
end for

Lemma 1 *Assume the outputs of $\mathcal{A}^1$ and $\mathcal{A}^2$ move slowly such that*

$$\|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\| \leq \frac{MD}{\lambda}, \text{ and } \|\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2\| \leq \frac{MD}{\lambda}, \ \forall t \in [T] \tag{23}$$

where $M \geq 0$ is some constant. Under Assumptions 1, 2 and 3, the meta-regret of $\mathcal{A}$ w.r.t. $\mathcal{A}^1$ over any interval $[r, s]$ satisfies

$$\begin{aligned} &\sum_{t=r}^{s} \big(f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\|\big) - \sum_{t=r}^{s} \big(f_t(\mathbf{w}_t^1) + \lambda G\|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\|\big) \\ \leq &-(1+M)GD\sum_{t=r}^{s}\big(w_t\ell_t - \lambda|w_t - w_{t+1}|\big) \end{aligned} \tag{24}$$

and the meta-regret of $\mathcal{A}$ w.r.t. $\mathcal{A}^2$ satisfies

$$\begin{aligned} &\sum_{t=r}^{s} \big(f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\|\big) - \sum_{t=r}^{s} \big(f_t(\mathbf{w}_t^2) + \lambda G\|\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2\|\big) \\ \leq &-(1+M)GD\sum_{t=r}^{s}\big(w_t\ell_t - \lambda|w_t - w_{t+1}| - \ell_t\big) \end{aligned} \tag{25}$$

where

$$\ell_t^1 = f_t(\mathbf{w}_t^1) + \lambda G\|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\| \overset{(21),(23)}{\in} [0, (1+M)GD], \tag{26}$$

$$\ell_t^2 = f_t(\mathbf{w}_t^2) + \lambda G\|\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2\| \overset{(21),(23)}{\in} [0, (1+M)GD], \tag{27}$$

$$\ell_t = \frac{\ell_t^1 - \ell_t^2}{(1+M)GD} \overset{(26),(27)}{\in} [-1, 1]. \tag{28}$$

Remark: By mapping w_t and ℓ_t in Lemma 1 to $g(x_t)$ and b_t in Corollary 2, we immediately see that the weight w_t in (22) can be determined by invoking DNP-cu, i.e., Algorithm 2 to process the scaled bit sequence

$$\frac{\ell_1}{\max(\sqrt{\lambda}, 1)}, \ldots, \frac{\ell_T}{\max(\sqrt{\lambda}, 1)}.$$

Then, we can utilize lower bounds in Corollary 2 to establish upper bounds for the regret in Lemma 1. We name the strategy of aggregating two algorithms by DNP-cu as Combiner, and summarize its procedure in Algorithm 3.

In the following, we will use online gradient descent (OGD) with constant step size [Zinkevich, 2003] as our expert-algorithm. OGD performs gradient descent to update the current solution $\mathbf{w}_t$:

$$\mathbf{w}_{t+1} = \Pi_{\mathcal{W}}\big[\mathbf{w}_t - \eta\nabla f_t(\mathbf{w}_t)\big]$$

Algorithm 4 Smoothed OGD

Require: Three parameters: K, M and Z

for $i = 1, \ldots, K$ **do**
  Set $n^{(i)} = T2^{1-i}$
  Let $\mathcal{A}^i$ be an instance of OGD with step size $\eta^{(i)}$ defined in (29)
  if $i = 1$ **then**
    Set $\mathcal{B}^1 = \mathcal{A}^1$
  else
    Let $\mathcal{B}^i$ be an instance of Combiner, i.e., Algorithm 3 which combines B^{i-1} and $\mathcal{A}^i$ with parameters M, $n^{(i)}$ and Z
  end if
end for
for $t = 1, \ldots, T$ **do**
  Run $\mathcal{B}^1, \ldots, \mathcal{B}^K$ sequentially for one step
  Predict the output of $\mathcal{B}^K$, denoted by $\mathbf{w}_t$
end for

where $\eta > 0$ is the step size, and $\Pi_{\mathcal{W}}[\cdot]$ denotes the projection onto $\mathcal{W}$. Notice that it is important to choose a *constant* step size, which makes it easy to analyze the regret, as well as the dynamic regret, over any interval $[r, s]$.

First, we create K instances of OGD, denoted by $\mathcal{A}^1, \ldots, \mathcal{A}^K$, where the value of K will be determined later. The step size of $\mathcal{A}^i$ is set to be

$$\eta^{(i)} = \frac{D}{G}\sqrt{\frac{1}{(1+2\lambda)n^{(i)}}} \tag{29}$$

where

$$n^{(i)} = T2^{1-i}. \tag{30}$$

Then, we use Combiner, i.e., Algorithm 3 to aggregate them sequentially. We will create a sequence of algorithms $\mathcal{B}^1, \ldots, \mathcal{B}^K$, where $\mathcal{B}^i$ is obtained by combining $\mathcal{B}^{i-1}$ with $\mathcal{A}^i$, and $\mathcal{B}^1 = \mathcal{A}^1$. The parameter n in Algorithm 3 is set to be $n^{(i)}$ when forming $\mathcal{B}^i$. In each iteration t, we invoke $\mathcal{B}^1, \ldots, \mathcal{B}^K$ sequentially for one step, and return the output of $\mathcal{B}^K$ as the prediction $\mathbf{w}_t$. The completed procedure is named as smoothed OGD, and summarized in Algorithm 4.

Remark: Our method has a similar structure with those of Cutkosky [2020] and Zhang et al. [2022], in the sense that we all combine multiple experts sequentially. In contrast, other approaches for adaptive regret use a two-level framework, where a meta-algorithm aggregates multiple experts (which is allowed to sleep) simultaneously [Hazan and Seshadhri, 2007, Daniely et al., 2015, Jun et al., 2017a,b, Zhang et al., 2019, 2021b]. On the other hand, all the previous works, including Cutkosky [2020] and Zhang et al. [2022], need to construct a set of sub-intervals, and maintain a sub-routine for each one. In this way, they can use a small number of sub-intervals to cover any possible interval, and attain a small regret on that interval. By comparison, our method does not rely on any special construction of sub-intervals, making it more elegant.

3.3 Theoretical Guarantees

Next, we provide the theoretical guarantee of smoothed OGD. We first characterize its regret with switching cost over any interval.

Theorem 3 *Assume*

$$T \geq \max(\sqrt{\lambda}\log_2 T, e) \tag{31}$$

and set

$$K = \left\lfloor \log_2 \frac{T}{32\max(\lambda, 1)\log 1/Z} \right\rfloor + 1, \tag{32}$$

$M = 2$ and $Z = 1/T$ in Algorithm 4. Under Assumptions 1, 2 and 3, we have

$$\begin{aligned}&\sum_{t=r}^{s} \left(f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\| - f_t(\mathbf{w})\right)\\ \leq& 2GD\sqrt{(1+\lambda)\tau} + 113GD\max(\sqrt{\lambda}, 1)\sqrt{\tau \log T} = O\left(\sqrt{(1+\lambda)\tau \log T}\right)\end{aligned}$$

for any interval $[r, s]$ with length τ, and any $\mathbf{w} \in \mathcal{W}$.

Remark: Our $O(\sqrt{(1+\lambda)\tau \log T})$ bound is on the same order as that of Zhang et al. [2022]. When $\lambda = 0$, we get the $O(\sqrt{\tau \log T})$ adaptive regret for general convex functions [Jun et al., 2017a].

Our proposed method is also equipped with nearly optimal dynamic regret with switching cost over any interval, as stated below.

Theorem 4 *Under the condition of Theorem 3, we have*

$$\begin{aligned}&\sum_{t=r}^{s} \left(f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\| - f_t(\mathbf{u}_t)\right)\\ \leq& 2GD\sqrt{(1+\lambda)\tau(1+2P_{r,s}/D)} + 120GD\max(\sqrt{\lambda}, 1)\sqrt{\tau(1+2P_{r,s}/D)\log T}\\ =& O\left(\sqrt{(1+\lambda)\tau(1+P_{r,s})\log T}\right)\end{aligned}$$

where τ is the interval length, and $P_{r,s}$, defined in (5), is the path-length of an arbitrary comparator sequence $\mathbf{u}_r, \ldots, \mathbf{u}_s \in \mathcal{W}$.

Remark: The above theorem shows that our method can minimize the dynamic regret with switching cost over any interval. According to the $\Omega(\sqrt{T(1+P_{1,T})})$ lower bound of dynamic regret [Zhang et al., 2018b, Theorem 2], we know that our upper bound is optimal, up to a logarithmic factor. Theorem 4 is very general and can be simplified in different ways.

1. If we choose a fixed comparator such that $P_{r,s} = 0$, Theorem 4 reduces to Theorem 3 and matches that of Zhang et al. [2022].
2. If we ignore the switching cost and set $\lambda = 0$, we obtain an $O(\sqrt{\tau(1+P_{r,s})\log T})$ bound for dynamic regret over any interval, which recovers the results of Zhang et al. [2020, Theorem 4] and Cutkosky [2020, Theorem 7].
3. When both $P_{r,s} = 0$ and $\lambda = 0$, we obtain the $O(\sqrt{\tau \log T})$ adaptive regret of Jun et al. [2017a].

4 Conclusion and Future Work

Based on a variant of Discounted-Normal-Predictor (DNP-cu), we design a novel algorithm, named as smoothed OGD for SOCO. Our algorithm combines multiple instances of OGD sequentially by DNP-cu, and thus is very simple. Theoretical analysis shows that it attains nearly optimal bounds for both adaptive regret and dynamic regret over any interval, in the presence of switching costs. We also conduct preliminary experiments to verify our theories, and present the results in Appendix B.

For adaptive regret and dynamic regret, we can obtain tighter bounds when the hitting cost exhibits additional curvature properties such as exponential concavity [Hazan and Seshadhri, 2007, Baby and Wang, 2021] and smoothness [Zhang et al., 2019, Zhao et al., 2020a]. It remains unclear whether DNP-cu can exploit such information to further improve the performance. For SOCO, it is common to consider the lookahead setting, and the problem is still nontrivial due to the coupling created by the switching cost [Chen et al., 2018]. For dynamic regret with switching cost, Zhang et al. [2021a] have demonstrated that Assumption 2 is unnecessary in the lookahead setting. It would be interesting to develop a lookahead version of DNP-cu, and investigate whether we can drop Assumption 2 as well.

