Optimal Gradient Sliding and its Application to Distributed Optimization Under Similarity

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Abstract

We study structured convex optimization problems, with additive objective $r := p + q$, where r is (μ-strongly) convex, q is L_q-smooth and convex, and p is L_p-smooth, possibly nonconvex. For such a class of problems, we proposed an inexact accelerated gradient sliding method that can skip the gradient computation for one of these components while still achieving optimal complexity of gradient calls of p and q, that is, $\mathcal{O}(\sqrt{L_p/\mu})$ and $\mathcal{O}(\sqrt{L_q/\mu})$, respectively. This result is much sharper than the classic black-box complexity $\mathcal{O}(\sqrt{(L_p + L_q)/\mu})$, especially when the difference between L_p and L_q is large. We then apply the proposed method to solve distributed optimization problems over master-worker architectures, under agents' function similarity, due to statistical data similarity or otherwise. The distributed algorithm achieves for the first time lower complexity bounds on *both* communication and local gradient calls, with the former having being a long-standing open problem. Finally the method is extended to distributed saddle-problems (under function similarity) by means of solving a class of variational inequalities, achieving lower communication and computation complexity bounds.

1 Introduction

We consider structured convex programming in the form [6, 13, 35]:

$$\min_{x \in \mathbb{R}^d} r(x) := q(x) + p(x), \tag{1}$$

where r is assumed to be convex and decomposed as the sum of a smooth, possibly nonconvex function p and a smooth convex function q. First order information of p and q is accessible separately. We are interested in scenarios where the cost of evaluating the gradient of the two functions is not even, but computing, say ∇p, is much more resource demanding than ∇q. The motivating application for this scenario is distributed optimization over master-worker systems, as discussed next.

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Consider the following distributed optimization problem over a network of n agents:

$$\min_{x\in\mathbb{R}^d} r(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x), \tag{2}$$

where f_i is the loss function of agent i, assumed to be convex, which is not known to other agents. Agents are embedded in a star-topology, with agent 1 being the master node, without loss of generality–this is the typical federated learning setup [24]. An instance of (2) of particular interest is the empirical risk minimization (ERM) whereby the goal is to minimize the average loss over some dataset, distributed across the nodes of the network, with f_i being the empirical risk of agent i, i.e., $f_i(x) = \frac{1}{m}\sum_{j=1}^{m} \ell\big(x; z_i^j\big)$, where $z_i^{(1)}, \ldots, z_i^{(m)}$ is the set of m samples owned by agent i, and $\ell(x; z_i^j)$ measures the loss of the model x on the sample z_i^j.

Several solutions methods have been proposed to solve (2); the prototype approach consists in interleaving local computations at the workers sides (nodes $i = 1, \ldots n$) with communications to/from the master node ($i = 1$), which maintains and updates the authoritative copy of the optimization variables, producing eventually the final solution estimate. Since the cost of communications is often the bottleneck in distributed computing (e.g., [7, 32]), a lot of research has been devoted to designing distributed algorithms that are *communication efficient*. Acceleration (in the sense of Nesterov) has been extensively investigated as a procedure to reduce the communication burden. For L-smooth and μ-strongly convex functions r in (2), linear convergence is certified by employing first-order methods, with computation (gradient evaluations) and communication complexities proportional to $\sqrt{\kappa}$ ($\kappa \triangleq L/\mu$ is the condition number of r). For ill-conditioned functions (κ very large), the polynomial dependence on κ may be unsatisfactory. This is, e.g., the typical setting of many ERM problems wherein the optimal regularization parameter for test predictive performance is very small.

Further improvements on the communication complexity can be obtained exploiting the extra structure typical in ERM problems, also known as *function similarity* (see, e.g., [4, 51, 44, 48]): $\|\nabla^2 f_i(x) - \nabla^2 f_j\| \leq \delta$, for all x in a proper domain of interest and all $i \neq j = 1, \ldots, n$, where $\delta > 0$ measures the degree of similarity between the Hessian matrices of the local losses. When data are i.i.d. among agents, f_i's reflect statistical similarities in local data, resulting in $\delta = \tilde{O}(1/\sqrt{m})$ with high-probability ($\tilde{O}$ hides log-factors and dependence on d). In this scenario, in general, $1 + \delta/\mu \ll \kappa$ [4]. This motivated a surge of studies aiming at exploiting function similarity coupled with acceleration to boost communication efficiency (see Sec. 1.2 for an overview of relevant works): linear convergence is certified with a number of communication steps (for nonquadratic losses) scaling with $\widetilde{\mathcal{O}}(\sqrt{\delta/\mu})$, where $\widetilde{\mathcal{O}}$ hides log-factors. This matches lower (communication) complexity bounds [4] *only up to log-factors*. Furthermore, these methods are not computationally optimal, yielding local gradients calls larger than lower complexity bounds $\mathcal{O}(\sqrt{\kappa})$. In fact, Table 1 shows that, to the date, there exists no distributed algorithm achieving the best of the two worlds, that is, optimal (lower bound) communication complexity *and* local gradient (oracle) complexity.

This paper fills this gap. Our starting point is the reformulation of (2) in the equivalent form

$$\min_{x\in\mathbb{R}^d} r(x) = \underbrace{f_1(x)}_{:=q(x)} + \underbrace{\frac{1}{n}\sum_{i=1}^{n}[f_i(x) - f_1(x)]}_{:=p(x)}, \tag{3}$$

which exploits function similarity at the agents' side via preconditioning. Problem (3) is an instance of (1): all f_i (thus q) are convex but p is nonconvex. Also, evaluating ∇q and ∇p has different costs; the former involves only local computations at the master node while the latter requires communications from/to master and workers nodes. At high-level the idea is then clear: one would like to design a distributed algorithm for (3) [or more generally for (1)] that skips gradient computations of ∇p (saving thus communications) without slowing down the overall optimal rate of convergence.

This naturally suggests the use of *gradient-sliding* techniques [19, 41, 10], yielding algorithms that skip from time to time computation of the gradient of one function in the summand objective. However, existing gradient-sliding algorithms are not applicable to (1) [and thus (3)] because they all require p and q to be *convex*. This calls for new designs, accounting for the nonconvexity of p.

1.1 Main contributions

Our contribution is threefold:

• **A new gradient-sliding algorithm for (1):** We propose a new Accelerated ExtraGradient sliding method that skips the computation of ∇p from time to time. The method builds on an inexact acceleration of a proximal envelop (outer-loop) coupled with a suitable termination criterion and inner-loop algorithm to approximately solve the proximal subproblem. When applied to (1), with r being (μ-strongly) convex, q being L_q-smooth and convex, and p being L_p-smooth (possibly nonconvex), the proposed algorithm achieves optimal complexity of gradient calls of q and p, that is,

r	∇q	∇p
strongly convex	$\mathcal{O}\left(\sqrt{\frac{L_q}{\mu}}\log\frac{1}{\varepsilon}\right)$	$\mathcal{O}\left(\sqrt{\frac{L_p}{\mu}}\log\frac{1}{\varepsilon}\right)$
convex	$\mathcal{O}\left(\sqrt{\frac{L_q}{\varepsilon}}\|x^0-x^*\|\right)$	$\mathcal{O}\left(\sqrt{\frac{L_p}{\varepsilon}}\|x^0-x^*\|\right)$

Notice that the above complexity bounds are sharper than the complexity bound obtained by the Nesterov's optimal first-order method for smooth (strongly) convex optimization applied to (1). For instance for strongly convex r, that would yield ∇q, ∇p complexity scaling as $\mathcal{O}(\sqrt{(L_q+L_p)/\mu})$, which is less favorable than our separate complexity bounds above. To the best of our knowledge, this is the first time that such bounds are achieved for nonconvex p.

• **Optimal complexity bounds for (2) under function similarity:** We customize the proposed accelerated gradient-sliding algorithm to the distributed optimization problem (2) under δ-function similarity. As showed in Table 1 for strongly convex r (see Sec. 3 for the case of convex r), the new distributed algorithm achieves lower complexity bounds on *both* the number of communications [4] and on the number of gradient computations (without logarithmic factors!) [37]. Achieving optimal communication complexity (for nonquadratic losses) was a long-standing open problem.

• **Gradient sliding for variational inequalities:** We extend the proposed gradient-sliding machinery to solve distributed saddle-points under similarity by means of solving a class of strongly-monotone Variational Inequalities (VI). We improve existing complexity bounds for such problems [9] achieving for the first time both optimal communication complexity and gradient oracle complexity–see Table 1.

Table 1: Existing convergence results for distributed (saddle point) optimization under δ-similarity.

		Reference	Communication complexity	Local gradient complexity	Order	Limitations
Minimization	Upper	DANE [42]	$\mathcal{O}\left(\frac{\delta^2}{\mu^2}\log\frac{1}{\varepsilon}\right)$	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\sqrt{\frac{\delta^3}{\mu^3}}\log^2\frac{1}{\varepsilon}\right)$ (2)	1st	quadratic
		DiSCO [51]	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}(\log\frac{1}{\varepsilon}+C^2\Delta F_0)\log\frac{L}{\mu}\right)$	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}(\log\frac{1}{\varepsilon}+C^2\Delta F_0)\log\frac{L}{\mu}\right)$	2nd	C - self-concordant (3)
		AIDE [40]	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}\log\frac{1}{\varepsilon}\log\frac{L}{\delta}\right)$	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\sqrt{\frac{\delta}{\mu}}\log\frac{1}{\varepsilon}\log\frac{L}{\delta}\right)$ (4)	1st	quadratic
		DANE-LS [50]	$\mathcal{O}\left(\frac{\delta}{\mu}\log\frac{1}{\varepsilon}\right)$	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\frac{\delta^{3/2}}{\mu^{3/2}}\log\frac{1}{\varepsilon}\right)$ (5)	1st/2nd	quadratic (6)
		DANE-HB [50]	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}\log\frac{1}{\varepsilon}\right)$	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\frac{\delta}{\mu}\log\frac{1}{\varepsilon}\right)$ (5)	1st/2nd	quadratic (6)
		SONATA [45]	$\mathcal{O}\left(\frac{\delta}{\mu}\log\frac{1}{\varepsilon}\right)$	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\sqrt{\frac{\delta}{\mu}}\log^2\frac{1}{\varepsilon}\right)$ (2)	1st	decentralized
		SPAG [21]	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\log\frac{1}{\varepsilon}\right)$ (1)	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\sqrt{\frac{L}{\delta}}\log^2\frac{1}{\varepsilon}\right)$ (1,2)	1st	M - Lipshitz hessian
		DiRegINA [12]	$\mathcal{O}\left(\frac{\delta}{\mu}\log\frac{1}{\varepsilon}+\sqrt{\frac{M\delta R_0}{\mu}}\right)$	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\sqrt{\frac{\delta}{\mu}}\log^2\frac{1}{\varepsilon}+\sqrt{\frac{MLR_0}{\mu}}\log\frac{1}{\varepsilon}\right)$ (2)	2nd	M -Lipshitz hessian
		ACN [1]	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}\log\frac{1}{\varepsilon}+\sqrt[3]{\frac{M\delta R_0}{\mu}}\right)$	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\log^2\frac{1}{\varepsilon}+\sqrt[3]{\frac{M\delta R_0}{\mu}}\sqrt{\frac{L}{\delta}}\log\frac{1}{\varepsilon}\right)$ (2)	2nd	M -Lipshitz hessian
		AccSONATA [46]	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}\log\frac{1}{\varepsilon}\log\frac{L}{\mu}\right)$	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}\log^2\frac{1}{\varepsilon}\log\frac{\delta}{\mu}\right)$ (2)	1st	decentralized
		This paper	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}\log\frac{1}{\varepsilon}\right)$	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\log\frac{1}{\varepsilon}\right)$	1st	
	Lower	[4]	$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}}\log\frac{1}{\varepsilon}\right)$	—		
		[37]	—	$\mathcal{O}\left(\sqrt{\frac{L}{\mu}}\log\frac{1}{\varepsilon}\right)$		non-distributed
Saddles	Upper	SMMDSA [9]	$\mathcal{O}\left(\frac{\delta}{\mu}\log\frac{1}{\varepsilon}\right)$	$\mathcal{O}\left(\frac{L}{\mu}\log\frac{1}{\varepsilon}\log\frac{L}{\mu}\right)$	1st	
		This paper	$\mathcal{O}\left(\frac{\delta}{\mu}\log\frac{1}{\varepsilon}\right)$	$\mathcal{O}\left(\frac{L}{\mu}\log\frac{1}{\varepsilon}\right)$	1st	
	Lower	[9]	$\mathcal{O}\left(\frac{\delta}{\mu}\log\frac{1}{\varepsilon}\right)$	—		
		[39]	-	$\mathcal{O}\left(\frac{L}{\mu}\log\frac{1}{\varepsilon}\right)$		non-distributed

(1) This is the worst-case complexity, as pointed out in the paper; the convergence of the method might be better, based upon an additional sequence G_t [21]; (2) proximal local computations (exact solution of local subproblems), we filled these cells assuming that the proximal operator is computed using Accelerated Gradient Descent [37] with accuracy ε^2; (3) from Lipschitzness of the Hessian and strong convexity follows self-concordance; (4) gradient complexity not provided, we derived it using [37]; (5) gradient complexity not provided, we derived it using [38]; (6) gradient complexity holds for nonquadratic functions;
Notation: δ = similarity parameter, L=smoothness constant of f_i, μ = strong convexity constant of r, ε =accuracy of the solution, $R_0 := \|x^0 - x^*\|$, $\Delta F_0 := r(x^0) - r(x^*)$.