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References

J. Abernethy, P. L. Bartlett, A. Rakhlin, and A. Tewari. Optimal strategies and minimax lower bounds for online convex games. In *Proceedings of the 21st Annual Conference on Learning Theory*, pages 415–423, 2008.

D. Adamskiy, W. M. Koolen, A. Chernov, and V. Vovk. A closer look at adaptive regret. In *Proceedings of the 23rd International Conference on Algorithmic Learning Theory*, pages 290–304, 2012.

A. Antoniadis and K. Schewior. A tight lower bound for online convex optimization with switching costs. In *Approximation and Online Algorithms*, pages 164–175, 2018.

A. Antoniadis, N. Barcelo, M. Nugent, K. Pruhs, K. Schewior, and M. Scquizzato. Chasing convex bodies and functions. In *Proceedings of the 12th Latin American Symposium on Theoretical Informatics*, pages 68–81, 2016.

C. Argue, S. Bubeck, M. B. Cohen, A. Gupta, and Y. T. Lee. A nearly-linear bound for chasing nested convex bodies. In *Proceedings of the 30th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 117–122, 2019.

C. Argue, A. Gupta, and G. Guruganesh. Dimension-free bounds for chasing convex functions. In *Proceedings of 33rd Conference on Learning Theory*, pages 219–241, 2020a.

C. Argue, A. Gupta, G. Guruganesh, and Z. Tang. Chasing convex bodies with linear competitive ratio. In *Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1519–1524, 2020b.

D. Baby and Y.-X. Wang. Optimal dynamic regret in exp-concave online learning. In *Proceedings of the 34th Conference on Learning Theory*, pages 359–409, 2021.

N. Bansal, A. Gupta, R. Krishnaswamy, K. Pruhs, K. Schewior, and C. Stein. A 2-competitive algorithm for online convex optimization with switching costs. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 96–109, 2015.

N. Bansal, M. Böhm, M. Eliáš, G. Koumoutsos, and S. W. Umboh. Nested convex bodies are chaseable. In *Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1253–1260, 2018.

O. Besbes, Y. Gur, and A. Zeevi. Non-stationary stochastic optimization. *Operations Research*, 63 (5):1227–1244, 2015.

S. Bubeck, Y. T. Lee, Y. Li, and M. Sellke. Competitively chasing convex bodies. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 861–868, 2019.

S. Bubeck, B. Klartag, Y. T. Lee, Y. Li, and M. Sellke. Chasing nested convex bodies nearly optimally. In *Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1496–1508, 2020.

N. Cesa-Bianchi and G. Lugosi. *Prediction, Learning, and Games*. Cambridge University Press, 2006.

N. Cesa-Bianchi and F. Orabona. Online learning algorithms. *Annual Review of Statistics and Its Application*, 8(1):165–190, 2021.

N. Chen, G. Goel, and A. Wierman. Smoothed online convex optimization in high dimensions via online balanced descent. In *Proceedings of the 31st Conference on Learning Theory*, pages 1574–1594, 2018.

C.-K. Chiang, T. Yang, C.-J. Lee, M. Mahdavi, C.-J. Lu, R. Jin, and S. Zhu. Online optimization with gradual variations. In *Proceedings of the 25th Annual Conference on Learning Theory*, pages 6.1–6.20, 2012.

A. Cutkosky. Parameter-free, dynamic, and strongly-adaptive online learning. In *Proceedings of the 37th International Conference on Machine Learning*, pages 2250–2259, 2020.

A. Cutkosky and F. Orabona. Black-box reductions for parameter-free online learning in Banach spaces. In *Proceedings of the 31st Conference On Learning Theory*, pages 1493–1529, 2018.

A. Daniely and Y. Mansour. Competitive ratio vs regret minimization: achieving the best of both worlds. In *Proceedings of the 30th International Conference on Algorithmic Learning Theory*, pages 333–368, 2019.

A. Daniely, A. Gonen, and S. Shalev-Shwartz. Strongly adaptive online learning. In *Proceedings of the 32nd International Conference on Machine Learning*, pages 1405–1411, 2015.

E. Even-Dar, M. Kearns, Y. Mansour, and J. Wortman. Regret to the best vs. regret to the average. In *Proceedings of the 20th Annual Conference on Learning Theory*, pages 233–247, 2007.

Y. Freund, R. E. Schapire, Y. Singer, and M. K. Warmuth. Using and combining predictors that specialize. In *Proceedings of the 29th Annual ACM Symposium on Theory of Computing*, pages 334–343, 1997.

J. Friedman and N. Linial. On convex body chasing. *Discrete & Computational Geometry*, 9:293–321, 1993.

G. Goel and A. Wierman. An online algorithm for smoothed regression and lqr control. In *Proceedings of the 22nd International Conference on Artificial Intelligence and Statistics*, pages 2504–2513, 2019.

G. Goel, N. Chen, and A. Wierman. Thinking fast and slow: Optimization decomposition across timescales. In *Proceedings of the 56th IEEE Conference on Decision and Control*, pages 1291–1298, 2017.

G. Goel, Y. Lin, H. Sun, and A. Wierman. Beyond online balanced descent: An optimal algorithm for smoothed online optimization. In *Advances in Neural Information Processing Systems 32*, pages 1875–1885, 2019.

P. Gradu, E. Hazan, and E. Minasyan. Adaptive regret for control of time-varying dynamics. *ArXiv e-prints*, arXiv:2007.04393, 2020.

A. György, T. Linder, and G. Lugosi. Efficient tracking of large classes of experts. *IEEE Transactions on Information Theory*, 58(11):6709–6725, 2012.

E. C. Hall and R. M. Willett. Dynamical models and tracking regret in online convex programming. In *Proceedings of the 30th International Conference on Machine Learning*, pages 579–587, 2013.

E. Hazan. Introduction to online convex optimization. *Foundations and Trends in Optimization*, 2 (3-4):157–325, 2016.

E. Hazan and C. Seshadhri. Adaptive algorithms for online decision problems. *Electronic Colloquium on Computational Complexity*, 88, 2007.

E. Hazan and C. Seshadhri. Efficient learning algorithms for changing environments. In *Proceedings of the 26th Annual International Conference on Machine Learning*, pages 393–400, 2009.

E. Hazan, A. Agarwal, and S. Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2-3):169–192, 2007.

A. Jadbabaie, A. Rakhlin, S. Shahrampour, and K. Sridharan. Online optimization: Competing with dynamic comparators. In *Proceedings of the 18th International Conference on Artificial Intelligence and Statistics*, pages 398–406, 2015.

K.-S. Jun, F. Orabona, S. Wright, and R. Willett. Improved strongly adaptive online learning using coin betting. In *Proceedings of the 20th International Conference on Artificial Intelligence and Statistics*, pages 943–951, 2017a.

K.-S. Jun, F. Orabona, S. Wright, and R. Willett. Online learning for changing environments using coin betting. *Electronic Journal of Statistics*, 11(2):5282–5310, 2017b.

M. Kapralov and R. Panigrahy. Prediction strategies without loss. *ArXiv e-prints*, arXiv:1008.3672, 2010.

M. Kapralov and R. Panigrahy. Prediction strategies without loss. In *Advances in Neural Information Processing Systems 24*, pages 828–836, 2011.

S.-J. Kim and G. B. Giannakis. Real-time electricity pricing for demand response using online convex optimization. In *Proceedings of the 2012 IEEE PES Conference on Innovative Smart Grid Technologies*, pages 1–5, 2014.

Y. Li and N. Li. Leveraging predictions in smoothed online convex optimization via gradient-based algorithms. In *Advances in Neural Information Processing Systems 33*, pages 14520–14531, 2020.

Y. Li, G. Qu, and N. Li. Using predictions in online optimization with switching costs: A fast algorithm and a fundamental limit. In *the 2018 Annual American Control Conference*, pages 3008–3013, 2018.

M. Lin, A. Wierman, L. L. Andrew, and E. Thereska. Dynamic right-sizing for power-proportional data centers. In *Proceedings of the 30th IEEE International Conference on Computer Communications*, pages 1098–1106, 2011.

M. Lin, Z. Liu, A. Wierman, and L. L. Andrew. Online algorithms for geographical load balancing. In *Proceedings of the 2012 International Green Computing Conference*, pages 1–10, 2012.

Y. Lin, G. Goel, and A. Wierman. Online optimization with predictions and non-convex losses. *Proceedings of the ACM on Measurement and Analysis of Computing Systems*, 4(1):18:1–18:32, 2020.

N. Littlestone and M. K. Warmuth. The weighted majority algorithm. *Information and Computation*, 108(2):212–261, 1994.

H. Luo and R. E. Schapire. Achieving all with no parameters: Adanormalhedge. In *Proceedings of the 28th Conference on Learning Theory*, pages 1286–1304, 2015.

A. Mokhtari, S. Shahrampour, A. Jadbabaie, and A. Ribeiro. Online optimization in dynamic environments: Improved regret rates for strongly convex problems. In *Proceedings of the 55th IEEE Conference on Decision and Control*, pages 7195–7201, 2016.

F. Orabona and D. Pál. Coin betting and parameter-free online learning. In *Advances in Neural Information Processing Systems 29*, pages 577–585, 2016.

F. Orabona, N. Cesa-Bianchi, and C. Gentile. Beyond logarithmic bounds in online learning. In *Proceedings of the 15th International Conference on Artificial Intelligence and Statistics*, pages 823–831, 2012.

A. Sani, G. Neu, and A. Lazaric. Exploiting easy data in online optimization. In *Advances in Neural Information Processing Systems 27*, pages 810–818, 2014.

M. Sellke. Chasing convex bodies optimally. In *Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1509–1518, 2020.

S. Shalev-Shwartz. Online learning and online convex optimization. *Foundations and Trends in Machine Learning*, 4(2):107–194, 2011.

S. Shalev-Shwartz, Y. Singer, and N. Srebro. Pegasos: primal estimated sub-gradient solver for SVM. In *Proceedings of the 24th International Conference on Machine Learning*, pages 807–814, 2007.