1.2 Related works

Gradient-Sliding: Since the seminal paper [28], the idea of gradient-sliding for structured convex optimization such as (1) has received significant attention as tool to skip gradient computations of one function in the summand; examples and generalization include first-order accelerated methods [28, 29, 31], zero-order (derivative-free) schemes [14, 22, 8, 43], high-order methods [25, 17, 2, 20], and slidings for saddle point problems and variational inequalities [3, 30, 47, 9]. Albeit applicable to more general classes of optimization problems than (1) (e.g., allowing either p and q to be nonsmooth), none of the existing methods provide guarantees when p is nonconvex. The proposed algorithm fills this gap. Furthermore, it achieves optimal lower complexity bounds on the calls of ∇p and ∇q.

Distributed optimization under function similarity: The literature of distributed optimization is vast; given the focus of this work, we comment next solution methods exploiting function similarity via proper preconditioning–Table 1 summarizes complexity results of existing distributed methods solving either minimization problems or saddle-point formulations, and is commented next.

The seminal paper [4] established lower communication complexity bounds for (2) under δ-similarity: ε-optimality cannot be achieve in less than $\Omega(\sqrt{\delta/\mu}\log 1/\varepsilon)$ communication rounds. Since then, a lot of effort has been devoted to design distributed schemes aiming at achieving optimal communication complexity. The authors in [42] proposed `DANE`, a mirror-descent based algorithm whereby workers perform a local data preconditioning via a suitably chosen Bregman divergence, and the master averages the solutions of the workers. For *quadratic* losses, `DANE` achieves communication complexity $\widetilde{\mathcal{O}}((\delta/\mu)^2\log 1/\varepsilon)$; this was later improved to $\mathcal{O}((\delta/\mu)\log 1/\varepsilon)$ for nonquadratic losses in [45], where the `SONATA` algorithm was proposed (also implementable over mesh-networks).

Improvements were achieved employing acceleration; efforts include: `DiSCO` [51], an inexact damped Newton method coupled with a preconditioned conjugate gradient (to compute the Newton direction), which achieves communication complexity $\widetilde{\mathcal{O}}(\sqrt{\delta/\mu})\log 1/\varepsilon)$ for self-concordant losses (see Table 1 for the log-factors hidden in the $\widetilde{\mathcal{O}}$); `AIDE` [40], which uses the Catalyst framework [33], matching the rate of `DiSCO` for quadratic losses; `DANE-HB` [50], a variant of `DANE` equipped with Heavy Ball momentum and matching for quadratic functions the communication complexity of `DiSCO` and `AIDE`; and `SPAG` [21], a preconditioned direct accelerated method, achieving for nonquadradic losses *asymptotically* the convergence rate $\mathcal{O}((1-1/\sqrt{\beta/\mu})^k)$ (k is the iteration index)–the worst-case rate is still $\mathcal{O}(\sqrt{L/\mu})\log 1/\varepsilon)$.

Finally, higher order methods employing preconditioning have been studied in [12, 1, 46]: [12] proposed DiRegINA, a decentralization of the cubic regularization of the Newton method, where workers build Newton direction sampling local Hessians; [1] introduced `ACN`, an inexact accelerated cubic-regularized Newton's method, with improved complexity with respect to [12]; and [46] extended the Catalyst framework [33] to the distributed setting (including mesh networks), proposing `Acc SONATA`–the communication complexity of these methods is reported in Table 1.

In summary, the above tour on the relevant literature shows that none of the existing methods can match lower communication complexity bounds for (non quadratic) optimization problems (2) under function similarity (all complexity bounds contain log-factors). The proposed distributed method achieves lower communication and computation complexity bounds.

2 Optimal Gradient Sliding for Minimization Problems

We study the minimization problem (1), under the following blanket assumptions.

Assumption 1. *$r(x)\colon \mathbb{R}^d \to \mathbb{R}$ is μ-strongly convex on $\mathbb{R}^d$.*

Assumption 2. *$q(x)\colon \mathbb{R}^d \to \mathbb{R}$ is convex and L_q-smooth on $\mathbb{R}^d$.*

Assumption 3. *$p(x)\colon \mathbb{R}^d \to \mathbb{R}$ is L_p-smooth on $\mathbb{R}^d$.*

The proposed Accelerated ExtraGradient sliding is formally introduced in Algorithm 1. Convergence of the outer loop is established in Theorem 1, while Theorem 2 determines complexity of solving the inner loop up to a suitable termination. Finally Theorem 3 provides the overall complexity merging inner and outer loop results. The proof of all the theorems can be found in Appendix A.1.

Algorithm 1 Accelerated Extragradient

1: **Input:** $x^0 = x_f^0 \in \mathbb{R}^d$
2: **Parameters:** $\tau \in (0,1], \eta, \theta, \alpha > 0, K \in \{1, 2, \dots\}$
3: **for** $k = 0, 1, 2, \dots, K-1$ **do**
4: $\quad x_g^k = \tau x^k + (1-\tau) x_f^k$
5: $\quad x_f^{k+1} \approx \arg\min_{x \in \mathbb{R}^d} \left[A_\theta^k(x) := p(x_g^k) + \langle \nabla p(x_g^k), x - x_g^k \rangle + \frac{1}{2\theta} \|x - x_g^k\|^2 + q(x) \right]$
6: $\quad x^{k+1} = x^k + \eta\alpha(x_f^{k+1} - x^k) - \eta \nabla r(x_f^{k+1})$
7: **end for**
8: **Output:** x^K

Theorem 1. *Consider Algorithm 1 for Problem 1 under Assumptions 1-3, with the following tuning:*

$$\tau = \min\left\{1, \frac{\sqrt{\mu}}{2\sqrt{L_p}}\right\}, \quad \theta = \frac{1}{2L_p}, \quad \eta = \min\left\{\frac{1}{2\mu}, \frac{1}{2\sqrt{\mu L_p}}\right\}, \quad \alpha = \mu;$$

and let x_f^{k+1} in line 5 satisfy

$$\|\nabla A_\theta^k(x_f^{k+1})\|^2 \le \frac{L_p^2}{3} \|x_g^k - \arg\min_{x \in \mathbb{R}^d} A_\theta^k(x)\|^2. \tag{4}$$

Then, for any

$$K \ge 2 \max\left\{1, \sqrt{\frac{L_p}{\mu}}\right\} \log \frac{\|x^0 - x^*\|^2 + \frac{2\eta}{\tau}\left[r(x^0) - r(x^*)\right]}{\varepsilon}, \tag{5}$$

we have the following estimate for the distance to the solution x^*:

$$\|x^K - x^*\|^2 \le \varepsilon. \tag{6}$$

2.1 Solving the auxiliary subproblems

At each iteration of Algorithm 1, one needs to solve the subproblem:

$$\min_{x \in \mathbb{R}^d} A_\theta^k(x) := p(x_g^k) + \langle \nabla p(x_g^k), x - x_g^k \rangle + \frac{1}{2\theta} \|x - x_g^k\|^2 + q(x). \tag{7}$$

According to Theorem 1, (7) need not be solved to arbitrary precision; inexact solutions x_f^{k+1} satisfying condition (4) suffice. Condition (4) means that gradient norm $\|\nabla A_\theta^k(x_f^{k+1})\|$ should be sufficiently small. Notice that $A_\theta^k(x)$ in (7) is $(2L_p + L_q)$-smooth and convex.