N. Srebro, K. Sridharan, and A. Tewari. Smoothness, low-noise and fast rates. In *Advances in Neural Information Processing Systems 23*, pages 2199–2207, 2010.

Y. Wan, W.-W. Tu, and L. Zhang. Strongly adaptive online learning over partial intervals. *SCIENCE CHINA Information Sciences*, 65(10):202101, 2022.

T. Yang, L. Zhang, R. Jin, and J. Yi. Tracking slowly moving clairvoyant: Optimal dynamic regret of online learning with true and noisy gradient. In *Proceedings of the 33rd International Conference on Machine Learning*, pages 449–457, 2016.

L. Zhang. Online learning in changing environments. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence*, pages 5178–5182, 2020. Early Career.

L. Zhang, T. Yang, J. Yi, R. Jin, and Z.-H. Zhou. Improved dynamic regret for non-degenerate functions. In *Advances in Neural Information Processing Systems 30*, pages 732–741, 2017.

L. Zhang, S. Lu, and Z.-H. Zhou. Adaptive online learning in dynamic environments. *ArXiv e-prints*, arXiv:1810.10815, 2018a.

L. Zhang, S. Lu, and Z.-H. Zhou. Adaptive online learning in dynamic environments. In *Advances in Neural Information Processing Systems 31*, pages 1323–1333, 2018b.

L. Zhang, T. Yang, R. Jin, and Z.-H. Zhou. Dynamic regret of strongly adaptive methods. In *Proceedings of the 35th International Conference on Machine Learning*, pages 5882–5891, 2018c.

L. Zhang, T.-Y. Liu, and Z.-H. Zhou. Adaptive regret of convex and smooth functions. In *Proceedings of the 36th International Conference on Machine Learning*, pages 7414–7423, 2019.

L. Zhang, S. Lu, and T. Yang. Minimizing dynamic regret and adaptive regret simultaneously. In *Proceedings of the 23rd International Conference on Artificial Intelligence and Statistics*, pages 309–319, 2020.

L. Zhang, W. Jiang, S. Lu, and T. Yang. Revisiting smoothed online learning. In *Advances in Neural Information Processing Systems 34*, pages 13599–13612, 2021a.

L. Zhang, G. Wang, W.-W. Tu, W. Jiang, and Z.-H. Zhou. Dual adaptivity: A universal algorithm for minimizing the adaptive regret of convex functions. In *Advances in Neural Information Processing Systems 34*, pages 24968–24980, 2021b.

Z. Zhang, A. Cutkosky, and I. C. Paschalidis. Adversarial tracking control via strongly adaptive online learning with memory. In *Proceedings of the 25th International Conference on Artificial Intelligence and Statistics*, pages 8458–8492, 2022.

P. Zhao and L. Zhang. Improved analysis for dynamic regret of strongly convex and smooth functions. In *Proceedings of the 3rd Annual Learning for Dynamics and Control Conference*, pages 48–59, 2021.

P. Zhao, Y.-J. Zhang, L. Zhang, and Z.-H. Zhou. Dynamic regret of convex and smooth functions. In *Advances in Neural Information Processing Systems 33*, pages 12510–12520, 2020a.

Y. Zhao, Q. Zhao, X. Zhang, E. Zhu, X. Liu, and J. Yin. Understand dynamic regret with switching cost for online decision making. *ACM Transactions on Intelligent Systems and Technology*, 11(3), 2020b.

M. Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the 20th International Conference on Machine Learning*, pages 928–936, 2003.

Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See the future work in Section 4.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] This work is mainly theoretical.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See the assumptions in Section 3.2.
   (b) Did you include complete proofs of all theoretical results? [Yes] The complete proofs can be found in Appendix A.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

A Analysis

In this section, we present the proof of all theorems.

A.1 Proof of Theorem 1

First, from the updating rule in Algorithm 2, we can prove that the derivation satisfies

$$-\mu \leq x_t \leq U(n) + \mu, \ \forall t \geq 1. \tag{33}$$

To see this, we first consider the upper bound in (33). Let k be any iteration such that $x_k \leq U(n)$ and $x_{k+1} > U(n)$. Then, we must have $x_{k+1} = \rho x_k + b_k$, because otherwise $x_{k+1} = \rho x_k < U(n)$. As a result,

$$x_{k+1} = \rho x_k + b_k \leq U(n) + \mu.$$

Now, we consider the next derivation x_{k+2}. Because $x_{k+1} > U(n)$, according to the conservative updating rule, we have

$$x_{k+2} = \begin{cases} \rho x_{k+1} + b_{k+1}, & b_{k+1} < 0; \\ \rho x_{k+1}, & \text{otherwise.} \end{cases}$$

which is always smaller than x_{k+1}. Repeating the above argument, we conclude that the subsequent derivations $x_{k+2}, x_{k+3}, \ldots$ keep decreasing until they become no bigger than $U(n)$. As a result, it is impossible for x_t to exceed $U(n) + \mu$.

The lower bound in (33) can be proved in a similar way. Let k be any iteration such that $x_k \geq 0$ and $x_{k+1} < 0$. Then, we must have $x_{k+1} = \rho x_k + b_k$, because otherwise $x_{k+1} = \rho x_k \geq 0$. As a result,

$$x_{k+1} = \rho x_k + b_k \geq b_k \geq -\mu.$$

Now, we consider the next derivation x_{k+2}. Because $x_{k+1} < 0$, according to the conservative updating rule, we have

$$x_{k+2} = \begin{cases} \rho x_{k+1} + b_{k+1}, & b_{k+1} > 0; \\ \rho x_{k+1}, & \text{otherwise.} \end{cases}$$

which is always bigger than x_{k+1}. Repeating the above argument, we conclude that the subsequent derivations $x_{k+2}, x_{k+3}, \ldots$ keep increasing until they become nonnegative. As a result, it is impossible for x_t to be smaller than $-\mu$.

Next, we make use of Algorithm 1 to analyze the reward of Algorithm 2. Following Kapralov and Panigrahy [2010], we construct the following bit sequence

$$\tilde{b}_t = \begin{cases} b_t, & \text{if Line 6 of Algorithm 2 is executed at round } t; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that the prediction $g(x_t)$, as well as the derivation x_t, of Algorithm 2 over the bit sequence $b_1, \ldots, b_T$ is exact the same as that of Algorithm 1 over the new sequence $\tilde{b}_1, \ldots, \tilde{b}_T$. Since Algorithm 1 is more simple, we will first establish the theoretical guarantee of Algorithm 1 over the new sequence, and then convert it to the reward of Algorithm 2 over the original sequence. We have the following theorem for Algorithm 1 [Daniely and Mansour, 2019].

Theorem 5 *Suppose $Z \leq \frac{1}{e}$ and $n \geq \max\{8e, 16 \log \frac{1}{Z}\}$. For any bit sequence $b_1, \ldots, b_T$ such that $|b_t| \leq \mu \leq 1$, the cumulative reward of Algorithm 1 over any interval $[r, s]$ with length τ satisfies*

$$\begin{aligned} &\sum_{t=r}^{s} \left(g(x_t) b_t - \frac{1}{\mu} |g(x_t) - g(x_{t+1})| \right) \\ \geq &\max \left(0, \sum_{t=r}^{s} b_t + x_r - \frac{\tau}{n} \left(U(n) + 2\mu \right) - U(n) \right) - \max(x_r, 0) - Z\tau \end{aligned} \tag{34}$$

where $U(n)$ is defined in (10). And the change of successive predictions satisfies

$$|g(x_t) - g(x_{t+1})| \leq \mu \sqrt{\frac{1}{n} \log \frac{1}{Z}} + \frac{Z\mu}{4}. \tag{35}$$

Theorem 5 can be extracted from the proofs of Lemmas 21 and 23 of Daniely and Mansour [2019]. For the sake of completeness, we provide its analysis in Appendix A.10. We can see that the lower bound in (34) depends on x_r, which explains the necessity of controlling its value.

Notice that μ is also the upper bound of the absolute value of the new sequence $\tilde{b}_1, \ldots, \tilde{b}_T$. According to Theorem 5, we directly obtain (13) from (35). From (34), we have

$$\begin{aligned}&\sum_{t=r}^{s}\left(g(x_t)\tilde{b}_t - \frac{1}{\mu}|g(x_t)-g(x_{t+1})|\right)\\ \geq &\max\left(0, \sum_{t=r}^{s}\tilde{b}_t + x_r - \frac{\tau}{n}\left(U(n)+2\mu\right) - U(n)\right) - \max(x_r, 0) - Z\tau.\end{aligned} \tag{36}$$

On the other hand, the reward in terms of the original sequence is

$$\begin{aligned}&\sum_{t=r}^{s}\left(g(x_t)b_t - \frac{1}{\mu}|g(x_t)-g(x_{t+1})|\right)\\ =&\sum_{t=r}^{s} g(x_t)(b_t-\tilde{b}_t) + \sum_{t=r}^{s}\left(g(x_t)\tilde{b}_t - \frac{1}{\mu}|g(x_t)-g(x_{t+1})|\right).\end{aligned} \tag{37}$$

So, we need to bound $\sum_{t=r}^{s} g(x_t)(b_t-\tilde{b}_t)$. Let k be any iteration such that $b_k \neq \tilde{b}_k$, i.e., Line 8 of Algorithm 2 is executed at round k, which also implies $\tilde{b}_k = 0$. From the updating rule, we must have

$$x_k < 0 \& b_k \leq 0 \text{ or } x_k > U(n) \& b_k \geq 0.$$

If $x_k < 0 \& b_k \leq 0$, we have

$$g(x_k)(b_k - \tilde{b}_k) = g(x_k)b_k = 0 \geq b_k = b_k - \tilde{b}_k$$

since $g(x_k) = 0$ and $\tilde{b}_k = 0$. Otherwise if $x_k > U(n) \& b_k \geq 0$, we have

$$g(x_k)(b_k - \tilde{b}_k) = b_k = b_k - \tilde{b}_k \geq 0$$

since $g(x_k) = 1$ and $\tilde{b}_k = 0$. So, we always have

$$g(x_k)(b_k - \tilde{b}_k) \geq \max\left(0, b_k - \tilde{b}_k\right), \text{ if } b_k \neq \tilde{b}_k. \tag{38}$$

As a result,

$$\begin{aligned}\sum_{t=r}^{s} g(x_t)(b_t-\tilde{b}_t) &= \sum_{t\in[r,s]\&b_t\neq\tilde{b}_t} g(x_t)(b_t-\tilde{b}_t)\\ &\overset{(38)}{\geq} \max\left(0, \sum_{t\in[r,s]\&b_t\neq\tilde{b}_t}\left(b_t-\tilde{b}_t\right)\right) = \max\left(0, \sum_{t=r}^{s}\left(b_t-\tilde{b}_t\right)\right).\end{aligned} \tag{39}$$

Combining (36), (37) and (39), we have

$$\begin{aligned}&\sum_{t=r}^{s}\left(g(x_t)b_t - \frac{1}{\mu}|g(x_t)-g(x_{t+1})|\right)\\ \geq &\max\left(0, \sum_{t=r}^{s}\tilde{b}_t + x_r - \frac{\tau}{n}\left(U(n)+2\mu\right) - U(n)\right) - \max(x_r, 0) - Z\tau\\ &+ \max\left(0, \sum_{t=r}^{s}\left(b_t-\tilde{b}_t\right)\right)\\ \geq &\max\left(0, \sum_{t=r}^{s} b_t + x_r - \frac{\tau}{n}\left(U(n)+2\mu\right) - U(n)\right) - \max(x_r, 0) - Z\tau.\end{aligned}$$

Then, we obtain (11) by using (33) to bound x_r, and obtain (12) based on $x_1 = 0$.