Problem 7 can be solved up to the termination (4) using any of the algorithms in [27, 26, 38]. We obtain the following complexity.

Theorem 2 ([38] Remark 1)**.** *There exists a certain algorithm such that, when applied to problem (7) with the starting point x_g^k, returns x_f^{k+1} satisfying*

$$\|\nabla A_\theta^k(x_f^{k+1})\| \le \frac{D^2 \cdot \max\{L_p, L_q\} \|x_g^k - \arg\min_{x \in \mathbb{R}^d} A_\theta^k(x)\|}{T^2}, \tag{8}$$

where $D > 0$ is some universal constant (independent of L_p, L_q, T etc) and T is the number of calls of ∇q by the algorithm.

2.2 Overall complexity of the optimal gradient-sliding

Theorem 2 suggests that, to satisfy condition (4) in Theorem 1, it is sufficient to choose the number T of iterations of the inner algorithm as

$$T = \sqrt[4]{3} D \max\left\{1, \sqrt{\frac{L_q}{L_p}}\right\}. \tag{9}$$

We can now determine the overal complexity of Algorithm 1. At each iteration of Algorithm 1 we call ∇p twice (at x_g^k – line 5 and at x_f^{k+1} – line 6), and ∇q is computed $T+1$ times (T times in the

auxiliary problem – line 5, and at x_f^{k+1} – line 6). Hence, to find an ε-solution of problem (1), i.e., to find $x^K \in \mathbb{R}^d$ that satisfies (6), Algorithm 1 requires K iterations as given in (5),

$$2 \times K = \mathcal{O}\left(\max\left\{1, \sqrt{\frac{L_p}{\mu}}\right\} \log \frac{1}{\varepsilon}\right) \text{ calls of } \nabla p(x), \text{ and}$$

$$(T+1) \times K = \mathcal{O}\left(\max\left\{1, \sqrt{\frac{L_q}{L_p}}, \sqrt{\frac{L_p}{\mu}}, \sqrt{\frac{L_q}{\mu}}\right\} \log \frac{1}{\varepsilon}\right) \text{ calls of } \nabla q(x).$$

Putting everything together we obtain the following final convergence result.

Theorem 3. *Consider Problem (1) under Assumptions 1 to 3, with $\mu \leq L_p \leq L_q$. Then, to reach an ε-solution, Algorithm 1 requires*

$$\mathcal{O}\left(\sqrt{\frac{L_q}{\mu}} \log \frac{1}{\varepsilon}\right) \text{ calls of } \nabla q(x) \text{ and } \mathcal{O}\left(\sqrt{\frac{L_p}{\mu}} \log \frac{1}{\varepsilon}\right) \text{ calls of } \nabla p(x).$$

This matches optimal complexity for the individual gradient calls.

We conclude this section providing convergence of variant of the proposed algorithm suitable for convex r in (1) (Assumption 1 with $\mu = 0$). The algorithm is described in Appendix A.2. Here we only provide the final convergence result, the analogous of Theorem 3.

Theorem 4. *Consider Problem (1) under Assumptions 1 (with $\mu = 0$)-3, with $L_p \leq L_q$. Then, to find an ε-solution of (1) (in objective value), Algorithm 3 in Appendix A.2 requires*

$$\mathcal{O}\left(\sqrt{\frac{L_q}{\varepsilon}} \|x^0 - x^*\|\right) \text{ calls of } \nabla q(x) \text{ and } \mathcal{O}\left(\sqrt{\frac{L_p}{\varepsilon}} \|x^0 - x^*\|\right) \text{ calls of } \nabla p(x).$$

3 Application to Distributed Optimization Under Similarity

In this section, we apply the proposed algorithm the the distributed optimization problem (2), under the following assumptions.

Assumption 4. *Each $f_i(x) \colon \mathbb{R}^d \to \mathbb{R}$ is convex and L-smooth.*

Assumption 5. *$r(x) \colon \mathbb{R}^d \to \mathbb{R}$ is μ-strongly convex.*

Assumption 6. *$f_1(x), \ldots, f_n(x)$ are δ-related: $\|\nabla^2 f_i(x) - \nabla^2 f_j(x)\| \leq \delta$, for all $i \neq j$ and $x \in \mathbb{R}^d$, and some $\delta > 0$.*

From the last assumption it is easy to get that for all i and $x \in \mathbb{R}^d$ we have $\|\nabla^2 f_i(x) - \nabla^2 r(x)\| = \|\frac{1}{n}\sum_{j=1}^n [\nabla^2 f_i(x) - \nabla^2 f_j(x)]\| \leq \frac{1}{n}\sum_{j=1}^n \|\nabla^2 f_i(x) - \nabla^2 f_j(x)\| \leq \delta$.

We leverage now Algorithm 1 to solve (2), using the equivalent reformulation (3). The algorithm applied to the distributed system can be described as follows. The server computes x_g^k and sends it to all the workers (line 4). Workers compute $\nabla f_i(x_g^k)$ and send it to the server. After collecting all $\nabla f_i(x_g^k)$, the server builds $\nabla p(x_g^k) = \nabla r(x_g^k) - \nabla f_1(x_g^k)$, and then solves (inexactly) the local problem A_θ^k (line 5). The inexact solution x_f^{k+1} is then broadcast to the workers, which update their own receives $\nabla f_i(x_f^{k+1})$ and send back to the server, which can then evaluate $\nabla r(x_f^{k+1})$ (line 6).

Using Assumptions 4 and 5, we infer that r is μ-strongly convex; and $q = f_1$ is L_q-smooth and convex, with $L_q = L$. It follows from Assumption 6 that $\|\nabla^2 p\| \leq \delta$. Therefore, p has L_p-Lipschitz gradient, with $L_p = \delta$. This shows that we can leverage Theorems 3 and 4 to establish convergence for strongly convex and convex r, as given next.

Theorem 5. *Let Assumptions 4 to 6 be satisfied with $\mu \leq \delta \leq L$. Then, to find ε-solution of the distributed optimization problem (2) Algorithm 1 requires*

$$\mathcal{O}\left(\sqrt{\frac{\delta}{\mu}} \log \frac{1}{\varepsilon}\right) \text{ communication rounds and } \mathcal{O}\left(\sqrt{\frac{L}{\mu}} \log \frac{1}{\varepsilon}\right) \text{ local gradient computations.}$$

Theorem 6. *Let Assumptions 4, 5 (with $\mu = 0$), 6 be satisfied and $\delta \leq L$. Then, to find ε-solution of the distributed optimization problem (2) Algorithm 1 requires*

$$\mathcal{O}\left(\sqrt{\frac{\delta}{\varepsilon}} \|x^0 - x^*\|\right) \text{ communication rounds and } \mathcal{O}\left(\sqrt{\frac{L}{\varepsilon}} \|x^0 - x^*\|\right) \text{ local computations.}$$

Such estimates are optimal from both communications [4] and local computations point of views [37]. It is important to remark that Algorithm 1 solves the local subproblem A_θ with some precision, while most of existing works (see Table 1) assume that local problems are solved with infinite precision (column Local gradient complexity), which is not practical. Note also that the subproblems in line 5 of Algorithm 1 do not necessarily have to be solved by a deterministic algorithm as in Theorem 2. Stochastic methods can also be used, as long as they guarantee that condition (4) is met.

4 Optimal Gradient Sliding for VIs

In this section we consider the composite variational inequality [16, 5] in the form:

$$\text{Find } x^* \in \mathbb{R}^d \ : \ R(x^*) = 0 \text{ with } R(x) := Q(x) + P(x), \tag{10}$$

where $Q(x), P(x) \colon \mathbb{R}^d \to \mathbb{R}^d$. Variational inequalities are a unified umbrella for a variety of problems–two examples follow.

Example 1 [Minimization]. Consider problem (1), choose $Q(x) = \nabla q(x)$ and $P(x) = \nabla p(x)$. Then the solution of the variational inequality (10) means that we need to find the point x^* where the operator $R(x^*) = \nabla r(x^*)$. For the convex function r, this is equivalent to finding the minimum.

Example 2 [Saddle point problems]. Consider the convex-concave saddle point problem

$$\min_{y \in \mathbb{R}^{d_y}} \max_{z \in \mathbb{R}^{d_z}} r(y, z) := q(y, z) + p(y, z). \tag{11}$$

If we take $Q(x) := Q(y, z) = [\nabla_y q(y, z), -\nabla_z q(y, z)]$ and $P(x) := P(y, z) = [\nabla_y p(y, z), -\nabla_z p(y, z)]$, then it can be proved that $x^* = (y^*, z^*)$ is a solution for (10) if and only if $x^* = (y^*, z^*)$ is a solution for (11), i.e.

$$r(y^*, z) \le r(y^*, z^*) \le r(y, z^*) \quad \text{for all } y \in \mathbb{R}^{d_y} \text{ and } z \in \mathbb{R}^{d_z}.$$

While minimization problems are widely considered separately from variational inequalities, saddle point problems are often analyzed under the VI lens. In recent years the popularity of saddles has grown, this is due to the fact that they have both classical [15] and new ML [18, 34] applications.

We study problem (10) under the following assumptions.

Assumption 7. *$R(x)$ is μ-strongly monotone: $\langle R(x_1) - R(x_2), x_1 - x_2 \rangle \ge \mu \|x_1 - x_2\|^2$, for all $x_1, x_2 \in \mathbb{R}^d$.*

Assumption 8. *$Q(x)$ is monotone and L_q-Lipschitz: $\langle Q(x_1) - Q(x_2), x_1 - x_2 \rangle \ge 0$ and $\|Q(x_1) - Q(x_2)\| \le L_q \|x_1 - x_2\|$, for all $x_1, x_2 \in \mathbb{R}^d$.*

Assumption 9. *$P(x)$ is L_p-Lipschitz: $\|P(x_1) - P(x_2)\| \le L_p \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^d$.*

For saddle point problems these assumptions are equivalent to (strong) convexity–(strong) concavity and Lipschitzness of gradients.

4.1 Sliding via Extragradient

This algorithm is a non-accelerated version of Algorithm 1. A similar non-accelerated sliding is used in [9]. Our version however has better theoretical and practical guarantees because of the effective stopping criterion (12). Convergence of the outer loop is established in Theorem 7; Theorem 8 establishes convergence of the inner loop up to the required termination; and finally Theorem 9 combine the two-loop complexity. The proofs of the theorems can be found in Appendix B.1.

Algorithm 2 Extragradient Sliding for VIs

1: **Input:** $x^0 \in \mathbb{R}^d$
2: **Parameters:** $\eta, \theta, \alpha > 0, K \in \{1, 2, \ldots\}$
3: **for** $k = 0, 1, 2, \ldots, K-1$ **do**
4: Find $u^k \approx \tilde{u}^k$ where $\tilde{u}^k$ is a solution for

$$\text{Find } \tilde{u}^k \in \mathbb{R}^d \ : \ B_\theta^k(\tilde{u}^k) = 0 \text{ with } B_\theta^k(x) := P(x^k) + Q(x) + \frac{1}{\theta}(x - x^k)$$

5: $x^{k+1} = x^k + \eta\alpha(u^k - x^k) - \eta R(u^k)$
6: **end for**
7: **Output:** x^K

Theorem 7. *Consider Algorithm 2 for Problem (10) under Assumptions 7-9, with the following tuning:*

$$\theta = \tfrac{1}{2L_p}, \quad \eta = \min\left\{\tfrac{1}{4\mu}, \tfrac{1}{4L_p}\right\}, \quad \alpha = 2\mu.$$

Assume that u^k (line 4) satisfies

$$\|B_\theta^k(u^k)\|^2 \le \tfrac{L_p^2}{3}\|x^k - \tilde{u}^k\|^2. \tag{12}$$

Then, for any

$$K \ge 2\max\left\{1, \tfrac{L_p}{\mu}\right\}\log\tfrac{\|x^0 - x^*\|^2}{\varepsilon}, \tag{13}$$

we have the following estimate for the distance to the solution x^*:

$$\|x^K - x^*\|^2 \le \varepsilon. \tag{14}$$

The proof is given in Appendix B.1.

4.2 Solving the auxiliary problem

As for Algorithm 1, we need an auxiliary solver for the subproblem in line 4, which ensures that (12) is met. One can observe that $B_\theta^k(x)$ is $(2L_p + L_q)$-Lipschitz and monotone. For this type of problem, the authors in [49] proposes an approach that guarantees convergence on $\|B_\theta^k(u^k)\|^2$.