A.2 Proof of Corollary 2

Notice that the magnitude of the scaled bit sequence is upper bounded by $\mu = 1/\max(\sqrt{\lambda}, 1)$. From Theorem 1, we have

$$\begin{aligned}
&\sum_{t=r}^{s}\left(g(x_t)\frac{b_t}{\max(\sqrt{\lambda},1)} - \max(\sqrt{\lambda},1)|g(x_t)-g(x_{t+1})|\right)\\
&\overset{(11)}{\geq} \max\left(0, \sum_{t=r}^{s}\frac{b_t}{\max(\sqrt{\lambda},1)} - \frac{\tau}{n}\left(U(n)+\frac{2}{\max(\sqrt{\lambda},1)}\right) - U(n) - \frac{1}{\max(\sqrt{\lambda},1)}\right)\\
&\quad - U(n) - \frac{1}{\max(\sqrt{\lambda},1)} - Z\tau.
\end{aligned} \tag{40}$$

Then, we can lower bound the cumulative reward as follows

$$\begin{aligned}
&\sum_{t=r}^{s}\left(g(x_t)b_t - \lambda|g(x_t)-g(x_{t-1})|\right)\\
&\geq \max(\sqrt{\lambda},1)\sum_{t=r}^{s}\left(g(x_t)\frac{b_t}{\max(\sqrt{\lambda},1)} - \max(\sqrt{\lambda},1)|g(x_t)-g(x_{t+1})|\right)\\
&\overset{(40)}{\geq} \max\left(0, \sum_{t=r}^{s} b_t - \max(\sqrt{\lambda},1)U(n)\left(\frac{\tau}{n}+1\right) - \frac{2\tau}{n} - 1\right) - \max(\sqrt{\lambda},1)U(n)\\
&\quad - 1 - \max(\sqrt{\lambda},1)Z\tau
\end{aligned}$$

which proves (15). The upper bound in (16) is a direct consequence of (13).

A.3 Proof of Theorem 3

First, we show that under our setting of parameters, all the preconditions in Corollary 2 and Lemma 1 are satisfied so that they can be exploited to analyze $\mathcal{B}^i$, which invokes Algorithm 2 to combine $\mathcal{B}^{i-1}$ and $\mathcal{A}^i$. From (31), we know that $Z = 1/T \leq 1/e$. From our definition of K, we have

$$n^{(i)} \geq T2^{1-K} \overset{(32)}{\geq} 32\max(\lambda,1)\log\frac{1}{Z} \geq 32\log\frac{1}{Z} \geq 32 \geq 8e,\ \forall i \in [K]. \tag{41}$$

Thus, the conditions about Z and n in Corollary 2 are satisfied. Furthermore, our choice of M ensures that (23) in Lemma 1 is true. To this end, we prove the following lemma.

Lemma 2 *For all $\mathcal{A}^i$'s and $\mathcal{B}^i$'s created in Algorithm 4, their outputs satisfy the condition in (23) with $M = 2$.*

Based on above discussions, we conclude that Corollary 2 and Lemma 1 can be used in our analysis.

Next, we introduce the following theorem about the regret of OGD with switching cost over any interval $[r, s]$, which will be used to analyze the performance of $\mathcal{A}^i$'s.

Theorem 6 *Let $\mathbf{w}_t$ be the outputs of OGD with step size η. Under Assumptions 1, 2 and 3, we have*

$$\sum_{t=r}^{s}\left(f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\| - f_t(\mathbf{w})\right) \leq \frac{D^2}{2\eta} + \frac{(1+2\lambda)\eta(s-r+1)G^2}{2}$$

for any $\mathbf{w} \in \mathcal{W}$.

Long Intervals We proceed to analyze the performance of Algorithm 4 over an interval $[r, s]$, and start with the case that the interval length

$$\tau = s - r + 1 \geq 32\max(\lambda,1)\log\frac{1}{Z}.$$

From our construction of $n^{(i)}$ in (30), there must exist a

$$k = \left\lfloor \log_2\frac{T}{\tau}\right\rfloor + 1 \leq K \tag{42}$$

such that

$$\frac{n^{(k)}}{2} \leq \tau \leq n^{(k)}. \tag{43}$$

Then, we divide the proof into two steps:

(i) We show that the algorithm $\mathcal{A}^k$ attains an optimal regret with switching cost over the interval $[r, s]$;

(ii) We demonstrate that the regret of $\mathcal{B}^K$ w.r.t. $\mathcal{A}^k$ is under control.

Let $\mathbf{w}_t^k$ be the output of $\mathcal{A}^k$ in the t-th iteration. From Theorem 6, we have

$$\begin{aligned}
&\sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^k) + \lambda G \|\mathbf{w}_t^k - \mathbf{w}_{t+1}^k\| - f_t(\mathbf{w})\right) \\
\leq & \frac{D^2}{2\eta^{(k)}} + \frac{(1+2\lambda)\eta^{(k)}\tau G^2}{2} \overset{(29)}{=} \frac{GD}{2}\sqrt{(1+2\lambda)n^{(k)}} + \frac{GD}{2}\tau\sqrt{\frac{1+2\lambda}{n^{(k)}}} \\
\overset{(43)}{\leq} & \frac{(\sqrt{2}+1)GD}{2}\sqrt{(1+2\lambda)\tau} \leq 2GD\sqrt{(1+\lambda)\tau}.
\end{aligned} \tag{44}$$

Let $\mathbf{v}_t^i$ be the output of $\mathcal{B}^i$ in the t-th iteration. We establish the following lemma to bound the regret of $\mathcal{B}^K$ w.r.t. $\mathcal{A}^k$.

Lemma 3 *For any interval $[r, s]$ with length $\tau \leq cn^{(k)}$, we have*

$$\begin{aligned}
&\sum_{t=r}^{s} \left(f_t(\mathbf{v}_t^K) + \lambda G \|\mathbf{v}_t^K - \mathbf{v}_{t+1}^K\|\right) - \sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^k) + \lambda G \|\mathbf{w}_t^k - \mathbf{w}_{t+1}^k\|\right) \\
\leq & GD\max(\sqrt{\lambda}, 1)\left((12c+53)\sqrt{n^{(k)}\log T} + 9 + 6c + 6(K-k)\right).
\end{aligned} \tag{45}$$

Based on Lemma 3, we have

$$\begin{aligned}
&\sum_{t=r}^{s} \left(f_t(\mathbf{v}_t^K) + \lambda G \|\mathbf{v}_t^K - \mathbf{v}_{t+1}^K\|\right) - \sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^k) + \lambda G \|\mathbf{w}_t^k - \mathbf{w}_{t+1}^k\|\right) \\
\overset{(43),(45)}{\leq} & GD\max(\sqrt{\lambda}, 1)\left(65\sqrt{2\tau\log T} + 15\right) + 6GD\max(\sqrt{\lambda}, 1)(K-k) \\
\leq & 107GD\max(\sqrt{\lambda}, 1)\sqrt{\tau\log T} + 6GD\max(\sqrt{\lambda}, 1)\log_2\tau \\
\overset{\log_2\tau \leq \sqrt{\tau\log\tau}}{\leq} & 113GD\max(\sqrt{\lambda}, 1)\sqrt{\tau\log T}
\end{aligned} \tag{46}$$

where in the penultimate step we make use of the following fact

$$\begin{aligned}
K - k \overset{(32),(42)}{=} & \left\lfloor \log_2 \frac{T}{32\max(\lambda, 1)\log 1/Z} \right\rfloor - \left\lfloor \log_2 \frac{T}{\tau} \right\rfloor \\
\leq & \log_2 \frac{\tau}{32\max(\lambda, 1)\log 1/Z} + 1 \leq \log_2 \tau.
\end{aligned}$$

Combining (44) and (46), we have

$$\begin{aligned}
&\sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^K) + \lambda G \|\mathbf{w}_t^K - \mathbf{w}_{t+1}^K\| - f_t(\mathbf{w})\right) \\
\leq & 2GD\sqrt{(1+\lambda)\tau} + 113GD\max(\sqrt{\lambda}, 1)\sqrt{\tau\log T}.
\end{aligned} \tag{47}$$

Short Intervals We study short intervals $[r, s]$ such that

$$\tau = s - r + 1 \leq 32\max(\lambda, 1)\log\frac{1}{Z}.$$

From Lemma 2, we know that the output of B^K moves slowly such that

$$\|\mathbf{w}_t^K - \mathbf{w}_{t+1}^K\| \leq \frac{2D}{\lambda}. \tag{48}$$

As a result, the regret of B^K over $[r, s]$ can be bounded by

$$\begin{aligned}
&\sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^K) + \lambda G \|\mathbf{w}_t^K - \mathbf{w}_{t+1}^K\| - f_t(\mathbf{w})\right) \leq \sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^K) + \lambda G \|\mathbf{w}_t^K - \mathbf{w}_{t+1}^K\|\right) \\
\overset{(21),(48)}{\leq}\ & 3\tau GD \leq 3GD\sqrt{\tau \cdot 32 \max(\lambda, 1) \log T} \leq 17GD \max(\sqrt{\lambda}, 1)\sqrt{\tau \log T}.
\end{aligned} \tag{49}$$

We complete the proof by combing (47) and (49).