Theorem 8 ([49] Corollary 2)**.** *There exists a certain algorithm that, applied to the subproblem (12) with starting point x^k, returns u^k satisfying*

$$\|B_\theta^k(u^k)\|^2 \le \frac{D^2 \cdot \max\{L_p^2, L_q^2\}\|x^k - \tilde{u}^k\|^2}{T^2}, \tag{15}$$

where $D > 0$ is some universal numerical constant (independent of L_p, L_q, T etc) and T is the number of calls of the operator Q.

4.3 Complexity of the optimal gradient sliding

Leveraging Theorems 7 and 8 while following the same reasoning as in Section 2.2, we obtain the following convergence (inner plus outer loops) for Algorithm 2.

Theorem 9. *Let Assumptions 7 to 9 be satisfied with $\mu \le L_p \le L_q$. Then, Algorithm 2 requires*

$$\mathcal{O}\left(\tfrac{L_q}{\mu}\log\tfrac{1}{\varepsilon}\right) \text{ calls of } Q(x) \quad \text{and} \quad \mathcal{O}\left(\tfrac{L_q}{\mu}\log\tfrac{1}{\varepsilon}\right) \text{ calls of } P(x)$$

to find an ε-solution of problem (10).

We also consider the case of monotone VIs (Assumption 7 with $\mu = 0$). For this we modify Algorithm 2 as in Appendix B.2 and obtain the following convergence result.

Theorem 10. *Let Assumption 7 (with $\mu = 0$), 8, 9 be satisfied and $L_p \le L_q$. Then, Algorithm 4, described in Appendix B.2, requires*

$$\mathcal{O}\left(\tfrac{L_q}{\varepsilon}\|x^0 - x^*\|^2\right) \text{ calls of } Q(x) \quad \text{and} \quad \mathcal{O}\left(\tfrac{L_p}{\varepsilon}\|x^0 - x^*\|^2\right) \text{ calls of } P(x)$$

to find an ε-solution of the problem (10). Here ε-solution is measured by the value of the gap function.

4.4 Application to distributed saddle-point problem under similarity

We apply now Algorithm 2 to solve a distributed saddle-point problem under statistical similarity, as introduced in [9]:

$$\min_{y\in\mathbb{R}^{d_y}} \max_{z\in\mathbb{R}^{d_z}} r(y,z) := \tfrac{1}{n}\textstyle\sum_{i=1}^n f_i(y,z). \tag{16}$$

Assumption 10. *Each $f_i(y,z)\colon \mathbb{R}^{d_y}\times\mathbb{R}^{d_z} \to \mathbb{R}$ is convex-concave and L-smooth on $\mathbb{R}^{d_y}\times\mathbb{R}^{d_z}$.*

Assumption 11. *$r(z,y)$ is μ-strongly convex (first argument)–μ-strongly concave (second argument).*

Assumption 12. *$f_1(y,z),\dots,f_n(y,z)$ are δ-related: for all $i \ne j$ and for all $y \in \mathbb{R}^{d_y}$ and $z \in \mathbb{R}^{d_z}$,*

$$\|\nabla_{yy}^2 f_i(y,z) - \nabla_{yy}^2 f_j(y,z)\| \le \delta, \|\nabla_{yz}^2 f_i(y,z) - \nabla_{yz}^2 f_j(y,z)\| \le \delta, \|\nabla_{zz}^2 f_i(y,z) - \nabla_{zz}^2 f_j(y,z)\| \le \delta.$$

Casting (16) into the VI formulation (10), by taking $Q(x) = Q(y,z) = [\nabla_y f_1(y,z), -\nabla_z f_1(y,z)]$ and $P(x) = P(y,z) = [\nabla_y [r - f_1](y,z), -\nabla_z [r - f_1](y,z)]$, we have that Q is monotone and L-Lipschitz, P is δ-Lipschitz, R is μ-strongly monotone. Therefore, we can apply Theorems 9 and 10 and obtain the following convergence results for Algorithm 2 applied to (16).

Theorem 11. *Let Assumptions 10 to 12 be satisfied with $\mu \leq \delta \leq L$. Then, to find ε-solution of the distributed saddle problem (16), Algorithm 2 requires*

$$\mathcal{O}\left(\frac{\delta}{\mu}\log\frac{1}{\varepsilon}\right) \text{ communication rounds and } \mathcal{O}\left(\frac{L}{\mu}\log\frac{1}{\varepsilon}\right) \text{ local gradient computations.}$$

Theorem 12. *Let Assumptions 10, 11 (with $\mu = 0$), 12 be satisfied and $\delta \leq L$. Then, to find ε-solution of the distributed saddle problem (16), Algorithm 2 requires*

$$\mathcal{O}\left(\frac{\delta}{\varepsilon}\|x^0 - x^*\|^2\right) \text{ communication rounds and } \mathcal{O}\left(\frac{L}{\varepsilon}\|x^0 - x^*\|^2\right) \text{ local computations.}$$

Our communication estimates are optimal, as in [9], but our algorithm also achieve optimal local complexity (see Table 1).

5 Experiments

5.1 Minimization

We consider the Ridge Regression problem

$$\min_{w\in\mathbb{R}^d}\left[f(w) := \frac{1}{2N}\sum_{i=1}^{N}(w^T x_i - y_i)^2 + \frac{\lambda}{2}\|w\|^2\right], \tag{17}$$

where w is the vector of weights of the model, $\{x_i, y_i\}_{i=1}^N$ is the training dataset, and $\lambda > 0$ is the regularization parameter.

We consider a network with 25 workers (simulated on a single-CPU machine), and use two types of datasets, namely: synthetic and real data. Synthetic data permit to control the similarity constant δ. To do so, we generate data on the server, say $\{\hat{x}_i, \hat{y}_i\}_{i=1}^{n=100}$. Data on the workers are generated by adding unbiased Gaussian noise to the server data. The lower the variance of this noise, the more similar the data, and thus the smaller δ. For simulations with real data, we considered the LIBSVM library [11] and give each agent a full dataset. Then, each device selects at random a part of size m from the full dataset. In Section C.2 we explain how the parameters L and δ are estimated. For the synthetic dataset we choose the noise level and the regularization parameter such that $L/\delta = 200$ and $L/\lambda = 10^5$. For the real datasets the regularization parameter is chosen such that $L/\lambda = 10^6$. See Table 2 for all values of L, δ, μ and m.

For comparison, we use distributed version of Accelerated Gradient Descent (AcGD) [37] as the basic, classical and optimal method for minimization problems without additional similarity assumptions, as well as state-of-the-art schemes for the similarity condition: DANE [42], DANE-HB [50], SPAG [21] and AccSONATA [46]. The settings of the methods are made as described in the original papers. For algorithms that assume an absolutely accurate solution of local problems (DANE, SPAG, AccSONATA), we use AcGD with an accuracy of 10^{-12} as a subsolver.

Results are summarized in Figure 1–the first two figures from the top left correspond to synthetic data while the other 6 on real data.

The results show that our method outperforms almost all methods in terms of communication (only in one experiment SPAG is slightly faster). In terms of local iterations, our method is slightly inferior to AcGD, but superior to all other methods.

In Appendix C.1, one can find a comparison of our method with a competitor for distributed saddle point problems under similarity assumption.

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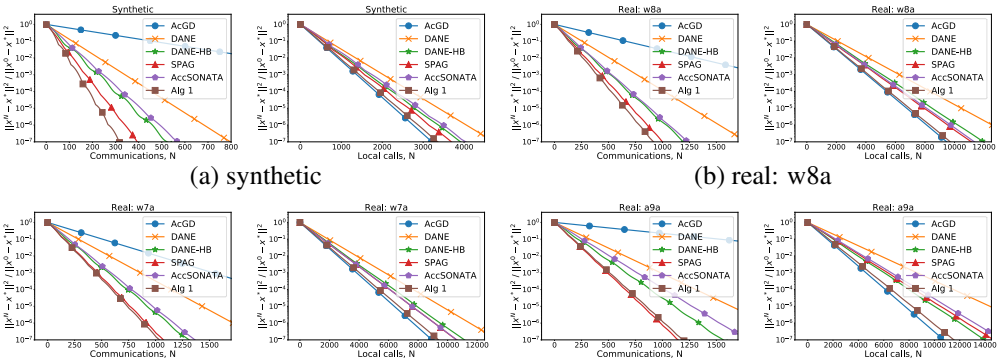

(a) synthetic (b) real: w8a

(c) real: w7a (d) real: a9a

Figure 1: Ridge regression problem (17): Comparison of state-of-the-art methods, under similarity; synthetic data (a) and real data (b,c,d). Distance from optimality vs. number of communications (first/third panel from the left in the both lines) and vs. number of local iterations (second/fourth panel from the left in the both lines).

References

[1] Artem Agafonov, Pavel Dvurechensky, Gesualdo Scutari, Alexander Gasnikov, Dmitry Kamzolov, Aleksandr Lukashevich, and Amir Daneshmand. An accelerated second-order method for distributed stochastic optimization. *arXiv preprint arXiv:2103.14392*, 2021.

[2] Masoud Ahookhosh and Yurii Nesterov. High-order methods beyond the classical complexity bounds, ii: inexact high-order proximal-point methods with segment search. *arXiv preprint arXiv:2109.12303*, 2021.

[3] Mohammad S Alkousa, Alexander Vladimirovich Gasnikov, Darina Mikhailovna Dvinskikh, Dmitry A Kovalev, and Fedor Sergeevich Stonyakin. Accelerated methods for saddle-point problem. *Computational Mathematics and Mathematical Physics*, 60(11):1787–1809, 2020.

[4] Yossi Arjevani and Ohad Shamir. Communication complexity of distributed convex learning and optimization. *Advances in neural information processing systems*, 28, 2015.

[5] Heinz Bauschke and Patrick Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. 01 2017.

[6] Amir Beck and Marc Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM journal on imaging sciences*, 2(1):183–202, 2009.

[7] R. Bekkerman, M. Bilenko, and J. Langford. *Scaling up Machine Learning: Parallel and Distributed Approaches*. Cambridge University Press, 2011.

[8] Aleksandr Beznosikov, Eduard Gorbunov, and Alexander Gasnikov. Derivative-free method for composite optimization with applications to decentralized distributed optimization. *IFAC-PapersOnLine*, 53(2):4038–4043, 2020.

[9] Aleksandr Beznosikov, Gesualdo Scutari, Alexander Rogozin, and Alexander Gasnikov. Distributed saddle-point problems under data similarity. *Advances in Neural Information Processing Systems*, 34, 2021.

[10] Ekaterina Borodich, Aleksandr Beznosikov, Abdurakhmon Sadiev, Vadim Sushko, Nikolay Savelyev, Martin Takáč, and Alexander Gasnikov. Decentralized personalized federated min-max problems. *arXiv preprint arXiv:2106.07289*, 2021.

[11] Chih-Chung Chang and Chih-Jen Lin. Libsvm: a library for support vector machines. *ACM transactions on intelligent systems and technology (TIST)*, 2(3):1–27, 2011.

[12] Amir Daneshmand, Gesualdo Scutari, Pavel Dvurechensky, and Alexander Gasnikov. Newton method over networks is fast up to the statistical precision. In *International Conference on Machine Learning*, pages 2398–2409. PMLR, 2021.

[13] John C Duchi, Shai Shalev-Shwartz, Yoram Singer, and Ambuj Tewari. Composite objective mirror descent. In *COLT*, volume 10, pages 14–26. Citeseer, 2010.

[14] Darina Dvinskikh, Sergey Omelchenko, Alexander Gasnikov, and Alexander Tyurin. Accelerated gradient sliding for minimizing a sum of functions. In *Doklady Mathematics*, volume 101, pages 244–246. Springer, 2020.

[15] F. Facchinei and J.S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Series in Operations Research and Financial Engineering. Springer New York, 2007.

[16] Francisco Facchinei and Jong-Shi Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Series in Operations Research. Springer, 2003.

[17] Alexander Gasnikov, Darina Dvinskikh, Pavel Dvurechensky, Dmitry Kamzolov, Vladislav Matyukhin, Dmitry Pasechnyuk, Nazary Tupitsa, and Alexey Chernov. Accelerated meta-algorithm for convex optimization problems. *Computational Mathematics and Mathematical Physics*, 61(1):17–28, 2021.

[18] Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. In *Advances in Neural Information Processing Systems (NIPS)*, 2014.

[19] Eduard Gorbunov, Darina Dvinskikh, and Alexander Gasnikov. Optimal decentralized distributed algorithms for stochastic convex optimization. *arXiv preprint arXiv:1911.07363*, 2019.

[20] Geovani Nunes Grapiglia and Yurii Nesterov. Adaptive third-order methods for composite convex optimization. *arXiv preprint arXiv:2202.12730*, 2022.

[21] Hadrien Hendrikx, Lin Xiao, Sebastien Bubeck, Francis Bach, and Laurent Massoulie. Statistically preconditioned accelerated gradient method for distributed optimization. In *International Conference on Machine Learning*, pages 4203–4227. PMLR, 2020.

[22] Anastasiya Ivanova, Evgeniya Vorontsova, Dmitry Pasechnyuk, Alexander Gasnikov, Pavel Dvurechensky, Darina Dvinskikh, and Alexander Tyurin. Oracle complexity separation in convex optimization. *arXiv preprint arXiv:2002.02706*, 2020.

[23] Anatoli Juditsky, Arkadi Nemirovski, and Claire Tauvel. Solving variational inequalities with stochastic mirror-prox algorithm. *Stochastic Systems*, 1(1):17–58, 2011.