A.4 Proof of Theorem 4

Since we focus on dynamic regret, so we need the following theorem regarding the dynamic regret of OGD with switching cost over any interval $[r, s]$.

Theorem 7 *Under Assumptions 1, 2 and 3, we have*

$$\sum_{t=r}^{s} \left(f_t(\mathbf{w}_t) + \lambda G \|\mathbf{w}_t - \mathbf{w}_{t+1}\| - f_t(\mathbf{u}_t)\right) \leq \frac{D^2}{2\eta} + \frac{D}{\eta} \sum_{t=r}^{s} \|\mathbf{u}_t - \mathbf{u}_{t+1}\| + \frac{(1+2\lambda)\eta(s-r+1)G^2}{2}$$

for any comparator sequence $\mathbf{u}_r, \ldots, \mathbf{u}_s \in \mathcal{W}$.

The proof is similar to that of Theorem 3, and we consider two scenarios: long intervals and short intervals. Here, we multiply the interval length τ by $1/(1 + 2P_{r,s}/D)$ to reflect the fact that the comparator is changing.

Long Intervals First, we study the case that

$$\frac{\tau}{1 + 2P_{r,s}/D} \geq 32 \max(\lambda, 1) \log \frac{1}{Z}.$$

From our construction of $n^{(i)}$ in (30), there must exist a

$$k = \left\lfloor \log_2 \frac{T(1 + 2P_{r,s}/D)}{\tau} \right\rfloor + 1 \leq K \tag{50}$$

such that

$$\frac{n^{(k)}}{2} \leq \frac{\tau}{1 + 2P_{r,s}/D} \leq n^{(k)}. \tag{51}$$

Next, we show that the dynamic regret of $\mathcal{A}^k$ with switching cost is almost optimal. From Theorem 7, we have

$$\begin{aligned}
&\sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^k) + \lambda G \|\mathbf{w}_t^k - \mathbf{w}_{t+1}^k\| - f_t(\mathbf{u}_t)\right) \\
\leq & \frac{D^2}{2\eta^{(k)}} + \frac{D}{\eta^{(k)}} P_{r,s} + \frac{(1+2\lambda)\eta^{(k)}\tau G^2}{2} \\
\overset{(29)}{=} & \frac{G(D + 2P_{r,s})}{2}\sqrt{(1+2\lambda)n^{(k)}} + \frac{GD\tau}{2}\sqrt{\frac{1+2\lambda}{n^{(k)}}} \\
\overset{(51)}{\leq} & \frac{(\sqrt{2}+1)GD}{2}\sqrt{(1+2\lambda)\tau(1+2P_{r,s}/D)} \leq 2GD\sqrt{(1+\lambda)\tau(1+2P_{r,s}/D)}.
\end{aligned} \tag{52}$$

Then, we prove that the regret of $\mathcal{B}^K$ w.r.t. $\mathcal{A}^k$ is roughly on the same order as (52). From Lemma 3, we have

$$\begin{aligned}
&\sum_{t=r}^{s}\left(f_t(\mathbf{v}_t^K)+\lambda G\|\mathbf{v}_t^K-\mathbf{v}_{t+1}^K\|\right)-\sum_{t=r}^{s}\left(f_t(\mathbf{w}_t^k)+\lambda G\|\mathbf{w}_t^k-\mathbf{w}_{t+1}^k\|\right)\\
\overset{(51),(45)}{\leq}& GD\max(\sqrt{\lambda},1)\left((65+24P_{r,s}/D)\sqrt{n^{(k)}\log T}+15+12P_{r,s}/D\right)+\\
&6GD\max(\sqrt{\lambda},1)(K-k)\\
\overset{(51)}{\leq}& GD\max(\sqrt{\lambda},1)\left(65\sqrt{2\tau(1+2P_{r,s}/D)\log T}+15+12P_{r,s}/D\right)\\
&+6GD\max(\sqrt{\lambda},1)(K-k)\\
\leq& 114GD\max(\sqrt{\lambda},1)\sqrt{\tau(1+2P_{r,s}/D)\log T}+6GD\max(\sqrt{\lambda},1)\log_2\tau\\
\overset{\log_2\tau\leq\sqrt{\tau\log\tau}}{\leq}& 120GD\max(\sqrt{\lambda},1)\sqrt{\tau(1+2P_{r,s}/D)\log T}
\end{aligned}\tag{53}$$

where in the penultimate step we use the following inequalities

$$\begin{aligned}
15+12P_{r,s}/D \overset{P_{r,s}\leq\tau D}{\leq} 15+12\sqrt{\tau P_{r,s}/D} \overset{a+b\leq\sqrt{2a^2+2b^2}}{\leq} 15\sqrt{2\tau(1+2P_{r,s}/D)},\\
K-k \overset{(32),(50)}{=} \left\lfloor\log_2\frac{T}{32\max(\lambda,1)\log 1/Z}\right\rfloor-\left\lfloor\log_2\frac{T(1+2P_{r,s}/D)}{\tau}\right\rfloor\leq\log_2\tau.
\end{aligned}$$

Combining (52) and (53), we can bound the dynamic regret of $\mathcal{B}^K$ with switching cost by

$$\begin{aligned}
&\sum_{t=r}^{s}\left(f_t(\mathbf{w}_t^K)+\lambda G\|\mathbf{w}_t^K-\mathbf{w}_{t+1}^K\|-f_t(\mathbf{u}_t)\right)\\
\leq& 2GD\sqrt{(1+\lambda)\tau(1+2P_{r,s}/D)}+120GD\max(\sqrt{\lambda},1)\sqrt{\tau(1+2P_{r,s}/D)\log T}.
\end{aligned}\tag{54}$$

Short Intervals We consider short intervals $[r,s]$ such that

$$\frac{\tau}{1+2P_{r,s}/D}\geq 32\max(\lambda,1)\log\frac{1}{Z}.$$

Following the analysis of Theorem 3, the dynamic regret of B^K over $[r,s]$ can be bounded by

$$\begin{aligned}
&\sum_{t=r}^{s}\left(f_t(\mathbf{w}_t^K)+\lambda G\|\mathbf{w}_t^K-\mathbf{w}_{t+1}^K\|-f_t(\mathbf{u}_t)\right)\\
\leq& 3\tau GD\leq 3GD\sqrt{\tau\cdot 32\max(\lambda,1)\log T\cdot(1+2P_{r,s}/D)}\\
\leq& 17GD\max(\sqrt{\lambda},1)\sqrt{\tau(1+2P_{r,s}/D)\log T}.
\end{aligned}\tag{55}$$

We complete the proof by combing (54) and (55).

A.5 Proof of Theorem 6

From the standard analysis of OGD [Zinkevich, 2003], we have the following regret bound

$$\sum_{t=r}^{s}\left(f_t(\mathbf{w}_t)-f_t(\mathbf{w})\right)\leq\frac{D^2}{2\eta}+\frac{\eta(s-r+1)G^2}{2}.\tag{56}$$

To bound the switching cost, we have

$$\begin{aligned}
&\sum_{t=r}^{s}\|\mathbf{w}_t-\mathbf{w}_{t+1}\|=\sum_{t=r}^{s}\left\|\mathbf{w}_t-\Pi_{\mathcal{W}}\left[\mathbf{w}_t-\eta\nabla f_t(\mathbf{w}_t)\right]\right\|\\
\leq&\sum_{t=r}^{s}\|-\eta\nabla f_t(\mathbf{w}_t)\|=\eta\sum_{t=r}^{s}\|\nabla f_t(\mathbf{w}_t)\|\overset{(19)}{\leq}\eta(s-r+1)G.
\end{aligned}\tag{57}$$

From (56) and (57), we have

$$\sum_{t=r}^{s} \big(f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\| - f_t(\mathbf{w})\big) \leq \frac{D^2}{2\eta} + \frac{\eta(s-r+1)G^2}{2} + \lambda\eta(s-r+1)G^2.$$

A.6 Proof of Theorem 7

From the dynamic regret of OGD [Zinkevich, 2003], in particular Theorem 6 of Zhang et al. [2018b], we have

$$\sum_{t=r}^{s} \big(f_t(\mathbf{w}_t) - f_t(\mathbf{u}_t)\big) \leq \frac{D^2}{2\eta} + \frac{D}{\eta}\sum_{t=r}^{s}\|\mathbf{u}_t - \mathbf{u}_{t+1}\| + \frac{\eta(s-r+1)}{2}G^2.$$

We complete the proof by combining the above inequality with (57).