[24] Peter Kairouz, H Brendan McMahan, Brendan Avent, Aurélien Bellet, Mehdi Bennis, Arjun Nitin Bhagoji, Kallista Bonawitz, Zachary Charles, Graham Cormode, Rachel Cummings, et al. Advances and open problems in federated learning. *Foundations and Trends® in Machine Learning*, 14(1–2):1–210, 2021.

[25] Dmitry Kamzolov, Alexander Gasnikov, and Pavel Dvurechensky. Optimal combination of tensor optimization methods. In *International Conference on Optimization and Applications*, pages 166–183. Springer, 2020.

[26] Donghwan Kim and Jeffrey A Fessler. Generalizing the optimized gradient method for smooth convex minimization. *SIAM Journal on Optimization*, 28(2):1920–1950, 2018.

[27] Donghwan Kim and Jeffrey A Fessler. Optimizing the efficiency of first-order methods for decreasing the gradient of smooth convex functions. *Journal of optimization theory and applications*, 188(1):192–219, 2021.

[28] Guanghui Lan. Gradient sliding for composite optimization. *Mathematical Programming*, 159(1):201–235, 2016.

[29] Guanghui Lan and Yuyuan Ouyang. Accelerated gradient sliding for structured convex optimization. *arXiv preprint arXiv:1609.04905*, 2016.

[30] Guanghui Lan and Yuyuan Ouyang. Mirror-prox sliding methods for solving a class of monotone variational inequalities. *arXiv preprint arXiv:2111.00996*, 2021.

[31] Guanghui Lan and Yi Zhou. Conditional gradient sliding for convex optimization. *SIAM Journal on Optimization*, 26(2):1379–1409, 2016.

[32] X. Lian, C. Zhang, H. Zhang, C.-J. Hsieh, W. Zhang, and J. Liu. Can decentralized algorithms outperform centralized algorithms? A case study for decentralized parallel stochastic gradient descent. In *Advances in Neural Information Processing Systems*, pages 5330–5340, 2017.

[33] Hongzhou Lin, Julien Mairal, and Zaid Harchaoui. Catalyst acceleration for first-order convex optimization: from theory to practice. *Journal of Machine Learning Research*, 18(1):7854–7907, 2018.

[34] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards deep learning models resistant to adversarial attacks. In *International Conference on Learning Representations (ICLR)*, 2018.

[35] Yu Nesterov. Gradient methods for minimizing composite functions. *Mathematical programming*, 140(1):125–161, 2013.

[36] Yurii Nesterov. Dual extrapolation and its applications to solving variational inequalities and related problems. *Mathematical Programming*, 109(2):319–344, 2007.

[37] Yurii Nesterov. *Lectures on convex optimization*, volume 137. Springer, 2018.

[38] Yurii Nesterov, Alexander Gasnikov, Sergey Guminov, and Pavel Dvurechensky. Primal–dual accelerated gradient methods with small-dimensional relaxation oracle. *Optimization Methods and Software*, pages 1–38, 2020.

[39] Yuyuan Ouyang and Yangyang Xu. Lower complexity bounds of first-order methods for convex-concave bilinear saddle-point problems. *Mathematical Programming*, 185(1):1–35, 2021.

[40] Sashank J Reddi, Jakub Konečnỳ, Peter Richtárik, Barnabás Póczós, and Alex Smola. Aide: Fast and communication efficient distributed optimization. *arXiv preprint arXiv:1608.06879*, 2016.

[41] Abdurakhmon Sadiev, Darina Dvinskikh, Aleksandr Beznosikov, and Alexander Gasnikov. Decentralized and personalized federated learning. *arXiv preprint arXiv:2107.07190*, 2021.

[42] Ohad Shamir, Nati Srebro, and Tong Zhang. Communication-efficient distributed optimization using an approximate newton-type method. In Eric P. Xing and Tony Jebara, editors, *Proceedings of the 31st International Conference on Machine Learning*, volume 32 of *Proceedings of Machine Learning Research*, pages 1000–1008, Bejing, China, 22–24 Jun 2014. PMLR.

[43] Ivan Stepanov, Artyom Voronov, Aleksandr Beznosikov, and Alexander Gasnikov. One-point gradient-free methods for composite optimization with applications to distributed optimization. *arXiv preprint arXiv:2107.05951*, 2021.

[44] Sebastian U Stich. Local sgd converges fast and communicates little. *arXiv preprint arXiv:1805.09767*, 2018.

[45] Ying Sun, Amir Daneshmand, and Gesualdo Scutari. Distributed optimization based on gradient-tracking revisited: Enhancing convergence rate via surrogation. *arXiv preprint arXiv:1905.02637*, 2019.

[46] Ye Tian, Gesualdo Scutari, Tianyu Cao, and Alexander Gasnikov. Acceleration in distributed optimization under similarity. *arXiv preprint arXiv:2110.12347*, 2021.

[47] Vladislav Tominin, Yaroslav Tominin, Ekaterina Borodich, Dmitry Kovalev, Alexander Gasnikov, and Pavel Dvurechensky. On accelerated methods for saddle-point problems with composite structure. *arXiv preprint arXiv:2103.09344*, 2021.

[48] Blake Woodworth, Kumar Kshitij Patel, Sebastian Stich, Zhen Dai, Brian Bullins, Brendan Mcmahan, Ohad Shamir, and Nathan Srebro. Is local sgd better than minibatch sgd? In *International Conference on Machine Learning*, pages 10334–10343. PMLR, 2020.

[49] TaeHo Yoon and Ernest K Ryu. Accelerated algorithms for smooth convex-concave minimax problems with o (1/kˆ 2) rate on squared gradient norm. In *International Conference on Machine Learning*, pages 12098–12109. PMLR, 2021.

[50] Xiao-Tong Yuan and Ping Li. On convergence of distributed approximate newton methods: Globalization, sharper bounds and beyond. *arXiv preprint arXiv:1908.02246*, 2019.

[51] Yuchen Zhang and Xiao Lin. Disco: Distributed optimization for self-concordant empirical loss. In Francis Bach and David Blei, editors, *Proceedings of the 32nd International Conference on Machine Learning*, volume 37 of *Proceedings of Machine Learning Research*, pages 362–370, Lille, France, 07–09 Jul 2015. PMLR.

Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] , theoretical paper
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [N/A] , we work with small models and simple optimisers, the experiments are easy to reproduce on any PC, but we will try to make the code user friendly and put it out in the public domain
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] , Section 5 gives information about the problem setup, all methods are tuned how it is done in the original papers
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A] , methods are deterministic
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] , Appendix C, but we make small experiments, they can be run on any PC
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] , Section 5.1, cite LIBSVM
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   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
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5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]