A.7 Proof of Lemma 1

Similar to the analysis of Daniely and Mansour [2019, Theorem 22], we decompose the weighted sum of hitting cost and switching cost as

$$\begin{aligned}
& f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\| \\
=& f_t\left((1-w_t)\mathbf{w}_t^1 + w_t\mathbf{w}_t^2\right) + \lambda G\left\|(1-w_t)\mathbf{w}_t^1 + w_t\mathbf{w}_t^2 - (1-w_{t+1})\mathbf{w}_{t+1}^1 - w_{t+1}\mathbf{w}_{t+1}^2\right\| \\
\leq& (1-w_t)f_t(\mathbf{w}_t^1) + w_t f_t(\mathbf{w}_t^2) + \lambda G\left\|(1-w_t)(\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1)\right\| + \lambda G\left\|w_t(\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2)\right\| \\
& + \lambda G\left\|(1-w_t)\mathbf{w}_{t+1}^1 - (1-w_{t+1})\mathbf{w}_{t+1}^1 + w_t\mathbf{w}_{t+1}^2 - w_{t+1}\mathbf{w}_{t+1}^2\right\| \\
=& (1-w_t)\left(f_t(\mathbf{w}_t^1) + \lambda G\|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\|\right) + w_t\left(f_t(\mathbf{w}_t^2) + \lambda G\|\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2\|\right) \\
& + \lambda G\left\|(w_t - w_{t+1})(\mathbf{w}_{t+1}^1 - \mathbf{w}_{t+1}^2)\right\| \\
\overset{(20)}{\leq}& (1-w_t)\left(f_t(\mathbf{w}_t^1) + \lambda G\|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\|\right) + w_t\left(f_t(\mathbf{w}_t^2) + \lambda G\|\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2\|\right) \\
& + \lambda GD|w_t - w_{t+1}|.
\end{aligned} \tag{58}$$

Then, the regret of $\mathcal{A}$ w.r.t. $\mathcal{A}^1$ over any interval $[r, s]$ can be upper bounded in the following way:

$$\begin{aligned}
& \sum_{t=r}^{s}\left(f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\|\right) - \sum_{t=r}^{s}\left(f_t(\mathbf{w}_t^1) + \lambda G\|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\|\right) \\
\overset{(58)}{\leq}& \sum_{t=r}^{s}\Big(w_t\left[\left(f_t(\mathbf{w}_t^2) + \lambda G\|\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2\|\right) - \left(f_t(\mathbf{w}_t^1) + \lambda G\|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\|\right)\right] \\
& + \lambda GD|w_t - w_{t+1}|\Big) \\
\overset{(26),(27)}{=}& \sum_{t=r}^{s}\left(w_t(\ell_t^2 - \ell_t^1) + \lambda GD|w_t - w_{t+1}|\right) \\
\overset{(28)}{=}& -(1+M)GD\sum_{t=r}^{s}\left(w_t\ell_t - \frac{\lambda}{1+M}|w_t - w_{t+1}|\right) \\
\leq& -(1+M)GD\sum_{t=r}^{s}\left(w_t\ell_t - \lambda|w_t - w_{t+1}|\right)
\end{aligned}$$

which proves (24). Similarly, the regret of $\mathcal{A}$ w.r.t. $\mathcal{A}^2$ over any interval $[r, s]$ can be upper bounded by

$$\begin{aligned}
&\sum_{t=r}^{s} \big(f_t(\mathbf{w}_t) + \lambda G\|\mathbf{w}_t - \mathbf{w}_{t+1}\|\big) - \sum_{t=r}^{s} \big(f_t(\mathbf{w}_t^2) + \lambda G\|\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2\|\big) \\
\overset{(58)}{\leq} &\sum_{t=r}^{s} (1 - w_t) \left[\big(f_t(\mathbf{w}_t^1) + \lambda G\|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\|\big) - \big(f_t(\mathbf{w}_t^2) + \lambda G\|\mathbf{w}_t^2 - \mathbf{w}_{t+1}^2\|\big)\right] \\
&+ \sum_{t=r}^{s} \lambda G D |w_t - w_{t+1}| \\
\overset{(26),(27)}{=} &\sum_{t=r}^{s} \big((1 - w_t)(\ell_t^1 - \ell_t^2) + \lambda G D |w_t - w_{t+1}|\big) \\
\overset{(28)}{=} &-(1 + M) G D \sum_{t=r}^{s} \left(w_t \ell_t - \frac{\lambda}{1 + M} |w_t - w_{t+1}| - \ell_t\right) \\
\leq &-(1 + M) G D \sum_{t=r}^{s} \big(w_t \ell_t - \lambda |w_t - w_{t+1}| - \ell_t\big)
\end{aligned}$$

which proves (25).

A.8 Proof of Lemma 2

We will prove that the outputs of $\mathcal{A}^i$'s and $\mathcal{B}^i$'s move slowly such that (23) holds. Let $\mathbf{w}_t^i$ be the output of $\mathcal{A}^i$ in the t-th iteration. From the updating rule of OGD, we have

$$\|\mathbf{w}_t^i - \mathbf{w}_{t+1}^i\| \leq \eta^{(i)} \left\|\nabla f_t(\mathbf{w}_t^i)\right\| \overset{(19)}{\leq} \eta^{(i)} G \overset{(29)}{=} D\sqrt{\frac{1}{(1 + 2\lambda) n^{(i)}}} \overset{(41)}{\leq} \frac{D}{\lambda}. \tag{59}$$

So, $\mathbf{w}_t^i$'s satisfy the condition in (23) when $M = 2$.

Let $\mathbf{v}_t^i$ be the output of $\mathcal{B}^i$ in the t-th iteration. We will prove by induction that

$$\|\mathbf{v}_t^i - \mathbf{v}_{t+1}^i\| \leq \frac{D}{\lambda} + \frac{D}{\max(\sqrt{\lambda}, 1)} \sum_{j=2}^{i} \left(\sqrt{\frac{1}{n^{(j)}} \log \frac{1}{Z}} + \frac{Z}{4}\right), \ \forall i \in [K]. \tag{60}$$

The above equation, together with the following fact

$$\begin{aligned}
&\frac{D}{\lambda} + \frac{D}{\max(\sqrt{\lambda}, 1)} \sum_{j=2}^{K} \left(\sqrt{\frac{1}{n^{(j)}} \log \frac{1}{Z}} + \frac{Z}{4}\right) \\
\overset{(30)}{=} &\frac{D}{\lambda} + \frac{D}{\max(\sqrt{\lambda}, 1)} \sqrt{\frac{1}{2T} \log \frac{1}{Z}} \sum_{j=2}^{K} \sqrt{2^j} + \frac{D}{\max(\sqrt{\lambda}, 1)} \frac{Z(K - 1)}{4} \\
\leq &\frac{D}{\lambda} + \frac{D}{\max(\sqrt{\lambda}, 1)} \sqrt{\frac{1}{2T} \log \frac{1}{Z}} \frac{2}{\sqrt{2} - 1} \sqrt{2}^{K-1} + \frac{D}{\max(\sqrt{\lambda}, 1)} \frac{Z(K - 1)}{4} \\
\overset{(32)}{\leq} &\frac{D}{\lambda} + \frac{D}{\max(\sqrt{\lambda}, 1)} \sqrt{\frac{1}{2T} \log \frac{1}{Z}} \frac{2}{\sqrt{2} - 1} \sqrt{\frac{T}{32\lambda \log 1/Z}} + \frac{D}{\max(\sqrt{\lambda}, 1)} \frac{Z}{4} \log_2 T \\
= &\frac{D}{\lambda} + \frac{(\sqrt{2} + 1) D}{4 \max(\sqrt{\lambda}, 1) \sqrt{\lambda}} + \frac{D \log_2 T}{4 \max(\sqrt{\lambda}, 1) T} \overset{(31)}{\leq} \frac{2D}{\lambda}
\end{aligned} \tag{61}$$

implies that $\mathbf{v}_t^i$'s meet the condition in (23) when $M = 2$.

Since $\mathcal{B}^1 = A^1$, we have

$$\|\mathbf{v}_t^1 - \mathbf{v}_{t+1}^1\| = \|\mathbf{w}_t^1 - \mathbf{w}_{t+1}^1\| \overset{(59)}{\leq} \frac{D}{\lambda}. \tag{62}$$

Thus, (60) holds when $i = 1$. Suppose (60) is true when $i = k$, and thus

$$\|\mathbf{v}_t^k - \mathbf{v}_{t+1}^k\| \leq \frac{D}{\lambda} + \frac{D}{\max(\sqrt{\lambda}, 1)} \sum_{j=2}^{k} \left(\sqrt{\frac{1}{n^{(j)}} \log \frac{1}{Z}} + \frac{Z}{4} \right) \overset{(61)}{\leq} \frac{2D}{\lambda}. \tag{63}$$

We proceed to bound the movement of $\mathbf{v}_t^{k+1}$, which is the output of B^{k+1}. Recall that B^{k+1} is an instance of Combiner which aggregates $\mathcal{B}^k$ and $\mathcal{A}^{k+1}$. From the procedure of Algorithm 3, we have

$$\mathbf{v}_t^{k+1} \overset{(22)}{=} (1 - w_t^{k+1})\mathbf{v}_t^k + w_t^{k+1}\mathbf{w}_t^{k+1}$$

where w_t^{k+1} is the weight generated by DNP-cu. Thus, the movement of $\mathbf{v}_t^{k+1}$ can be bounded by

$$\begin{aligned}
&\|\mathbf{v}_t^{k+1} - \mathbf{v}_{t+1}^{k+1}\| = \left\|(1 - w_t^{k+1})\mathbf{v}_t^k + w_t^{k+1}\mathbf{w}_t^{k+1} - \left((1 - w_{t+1}^{k+1})\mathbf{v}_{t+1}^k + w_{t+1}^{k+1}\mathbf{w}_{t+1}^{k+1}\right)\right\| \\
&\leq \left\|(1 - w_t^{k+1})\mathbf{v}_t^k - (1 - w_{t+1}^{k+1})\mathbf{v}_t^k + w_t^{k+1}\mathbf{w}_t^{k+1} - w_{t+1}^{k+1}\mathbf{w}_t^{k+1}\right\| \\
&\quad + \left\|(1 - w_{t+1}^{k+1})\mathbf{v}_t^k + w_{t+1}^{k+1}\mathbf{w}_t^{k+1} - \left((1 - w_{t+1}^{k+1})\mathbf{v}_{t+1}^k + w_{t+1}^{k+1}\mathbf{w}_{t+1}^{k+1}\right)\right\| \\
&\leq |w_t^{k+1} - w_{t+1}^{k+1}| \|\mathbf{v}_t^k - \mathbf{w}_t^{k+1}\| + (1 - w_{t+1}^{k+1})\|\mathbf{v}_t^k - \mathbf{v}_{t+1}^k\| + w_{t+1}^{k+1}\|\mathbf{w}_t^{k+1} - \mathbf{w}_{t+1}^{k+1}\| \\
&\overset{(20),(59)}{\leq} D|w_t^{k+1} - w_{t+1}^{k+1}| + (1 - w_{t+1}^{k+1})\|\mathbf{v}_t^k - \mathbf{v}_{t+1}^k\| + w_{t+1}^{k+1}\frac{D}{\lambda} \\
&\leq D|w_t^{k+1} - w_{t+1}^{k+1}| + \max\left(\|\mathbf{v}_t^k - \mathbf{v}_{t+1}^k\|, \frac{D}{\lambda}\right).
\end{aligned} \tag{64}$$

From (59) and (63), we know that the outputs of $\mathcal{B}^k$ and $\mathcal{A}^{k+1}$ satisfy (23). Thus, we can apply Corollary 2 to bound the change of w_t^{k+1}:

$$|w_t^{k+1} - w_{t+1}^{k+1}| \overset{(16)}{\leq} \frac{1}{\max(\sqrt{\lambda}, 1)} \left(\sqrt{\frac{1}{n^{(k+1)}} \log \frac{1}{Z}} + \frac{Z}{4} \right). \tag{65}$$

From (64) and (65), we have

$$\begin{aligned}
\|\mathbf{v}_t^{k+1} - \mathbf{v}_{t+1}^{k+1}\| &\leq \frac{D}{\max(\sqrt{\lambda}, 1)} \left(\sqrt{\frac{1}{n^{(k+1)}} \log \frac{1}{Z}} + \frac{Z}{4} \right) + \max\left(\|\mathbf{v}_t^k - \mathbf{v}_{t+1}^k\|, \frac{D}{\lambda}\right) \\
&\overset{(63)}{\leq} \frac{D}{\lambda} + \frac{D}{\max(\sqrt{\lambda}, 1)} \sum_{j=2}^{k+1} \left(\sqrt{\frac{1}{n^{(j)}} \log \frac{1}{Z}} + \frac{Z}{4} \right)
\end{aligned}$$

which shows that (60) holds when $i = k + 1$.

A.9 Proof of Lemma 3

The regret of $\mathcal{B}^K$ w.r.t. $\mathcal{A}^k$ can be decomposed as

$$\begin{aligned}
&\sum_{t=r}^{s} \left(f_t(\mathbf{v}_t^K) + \lambda G\|\mathbf{v}_t^K - \mathbf{v}_{t+1}^K\|\right) - \sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^k) + \lambda G\|\mathbf{w}_t^k - \mathbf{w}_{t+1}^k\|\right) \\
&= \underbrace{\sum_{t=r}^{s} \left(f_t(\mathbf{v}_t^k) + \lambda G\|\mathbf{v}_t^k - \mathbf{v}_{t+1}^k\|\right) - \sum_{t=r}^{s} \left(f_t(\mathbf{w}_t^k) + \lambda G\|\mathbf{w}_t^k - \mathbf{w}_{t+1}^k\|\right)}_{:=U} \\
&+ \sum_{i=k+1}^{K} \left(\underbrace{\sum_{t=r}^{s} \left(f_t(\mathbf{v}_t^i) + \lambda G\|\mathbf{v}_t^i - \mathbf{v}_t^{i+1}\|\right) - \sum_{t=r}^{s} \left(f_t(\mathbf{v}_t^{i-1}) + \lambda G\|\mathbf{v}_t^{i-1} - \mathbf{v}_{t+1}^{i-1}\|\right)}_{:=V^i} \right)
\end{aligned} \tag{66}$$

where U is the regret of $\mathcal{B}^k$ w.r.t. $\mathcal{A}^k$, and V^i is the regret of $\mathcal{B}^i$ w.r.t. $\mathcal{B}^{i-1}$. Next, we make use of Corollary 2 and Lemma 1 to bound these quantities.

To bound U, we have

$$
\begin{aligned}
U \overset{(15),(25)}{\leq} & 3GD\Bigg(\max(\sqrt{\lambda},1)U(n^{(k)})\left(\frac{\tau}{n^{(k)}}+1\right)+\frac{2\tau}{n^{(k)}}+1+\max(\sqrt{\lambda},1)U(n^{(k)}) \\
& +1+\max(\sqrt{\lambda},1)\frac{\tau}{T}\Bigg) \\
\overset{\tau\leq cn^{(k)}}{\leq} & 3GD\left((2+c)\max(\sqrt{\lambda},1)U(n^{(k)})+2+2c+\max(\sqrt{\lambda},1)\right) \\
\overset{(10)}{\leq} & 3GD\left((2+c)\max(\sqrt{\lambda},1)\sqrt{16n^{(k)}\log T}+2+2c+\max(\sqrt{\lambda},1)\right) \\
\leq & 3GD\max(\sqrt{\lambda},1)\left(4(2+c)\sqrt{n^{(k)}\log T}+3+2c\right).
\end{aligned} \tag{67}
$$

To bound the summation of V^i, we have

$$
\begin{aligned}
\sum_{i=k+1}^{K} V^i \overset{(15),(24)}{\leq} & 3GD\sum_{i=k+1}^{K}\left(\max(\sqrt{\lambda},1)U(n^{(i)})+1+\max(\sqrt{\lambda},1)\frac{\tau}{T}\right) \\
\leq & 3GD\max(\sqrt{\lambda},1)\sum_{i=k+1}^{K}U(n^{(i)})+6GD\max(\sqrt{\lambda},1)(K-k) \\
\overset{(10)}{\leq} & 3GD\max(\sqrt{\lambda},1)\sum_{i=k+1}^{K}\sqrt{16n^{(i)}\log T}+6GD\max(\sqrt{\lambda},1)(K-k) \\
\overset{(30)}{=} & 12GD\max(\sqrt{\lambda},1)\sqrt{n^{(k)}\log T}\sum_{i=1}^{K-k}\sqrt{2^{-i}}+6GD\max(\sqrt{\lambda},1)(K-k) \\
\leq & 12GD\max(\sqrt{\lambda},1)\sqrt{n^{(k)}\log T}\frac{1}{\sqrt{2}-1}+6GD\max(\sqrt{\lambda},1)(K-k) \\
\leq & 29GD\max(\sqrt{\lambda},1)\sqrt{n^{(k)}\log T}+6GD\max(\sqrt{\lambda},1)(K-k).
\end{aligned} \tag{68}
$$

We complete the proof by substituting (67) and (68) into (66).

A.10 Proof of Theorem 5

Our purpose is to provide a general analysis of Algorithm 1 over any bit sequence, so we do not make use of the range of x_t in (33). As an alternative, we use the following simple upper bound

$$
|x_t| \leq \mu n,\ \forall t \geq 1 \tag{69}
$$

which can be proved by induction. From the initialization, we have $|x_1| = 0 \leq \mu n$. Now, suppose $|x_k| \leq \mu n$. Then, we have

$$
|x_{k+1}| \leq |\rho x_k| + |b_k| \overset{\rho=1-1/n}{\leq} \left(1-\frac{1}{n}\right)\mu n + \mu = \mu n.
$$

Then, we can bound the difference between any two consecutive derivations by 2:

$$
|x_t - x_{t+1}| = |(1-\rho)x_t - b_t| \leq \frac{1}{n}|x_t| + |b_t| \overset{(69)}{\leq} 2\mu \leq 2. \tag{70}
$$

Next, we introduce Lemma 19 of Daniely and Mansour [2019], which characterizes the derivative of $g(\cdot)$ over short intervals.

Lemma 4 *Suppose* $\log\frac{1}{Z} \leq \frac{n}{16}$, $Z \leq \frac{1}{e}$ *and* $n \geq 8e$. *For every segment* $\mathcal{I} \subset \mathbb{R}$ *of length* ≤ 2 *and every* $x \in \mathcal{I}$, *we have*

$$
4\max_{s\in\mathcal{I}}|g'(s)| \leq \frac{1}{n}xg(x) + Z.
$$

Then, we can apply the above lemma to bound the derivative of $g(\cdot)$ over the interval $[x_t, x_{t+1}]$,[4] whose length is smaller than 2. Under the conditions of Lemma 4, we have

$$4 \max_{s\in[x_t,x_{t+1}]} |g'(s)| \leq \frac{1}{n} x_t g(x_t) + Z. \tag{71}$$

Since $g(x) = 0$ if $x \leq 0$, we have

$$4 \max_{s\in[x_t,x_{t+1}]} |g'(s)| \leq Z, \text{ if } x_t \leq 0. \tag{72}$$

Furthermore, we know that $g'(x) = 0$, if $x \geq U(n)$. When $x_t \geq U(n) + 2\mu$, from (70) we have

$$[x_t, x_{t+1}] \subset [U(n), \infty).$$

Thus,

$$\max_{s\in[x_t,x_{t+1}]} |g'(s)| = 0, \text{ if } x_t \geq U(n) + 2\mu. \tag{73}$$

Let $\mathbb{I}(x)$ be the indicator function of the interval $[0, U(n) + 2\mu]$. We can summarize the general result in (71) and the special cases in (72) and (73) as

$$4 \max_{s\in[x_t,x_{t+1}]} |g'(s)| \leq \frac{1}{n} x_t g(x_t) \mathbb{I}(x_t) + Z. \tag{74}$$

We proceed to use the following potential function

$$\Phi_t = \int_0^{x_t} g(s) ds$$

to analyze the reward of Algorithm 1. It is easy to verify that

$$\max(0, x_t - U(n)) \leq \Phi_t = \int_0^{x_t} g(s) ds \leq \max(x_t, 0). \tag{75}$$

To bound the change of the potential function, we need the following inequality for piece-wise differential functions $f : [a, b] \mapsto \mathbb{R}$ [Kapralov and Panigrahy, 2010, Daniely and Mansour, 2019]

$$\int_a^b f(x)dx \leq f(a)(b-a) + \max |f'(z)| \frac{1}{2}(b-a)^2. \tag{76}$$

We have

$$\begin{aligned}
\Phi_{t+1} - \Phi_t = &\int_{x_t}^{x_{t+1}} g(s) ds \\
&\overset{(76)}{\leq} g(x_t)(x_{t+1} - x_t) + \frac{1}{2}(x_{t+1} - x_t)^2 \max_{s\in[x_t,x_{t+1}]} |g'(s)| \\
&\overset{(70)}{\leq} g(x_t)\left(-\frac{1}{n}x_t + b_t\right) + 2 \max_{s\in[x_t,x_{t+1}]} |g'(s)| \\
&= g(x_t)\left(-\frac{1}{n}x_t + b_t\right) - 2 \max_{s\in[x_t,x_{t+1}]} |g'(s)| + 4 \max_{s\in[x_t,x_{t+1}]} |g'(s)| \\
&\leq g(x_t)\left(-\frac{1}{n}x_t + b_t\right) - 2\left|\frac{g(x_t) - g(x_{t+1})}{x_t - x_{t+1}}\right| + 4 \max_{s\in[x_t,x_{t+1}]} |g'(s)| \\
&\overset{(70)}{\leq} g(x_t)\left(-\frac{1}{n}x_t + b_t\right) - \frac{1}{\mu}|g(x_t) - g(x_{t+1})| + 4 \max_{s\in[x_t,x_{t+1}]} |g'(s)| \\
&\overset{(74)}{\leq} g(x_t)\left(-\frac{1}{n}x_t + b_t\right) - \frac{1}{\mu}|g(x_t) - g(x_{t+1})| + \frac{1}{n}x_t g(x_t)\mathbb{I}(x_t) + Z \\
&= g(x_t)b_t - \frac{1}{\mu}|g(x_t) - g(x_{t+1})| + \frac{1}{n}x_t g(x_t)\left(\mathbb{I}(x_t) - 1\right) + Z
\end{aligned} \tag{77}$$

[4] With a slight abuse of notation, we will write $[a, b]$ to denote $[\min\{a, b\}, \max\{a, b\}]$.

where the 3rd inequality is due to the mean value theorem. To bound the cumulative reward over any interval $[r, s]$, we sum (77) from $t = r$ to $t = s$, and obtain

$$\Phi_{s+1} - \Phi_r \leq \sum_{t=r}^{s} \left(g(x_t)b_t - \frac{1}{\mu}|g(x_t) - g(x_{t+1})| \right) + \sum_{t=r}^{s} \frac{1}{n} x_t g(x_t) \left(\mathbb{I}(x_t) - 1 \right) + Z\tau.$$

Thus,

$$\sum_{t=r}^{s} \left(g(x_t)b_t - \frac{1}{\mu}|g(x_t) - g(x_{t+1})| \right) \geq \Phi_{s+1} + \sum_{t=r}^{s} \frac{1}{n} x_t g(x_t) \left(1 - \mathbb{I}(x_t) \right) - \Phi_r - Z\tau. \quad (78)$$

First, we use the simple fact that

$$x_t g(x_t) \left(1 - \mathbb{I}(x_t) \right) \geq 0,$$

to simplify (78), and have

$$\sum_{t=r}^{s} \left(g(x_t)b_t - \frac{1}{\mu}|g(x_t) - g(x_{t+1})| \right) \geq -\Phi_r - Z\tau. \quad (79)$$

Second, we lower bound the cumulative reward by the summation of the bit sequence. From (75), we have

$$\Phi_{s+1} \geq x_{s+1} - U(n). \quad (80)$$

We also have

$$x_t g(x_t) \left(1 - \mathbb{I}(x_t) \right) \geq x_t - U(n) - 2\mu. \quad (81)$$

That is because if $x_t \geq U(n) + 2\mu$, we have

$$x_t g(x_t) \left(1 - \mathbb{I}(x_t) \right) = x_t \geq x_t - U(n) - 2\mu;$$

otherwise,

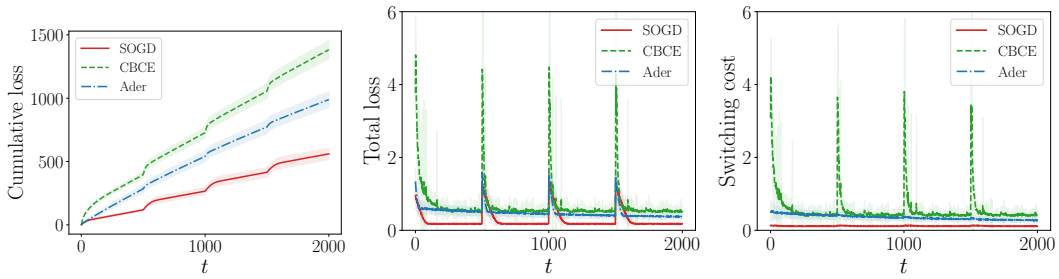
$$x_t g(x_t) \left(1 - \mathbb{I}(x_t) \right) \geq 0 \geq x_t - U(n) - 2\mu.$$

Based on (80) and (81), we have

$$\begin{aligned}
&\Phi_{s+1} + \sum_{t=r}^{s} \frac{1}{n} x_t g(x_t) \left(1 - \mathbb{I}(x_t) \right) \\
\geq & x_{s+1} - U(n) + \frac{1}{n} \sum_{t=r}^{s} \left(x_t - U(n) - 2\mu \right) = x_{s+1} + \frac{1}{n} \sum_{t=r}^{s} x_t - \frac{\tau}{n} \left(U(n) + 2\mu \right) - U(n) \\
= & \rho^{\tau} x_r + \sum_{j=r}^{s} \rho^{s-j} b_j + \frac{1}{n} \sum_{t=r}^{s} \left(\rho^{t-r} x_r + \sum_{j=r}^{t-1} \rho^{t-1-j} b_j \right) - \frac{\tau}{n} \left(U(n) + 2\mu \right) - U(n) \\
= & \rho^{\tau} x_r + \sum_{j=r}^{s} \rho^{s-j} b_j + \frac{x_r}{n} \sum_{t=r}^{s} \rho^{t-r} + \sum_{j=r}^{s} \frac{b_j}{n} \sum_{t=j+1}^{s} \rho^{t-1-j} - \frac{\tau}{n} \left(U(n) + 2\mu \right) - U(n) \\
= & \rho^{\tau} x_r + \sum_{j=r}^{s} \rho^{s-j} b_j + \frac{x_r}{n} \frac{1 - \rho^{\tau}}{1 - \rho} + \sum_{j=r}^{s} \frac{b_j}{n} \frac{1 - \rho^{s-j}}{1 - \rho} - \frac{\tau}{n} \left(U(n) + 2\mu \right) - U(n) \\
\overset{\rho = 1 - 1/n}{=} & x_r + \sum_{t=r}^{s} b_t - \frac{\tau}{n} \left(U(n) + 2\mu \right) - U(n).
\end{aligned}$$

Combining the above inequality with (78), we have

$$\sum_{t=r}^{s} \left(g(x_t)b_t - \frac{1}{\mu}|g(x_t) - g(x_{t+1})| \right) \geq \sum_{t=r}^{s} b_t + x_r - \frac{\tau}{n} \left(U(n) + 2\mu \right) - U(n) - \Phi_r - Z\tau. \quad (82)$$



(a) Cumulative loss over t rounds (b) Total loss in each round (c) Switching cost in each round

Figure 1: Performance of different methods versus the number of rounds.

Third, from (79) and (82), we have

$$\begin{aligned}
&\sum_{t=r}^{s}\left(g(x_t)b_t - \frac{1}{\mu}|g(x_t)-g(x_{t+1})|\right)\\
\geq &\max\left(0, \sum_{t=r}^{s} b_t + x_r - \frac{\tau}{n}\left(U(n)+2\mu\right) - U(n)\right) - \Phi_r - Z\tau\\
\overset{(75)}{\geq} &\max\left(0, \sum_{t=r}^{s} b_t + x_r - \frac{\tau}{n}\left(U(n)+2\mu\right) - U(n)\right) - \max(x_r, 0) - Z\tau
\end{aligned}$$

which proves (34).

Finally, to bound the change of successive predictions, we have

$$|g(x_t) - g(x_{t+1})| \leq |x_t - x_{t+1}| \max_s |g'(s)| \overset{(70)}{\leq} 2\mu \max_s |g'(s)|. \tag{83}$$

Following the analysis of Lemma 23 of Daniely and Mansour [2019], we know that $g'(\cdot)$ is nondecreasing in $[0, U(n)]$ and is 0 outside, and thus

$$\max_s |g'(s)| = g'(U(n)) = \frac{U(n)g(U(n))}{8n} + \frac{Z}{8} = \frac{U(n)}{8n} + \frac{Z}{8} \overset{(10)}{\leq} \frac{\sqrt{16n\log\frac{1}{Z}}}{8n} + \frac{Z}{8} \tag{84}$$

where the 2nd equality is due to the property of the confidence function [Daniely and Mansour, 2019, Lemma 18]. We obtain (35) by combining (83) and (84).

B Experiments

In this section, we implement online linear regression on synthetic data to evaluate our method, i.e., smoothed OGD (SOGD). In each round t, a batch of data points $\{(\mathbf{x}_{t,1}, y_{t,1}), \ldots, (\mathbf{x}_{t,n}, y_{t,n})\}$ arrive, where $\mathbf{x}_{t,i}$ is sampled randomly from $[-1, 1]^d$. The target value $y_{t,i}$ is generated by $y_{t,i} = \mathbf{w}^\top \mathbf{x}_{t,i} + \epsilon$, where $\epsilon \sim \mathcal{N}(0, 0.1)$ is a zero-mean Gaussian noise with standard deviation 0.1. The unknown parameter $\mathbf{w}$ is sampled randomly from $[-1, 1]^d$, and would be re-sampled every 500 rounds to simulate changing environments. After predicting $\mathbf{w}_t$, the online learner suffers the following total loss

$$\sum_{i=1}^{n} |\mathbf{w}_t^\top \mathbf{x}_{t,i} - y_{t,i}| + \lambda G \|\mathbf{w}_t - \mathbf{w}_{t-1}\|$$

which includes both the hitting cost and the switching cost.

In the experiment, we set $n = 64$, $d = 10$, $\lambda = 1$, $D = 2\sqrt{10}$, and $G = \sqrt{10}$. We compare our method with CBCE [Jun et al., 2017a] and Ader [Zhang et al., 2018a], which obtain optimal adaptive regret and dynamic regret respectively, but do not consider the switching cost.

The whole experiment is conducted on a personal laptop equipped with an Intel i7-10750H CPU and 16G memory. We repeat the experiment 100 times and plot the average cumulative loss, total loss and switching cost in Fig. 1. As can be seen, all three methods can deal with changing environments and adapt quickly when the underlying parameter $\mathbf{w}$ changes. Among them, our SOGD suffers the smallest cumulative loss, and incurs the least total loss in most rounds. As indicated in Fig. 1(c), both Ader and CBCE have much higher switching cost compared with our method, and CBCE suffers huge switching loss when $\mathbf{w}$ is re-sampled. In contrast, SOGD maintains the lowest switching cost in all rounds, since it explicitly takes the switching cost into consideration.